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ON SURFACES AND HEEGAARD SURFACES

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ABSTRACT. This paper is concerned with the intersection of surfaces and Heegaard surfaces in closed orientable 3-manifolds M. Given a Heegaard decomposition (M, V_1, V_2) it will be shown that any surface (orientable or not) in M is equivalent to a surface which intersects V_1 in discs whose total number is limited from above by some function in the genus of ∂V_1 alone. The equivalence relation in question is generated by disc- and annulus-compressions.

1. INTRODUCTION

This paper is concerned with estimates for the intersection of surfaces with Heegaard surfaces in closed and orientable 3-manifolds.

Recall that a *Heegaard surface* in a closed 3-manifold M is, by its very definition, a surface which splits M into two handlebodies ($\cong$ small neighborhoods of finite graphs in Euclidean space). Any closed 3-manifold admits a Heegaard surface and so any estimate for the intersection of a Heegaard surface with other surfaces is of interest for theoretical and computational aspects in the realm of 3-manifold theory—at least in the case of 3-manifolds with given Heegaard splittings. The search for such estimates goes back to Haken [Ha 2], who originally considered 2-spheres, and was recently continued in [Och, Ko 1], where projective planes, resp. nonseparating tori, in Heegaard genus two 3-manifolds have been considered. Given any Heegaard surface, it has been shown there that the small surfaces above can always be chosen as to intersect the Heegaard surface in one curve only. This strong statement cannot be expected to hold in general. On the other hand, however, it follows from Haken's normal surface theory [Ha 1] that in simple 3-manifolds [Joh 1] the intersection of all incompressible surfaces with a given Heegaard surface is limited (modulo isotopy) by some known function in the Euler characteristics of the incompressible surfaces in question. In fact, this is already true for normal surfaces since, according to Haken's theory, all normal surfaces in M can be obtained from a finite set of fundamental surfaces, using cut-and-paste along simple closed curves. It is the goal of this paper to show that in the above context the construction of fundamental surfaces can be avoided. Indeed, instead of referring to fundamental surfaces (which are difficult to construct), we rather suggest to refine the

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concept of normal surfaces to that of "strictly normal surfaces". We will show that (incompressible) normal surfaces can be isotoped into strictly normal ones, and that moreover the intersection of strictly normal surfaces with a given Heegaard surface is very well behaved. As one striking feature we will show that all strictly normal 2-spheres (bounding a 3-ball or not) intersect Heegaard surfaces in one curve alone. We will show, moreover, how to use our method in order to improve the results mentioned above (see e.g. 5.4 below).

To give estimates for larger surfaces, let M denote a closed (orientable) 3-manifold and let S be an arbitrary closed 2-manifold (possibly compressible, disconnected or nonorientable) in M. Then, given a disc or annulus A in M with $A \cap S = \partial A$, define

$$S' := (S - U(A))^{-} \cup A_1 \cup A_2,$$

where $U(A)$ denotes the regular neighborhood of A in M and where A_1 and A_2 are the two copies of A in $\partial U(A)$. We say that S' has been obtained from S by a *disc-* resp. *annulus-compression* according whether A is a disc or an annulus. Moreover, such a compression will be called *essential*, if each one of the components of ∂A is essential in S, i.e. not the boundary of any disc in S. Observe that the Euler characteristic of S will indeed not be diminished neither by a disc-compression nor by an essential annulus-compression. Furthermore, these compressions do not change the $\mathbb{Z}_2$-homology class of S, although one always has to be aware of the fact that an annulus-compression might possibly change the orientability-type of S. Finally, we say that S can be *compressed into* a closed 2-manifold S', provided there is a finite sequence of closed 2-manifolds

$$S := S_1, S_1, \dots, S_n =: S'$$

such that S_{i+1} is obtained from S_i by one of the following operations: (1) isotopy, (2) disc-compression or essential annulus-compression, or (3) removing a sphere-component, provided it is inessential or another one is parallel to it. (Recall that, by definition, a 2-sphere in M is *inessential*, if it is the boundary of a 3-ball in M, and that a closed 2-manifold in M is *essential* if it consists of incompressible surfaces and essential 2-spheres.)

Keeping the previous notation in mind, it is the main object of this paper to prove the following result:

1.1. **Theorem.** *Let M be a closed and orientable 3-manifold with Heegaard-splitting (M, V_1, V_2) of genus $g \geq 2$, and let S be any closed (possibly disconnected or nonorientable) 2-manifold in M. Then S can be compressed into some closed 2-manifold which intersects V_1 in discs whose number is at most $n = 6g - 6$.*

In addition, if S is orientable, the estimate can be taken to be $n = 6g - 11$.

Remarks. (1) The estimate given in the above theorem is in general not best possible. Indeed, it is known that it can be refined in special cases, notably when S is a 2-sphere (see [Ha 2, §7] or Proposition 3.2 below). Moreover, using a similar argument as in §4 below, it can be seen that, in the case when $g \geq 3$, the estimate can be refined to $n = 6g - 13$ for all orientable 2-manifolds. For nonorientable 2-manifolds in turn a slightly better estimate can be obtained, if we relax the condition that the resulting 2-manifold has to intersect the handlebody V_1 in discs and if we count the number of intersection curves with ∂V_1 instead. For incompressible surfaces in Haken 3-manifolds there is yet another relevant estimate which will be discussed in [Joh 3].

(2) If M is irreducible, the above theorem says, in particular, that the homology classes of $H_2(M, \mathbb{Z}_2)$ can be realized by 2-manifolds which intersect a given Heegaard surface in a small number of curves. A result which is related to the Thurston (semi) norm and which is further explored in [Joh 3].

