# THE SYMBOLIC REPRESENTATION OF
# BILLIARDS WITHOUT BOUNDARY CONDITION

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ABSTRACT. We consider a dynamical system with elastic reflections in the whole plane and show that such a dynamical system can be represented as a symbolic flow over a mixing subshift of finite type. This fact enables us to prove an analogue of the prime number theorem for the closed orbits of such a dynamical system.

## INTRODUCTION

Let $O_1, O_2, \ldots, O_L$ $(L \geq 3)$ be a finite number of bounded domains in $\mathbb{R}^2$ with smooth boundaries $\partial O_1, \partial O_2, \ldots, \partial O_L$. We assume that the closures $\overline{O_j} = O_j \cup \partial O_j$ of $O_j$ are strictly convex and mutually disjoint. Consider the motion of a particle in the exterior domain $O = \mathbb{R}^2 \setminus \bigcup_{j=1}^{L} \overline{O_j}$, which obeys the law of reflection: "the particle moves along the straight line with unit speed in $O$ and reflects at the boundary $\partial O = \bigcup_{j=1}^{L} \partial O_j$ so that the angle of the reflection coincides with the angle of the incidence." We can describe this motion of a particle by a dynamical system (a flow) $S_t$ on the unit tangent bundle over $O$. We call the flow $S_t$ a billiard without boundary condition in the light of the Sinai's billiard in [6], which is defined on the unit tangent bundle over 2-torus $T^2$, i.e., the billiard with periodic boundary condition.

The purpose of this paper is to prove the following theorems:

**Theorem 1.** *Under the hypotheses* (H.1) *and* (H.2) *(see* §2), *the flow* $S_t$ *restricted to the nonwandering set can be represented as a symbolic flow* $\sigma_t$ *over an appropriate subshift of finite type so that the corresponding closed orbits have the same period (see Proposition* 3.1).

**Theorem 2.** *Under the hypotheses* (H.1) *and* (H.2), *there is a positive constant* $h$ *such that an analogue of the prime number theorem*

$$\#\{\gamma \,;\, \exp[hT_\gamma] \leq x\} \cdot \frac{\log x}{x} \to 1 \qquad (x \to \infty)$$

*holds, where* $\gamma$ *and* $T_\gamma$ *denote the prime closed orbit of* $S_t$ *and its period respectively.*

It is not hard to see that Theorem 2 is obtained by combining Theorem 1 and the result in Parry and Pollicott [5] (see §3).

It will be meaningful to note that the closed orbits of the dynamical system $S_t$ make the essential contribution to the singular support of the distributional function $\sum_{\lambda_i \in \mathrm{Spec}\,\Delta} \cos \lambda_i^{1/2} t$ (see [1]) and they are closely related to the poles of the scattering matrix as mentioned in Ikawa [2] and [3] etc., where $\mathrm{Spec}\,\Delta$ denotes the set of eigenvalues of the selfadjoint realization of the Laplace operator with the appropriate boundary condition.

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## 1. Preliminaries

In this section we prepare the basic notions for the later convenience.

Let $O_1, O_2, \ldots, O_L$ $(L \geq 3)$ be a finite number of bounded domains in $\mathbb{R}^2$ as in the beginning of Introduction. We denote by $S\mathbb{R}^2 = \mathbb{R}^2 \times S^1 = \{(q, v) \in \mathbb{R}^2 \times \mathbb{R}^2 ; |v| = 1\}$ the unit tangent bundle over $\mathbb{R}^2$ and $\pi : S\mathbb{R}^2 \to \mathbb{R}^2$ the natural projection, where $|\cdot|$ denotes the usual Euclidean norm. Choose a point $q_j \in \partial O_j$ for each $j$ and define the following quantities for $x = (q, v) \in \partial O = \bigcup_{j=1}^{L} \partial O_j$:

(1.1)
$$\begin{cases} \xi_0(x) = j, \quad \text{if } q \in \partial O_j; \\ r(x) = \text{the arclength between } q_{\xi_0(x)} \text{ and } q, \\ \qquad \text{measured clockwise along the curve } \partial O_{\xi_0(x)}; \\ \phi(x) = \text{the angle between the vector } v \text{ and the} \\ \qquad \text{unit innernormal } n(q) \text{ of } \partial O_{\xi_0(x)} \text{ at } q, \\ \qquad \text{measured unitclockwise}. \end{cases}$$

