

## COVERING MOVES

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**ABSTRACT.** In this paper we give a positive answer to a long standing question posed by Montesinos, by introducing new covering moves, in order to relate any two colored diagrams representing the same 3-manifold as simple branched 3-covering of  $S^3$ .

### INTRODUCTION

By a well-known theorem of Hilden and Montesinos ([6] and [7]), every closed 3-manifold is a simple 3-covering of  $S^3$  branched over a link.

Following R. Fox (cf. [5]), such a covering can be described by coloring each bridge of a planar diagram of the branch link with  $R$  = red,  $G$  = green or  $B$  = blue, according to whether the monodromy of the corresponding meridian is the transposition (12), (23) or (13) of  $S_3$ .

Then we can use colored diagrams of links to represent 3-manifolds, and, of course, the natural question which immediately arises is to determine what different diagrams represent the same manifold.

Montesinos long ago posed the problem, more or less, in the following way:

*Find a set of moves on colored link diagrams, such that any two diagrams represent the same manifold iff they are related (up to colored Reidemeister moves), by a finite sequence of such moves.*

The local move I, described in Figure 1, was considered a possible answer for a long time (cf. [9]), up to when Montesinos found a counterexample (cf. [8]).

In this work we give a solution to the problem (equivalence theorem of §1), by providing three new moves (see Figure 7), which together with the previous one form a complete set of moves as required above.

That is obtained by normalizing the branch links as colored braids, corresponding to Heegaard splittings of the covering manifolds (cf. [1]), in such a way that we can apply results of [10] and [2], respectively in order to realize stable equivalence of splittings and to relate different braids representing the same splitting homeomorphism.

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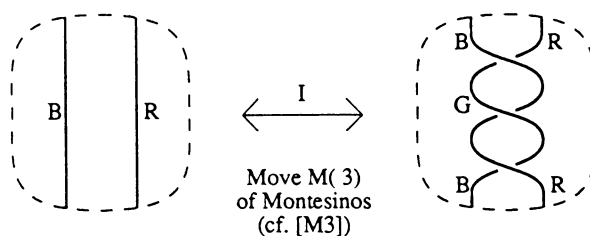


FIGURE 1

An application of our moves will be given in a forthcoming paper, in which we prove that any 4-manifold can be represented as a 4-fold branched covering of  $S^4$ .

## 0. PRELIMINARIES

By an  $n$ -fold *branched covering* we mean a nondegenerate pl map  $p: C \rightarrow X$  between compact pl manifolds, for which there exist subpolyhedra  $S \subset C$  and  $B \subset X$  (the *branch set*) of codimension 2, such that  $p|_{C-S}: C - S \rightarrow X - B$  is an ordinary  $n$ -fold covering.

If  $B$  is a locally flat submanifold of  $X$ , then any  $n$ -fold ordinary covering of  $X - B$  uniquely extends to an  $n$ -fold branched covering of  $X$ . Thus,  $n$ -fold branched coverings of  $X$  with branch set  $B$  are in 1-1 correspondence with the representations of  $\pi_1(X - B)$  into  $S_n$ , the symmetric group of degree  $n$ .

For a branch link  $L \subset S^3$ , any such representation can be given by considering a planar diagram of  $L$ , and associating to each bridge of the diagram a permutation representing the corresponding Wirtinger generator of  $\pi_1(S^3 - L)$ . Moreover, in the case of a 3-fold branched covering, which is *simple* (that is all the Wirtinger generators are represented by transpositions), we can associate to each bridge of the diagram the color  $R = \text{red}$ ,  $G = \text{green}$  or  $B = \text{blue}$ , instead of the transposition  $(12)$ ,  $(23)$  or  $(13)$ .

A link diagram whose bridges are colored in this way is called a *colored diagram*. Of course, the coloration of a colored diagram is not completely arbitrary, because of the Wirtinger relations. By directly substituting in any order the transpositions  $(12)$ ,  $(23)$ ,  $(13)$  for  $x, y, z$  in the typical form  $x = yz\bar{y}$  of Wirtinger relators, we see that a coloration of a link diagram gives a colored diagram if and only if the colors of the three local arcs meeting at any overcrossing point are all the same or all distinct.

We define *colored Reidemeister moves*, by recoloring the part of the diagram modified by ordinary Reidemeister moves in the unique way compatible with the 'all the same all distinct' rule. Then, a *colored isotopy* is given as a sequence of colored Reidemeister moves.

By a *normalized diagram* we mean a colored plat (= braid closed at the top and the bottom by simple arcs) diagram, such that all the top and bottom arcs are red except for the leftmost ones which are blue (cf. Figure 2).

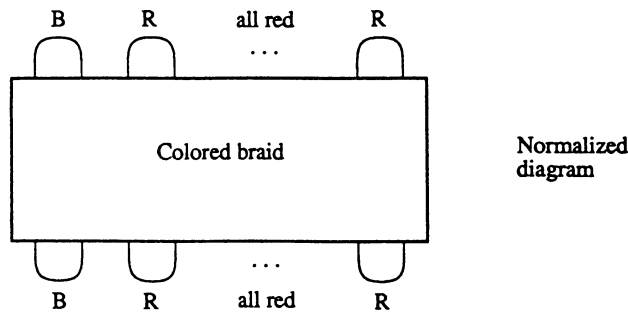


FIGURE 2

Recall that an  $m$ -braid on  $S^2$  is an element of the loop space  $\Lambda(P_m S^2, *)$ , where  $P_m S^2$  denotes the set of all subsets of  $S^2$  of order  $m$ , and  $*$  is any such subset. Two braids are said to be *equivalent* if they are homotopy equivalent as based loops, that is if they represent the same element of the group  $B(m) = \pi_1(P_m S^2, *)$ , which is called the *braid group* of degree  $m$  of  $S^2$ .

A braid  $b \in \Lambda(P_m S^2, *)$  can also be represented geometrically in  $S^2 \times [0, 1]$ , by considering the union of all the sets  $b(t) \times \{t\}$ , with  $t \in [0, 1]$ . Then, two braids are equivalent if and only if their geometric representations are isotopically equivalent in  $S^2 \times [0, 1]$  by means of an ambient isotopy fixing the boundary.

Classical braids in  $R^3$  between planes  $z = z_0$  and  $z = z_1$ , such as the one in Figure 2, can be seen as geometric representations of braids on  $S^2$ , considering that  $R^2 \times [z_0, z_1] \cong R^2 \times [0, 1] \subset S^2 \times [0, 1]$ , the inclusion being induced by the one-point compactification of  $R^2$ .

Given any surface  $F$ , we denote by  $M(F)$  the *mapping class group* of  $F$ , that is the group of all isotopy classes of homeomorphisms of  $F$  onto itself.  $M(g)$  will be used instead of  $M(F_g)$ , where  $F_g \subset R^3$  is the closed surface of genus  $g$  shown in Figure 4.