2. Normal positions

We work in the *PL*-category (see e.g. [He] for background). Throughout this paper M will denote a closed, orientable 3-manifold, and (M, V_1, V_2) will be a Heegaard splitting of M, i.e. V_1, V_2 are handlebodies in M with $V_1 \cup V_2 = M$ and $V_1 \cap V_2 = \partial V_1 = \partial V_2$. Generalizing an original idea of Kneser, Haken introduced in [Ha 1] the concept of "normal surfaces" for Heegaard splittings which subsequently proved to be of central importance for the study of 3-manifolds. Deviating slightly from [Ha 1], we here call a closed 2-manifold (orientable or not) S^+ in M *normal* with respect to the Heegaard-splitting (M, V_1, V_2) if

(1) $S^+ \cap V_1$ is a system of essential discs in the handlebody V_1, and

(2) each component of $S := S^+ \cap V_2$ is either a disc, or an incompressible surface in V_2.

Here a disc in a handlebody is called *essential*, if it is not boundary-parallel. Since S^+ may be nonorientable, we further note that "incompressible" is meant here in the geometric sense, i.e. a 2-manifold S in V_2 is *incompressible* if the boundary of every disc D in V_2, with $D \cap S = \partial D$, bounds a disc in S.

Every incompressible surface in an irreducible 3-manifold can be isotoped into a normal 2-manifold (but not every normal surface is incompressible) and our discussion will center around normal surfaces. Specifically, Theorem 1.1 will follow from properties of normal surfaces.

As indicated in the introduction, it is our goal to refine the concept of normal surfaces. For this it is convenient to introduce the following notions first. Let S^+ be any closed 2-manifold (possibly disconnected and nonorientable) in M which intersects V_1 in discs, let $S := S^+ \cap V_2$, and let b be any simple arc in S with $b \cap \partial S = \partial b$. Then b is called *essential in* S, resp., *in* S^+, provided b cannot be deformed (fixing ∂b) in S, resp. in S^+, into ∂S. The arc b is called a *recurrent arc*, if both its end-points lie in one component of ∂S. We say b is a *compression-arc* (for S), if b is essential in S and inessential in V_2, i.e. if there is a disc D in V_2, but no disc D' in S, such that $(\partial D - \partial V_2)^- = b = (\partial D' - \partial S)^-$. The arc $b' := \partial D \cap \partial V_2$ is then called a *compagnion for* b.

We now turn our attention to a crucial property of nonrecurrent compression-arcs. To formulate it, let S^+ be again a closed 2-manifold in M which intersects V_1 in discs and let b be a compression-arc in S. Let b' be any compagnion of b, and let r be one of the boundary-curves of S containing a point from $\partial b = \partial b'$. Then the following holds:

2.1. **Lemma.** *If b is nonrecurrent and if $b' \cap r$ consists of one point only, then S^+ can be isotoped into a 2-manifold S' which intersects V_1 in discs, but in one less than S^+.*

Proof. Let D^+ be a disc in V_2 with $D = (D^+ - U(\partial \mathscr{D}^+ - \partial V_2))^-$ and set $b^* := (\partial D^+ - \partial V_2)^-$. Moreover, let B be that component of $S^+ \cap V_1$ whose boundary is r, and set $U := U(B) \cup U(D)$, where the regular neighborhood is taken in V_1. Finally, set

$$W_1 := (V_1 - U)^- \cup U(b^*) \quad \text{and} \quad W_2 := (M - W_1)^- .$$

Then clearly (M, W_1, W_2) is a Heegaard-splitting which is ambient isotopic to (M, V_1, V_2) $(U(B) \cup U(D)$ is a 3-ball since $\partial B \cap \partial D$ is a point). It follows the existence of an isotopy which pushes S^+ into a 2-manifold intersecting V_1 in a system of discs whose number equals the number of components from $S^+ \cap (V_1 - U)$. But, by our choice of U, the latter system obviously has one less component than $S^+ \cap V_1$. □

The previous result indicates the importance of certain compression-arcs whose absence will indeed have a number of interesting consequences. It therefore suggests to single out the relevant property by means of the following definition:

2.2. **Definition.** *Let (M, V_1, V_2) be given as before, and let $\mathscr{D}$ be a system of essential discs in V_2 splitting V_2 into a 3-ball. A strongly normal 2-manifold S^+ in M is called a* <u>*strictly normal 2-manifold*</u> *(with respect to $\mathscr{D}$), if, for every component D_i, $1 \leq i \leq n$, from $\mathscr{D}$ the following holds:*

(1) D_i intersects $S = S^+ \cap V_2$ in arcs alone and every arc from $D_i \cap S$ is essential in S,

(2) if b is an arc from $D_i \cap S$ which joins two different boundary curves of S, then each (open) component of $\partial D_i - \partial b$ intersects both the latter curves.

If no disc-system has been specified in advance, then S^+ is *strictly normal* if there is some disc system $\mathscr{D}$ in V_2, splitting V_2 into a 3-ball, such that S^+ is strictly normal with respect to $\mathscr{D}$.

2.3. **Proposition.** *Let (M, V_1, V_2) be a Heegaard splitting and let S^+ be a normal 2-manifold in M. Then either S^+ is strictly normal (with respect to (M, V_1, V_2)), or it can be isotoped into a 2-manifold which intersects V_1 in discs, but in strictly fewer discs than S^+.*

Remark. It follows that every incompressible surface in an irreducible 3-manifold can be isotoped into a strictly normal surface.

Proof. Since V_2 is a handlebody and since $S = S^+ \cap V_2$ is supposed to be a system of discs and incompressible surfaces (S^+ is normal), there is a system $\mathscr{D}$ of nonseparating discs in V_2 such that $\mathscr{D}$ splits V_2 into a 3-ball and that $\mathscr{D} \cap S$ consists of arcs (innermost-disc-argument).