Therefore $\pi^{-1}(\partial O)$ is parametrized as

(1.2)
$$\pi^{-1}(\partial O) = \{(j, r, \phi); \ 1 \leq j \leq L, \\ 0 \leq r < \text{ the perimeter of } \partial O_j, \text{ and } 0 \leq \phi < 2\pi\}.$$

Put

(1.3)
$$\begin{cases} M_- = \{x \in \pi^{-1}(\partial O), \ \frac{\pi}{2} \leq \phi(x) \leq \frac{3}{2}\pi\}, \text{ and} \\ M_+ = \{x \in \pi^{-1}(\partial O), \ \frac{\pi}{2} < \phi(x) + \pi < \frac{3}{2}\pi \bmod 2\pi\}. \end{cases}$$

We introduce the following equivalence relation '$\sim$' to $\pi^{-1}(\partial O)$:

(1.4)
$$x \sim y \quad \text{if and only if } \mathrm{Inv}(x) = y \text{ or } x = y,$$

where $\mathrm{Inv} : \pi^{-1}(\partial O) \to \pi^{-1}(\partial O)$ is defined by

(1.5)
$$\mathrm{Inv}(j, r, \phi) = (j, r, \pi - \phi) \bmod 2\pi.$$

It is natural to identify $(\pi^{-1}(\partial O))/\sim$ with $M_-$ and we often use this identification without specification. Put

(1.6) $$M = \pi^{-1}(O) \cup (\pi^{-1}(\partial O)/\sim) = \pi^{-1}(O) \cup M_- .$$

Now we recall the notion of the billiard without boundary condition. Consider the motion of a particle which moves along the straight line with unit speed in $O = \mathbb{R}^2 \setminus \bigcup_{j=1}^{L} \overline{O_j}$ and reflects at the boundary $\partial O = \bigcup_{j=1}^{L} \partial O_j$ according to the law of reflection: the angle of reflection coincides with that of incidence. Then the motion determines a dynamical system (a flow) on $M$ in a canonical way (see Remark 1.2 below). We call it a billiard without boundary condition.

*Remark* 1.1. It is easy to see that $M_-$ and $M_+$ denote the set of the incidental vectors and the set of the reflection vectors respectively.

We define the first collision time $\tau_+$ and the last collision time $\tau_-$ by

(1.7) $$\begin{cases} \tau_+(x) = \inf\{t > 0, \ \pi(S_t x) \in \partial O\} \\ \tau_-(x) = \sup\{t < 0, \ \pi(S_t x) \in \partial O\}. \end{cases}$$

Here we regard $\tau_+(x)$ (resp. $\tau_-(x)$) as $+\infty$ (resp. $-\infty$) if the set in the definition is empty.

*Remark* 1.2. Let $x = (q, v)$, $M = \pi^{-1}(O) \cup M_-$. We note that the flow $S_t$ is defined so that

$$S_t x = \begin{cases} (q + tv, v), & (x \in \pi^{-1}(O)), \\ (q + t\tilde{v}, \tilde{v}), & (x \in M_-) \end{cases}$$

if $0 < t < \tau_+(x)$, where $\tilde{v}$ is determined by the formula $\text{Inv} x = (q, \tilde{v})$.
Put

(1.8) $$\Omega = \{x \in M, \ \pi(S_t x) \in \partial O \text{ for both infinitely many}$$
$$t > 0 \text{ and infinitely many } t < 0\}.$$

Clearly, $\Omega$ coincides with the nonwandering set of the flow $S_t$. Put

(1.9) $$\Omega_0 = \pi^{-1}(O) \cap \Omega \quad \text{and} \quad \Omega_- = M_- \cap \Omega.$$

We define the local maps $T$ and $T^{-1}$ by

(1.10) $$\begin{cases} T(x) = S_{\tau_+(x)}(x) & \text{if } \tau_+(x) < +\infty, \\ T^{-1}(x) = S_{\tau_-(x)}(x) & \text{if } \tau_-(x) > -\infty, \end{cases}$$

respectively.

It is not hard to see that the above notation $T^{-1}$ is compatible with the definition of the inverse map of $T$ and $T$ is locally diffeomorphic.

*Remark* 1.3. Consider the flow $S_t$ restricted to $\Omega$. The set $\Omega_-$ and the first collision time $\tau_+$ play the role of the Poincaré section and the Poincaré map respectively.