Now, we say that an ambient isotopy  $\{h_t\}_{0 \leq t \leq 1}$  of  $S^2$  *realizes* a braid  $b$  if  $h_t(*) = b(t)$ , for every  $0 \leq t \leq 1$ . It is a classical result (cf. [3]), that any braid can be realized by an (obviously not unique) ambient isotopy of  $S^2$ , and that the correspondence  $b \rightarrow h_{1|S^2-*}$ , induces a group epimorphism

$$\eta: B(g) \rightarrow M(S^2 - *).$$

Any normalized colored diagram induces a Heegaard splitting  $M = H_1 \cup_h H_2$  of the covering manifold, as sketched in Figure 3.

In order to describe the splitting, we observe that the handlebodies  $H_1$  and  $H_2$  cover (by restriction) the two 3-cells  $B_1$  and  $B_2$  in a standard way, which depends only on the genus of the splitting.

Therefore we can identify  $H_1$  and  $H_2$  with a fixed handlebody  $T_g \subset R^3$ , for all the splittings of genus  $g$ , induced by normalized diagrams. The covering after this identification is described in Figure 4.

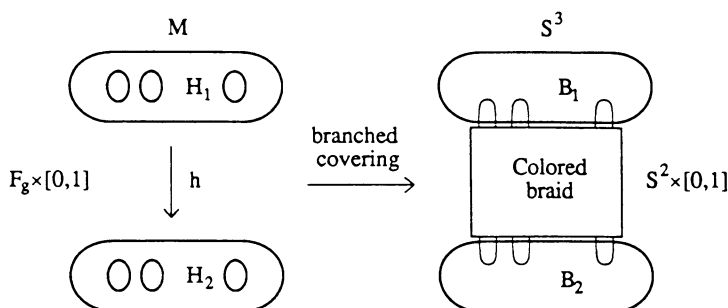


FIGURE 3

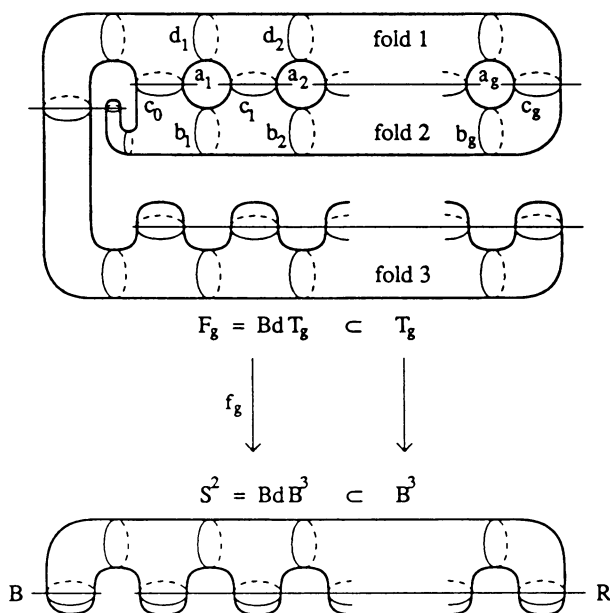


FIGURE 4

Let  $f_g: F_g \rightarrow S^2$  be the restriction of such a covering to the boundary. Then the splitting homeomorphism  $h: \text{Bd } H_1 \rightarrow \text{Bd } H_2$  is (up to isotopy) the lifting with respect to  $f_g$  of the homeomorphism  $h_1$  of any isotopy which realizes the  $(2g+4)$ -braid of the diagram.

We remark that this construction cannot be applied to any  $(2g+4)$ -braid  $b$ . In fact  $h_1$  is liftable with respect to  $f_g$  if and only if  $b$  is colorable as the braid of a normalized diagram. In this case we say that  $b$  is *liftable*. By using colored isotopy, we see that liftability is invariant under equivalence of braids.

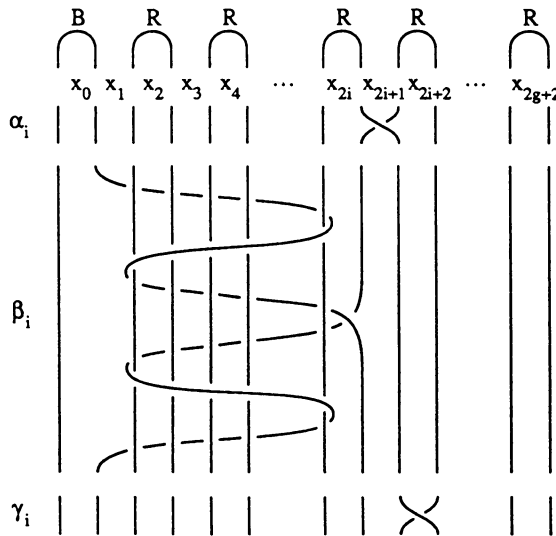


FIGURE 5

Then we denote by  $L(2g+4) \subset B(2g+4)$  the *liftable braid group* consisting of all liftable equivalence classes of braids.

Moreover, since lifting can be obviously defined in terms of the epimorphism  $\eta$ , we have a lifting group homomorphism

$$\lambda: L(2g+4) \rightarrow M(g).$$

It turns out (cf. [1] or [4]) that any Heegaard splitting (hence any 3-manifold) can be represented in the above way, by means of a normalized diagram. This follows immediately from the fact that the homomorphism  $\lambda$  is onto. In Figure 5, are shown braids representing (= lifting to) the left-handed Dehn twists  $\alpha_i$ ,  $\beta_i$  and  $\gamma_i$  about the loops  $a_i$ ,  $b_i$  and  $c_i$  drawn in Figure 4, which, by a well-known theorem of Lickorish (cf. [3]), generate the mapping class group  $M(g)$ .

Namely, if  $x_0, \dots, x_{2g+2}$  are the standard generators for the braid group  $B(2g+4)$  (cf. Figure 5), and  $[x]y$  denote the braid  $y^{-1}xy$  then:

$\alpha_i$  is represented by  $x_{2i+1}$ ;

$\beta_i$  is represented by  $[x_{2i}](x_{2i-1} \cdots x_2 x_1^2 x_2 \cdots x_{2i-2} x_{2i-1}^2 x_{2i-2} \cdots x_1)$ , in fact this braid corresponds to the halftwist about the arc  $p_i \subset S^2$  shown in Figure 6, whose lifting is, up to isotopy,  $\beta_i$  (cf. [2] for  $\beta_2$ );

$\gamma_i$  is represented by  $x_{2i-2}$ .

We warn the reader that, in drawing colored diagrams, we usually omit superfluous color labels; they can be deduced, by the 'all the same or all distinct' rule.