If one of these arcs, say b, is inessential in S, then w.l.o.g. it separates a disc D_0 from S with $D_0 \cap \mathscr{D} = b$. Now, let D_1 be the disc-component from $\mathscr{D}$ containing the arc b. Then $\mathscr{D} - D_1$ is a system of discs which splits V_2 into a solid torus W containing the regular neighborhood $U(D_0 \cup D_1)$ of $D_0 \cup D_1$ in V_2. The surface $(\partial(D_0 \cup D_1) - \partial W)^-$ consists of three proper discs in W. Two of them are nonseparating discs in W since D_1 is nonseparating in W. One of the latter discs is nothing but a copy of D_1; let D_1' be the other one. Then, replacing D_1 by D_1', we clearly obtain from $\mathscr{D}$ a new system of nonseparating discs in V_2 which splits V_2 into a 3-ball, but which intersects S in strictly fewer arcs than $\mathscr{D}$. Applying finitely many of such steps if necessary, we may suppose that $\mathscr{D}$ is chosen so that the arcs from $\mathscr{D} \cap S$ are all essential in S.

By what we have just seen, the disc-system $\mathscr{D}$ can always be chosen in V_2 so that (1) of Definition 2.2 is satisfied. Thus, if S^+ is not strictly normal, it cannot have property (2) of that definition. Thus we assume the existence of an arc b from $\mathscr{D} \cap S$ which joins two different boundary curves of S, say r_1, r_2, such that $r_1 \cap b_1 = \varnothing$, for some component b_1 of $\partial\mathscr{D} - \partial b$. Notice that b_1 is a compagnion for b and so, by Lemma 2.1, the 2-manifold S^+ can be isotoped into a 2-manifold which intersects V_1 in discs, but in strictly fewer discs than S^+. □

3. Properties of strictly normal surfaces

In this section we collect some properties of strictly normal surfaces and show for instance how a strong form of Haken's result on 2-spheres can directly be derived from these properties. Again let (M, V_1, V_2) be a fixed Heegaard splitting of the 3-manifold M. Furthermore, let $\mathscr{D}$ be a system of discs in V_2, $\mathscr{D} \cap \partial V_2 = \partial\mathscr{D}$, which splits V_2 into 3-balls.

3.1. **Lemma.** *Let S^+ be a 2-manifold which is strictly normal with respect to the disc-system $\mathscr{D}$. Let r be one boundary curve of $S := S^+ \cap V_2$. Then either* (1) *or* (2) *holds:*

(1) *r is the boundary of one component of S, or*

(2) *at least one nonrecurrent as well as at least one recurrent compression-arc from $\mathscr{D} \cap S$ has an end-point in r.*

Proof. Suppose r is not the boundary of one component of S.

Let U be a regular neighborhood of $(\mathscr{D} \cap S) \cup \partial S$ in S. Then each component of $\partial U - \partial S$ is contained in $V_2 - \mathscr{D}$, and therefore contractible in $V_2 - \mathscr{D}$

since $\mathscr{D}$ splits V_2 into 3-balls. Now, recall that S is incompressible in V_2 and that a 3-ball contains no proper nonorientable surface. Thus, by a standard argument involving the loop-theorem, it follows that each component of $\partial U - \partial S$ bounds a discs in S. The existence of the required nonrecurrent compression-arc then follows immediately since ∂S is supposed to be disconnected.

By what we have seen so far, there has to be at least one arc b from $\mathscr{D} \cap S$ which is a nonrecurrent compression-arc in S with one end-point in the boundary curve r. Let D_0 be one of the two discs in which a disc from $\mathscr{D}$ is separated by b, and denote $b' := D_0 \cap \partial\mathscr{D}$. Since S^+ is strictly normal with respect to $\mathscr{D}$, it follows that the interior of b' has to intersect r as well, and let x denote one point of this intersection. Then the point x in turn is the end-point of some arc k from $\mathscr{D} \cap S$. Without loss of generality, k is recurrent in S, for otherwise recall that one end-point of k, namely x itself, lies in r and so we could replace b by k and argue as before. This completes the proof of Lemma 3.1. □

As a first consequence of Lemma 3.1 we obtain the following form of Haken's 2-sphere result (see [Ha 2 and Och]).

3.2. **Proposition.** *Let S^+ be any (possibly mixed) system of normal, essential 2-spheres and normal projective planes in the 3-manifold M. Suppose no component of $S := S^+ \cap V_2$ is boundary-parallel in V_2. Then S^+ can be disc-compressed into a similar system which intersects V_1 in exactly one disc.*

Proof. By Proposition 2.3 we may suppose that S^+ is disc-compressed into a strictly normal 2-manifold, and so the proposition follows from the next lemma.

3.3. **Lemma.** *Let S^+ be a strictly normal system of 2-spheres and projective planes in M. Then each component of S^+ intersects the handlebody V_1 in exactly one disc.*

Proof of lemma. By Lemma 3.1, any component of $S := S^+ \cap V_2$ either has connected boundary, or contains a system of essential and recurrent arcs which meets *each* boundary curve. But the second alternative of this conclusion is impossible since every component of S is either a disc with holes or a Möbius band with holes. This proves Lemma 3.3, and so Proposition 3.2. □

As another consequence of Lemma 3.1 we here note further the following property of strictly normal surfaces which will be utilized in the next section.