For $x \in M_-$ $(= \pi^{-1}(\partial O)/\sim)$, we put

$$(1.11) \qquad\qquad \xi_j(x) = \xi_0(T^j x) \quad \text{if } T^j \text{ is defined}.$$

We call the sequence $\xi = (\xi_j)_{j=-\infty}^{\infty}$ the itinarary of $x \in \Omega_-$ if $\xi_j = \xi_j(x)$ and write $\xi$ as $\xi(x)$.

We conclude this section by stating the following lemma. The proof is due to elementary calculation of the Jacobi matrix of $T$, and it can be found in [4]. Therefore we omit the proof.

**Lemma 1.1.** *Let $C$ be a curve of class $C^1$ in $M_-$ which is represented as $\{(j, r, \phi); \phi = \psi(r), a \le r \le b\}$, where $\psi$ is a $C^1$-function in $r$. Assume that $T$ (resp. $T^{-1}$) is defined and continuous on $C$. If the image $C_1 = TC$ (resp. $C_{-1} = T^{-1}C$) is represented as $\{(j_1, r_1, \phi_1); \phi_1 = \psi_1(r_1), a_1 \le r_1 \le b_1\}$ (resp. $\{(j_{-1}, r_{-1}, \phi_{-1}); \phi_- = \psi_-(r_{-1}), a_{-1} \le r_{-1} \le b_{-1}\}$), where $\psi_1$ (resp. $\psi_{-1}$) is $C^1$-function in $r_1$ (resp. $r_{-1}$), then we have:*

$$(1.12)$$

$$\frac{d\psi_1}{dr_1} = k(r_1) - \frac{\cos\psi_1}{\cos\psi} \left( \frac{\tau_+(j, r, \phi)}{\cos\psi} - \left( \frac{d\psi}{dr} + k(r) \right)^{-1} \right)^{-1}$$

$$\left( resp. \; \frac{d\psi_{-1}}{dr_{-1}} = -k(r_{-1}) - \frac{\cos\psi_{-1}}{\cos\psi} \left( \frac{\tau_-(j, r, \phi)}{\cos\psi} - \left( \frac{d\psi}{dr} + k(r) \right)^{-1} \right)^{-1} \right),$$

$$(1.13) \quad \frac{dr_1}{dr} = -\frac{\cos\psi}{\cos\psi_1} \left( 1 - \frac{\tau_+(j, r, \phi)}{\cos\psi} \left( \frac{d\psi}{dr} + k(r) \right) \right)$$

$$\left( resp. \; \frac{dr_{-1}}{dr} = -\frac{\cos\psi}{\cos\psi_{-1}} \left( 1 - \frac{\tau_-(j, r, \phi)}{\cos\psi} \left( \frac{d\psi}{dr} - k(r) \right) \right) \right),$$

*where $k(r)$ denotes the curvature of $\partial O_j$ at $(j, r, \phi)$, etc.*

## 2. WELL-POSEDNESS OF ITINARARY PROBLEM

From now on we assume:

**(H.1)** (convexity). *For each $j = 1, 2, \ldots, L$ boundary $\partial O_j$ of $O_j$ is a simple closed curve with nonvanishing curvature.*

**(H.2)** (no eclipse). *For any triple $(j_1, j_2, j_3)$ of distinct indices,*

$$\text{conv}[O_{j_1} \cup O_{j_2}] \cap O_{j_3} = \varnothing,$$

*where $\text{conv}[B]$ denotes the convex hull of the set $B$.*

We introduce the following shift dynamical system. Let $A$ be an $L \times L$-matrix with entries $A(i, j) = (1 - \delta_{ij})$, $1 \le i, j \le L$, where $\delta_{ij}$ denotes the

Kronecker's delta. Put

$$(2.1) \quad \Sigma = \Sigma_A = \left\{ \xi = (\xi_j)_{j=-\infty}^{\infty} \in \prod_{j=-\infty}^{\infty} \{1, 2, \ldots, L\} ; \right.$$

$$\left. A(\xi_j, \xi_{j+1}) = 1 \text{ for any } j \right\}.$$

We define $d_\rho : \Sigma \times \Sigma \to R$ by

$$(2.2) \quad d_\rho(\xi, \eta) = \rho^n \text{ if } \xi_j = \eta_j \text{ for } |j| < n \text{ and } \xi_n \neq \eta_n \text{ or } \xi_{-n} \neq \eta_{-n}.$$