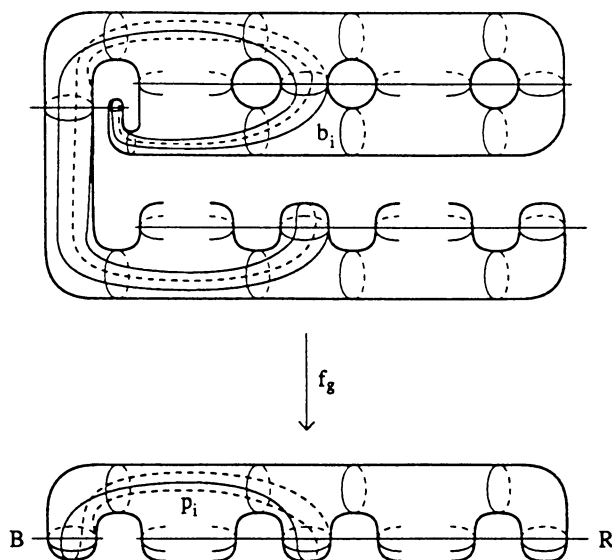


FIGURE 6

### 1. THE NEW MOVES AND THE EQUIVALENCE THEOREM

Let us begin this section by describing the new moves we introduced, in order to relate colored diagrams representing the same manifold. We do that in Figure 7, where all the diagrams consist of a colored braid (trivial on the left side) joining two arbitrary colored links  $L$  and  $L'$ .

We remark that such moves, in contrast with the Montesinos move, are not local. In fact they involve the global structure of the diagram. (In Figure 7 all the diagrams are completely drawn.)

The fact that they do not change the covering manifold (up to homeomorphism) will be proved in §3, by showing that all the braids on the right side of Figure 7 are equivalent to braids which represent the identity in  $M(g)$ .

Finally, we note that the new moves, together with the Montesinos move, might be not completely independent. In fact we conjecture that this is the case, but at present we have no results in this direction.

Now, we can state the main result of our work:

**The Equivalence Theorem.** *Two colored link diagrams represent the same manifold iff they can be related (up to colored Reidemeister moves) by a finite sequence of moves of the types I, ..., IV (described in Figures 1 and 7).*

We prove the theorem in three steps: (1) normalization of any colored diagram by means of colored isotopy, or what is the same thing, colored Reidemeister moves; (2) realization of stable equivalence of Heegaard splittings

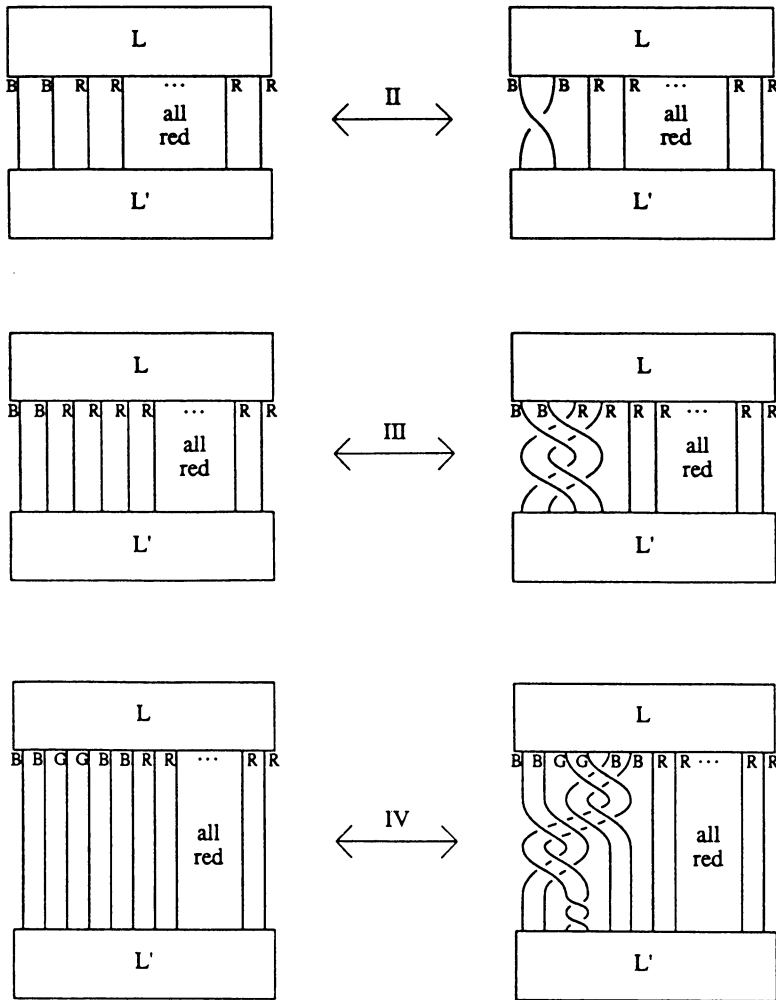


FIGURE 7

represented by normalized diagrams, by using colored isotopy and move I; (3) connection of normalized diagrams representing the same splitting homeomorphism.

The first step is quite elementary and we conclude this section by proving it. The other two steps are more complicated and will be carried out respectively in §2 and §3.

In order to normalize a colored link diagram, we can assume that the diagram is a plat, since any link has a plat presentation (cf. [4]).

Then we need only to get the right colors for the top and bottom arcs of the plat. This can be done separately for the top and the bottom, by using the

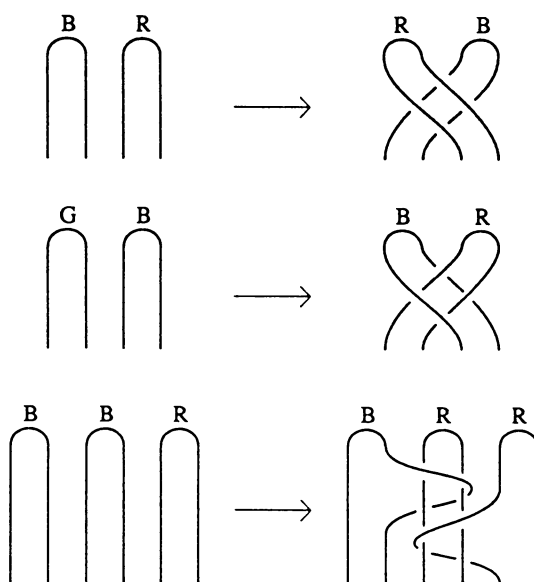


FIGURE 8

local isotopies shown in Figure 8, if we assume (as we can easily do) that all the colors are present there. The first kind of isotopy allows us to reorder the colors, the second one can be used for eliminating the green arcs, and the third one makes it possible to eliminate all the blue arcs but the leftmost.

## 2. REALIZING STABLE EQUIVALENCE OF HEEGAARD SPLITTINGS

In this section we prove that any two normalized diagrams representing the same manifold, can be modified by using colored isotopy and move I, in such a way that they induce the same splitting homeomorphism.