3.4. **Lemma.** *Let S^+ be a 2-manifold which is strictly normal with respect to $\mathscr{D}$. Then each recurrent arc from $\mathscr{D} \cap S$ is essential in S^+, where $S := S^+ \cap V_2$.*

Proof. Assume the converse. Then there is component S_i of S and at least one recurrent arc k from $\mathscr{D} \cap S_i$ which is inessential in S^+. Let B be that component from $(S^+ - S)^- = S^+ \cap V_1$ which contains ∂k. Then k separates a disc D_0 from $(S^+ - B)^-$. But D_0 contains at least one boundary curve of S_i since S^+ is strictly normal with respect to $\mathscr{D}$, and so, in particular, k is essential in S. On the other hand, by Lemma 3.1, there is a system of essential *and* recurrent arcs in $S \cap D_0$ which meets every component of ∂S_i contained in D_0. It is easily checked that this is impossible since D_0 is a disc. □

4. Existence of annulus-compressions

Again let (M, V_1, V_2) denote a Heegaard splitting of the closed 3-manifold M. Let S^+ be a strictly normal 2-manifold, i.e. more precisely a closed 2-manifold in M which is strictly normal with respect to some appropriate disc-system $\mathscr{D}$ in V_2 which splits V_2 into 3-balls. Denote $S := S^+ \cap V_2$.

Given the previous setting, consider the following situation.

Let A be any system of (pairwise disjoint) annuli in ∂V_2 such that $\partial a \cap \partial S = \emptyset$ and that, in addition, each component of A contains at least one component of ∂S. Furthermore, suppose each component of $A \cap \partial S$ is parallel in A to a component of ∂A.

Finally, denote $I := A \cap \partial\mathscr{D}$, and suppose I is a system of essential arcs in A which intersects ∂S in a minimal number of points.

4.1. **Lemma.** *In the situation above, suppose that $\partial\mathscr{D}$ intersects A nontrivially. Then there is at least one component I_0 of I with the following property: Every arc from $\mathscr{D} \cap S$ which has one end-point in I_0 is a recurrent arc in S joining I_0 with one of those components of I neighboring I_0 (in $\partial\mathscr{D}$).*

Proof. Before starting the proof note that no arc from $\mathscr{D} \cap S$ has both its end-points in one component of I (S^+ is strictly normal) and that w.l.o.g. we may suppose that some component of S, meeting A, has disconnected boundary (see below).

Now, to show Lemma 4.1, we first have to face the problem that not necessarily all arcs from $\mathscr{D} \cap S$ have their end-points in I. In order to overcome this problem we may refer to Lemma 3.1. Indeed, by Lemma 3.1, there is at least one nonrecurrent arc k from $\mathscr{D} \cap S$ which has at least one end-point in I.

Moreover, let D' be one of the two discs in which that component of $\mathscr{D}$ containing k is separated by k. W.l.o.g. we may suppose that k and D' are both chosen in such a way that D' contains no nonrecurrent arc from $\mathscr{D} \cap S$ with an end-point in I. Define I' to be the union of all those components from I which are entirely contained in the arc $D' \cap \partial\mathscr{D}$. Recalling our choice of k, it is easily checked that I' is nonempty (S^+ is strictly normal). Given I', let $\mathscr{K}$ denote the union of all those arcs from $\mathscr{D} \cap S$ which have an end-point in I'. Then $\mathscr{K}$ is nonempty since I' is nonempty and since, by hypothesis, each component of A contains at least one component of ∂S (parallel to a component of ∂A). Furthermore, by our choice of k, it follows that $\mathscr{K}$ consists of recurrent arcs in S. In particular, $\mathscr{K}$ consists of arcs whose end-points lie in I.

We claim the required component I_0 of I can be found in I'. If this were not the case, there were at least one arc, say k_0, from $\mathscr{K}$, joining two nonneighboring components of I. Let k_0 be chosen to be minimal, i.e. in such a way that k_0 separates a disc D_0 from D' not meeting k and with the property that every arc from $D_0 \cap S$, different from k_0, joins neighboring components of I. Then it is easily checked that $D_0 \cap \partial\mathscr{D}$ has to contain the required component I_0, proving the claim. □

The previous result describes a crucial property for strictly normal surfaces. Here we are going to use it for establishing the first part of Theorem 1.1. Another application of it yields various finiteness theorems for Haken 3-manifolds. In particular, properties of strictly normal surfaces offer a common explanation of two seemingly unrelated results of Haken (see [Joh 3] for details).

4.2. **Proposition.** *Let (M, V_1, V_2) be a Heegaard splitting of genus g and let S^+ be a 2-manifold. Then S^+ can be compressed into some closed (possibly nonorientable) 2-manifold S' such that S' intersects V_1 in discs whose number does not exceed $n = 6g - 6$.*

Remark. See beginning of §2 for our convention concerning "incompressibility".

Proof. For every 2-manifold S^+ in M (which intersects V_1 in discs alone), let $a(S^+)$ denote the negative Euler characteristic of the union of all components of S^+ different from 2-spheres and projective planes, and let $b(S^+)$ denote the number of components of $S^+ \cap V_1$. Moreover, let the *complexity of* S^+ defined to be

$$c(S^+) := (a(S^+), b(S^+))$$

with respect to the lexicographical order. Since the integers $a(S^+)$ and $b(S^+)$ are both always nonnegative, we may suppose that S^+ is disc-compressed in such a way that its complexity $c(S^+)$ is as small as possible. In this case, S^+ is a system (possibly empty) of 2-spheres and incompressible surfaces. It remains to show that S^+ can be further compressed, without enlarging $c(S^+)$, so that it satisfies the estimates of the proposition. For this purpose we proceed as follows.

First, observe that S^+ may be supposed to be strictly normal since, in view of Proposition 2.3, it can be disc-compressed into such a 2-manifold without enlarging $c(S^+)$. Thus, by definition, there is a system $\mathscr{D}$ of discs in V_2 which splits V_2 into 3-balls and such that S^+ is strictly normal with respect to $\mathscr{D}$. Let us further choose a system A of pairwise disjoint annuli in ∂V_2 which contains $S^+ \cap \partial V_2$ and whose number of components is as small as possible.