It is easy to see that $d_\rho$ becomes a metric on $\Sigma$ which defines the same topology as the induced topology of $\Sigma$ as a subset of the product space

$$\prod_{j=-\infty}^{\infty} \{1, 2, \ldots, L\}.$$

The shift transformation $\sigma : \Sigma \to \Sigma$, $(\sigma\xi)_j = \xi_{j+1}$, $j \in Z$ is well defined and the shift dynamical system $(\Sigma, \sigma)$ is a typical example of a mixing subshift of finite type (see [5]). Consider the billiard without boundary condition $S_t$. We say that a point $x \in \Omega_-$ solves the itinarary problem

$$(2.3) \quad \xi(y) = \xi \in \Sigma$$

or $x$ is a solution of the itinarary problem (2.3) if the itinarary $\xi(x)$ of $x$ coincides with the sequence $\xi$.

Under the hypotheses (H.1) and (H.2) we prove that the itinarary problem (2.3) is well-posed in the following sense.

**Theorem 0** (the Lipschitz well-posedness of the itinarary problem). *If the hypotheses (H.1) and (H.2) are satisfied, there exists a unique $x \in \Omega_-$ which solves the itinarary problem (2.3) for any $\xi \in \Sigma$. In addition, if we denote by $x(\xi)$ the solution of the itinarary problem (2.3), there exist constants $C > 0$ and $0 < \rho < 1$ such that*

$$(2.4) \quad |\tau_+(x(\xi)) - \tau_+(x(\eta))| < C d_\rho(\xi, \eta) \quad \text{for any } \xi, \eta \in \Sigma,$$

*where $d_\rho$ denotes the metric on $\Sigma$ defined by (2.2).*

We prepare an a priori estimate for the proof of Theorem 0.

**Lemma 2.1** (a priori estimate). *Let $x$ and $y$ be elements in $M_-$. Assume $T^j$ is well defined and $\xi_j(x) = \xi_j(y)$ for each $j$ with $-n \leq j \leq n$ $(n \geq 1)$. Then the arclength $r(x, y)$ between $\pi(x)$ and $\pi(y)$ satisfies*

$$(2.5) \quad r(x, y) \leq c_0 l (1 + \eta)^{-n},$$

*where $c_0$ is a positive constant independent of $x$ and $y$,*

$$l = \max\{\text{the perimeter of } \partial O_j, \ j = 1, 2, \ldots, L\} \quad \text{and}$$

$$\eta = \min\{\text{the distance between } \overline{O_i} \text{ and } \overline{O_j}, \ 1 \leq i < j \leq L\}$$

$$\times \min\{k(q), \ q \leq \partial O_j, \ j = 1, 2, \ldots, L\}.$$

*Proof.* Let $C$ be a $C^1$-curve in $M_-$ as in Lemma 1.1. We call it an increasing (resp. decreasing) curve if $\frac{d\psi}{dr} \geq 0$ (resp. $\frac{d\psi}{dr} \leq 0$). Assume that $x$, $y \in M_-$ satisfy the assumption of Lemma 2.1. Then we may write $T^j x = (r_j(x), \phi_j(x))$, $j = -n, -(n-1), \ldots, n-1, n$ without confusion. First we connect $x$ and $y$ by a line segment $C_0$ in $M_-$. We may assume that $r_0(x) < r_0(y)$. If $\phi_0(x) \leq \phi_0(y)$, $C_0$ becomes an increasing curve in $M_-$. Therefore it is not hard to show that $T^j$ is continuous on $C_0$ and the image $C_j = T^j C_0$ turns out to be an increasing curve for each $j = 1, 2, \ldots, n$ in the same way as in the proof of Lemma 4.1 in [4]. Thus $C_j$ can be expressed as $\{(r_j, \phi_j), \phi_j = \psi_j(r_j), a_j \leq r_j \leq b_j\}$ with $d\psi_j/dr_j \geq 0$ for each $j$. In virtue of the formula (1.13), we obtain

$$(2.6) \qquad \frac{dr_n}{dr_0} = \frac{dr_n}{dr_{n-1}} \frac{dr_{n-1}}{dr_{n-2}} \cdots \frac{dr_1}{dr_0} = (-1)^n \frac{\cos \psi_n}{\cos \psi_0} \prod_{j=1}^{n} b_j,$$

where $b_j = (1 - \tau_+(T^j(r_0, \phi_0)))(d\psi_j/dr_j + k(r_j))$. Since $d\psi_j/dr_j \geq 0$ for all $j = 0, 1, \ldots, n-1$, we have

$$(2.7) \qquad\qquad |b_j| \geq 1 + \eta$$

from the formula (1.12). Thus we have

$$(2.8) \qquad\qquad \left| \frac{dr_n}{dr_0} \right| \geq |\cos \phi_0|(1 + \eta)^n.$$