Let  $D$  and  $D'$  be two normalized diagrams representing the same 3-manifold  $M$ , then the corresponding Heegaard splittings  $M = H_1 \cup_h H_2$  and  $M = H'_1 \cup_{h'} H'_2$  are stably equivalent (cf. [11]). That is, there exist stabilizations:

$$H''_1 \cup_{\tilde{h}} H''_2 = (H_1 \cup_h H_2) \# (\#_n(T \cup T)),$$

and

$$H''_1 \cup_{\tilde{h}'} H''_2 = (H'_1 \cup_{h'} H'_2) \# (\#_{n'}(T \cup T)),$$

of them, where  $(T \cup T)$  denote the genus 1 splitting of  $S^3$ , and there exists a commutative diagram as follows:

$$\begin{array}{ccc} H''_1 \supset \text{Bd } H''_1 & \xrightarrow{\tilde{h}} & \text{Bd } H''_2 \subset H''_2 \\ \downarrow & \downarrow h_1 & \downarrow h_2 \quad \downarrow \\ H''_1 \supset \text{Bd } H''_1 & \xrightarrow{\tilde{h}'} & \text{Bd } H''_2 \subset H''_2 \end{array}$$

where all the vertical arrows are homeomorphisms. (This means that the two homeomorphisms  $h_1$  and  $h_2$  between the boundaries of the handlebodies extend to homeomorphisms between the handlebodies.)



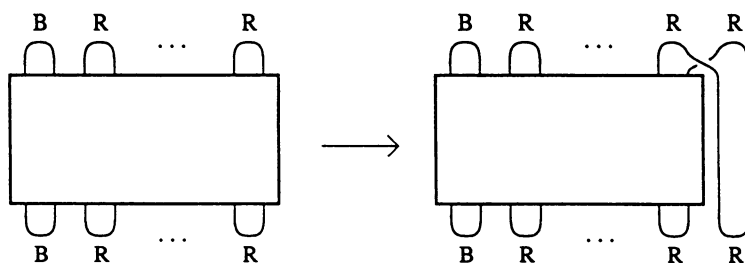


FIGURE 9

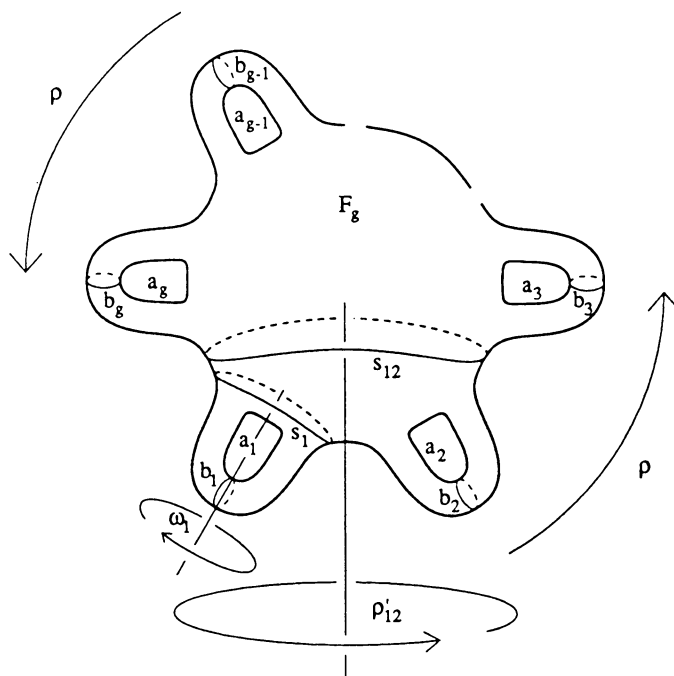


FIGURE 10

First of all, we observe that stabilization can be represented in terms of normalized diagrams, by iterating the modification described in Figure 9 (which corresponds to making a connected sum with a copy of  $T \cup T$ ).

Then, by means of colored isotopy, we can change  $D$  and  $D'$  into normalized diagrams  $\tilde{D}$  and  $\tilde{D}'$  representing the stabilized splittings  $M = H_1'' \cup_{\tilde{h}} H_2''$  and  $M = H_1'' \cup_{\tilde{h}'} H_2''$ .

Now, if we add on top and on bottom of  $\tilde{D}$  two colored braids, respectively lifting to the homeomorphisms  $h_1^{-1}$  and  $h_2$ , we get a new normalized diagram inducing the same splitting homeomorphism  $\tilde{h}'$  induced by  $\tilde{D}'$ .

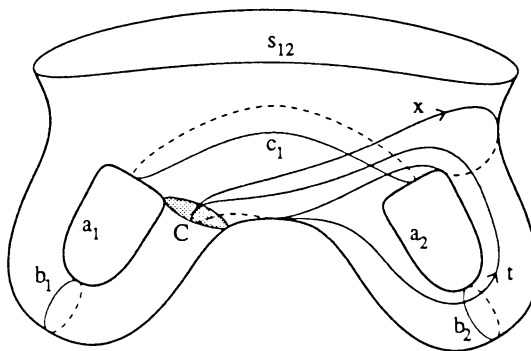


FIGURE 11

The rest of this section is entirely devoted to show how to add these braids, by means of the prescribed moves.

Denote by  $M^*(g)$  the subgroup of  $M(g)$  consisting of all isotopy classes of the homeomorphisms of  $F_g$  onto itself, which extend to homeomorphisms of the handlebody  $T_g$  onto itself (cf. [10]).

We will prove that any element of  $M^*(g)$  is represented by a colored braid which can be added on top (and hence on bottom) of a normalized diagram by using only colored isotopy and move I.

In [10], Suzuki proved that  $M^*(g)$  is generated by the isotopy classes of the following homeomorphisms (we drop the dots in Suzuki's notation):

$$\rho, \rho_{12}, \omega_1, \tau_1, \theta_{12}, \text{ and } \xi_{12},$$

where  $\rho$  is the cyclic permutation, by rotation of  $2\pi/g$  radians, of the  $g$  knobs of  $F_g$  embedded in  $R^3$  as in Figure 10, and moreover, by looking at Figures 10 and 11, and denoting by  $\sigma_i$  and  $\sigma_{i,i+1}$  the Dehn twists about the loops  $s_i = \rho^{i-1}(s_1)$  and  $s_{i,i+1} = \rho^{i-1}(s_{12})$ :

- $\tau_i$  is the right-handed twist of the  $i$ th handle, that is  $\beta_i^{-1}$ ;
- $\omega_i$  is the half twist of the  $i$ th knob, whose square is  $\sigma_i^{-1}$ ;
- $\rho_{i,i+1} = \rho'_{i,i+1} \omega_i \omega_{i+1}$  where  $\rho'_{i,i+1}$  interchanges the  $i$ th and  $(i+1)$ th knobs, by means of a half twist whose square is  $\sigma_{i,i+1}$ ;
- $\theta_{12}$  is (modulo  $\beta_1$ ) the sliding of the attaching loop  $\text{Bd } C$  of the first handle (to  $S^2$ ) about the oriented simple loop  $t \subset F_g \cup C$ ;
- $\xi_{12}$  is (modulo  $\beta_1$ ) the sliding of the same attaching loop about the oriented simple loop  $x \subset F_g \cup C$ .