Without loss of generality, $A \cap \partial\mathscr{D}$ is a system of essential arcs in A. Moreover, by our minimality condition on A, every component of A contains at least one component of ∂S, where $S := S^+ \cap V_2$. Finally, note that every component of ∂S is parallel in A to some component of ∂A, for $S^+ \cap V_1$ consists of discs which are essential in V_1 (S^+ is strictly normal).

By what we have checked so far, we are in the situation described in the beginning of §4. Moreover, observe that A has at most $3g-3$ components since this number is the upper bound for the number of components of any system of essential, pairwise disjoint and pairwise nonparallel, simple closed curves in an orientable surface of genus $g \geq 2$ (such as ∂V_2). Thus the proposition is a consequence of the following lemma.

4.3. **Lemma.** *In the situation given in the beginning of* §4 *suppose* ∂S *is contained in* A. *Then* S^+ *can be compressed into a closed* 2*-manifold* S' *such that* $S' \cap V_1$ *is a system of discs contained in* $S^+ \cap V_1$ *and that every disc from* $S' \cap V_1$ *is parallel* (*in* V_1) *to at most one other disc from* $S' \cap V_1$.

Proof of lemma. We divide the proof of this lemma into two cases.

Case 1. S^+ *has no sphere-component.*

Consider $S := S^+ \cap V_2$. Since we are in Case 1, no component of S is a disc. Since S^+ is strictly normal, S consists of incompressible surfaces. Thus, by the innermost-disc-argument, no boundary curve of S can be the boundary of a disc in V_2. Therefore every boundary curve of S has to meet $\mathscr{D}$ since, by our choice of $\mathscr{D}$, the system $\mathscr{D}$ splits V_2 into 3-balls. But, by our choice of A, every component of A contains a component of ∂S which is essential in A, and so $\partial\mathscr{D}$ has to intersect each component of A nontrivially.

Let A_i, $i \geq 1$, be any component of A. Then, by what we have seen above, A_i is an annulus which satisfies the hypothesis of Lemma 4.1. Let I_i be that special interval of $A_i \cap \partial\mathscr{D}$ as given by this lemma, and let $\mathscr{K}_i$ denote the subsystem of all those arcs from $\mathscr{D} \cap S$ with an end-point in I_i. Then $\mathscr{K}_i$ consists of recurrent arcs (in S) joining I_i with one of the (at most) two components of $A_i \cap \partial\mathscr{D}$ neighboring I_i (in $\partial\mathscr{D}$).

Every one of the above subsystems $\mathscr{K}_i$, $i \geq 1$, gives rise to a (possibly empty) system Q_i of pairwise disjoint discs (squares) in $\mathscr{D}$ with the following properties: (1) each component of Q_i intersects $\partial\mathscr{D}$ in precisely two components, (2) $(\partial Q_i - \partial\mathscr{D})^- = Q_i \cap \mathscr{K}_i$, and (3) the number of components of Q_i is as large as possible.

Observe that $Q_i \cap Q_j = \emptyset$ if $i \neq j$, and that, for every $i \geq 1$, at most two arcs from $\mathscr{K}_i$ are not contained in Q_i. But the number of arcs from $\mathscr{K}_i$ equals the number of components of $S^+ \cap \partial V_2$ contained in the annulus A_i. Thus it remains to show that Q_i is empty.

Assume the converse and consider a component C from Q_i. Let k_1, k_2 denote the two components from $(\partial C - \partial\mathscr{D})^-$. Then k_1 as well as k_2 is recurrent in S (they are both components of $\mathscr{K}_i$ and see Lemma 4.1), and denote by B_j, $j = 1, 2$, the disc from $S^+ \cap V_1 = (S^+ - S)^-$ which contains both the end-points of k_j. Since ∂B_1 and ∂B_2 are parallel in A and since V_1 is irreducible, it follows that B_1 and B_2 are parallel in V_1, i.e. $B_1 \cup B_2$ separates a 3-ball E from V_1 with $(\partial E - \partial V_1)^- = B_1 \cup B_2$. Then we find an annulus, B, in $E \cup C$ with $B \cap S^+ = \partial B$. Moreover, B can be easily chosen so that, in addition, ∂B is essential in S^+ (see Lemma 3.4). Using the (essential) annulus-compression along the annulus B and afterwards a small general-position isotopy, the 2-manifold S^+ is compressed into a 2-manifold S^* which intersects V_1 in strictly less discs than S^+, i.e. $b(S^*) < b(S^+)$. Then $a(S^*) > a(S^+)$, by our minimality condition on $c(S^+)$. But this is impossible since essential annulus-compressions do not alter the Euler characteristic at all.

Case 2. *S^+ has at least one sphere component.*

Let S' denote the union of all sphere-components of S^+ and let $S'' := S^+ - S'$. Then certainly S' as well as S'' are strictly normal with respect to the disc-system $\mathscr{D}$ since S^+ is. Thus, by Lemma 3.3, $S' \cap V_2$ consists of discs. This implies that every component of A which contains a component of $S' \cap \partial V_2$ does not contain a component of $S'' \cap \partial V_2$ and vice versa (recall $S^+ \cap V_2$ consists of discs and incompressible surfaces since S^+ is strictly normal). Thus A can be written as the disjoint union of two systems A' and A'' of annuli from A such that A' and A'' contain $S' \cap \partial V_2$, resp., $S'' \cap \partial V_2$.

By definition, S'' has no sphere-components, and so, replacing S^+ and A by S'' and A'', respectively, we are in Case 1 above. Thus S'' can be compressed into a 2-manifold which intersects V_1 in discs whose number does not exceed twice the number of components of A''. By the construction given in Case 1, this can be achieved by using compressions along annuli which do not meet S' (to check this note that $\mathscr{D} \cap S' = \emptyset$ since $S' \cap V_2$ consists of discs and since S' is strictly normal with respect to $\mathscr{D}$). On the other hand observe that any two components of $S' \cap \partial V_2$ contained in one component of A', lie in parallel sphere-components of S^+ and we are allowed to remove one of the latter. Therefore S' can be compressed, without changing S'' at all, so that afterwards the number of components of S' equals that of A'. Altogether, this proves Lemma 4.3 in Case 2.