Therefore we obtain

$$(2.9) \qquad\qquad r(x, y) \leq |\cos \phi_0|^{-1} l(1 + \eta)^{-n}.$$

On the other hand it is easy to show that $|\cos \phi_0|$ is bounded below by a positive constant which is independent of $x$ and $y$ in virtue of the hypotheses (H.1) and (H.2). Hence we have proved the inequality (2.5) when $\phi_0(x) \leq \phi_0(y)$. If $\phi_0(x) > \phi_0(y)$, we can prove the estimate (2.5) in the same manner, by using $T^j$, $-n \leq j \leq -1$, instead of $T^j$, $1 \leq j \leq n$. $\square$

Now we can prove Theorem 0. The Lipschitz continuity (2.4) of the first collision time is an immediate consequence of Lemma 2.1 if we take $(1 + \eta)^{-1}$ as $\rho$. The uniqueness of the itinarary problem follows from the estimate (2.4). Therefore it suffices to show the existence. First we assume that $\xi$ is periodic, i.e., $\xi_{n+m} = \xi_n$ for some $m > 0$, for all $n \in Z$. Consider the following minimal value problem

$$(2.10) \quad l(q^0, q^1, \ldots, q^{m-1}) = \sum_{j=0}^{m-1} |q^j - q^{j-1}|,$$

$$q^j \in \partial O_{\xi_j}, \quad j = 0, 1, \ldots, m-1,$$

where $q^{-1} = q^{m-1}$.

Hypotheses (H.1) and (H.2) imply that there exists $(p^0, p^1, \ldots, p^{m-1})$ which minimizes $l(q^0, q^1, \ldots, q^{m-1})$ in virtue of the Borzano Weierstrass theorem. The points $p^0, p^1, \ldots, p^{m-1}$ have to satisfy the equations

$$(2.11) \qquad \frac{\partial l}{\partial q_k^j} = \lambda_j \frac{\partial f_j}{\partial q_k^j}(p^j), \qquad j = 0, 1, \ldots, m-1 \quad \text{and} \quad k = 1, 2, \ldots,$$

where the curves $\partial O_{\xi_j}$ are assumed to be represented as $f_j(q^j) = 0$ in the neighborhood of $p^j$, and $\lambda_j$ denote the Lagrange multipliers.

The equations (2.11) are nothing but the law of reflections. Therefore the existence of the solution of the itinarary problem has been proved when $\xi$ is periodic.

Let $\xi$ be an element in $\Sigma$ which is not periodic. Choose $\xi^m \in \Sigma$ which is periodic and $d_\rho(\xi^m, \xi) \to 0$, $(m \to \infty)$. Let $x^m$ be the unique solution of the itinarary problem $\xi(x) = \xi^m$. The estimate (2.5) in Lemma 2.1 implies that

$$r(T^j x^m, T^j x^{m+1}) \le C\rho^{(\text{the period of } \xi^m) - |j|}$$

for $|j| \le$ the period of $\xi^m$. Therefore $x^m$ converges to some $x \in M_-$ and $x$ satisfies $\xi(x) = \xi$. Now the proof of Theorem 0 is complete.

*Remark* 2.1. It is not necessary to use the inequality (2.5) to show the existence of the solution of the itinarary problem. One can show it by use of the diagonal argument.

## 3. PROOFS OF RESULTS

The purpose of this section is to complete the proofs of Theorem 1 and Theorem 2 in Introduction. Note that we always assume the hypotheses (H.1) and (H.2).