Let us write all these homeomorphisms in terms of the standard generators of  $M(g)$ .

First of all, since the loop  $x$  of Figure 11 is (up to isotopy in  $F_g \cup C$  modulo  $C$ ) the core of the annulus  $A \subset F_g \cup C$  bounded by  $\omega_2^{-1}(c_1)$  on the left side

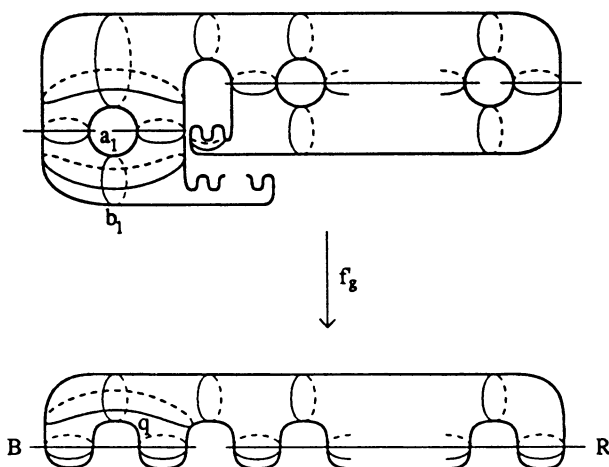


FIGURE 12

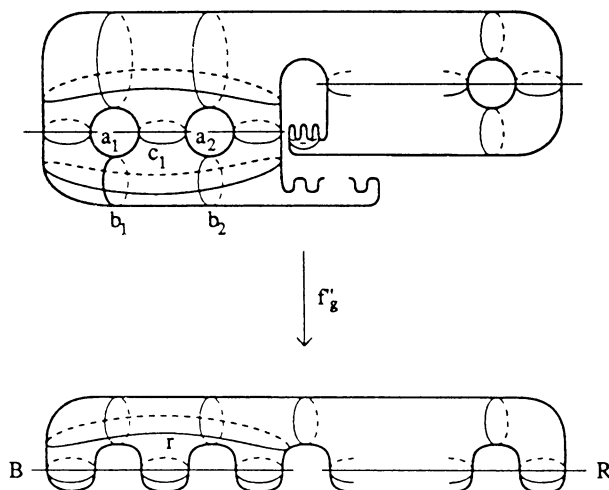


FIGURE 13

(with respect to the orientation of  $x$ ) and by  $\omega_2^{-1}(b_2)$  on the right side, we easily get the sliding  $\omega_2\gamma_1\beta_2^{-1}\omega_2^{-1}$  for  $\xi_{12}$ .

Analogously, by considering the loop  $t$  and the annulus  $A' \subset F_g \cup C$  bounded by  $(\alpha_2\beta_2)(b_2)$  on the left side and by  $(\alpha_2\beta_2)(c_1)$  on the right side, we get the sliding  $\beta_2^{-1}\alpha_2^{-1}\beta_2\gamma_1^{-1}\alpha_2\beta_2$  for  $\theta_{12}$ .

Moreover, by looking at the branched covering  $f'_g: F_g \rightarrow S^2$  of Figure 12, we see that  $\omega_1$  is the lifting of a half twist of the disk containing the blue branch points and bounded by the simple loop  $q$ , and it can be represented by the (blue) braid  $(x_0x_1x_2x_0x_1x_0)^{-1}$ , which obviously also represent  $(\alpha_1\beta_1)^{-3} = (\alpha_1^2\beta_1\alpha_1^2\beta_1)^{-1}$ ; so we have  $\omega_1 = (\alpha_1^2\beta_1\alpha_1^2\beta_1)^{-1}$ .