Thus the proof of Lemma 4.3 and so of Proposition 4.2 is complete. □

5. Improving the estimate

In order to finish the proof of our theorem it remains to sharpen the estimate given in the previous section in the case of orientable 2-manifolds. This in turn will be made possible by our next two results. To formulate them we introduce the notion of a "good" system of 2-handles.

Recall that a *system of* 2-*handles* in the handlebody V_1 is, by its very definition, a system of 3-balls in V_1 each of which meeting ∂V_1 in an essential annulus. That means a system of 2-handles in V_1 is nothing more but a regular neighborhood of a system of essential discs in V_1. A system $\mathscr{E}$ of pairwise nonparallel 2-handles in V_1 is called a *good* (resp. a *very good*) system, provided each component of $(V_1 - \mathscr{E})^-$ is either a 3-ball, or a solid torus which meets $\mathscr{E}$ in at most two (resp. one) component(s).

5.1. **Lemma.** *Let g denote the genus of ∂V_1 and suppose $g \geq 2$. Then any system $\mathscr{E}$ of pairwise nonparallel* 2-*handles in V_1 is good* (*resp. very good*) *whose number of* 2-*handles exceeds $3g - 6$* (*resp.* $3g - 5$).

Proof. Since V_1 is irreducible and since ∂V_1 contains at most $3g-3$ pairwise disjoint and pairwise nonparallel, essential, simple closed curves, it follows that $\mathscr{E}$ is very good if it has $3g-3$ components. Now, observe that there is always a system $\mathscr{E}'$ of pairwise nonparallel 2-handles in V_1 which contains $\mathscr{E}$ as a subsystem and whose number of 2-handles equals $3g-3$. Then, by the previous argument, the boundary of $(V_1 - \mathscr{E}')^-$ consists of 2-spheres alone. More precisely, each component of $(\partial V_1 - \mathscr{E}')^-$ is a thrice punctured sphere.

If $\mathscr{E} \neq \mathscr{E}'$, we reobtain $\mathscr{E}$ from $\mathscr{E}'$ by removing one or two 2-handles, say E_1 and E_2 (possibly $E_2 = \emptyset$), from $\mathscr{E}'$, provided the number of 2-handles from $\mathscr{E}$ exceeds $3g-6$. Consider the boundary of $(V_1 - (\mathscr{E}' - E_1))^-$. This is either a system of 2-spheres or the union of one torus with a system of 2-spheres. Since $(\partial V_1 - \mathscr{E}')^-$ consists of thrice punctured 2-spheres and since $g \geq 2$, it follows that the previous torus has to meet $E' - E_1$ in precisely one component. Thus $\mathscr{E}' - E_1$ is very good. By the same counting argument, it also follows that $\mathscr{E}' - E_1 \cup E_2$ is good. □

5.2. **Lemma.** *Let S^+ be a strictly normal and incompressible 2-manifold in M, but not a system of 2-spheres. Let $\mathscr{E}$ be a minimal system of 2-handles in V_1 containing $S^+ \cap V_1$. Then* (1), (2) *and* (3) *holds:*

(1) *If S^+ is orientable, at least one component of $(V_1 - \mathscr{E})^-$ is different from a 3-ball.*

(2) *If S^+ is orientable and $\mathscr{E}$ is very good, then S^+ can be compressed into a strictly normal and orientable 2-manifold which intersects V_1 in strictly less discs than S^+.*

(3) *If $\mathscr{E}$ is good, then S^+ can be compressed into a 2-manifold which intersects V_1 in discs whose number is strictly less than twice the number of components of $\mathscr{E}$.*

Remark. Note that in (3) the 2-manifold S^+ need not be orientable.

Proof. Observe that, by our choice of $\mathscr{E}$, the intersection

$$A := \mathscr{E} \cap \partial V_1 = \mathscr{E} \cap \partial V_2$$

is a system of essential annuli in ∂V_2. Furthermore, ∂S is contained in the interior of A (where again $S := S^+ \cap V_2$), and every component of ∂S is parallel in A to some component of ∂A since $S^+ \cap V_1$ consists of discs which are essential in V_1 (S^+ is strictly normal). Finally, by our minimality condition on $\mathscr{E}$, each component of A contains at least one component of ∂S. Let $\mathscr{D}$ be any system of discs in V_2 which splits V_2 into 3-balls and such that S^+ is strictly normal with respect to $\mathscr{D}$. Such a system exists since S^+ is strictly normal. It follows that $I := A \cap \partial \mathscr{D}$ is nonempty, for otherwise S^+ has to be a system of 2-spheres which is excluded (see Case 1 of the proof of Lemma 4.3). More precisely, I is a nonempty system of essential arcs in A which intersects each component of ∂S in one point.

Thus, by what we have verified so far, we are again in the situation of Lemma 4.1. Let I_0 be that special component of $I := A \cap \partial\mathscr{D}$ as provided by Lemma 4.1, let A_0 be that component of A containing I_0, and let E_0 be that 2-handle from $\mathscr{E}$ with $E_0 \cap \partial V_2 = A_0$. Finally, let $\mathscr{K}_0$ be the subsystem of all those arcs from $\mathscr{D} \cap S$ with one end-point in I_0.