Define a function $f$ on $\Sigma$ by

$$(3.1) \qquad\qquad f(\xi) = \tau_+(x(\xi)), \quad \text{for } \xi \in \Sigma,$$

where $x(\xi)$ denotes the unique solution of the itinarary problem (2.3) as before. We denote by $(\Sigma^f, \sigma_t)$ (simply $\sigma_t$) the symbolic flow over $\sum$ with ceiling function $f$. Precisely, $\Sigma^f$ is the set $\{(\xi, s); \xi \in \Sigma, 0 \le s < f(\xi)\}$ with the identification $(\xi, f(\xi)) = (\sigma\xi, 0)$ for any $\xi \in \Sigma$, and the flow $\sigma_t$ on $\Sigma^f$ is defined so that

$$(3.2) \qquad \sigma_t(\xi, s) = (\sigma^k \xi, u), \quad \text{if } \sum_{j=0}^{k-1} f(\sigma^j \xi) \le t + s < \sum_{j=0}^{k} f(\sigma^j \xi),$$

where $u = t + s - \sum_{j=0}^{k-1} f(\sigma^j \xi)$ and so on.

Theorem 1 and Theorem 2 follow from Proposition 3.1 and Proposition 3.2 below, respectively.

**Proposition 3.1.** *The map* $h : \Sigma^f \to \Omega$ *defined by* $h(\xi, s) = S_s(x(\xi))$ *gives the conjugacy between the flows* $\sigma_t$ *and* $S_t$ *restricted to* $\Omega$ *so that the corresponding closed orbits have the same period.*

*Proof.* Theorem 0 in §2 implies that the map $h_0 : \Sigma \to \Omega_-$ defined by $h_0(\xi) = x(\xi)$ is a homeomorphism with $h_0(\sigma\xi) = x(\sigma\xi) = T(x(\xi)) = T(h_0(\xi))$. Therefore $h_0$ gives a conjugacy between the Poincaré maps $\sigma$ of $\sigma_t$ and $T$ of $S_t$ restricted to $\Omega$. On the other hand, the corresponding points $\xi$ and $h_0(\xi)$ have the same return time to $\Sigma$ and $\Omega_-$ respectively from the definition of $f$. Hence $h$ gives a conjugacy between $\sigma_t$ and $S_t$ restricted to $\Omega$. Obviously the corresponding closed orbits have the same period. $\square$

**Proposition 3.2.** *The ceiling function* $f$ *cannot be represented as*

$$(3.3) \qquad\qquad f = g \circ \sigma - g + aK,$$

*where* $g$ *denotes a real valued function,* $K$ *an integer valued function, and* $a$ *a positive constant.*

*Remark* 3.1. Proposition 3.2 implies that the symbolic flow $\sigma_t$ is topologically weak mixing. On the other hand the estimate (2.4) shows that the celling function $f$ is Lipschitz continuous with respect to the metric $d_\rho$. Therefore we can obtain an analogue of the prime number theorem for the distribution of the prime closed orbits of the flow $S_t$ by use of the zeta function

$$(3.4) \qquad \begin{aligned} \zeta(s) &= \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} \sum_{\sigma^n\xi=\xi} \exp\left[-s \sum_{j=0}^{n-1} f(\sigma^j\xi)\right]\right) \\ &= \prod_{\gamma}(1 - \exp[-sT_\gamma])^{-1}, \end{aligned}$$

where $\gamma$ denotes a prime closed orbit of $S_t$ and $T_\gamma$ is its period (see Parry and Pollicott [5]).

*Proof of Proposition* 3.2. Suppose that $f$ can be represented as in (3.3). By using the similarity transformation we may assume that $a = 1$, i.e.,

$$(3.5) \qquad\qquad f = g \circ \sigma - g + K,$$

where $g$ is a real valued function and $K$ is an integer valued function.

Since we already established the conjugacy in Proposition 3.1 we can identify the symbolic flow $\sigma_t$ and the flow $S_t$ restricted to $\Omega$ without confusion. The assumption (3.5) yields that every closed orbit of $S_t$ has an integer period. Now we restrict ourselves to three domains $O_1$, $O_2$, and $O_3$.

For each $n \geq 1$, let $\xi^n = (\xi_j^n)_{j=-\infty}^{\infty}$ be the sequence in $\sum$ so that $\xi_0^n = 1$; $\xi_j^n = 2$ for odd $j \leq 4n-1$; $\xi_j^n = 3$ for even $j \leq 4n-2$; and $\xi_{m+4n}^n = \xi_m^n$ for any $m \in Z$. Let $\xi^0$ be the sequence in with $\xi_j^0 = 3$ for even $j$ and $\xi_j^n = 2$ for odd $j$. We denote by $x^{n,j}$ the unique element in $\Omega_-$ which solves $\xi(x) = \sigma^j\xi^n$, and $q^{n,j} = \pi(x^{n,j})$. Namely, $q^{n,j}$ denotes the position where the $j$th collision