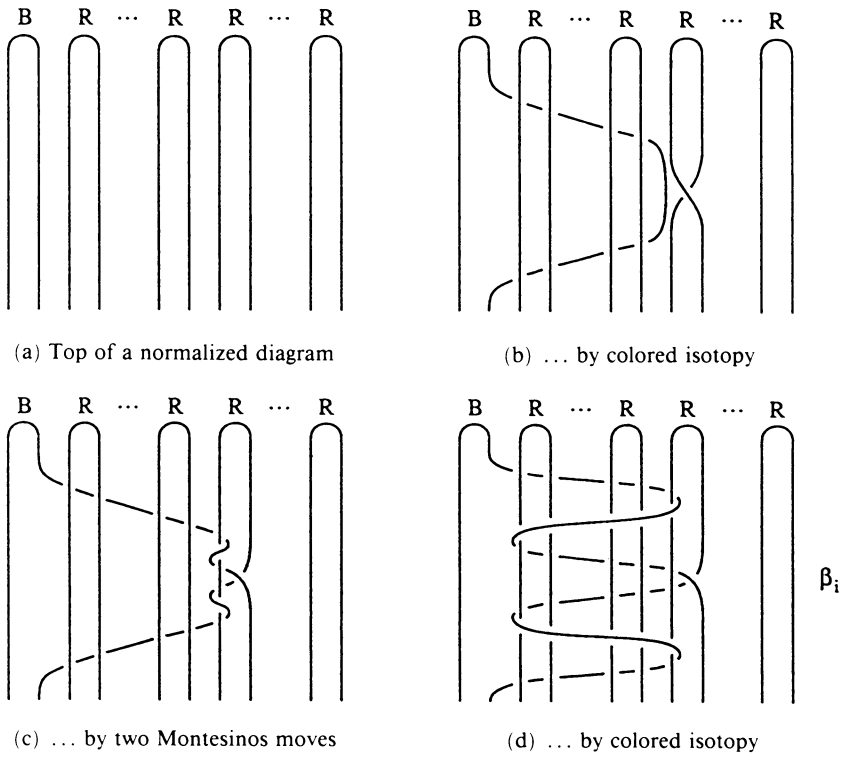


FIGURE 14

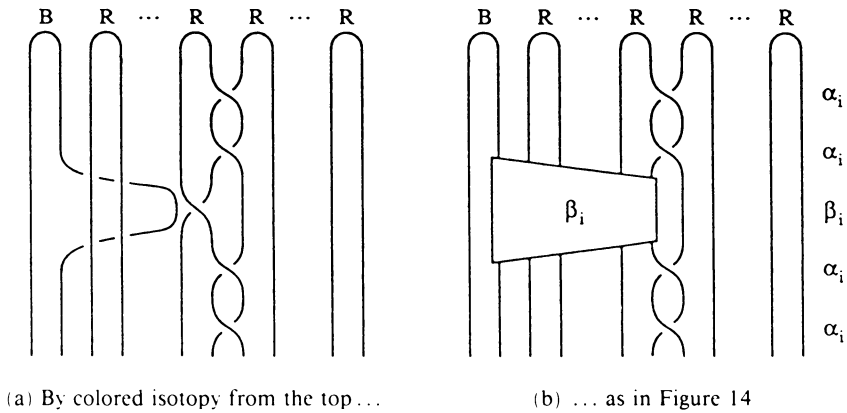
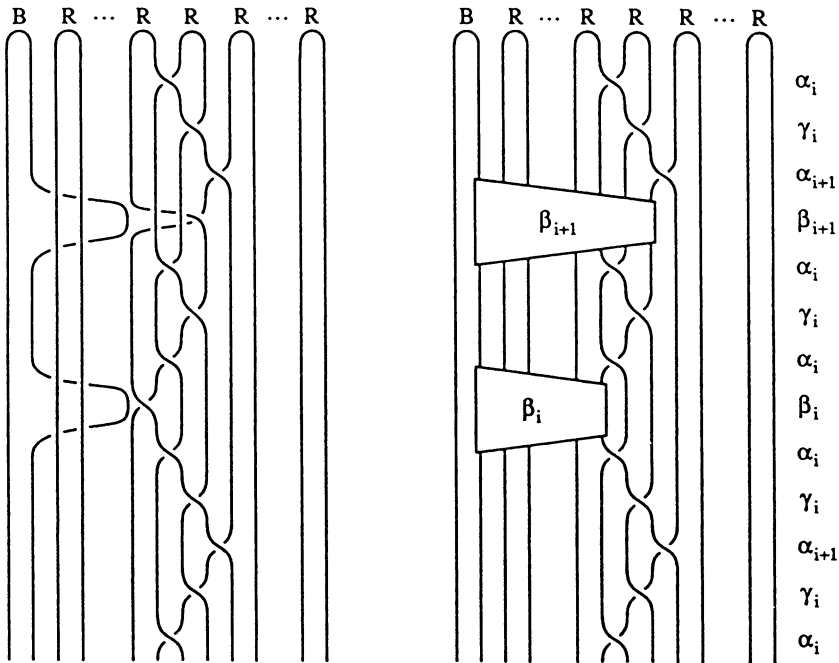


FIGURE 15

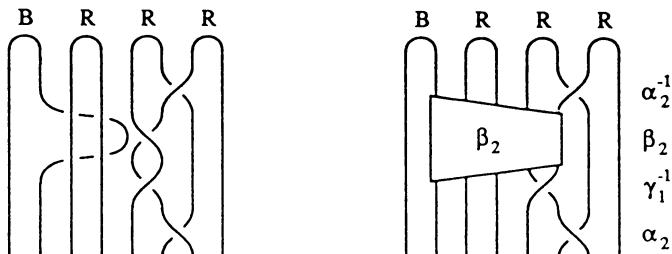
In the same way, by looking at the branched covering  $f_g'': F_g \rightarrow S^2$  of Figure 13, we see that  $\rho'_{12}$  is the lifting of a half twist of the disk containing the blue branch points and bounded by the simple loop  $r$ , and we have (by using standard identities for getting a more convenient form):

$$\begin{aligned}
 \rho'_{12} &= \beta_1 \alpha_1 \gamma_1 \alpha_2 \beta_2 \beta_1 \alpha_1 \gamma_1 \alpha_2 \beta_1 \alpha_1 \gamma_1 \beta_1 \alpha_1 \beta_1 \\
 &= \beta_1 \alpha_1 \gamma_1 \alpha_2 \beta_2 \beta_1 \alpha_1 \beta_1 \gamma_1 \alpha_1 \alpha_2 \gamma_1 \beta_1 \alpha_1 \beta_1 \\
 &= \beta_1 \alpha_1 \gamma_1 \alpha_2 \beta_2 \alpha_1 \beta_1 \alpha_1 \gamma_1 \alpha_1 \alpha_2 \gamma_1 \beta_1 \alpha_1 \beta_1 \\
 &= \beta_1 \alpha_1 \gamma_1 \alpha_2 \beta_2 \alpha_1 \beta_1 \gamma_1 \alpha_1 \gamma_1 \alpha_2 \gamma_1 \beta_1 \alpha_1 \beta_1 \\
 &= \beta_1 \alpha_1 \gamma_1 \alpha_2 \beta_2 \alpha_1 \gamma_1 \beta_1 \alpha_1 \beta_1 \gamma_1 \alpha_2 \gamma_1 \alpha_1 \beta_1 \\
 &= \beta_1 \alpha_1 \gamma_1 \alpha_2 \beta_2 \alpha_1 \gamma_1 \alpha_1 \beta_1 \alpha_1 \gamma_1 \alpha_2 \gamma_1 \alpha_1 \beta_1.
 \end{aligned}$$



(a) By colored isotopy from the top ... (b) ... as in Figure 14 (+ isotopy for  $\beta_{i+1}$ )

FIGURE 16



(a) By colored isotopy from the top ... (b) ... as in Figure 14

FIGURE 17

Finally, a straightforward computation of the actions on  $\pi_1(F_g)$ , shows that  $\rho = \rho_{g-1,g} \rho_{g-2,g-1} \cdots \rho_{12} \omega_1^2$ ; in fact, by referring to the notation of [10], we have that the induced homomorphisms coincide, up to conjugation by the commutator  $s_1$ .

So, we have the following identities:

$$\begin{aligned} \tau_i &= \beta_i^{-1}, & \omega_i &= (\alpha_i^2 \beta_i \alpha_i^2 \beta_i)^{-1}, & \rho &= \rho_{g-1,g} \rho_{g-2,g-1} \cdots \rho_{12} \omega_1^2, \\ \rho_{i,i+1} &= \beta_i \alpha_i \gamma_i \alpha_{i+1} \beta_{i+1} \alpha_i \gamma_i \alpha_i \beta_i \alpha_i \gamma_i \alpha_{i+1} \gamma_i \alpha_i \beta_i \text{ (modulo } \omega_i \omega_{i+1}), \\ \xi_{12} &= \omega_2 \gamma_1 \beta_2^{-1} \omega_2^{-1} \text{ and } \theta_{12} = \beta_2^{-1} \alpha_2^{-1} \beta_2 \gamma_1^{-1} \alpha_2 \beta_2 \text{ (modulo } \beta_1), \end{aligned}$$

the second and the fourth of which, are obtained by conjugating by  $\rho^{i-1}$  the analogous ones given above for  $i = 1$ .