We now first prove (1) and (2) of Lemma 5.2, and leave the discussion of conclusion (3) for later. For this recall from Lemma 4.1 that all the arcs from $\mathscr{K}_0$ are joining neighboring components of I (in $\partial\mathscr{D}$). Therefore at least one arc, say k_0, from $\mathscr{K}_0$ separates a disc D_0 from $\mathscr{D}$ such that k_0 equals $D_0 \cap S$ and that, moreover, the intersection of the arc $k_0^* := D_0 \cap \partial\mathscr{D}$ with I is a regular neighborhood of points from ∂I in I.

The existence of the arc k_0 with the previously described properties gives rise to the following construction of two annuli—one contained in S^+ and the other one contained in the boundary of $V_2 \cup E$.

To construct these annuli recall from Lemma 4.1 that k_0 is recurrent in S, and let B_0 denote that component of $S^+ \cap V_1 = (S^+ - S)^-$ which contains ∂k_0. Define

$$B_0' := U(B_0 \cup k_0),$$

where the regular neighborhood is taken in S^+. Then B_0' is an annulus (B_0 is a disc and k_0 is recurrent) which is essential in S^+ (it follows from Lemma 3.4 that k_0 is essential in S^+). Now, to construct the second annulus, consider the arc $k_0' = (k_0^* - I)^-$. Since k_0 is recurrent and since S^+ is supposed to be orientable (we are in the proof for (1) and (2)), it follows that k_0' is an arc in the surface $(\partial V_2 - A)^-$ whose both end-points are contained in one component of ∂A_0, and so in one component, say F_0, of $(\partial E_0 - \partial V_1)^-$. Define

$$F_0' := U(F_0 \cup k_0'),$$

where this time the regular neighborhood is taken in the boundary of $V_2^+ := V_2 \cup \mathscr{E}$. Then F_0' is certainly an annulus again.

Pushing k_0 across the disc D_0 and into k_0^*, we isotop S^+ in V_2^+ and without changing $S^+ \cap \operatorname{int}(V_1)$ (in S^+) so that afterwards $B_0' = S^+ \cap \partial V_2^+ = F_0'$. Since B_0' is an essential annulus in S^+ and since, by hypothesis, the 2-manifold S^+ is incompressible in M, it follows that F_0' has to be essential in ∂V_2^+. In particular, at least one component of $(V_1 - \mathscr{E})^-$ has to be different from a 3-ball, proving (1) of Lemma 5.2. Thus we may suppose we are in the proof of (2). In this case, however, recall that $\mathscr{E}$ is supposed to be very good. Therefore the annulus F_0' lies in some component T_0 of ∂V_2^+ which is a torus. Moreover, $F_0 = T_0 \cup \mathscr{E}$ since T_0 intersects $\mathscr{E}$ in precisely one component ($\mathscr{E}$ is very good). Thus $(T_0 - F_0')^-$ is an annulus which does not meet $\mathscr{E}$ at all, and so

$$S' := (S^+ - B_0') \cup (T_0 - F_0')^-$$

intersects $\mathscr{E}$ in strictly less components than S^+.

Since T_0 bounds a solid torus in V_1, it is easily checked that S' is orientable again (S^+ is orientable). In order to complete the proof of (2), it now still remains to show that S^+ can be compressed into S'. To see this simply observe that the compression along the annulus $(T_0 - F_0')^-$ is an essential annulus-compression of S^+ which results in the union of S' with T_0. But T_0 is contained in the handlebody V_1 and therefore can be compressed into the empty set, proving the claim.

Having established (1) and (2) we now finally turn our attention to (3). This time neither S^+ nor the resulting 2-manifold is required to be orientable. Thus we may apply Lemma 4.3. Hence we may suppose that S^+ has been compressed so that every component of A contains at most two components of ∂S. Without loss of generality we may further suppose that every component of A contains precisely two components, for otherwise (3) follows immediately. In this case, however, the system $\mathscr{K}_0$ (see above) consists of precisely two arcs, say k_1, k_2. Without loss of generality these two arcs are not parallel in $\mathscr{D}$, for otherwise the intersection $S^+ \cap V_1$ can be reduced, using compressions (see proof of Lemma 4.3) and (3) again follows.

Now, we proceed as in the above proof of (2) again: As there observe that the arcs k_1 and k_2 give rise to annuli B_1', resp. B_2', in S^+ as well as annuli F_1', resp. F_2', in ∂V_2^+ which are disjoint since k_1 and k_2 are not parallel. Let T_1 resp. T_2 be the components of ∂V_2^+ containing F_1' resp. F_2'. Then possibly $T_1 = T_2$, but in any case $(F_1' \cup F_2') \cap E_0 = (T_1 \cup T_2) \cap E_0$.

Let $C := (T_1 \cup T_2 - F_1' \cup F_2')^-$. Then again C is a system of annuli since F_i', $i = 1, 2$, is an essential annulus in the torus T_i. Furthermore, $C \cap E_0 = \varnothing$ and $C \cap \mathscr{E}$ has not more components than $(F_1' \cup F_2') \cap \mathscr{E}$ ($\mathscr{E}$ is good). Define

$$S' := (S^+ - B_1' \cup B_2') \cup C.$$

Then S' is contained in $V_2^+ - E_0$ and intersects V_1 in no more discs than S^+. In view of Proposition 2.3, it follows either that $S' \cap V_1$ can be reduced, using an isotopy of S' in M, or that S' can be compressed into a strictly normal 2-manifold without changing $S' \cap V_1$. In the first case we are done and in the second we may apply Lemma 4.3. Now, recall $S' \cap V_1$ is contained in $\mathscr{E} - E_0$, and so, by Lemma 4.3, S' can be compressed into a closed 2-manifold which intersects V_1 in discs whose number does not exceed twice the number of components of $\mathscr{E} - E_0$. Property (3) then follows immediately since S^+ can be compressed into S' (by the same argument as given in the proof of (2) above). □

We are now finally in the position to prove the additional remark of Theorem 1.1.