occurs along the closed orbit $\gamma_n = (S_t x^{n,0})_t$ starting from $x^{n,0}$, for $n \geq 0$. We note that the period $T_n$ of $\gamma_n$ has the minimal property which appeared in the proof of Theorem 0 (see the minimal value problem (2.11) in §2). The minimal property of $T_n$ and the uniqueness of the solution of the itinarary problem imply that $\gamma_n$ must by symmetric, i.e., $q^{n,2n+j} = q^{n,2n-j}$ for $j \geq 1$. We claim that

$$(3.6) \qquad T_{n+1} \geq T_n + 2T_0 + 1, \quad \text{for } n \geq 1.$$

Consider a fictitious motion of a particle such that the particle moves along the orbit $\gamma_{n+1}$ until it collides at $q^{n+1,2n}$ and after that it returns to $q^{n+1,0}$ taking the same way as it has taken to reach $q^{n+1,2n}$. It will be more convenient to introduce the following notation:

$$q^{n+1,0} \to q^{n+1,1} \to \cdots \to q^{n+1,2n} \to q^{n+1,2n-1} \to \cdots \to q^{n+1,1} \to q^{n+1,0},$$

where $p \to q$ denotes that the fictitious particle moves from $p$ to $q$.

Now we obtain a fictitious closed orbit $\gamma_n'$ whose period $T_n'$ is

$$l(q^{n+1,0}, q^{n+1,1}, \ldots, q^{n+1,2n}, q^{n+1,2n-1}, \ldots, q^{n+1,1}).$$

Therefore, $T_n < T_n'$ in virtue of the minimal value problem (2.11). Thus we have

$$T_{n+1} \geq T_n' + 2T_0 > T_n + 2T_0.$$

We used the fact that $\gamma_{n+1}$ is symmetric to see the first inequality in the above. But $T_n$'s are all integers by our assumption. Hence we obtain (3.6).

On the other hand we can show

$$(3.7) \qquad T_{n+1} \leq T_n + 2T_0 + C'\rho^{2n}$$

where $C'$ is a positive constant which is independent of $n$ and $\rho = (1+\eta)^{-1}$ as before. Clearly the inequality (3.7) contradicts the inequality (3.6). Therefore the celling function $f$ cannot be represented as in (3.5). This completes the proof of Proposition 3.2.

It remains to prove the inequality (3.7). We consider the following fictitious motion of a particle:

$$q^{n,0} \to q^{n,1} \to \cdots \to q^{n,2n} \to q^{0,0} \to q^{0,1} \to q^{0,0} \to q^{n,2n}$$
$$\to q^{n,2n-1} \to \cdots \to q^{n,1} \to q^{n,0}.$$

Then we obtain the fictitious closed orbit $\gamma_{n+1}'$ whose fictitious period $T_{n+1}'$ is

$$l(q^{n,0}, q^{n,1}, \ldots, q^{n,2n}, q^{0,0}, q^{0,1}, q^{0,1}, q^{n,2n}, q^{n,2n-1}, \ldots, q^{n,1}).$$

On the other hand we have

$$(3.8) \qquad T_{n+1}' \leq T_n + T_0 + 2|q^{n,2n} - q^{0,0}|.$$

Here we used the fact that $\gamma$ is symmetric. From the definition of $\gamma_n$ and $\gamma_0$, $T^j x^{n,2n}$ and $T^j x^{0,1}$ belong to $\partial O_{\zeta_{j+1}^0}$ for $|j| \leq 2n - 1$. Therefore the

arclength $r(x^{0,1}, x^{n,2n})$ between $q^{0,1} = \pi(x^{0,1})$ and $q^{n,2n} = \pi(x^{n,2n})$ is less than or equal to $C\rho^{2n-1}$ by the a priori estimate (2.5). Thus we obtain

$$(3.9) \qquad\qquad 2|q^{n,2n} - q^{0,0}| \le T_0 + 2C\rho^{2n-1}$$

in virtue of the triangle inequality.

The inequalities (3.8) and (3.9) imply the inequality (3.7) in virtue of the minimal property of $T_{n+1}$.

Now the proof is complete.  □

*Remark* 3.2.  In the proof of inequality (3.7) we used the fact that $\pi\gamma_0$ and $\pi\gamma_n$ cannot intersect. We restrict ourselves to note that it is an easy consequence of our hypotheses (H.1) and (H.2).

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