Hence, it is clear that we need only show how to add on top of a normalized diagram colored braids representing the homeomorphisms:

$$\begin{aligned} &\beta_i \text{ and } \alpha_i^2 \beta_i \alpha_i^2 \text{ for } i = 1, \dots, g, \\ &\alpha_i \gamma_i \alpha_{i+1} \beta_{i+1} \alpha_i \gamma_i \alpha_i \beta_i \alpha_i \gamma_i \alpha_{i+1} \gamma_i \alpha_i \text{ for } i = 1, \dots, g-1, \\ &\gamma_1 \text{ and } \alpha_2^{-1} \beta_2 \gamma_1^{-1} \alpha_2. \end{aligned}$$

We do that (except for the trivial case of  $\gamma_1$ ) in Figures 14 to 17, which conclude this section.

### 3. REPRESENTING THE IDENTITY SPLITTING HOMEOMORPHISM

In this section we prove that the new moves introduced in Figure 7 do not change the covering manifold (up to homeomorphism), and then we use them to relate normalized diagrams inducing the same splitting homeomorphism.

Considering the homomorphism  $\lambda: L(2g+4) \rightarrow M(g)$  defined in §0, we have that two normalized diagrams induce the same splitting homeomorphism if and only if the corresponding braids differ by an element of  $\ker \lambda$ .

Then the two aims of this section can be achieved, by proving that: (1) all the braids on right side of Figure 7 are (up to colored isotopy and Montesinos moves) in  $\ker \lambda$ ; (2) elements of  $\ker \lambda$  can be added on top of a normalized diagram by means of moves I–IV.

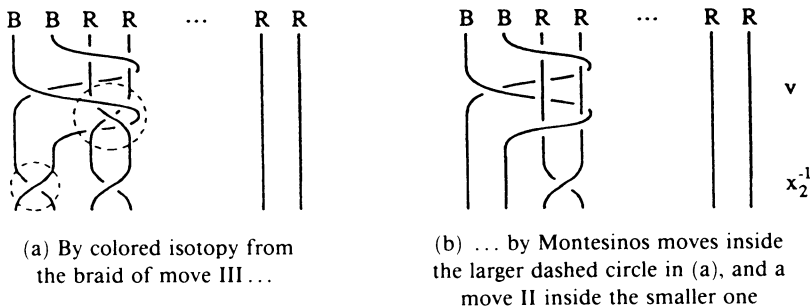
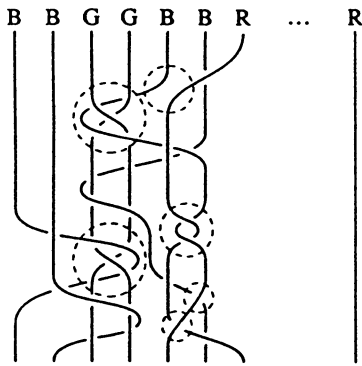
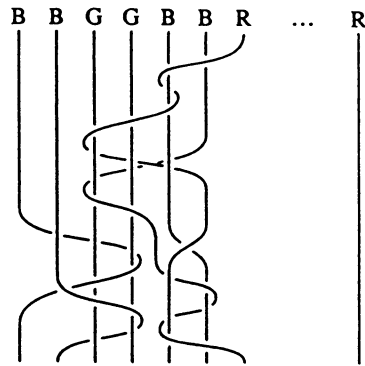


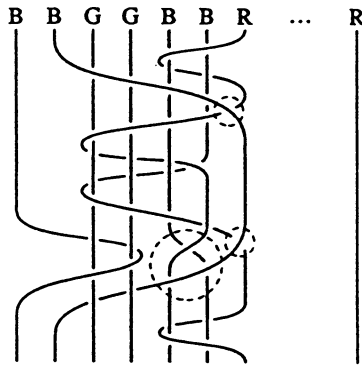
FIGURE 18



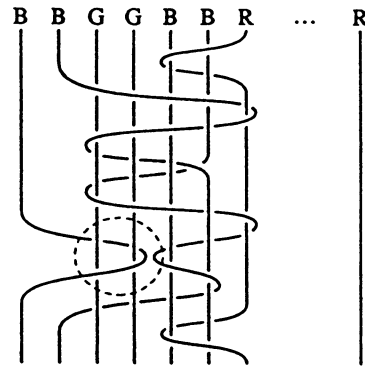
(a) By colored isotopy from the braid of move IV...



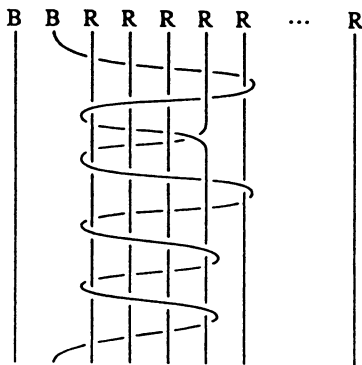
(b) ... by Montesinos moves inside dashed circles in (a)



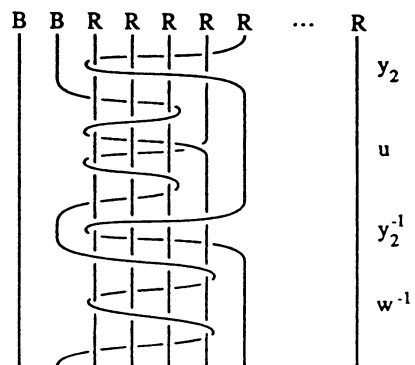
(c) ... by colored isotopy



(d) ... by Montesinos moves inside dashed circles in (c)

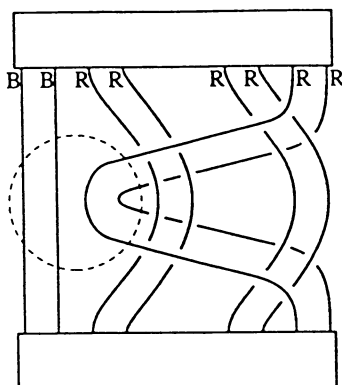


(e) ... by conjugation, and a move III inside the dashed circle in (d)

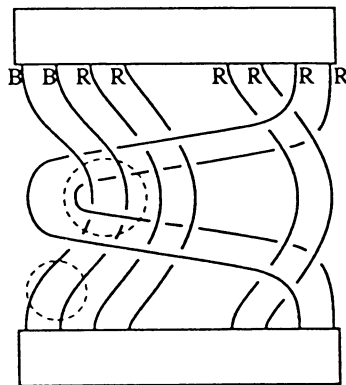


(f) ... by colored isotopy

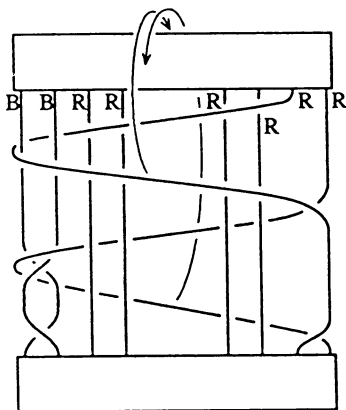
FIGURE 19



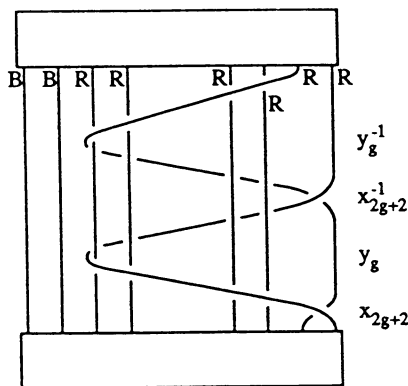
(a) By colored isotopy from the trivial braid ...



(b) ... by a move III inside the dashed circle in (a)



(c) ... by Montesinos moves inside the large circle in (b), and a move II inside the small one



(d) ... by colored isotopy as shown in (c)

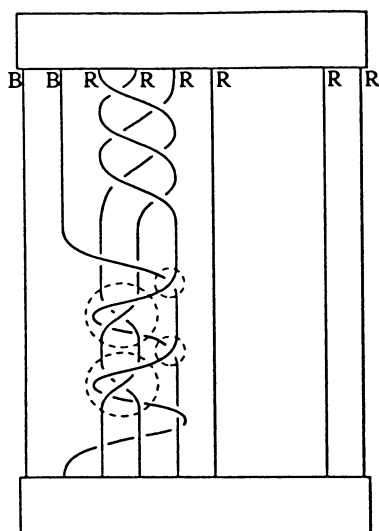
FIGURE 20

The first fact is well known for the braid  $x_0$  corresponding to move II (cf. [1] or [4]). Moreover, it is easy to prove for the braid corresponding to move III, since this braid is equivalent, as shown in Figure 18, to the braid  $v x_2^{-1}$ , where  $v = [x_0](x_1 x_2^2 x_1)^{-1}$  represents the Dehn twist  $\delta_1$  about the loop  $d_1$  drawn in Figure 4 (cf. [1], where  $\delta_1$  is denoted by  $t(b_1)$ ).

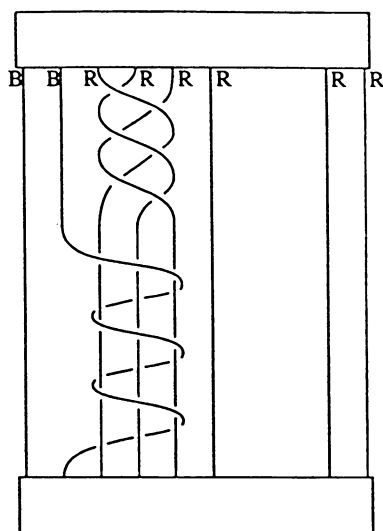
Finally, in Figure 19, we show how to modify the braid corresponding to move IV, in order to get the braid  $[u] y_2^{-1} w^{-1}$ , where  $u = [x_4](x_3 x_2 x_1^2 x_2 x_3^2 x_2 x_1)$ ,  $y_2 = x_5 x_4 x_3 x_2^2 x_3 x_4 x_5$ , and  $w = (x_1 x_2 x_3 x_4^2 x_3 x_2 x_1)^{-1}$ .

This last braid belongs to  $\ker \lambda$ , since  $w$  represents the Dehn twist  $\delta_2$  about the loop  $d_2$  drawn in Figure 4 (cf. [4], where  $\delta_2$  is denoted by  $\beta_2$ ),  $u$  represents

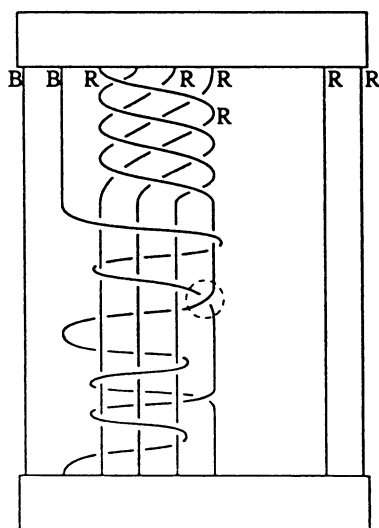




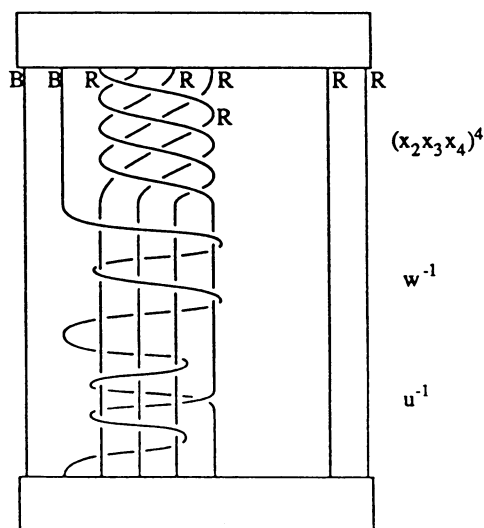
(a) By colored isotopy from the trivial braid...



(b) ... by Montesinos moves inside the dashed circles in (a)



(c) ... by colored isotopy



(d) ... by Montesinos moves inside the dashed circle in (c)

FIGURE 21

$\beta_2$  (cf. §0), and the homeomorphism  $\alpha_2 \gamma_1 \alpha_1 \gamma_0^2 \alpha_1 \gamma_1 \alpha_2$ , given by  $\gamma_2$ , makes  $b_2$  into  $d_2$  and vice versa.

In order to prove the second fact, let us consider the characterization of  $\ker \lambda$  (given in [2], where the homomorphism  $\lambda$  is denoted by  $\tilde{\lambda}$ ) as the smallest normal subgroup of  $L(2g+4)$  containing the following elements:

$$x_0, \quad x_1^3, \quad B' = (x_2 x_3 x_4)^4 [u^{-1}] y_2^{-1} u^{-1}, \quad y_g^{-1} x_{2g+2}^{-1} y_g x_{2g+2}$$

where  $y_g = x_{2g+1} \cdots x_3 x_2^2 x_3 \cdots x_{2g+1}$ .

Then, we are reduced to showing how to insert such four braids in a normalized diagram (that is something more than we really need, so this is a point where our work could be possibly improved).

For  $x_0$  and  $x_1^3$  we do not need to prove anything, since they can be obviously realized by move I and II. On the other hand, Figure 20 shows how to get the braid  $y_g^{-1} x_{2g+2}^{-1} y_g x_{2g+2}$ .

Figure 21 concludes the proof of the theorem, by showing how to realize the braid  $(x_2 x_3 x_4)^4 w^{-1} u^{-1}$ , which is equivalent to  $B'$  up to moves, since what we have proved above (cf. Figure 19).

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