5.3. **Proposition.** *Let (M, V_1, V_2) be a Heegaard-splitting and let S^+ be an orientable 2-manifold. Then S^+ can be compressed either into a system of 2-spheres, or into a (possibly nonorientable) 2-manifold which intersects V_1 in discs whose number is at most $6g - 11$.*

Remark. This proposition has special interest in the case when $g = 2$.

Proof. By Proposition 2.3, we may suppose w.l.o.g. that S^+ is strictly normal and not compressible. If S^+ is a system of 2-spheres, we are done. So we assume the converse. Then observe that S^+ satisfies the hypothesis of Lemma 5.2 and let $\mathscr{E}$ be a minimal system of 2-handles containing $S^+ \cap V_1$. If now $\mathscr{E}$ is not good, then, by Lemma 5.1, $\mathscr{E}$ has at most $3g - 6$ components, and it is easily checked that the proposition follows from Lemma 4.3. If, however, $\mathscr{E}$ is good, then, by Lemma 5.1 and (2) of Lemma 5.2, we may suppose $\mathscr{E}$ has precisely $3g - 5$ components and the proposition follows from (3) of Lemma 5.2. □

The proof of Theorem 1.1 is now complete.

6. Heegaard genus two 3-manifolds

As an illustration of our method, we conclude with a result on incompressible tori in Heegaard genus two 3-manifolds. (Recall the Heegaard genus of a 3-manifold M is defined to be the genus of the smallest Heegaard surface in M.) Incompressible tori in Seifert fibre spaces over the projective plane with two exceptional fibres constitute concrete examples of this situation. The following proposition extends the main result of [Ko 1] where the case of nonseparating tori has been considered (see [Ko 2] for more information on the intersection of tori with Heegaard surfaces).

6.1. **Theorem.** *Let M be an orientable and irreducible 3-manifold with Heegaard genus two. Suppose M contains an incompressible torus T. Then there is a genus two Heegaard splitting (M, V_1, V_2) and an incompressible torus T' such that $T' \cap V_1$ consists of exactly one disc.*

In addition, (1) T' may be chosen to be separating or not according whether T is separating or not, and (2) the conclusion of Proposition 6.1 holds true for all genus two Heegaard splittings of M if T is nonseparating.

Remarks. (1) A 2-manifold S in M will be called *separating* if there are two submanifolds X, Y in M with $X \cap Y = S$ and $X \cup Y = M$; otherwise S is nonseparating. Note that a nonseparating 2-manifold remains nonseparating after disc- as well as annulus-compressions.

(2) By Proposition 5.3, a similar result holds true for Klein bottles as well.

Proof. To begin with, let a genus two Heegaard splitting (M, V_1, V_2) be fixed in advance. Moreover, let an incompressible torus be chosen in M which is nonseparating iff T is and which, in addition, intersects V_1 in the smallest possible number of disc. Without loss of generality, we may suppose that T is already chosen to be this torus. Since T is incompressible, it has to intersect V_1 in at least one disc. If T intersects V_1 in one disc, we are done. Thus we assume the converse, and we have to show that this assumption leads to contradictions.

Case 1. *T is nonseparating.*

By Proposition 5.3, the torus T can be compressed into a 2-manifold T' which intersects V_1 in strictly fewer discs than T. This compression cannot involve a disc-compression, for otherwise T can be isotoped into a torus which intersects V_1 in fewer discs (T is incompressible and M is irreducible); contradicting our minimality condition on $T \cap V_1$. Therefore we may suppose it is an essential annulus-compression, along an annulus A say (see constructions in 4.3 and 5.2).

Since T is a torus (and not a Klein bottle) and since $T \cap U(A)$ consists of exactly two different annuli, say A_1, A_2, it follows that also $(T - U(A))^-$ consists of two different annuli, say B_1, B_2. The annulus B_1 meets either one or two components of $(\partial U(A) - T)^-$.

If B_1 meets both components of $(\partial U(A) - T)^-$, then T' is connected with $\chi(T') = 0$. A simple counting-argument (involving the directions of the normal vectors) shows that T' is 2-sided, and so a torus (and not a Klein bottle). More precisely, it is a separating torus (see our minimal condition on $T \cap V_1$), and so it separates M into two submanifolds, say M_1, M_2. Without loss of generality we may suppose the indices have been chosen so that M_1 contains $U(A)$. Then it follows from the construction of T' that the torus T bounds the submanifold $M_2 \cup U(A)$. This, however, contradicts the fact that we are in Case 1.

If ∂B_1 is contained in one component of $(\partial U(A) - T)^-$, then T' consists of two different components. No component of T' can be a nonseparating torus, for every nonseparating torus in M is incompressible (M is irreducible), and so we would have a contradiction to our minimality condition on $T \cap V_1$. In particular, no component of T' can be null-homologous since then T is homologous to the other component of T'. It therefore follows that both components of T' have to be Klein bottles. This in turn is only possible, if the compression-annulus A meets both sides of the torus T. It follows that T' is a *good pair* of Klein bottles in the sense that there is a torus (near T) intersecting every one of these Klein bottles in a single, nonseparating curve (nonseparating in the torus as well as the Klein bottles). Now, it follows from Proposition 5.3 (see again the constructions in 4.3 and 5.2) the existence of a sequence $T' =: T'_1, T'_2, \dots, T'_n$ of 2-manifolds such that (1) T'_{i+1} is obtained from T'_i by an essential annulus-compression, along some annulus C_i say, (2) T'_{i+1} meets V_1 in no more discs than T'_i, and (3) $T'_n \cap V_1$ consists of at most one disc. By (2) and our minimal condition on $T \cap V_1$, no component of T'_i, $1 \leq i \leq n$, is a nonseparating torus. By (3) and since the handlebody V_2 contains no Klein bottle, we conclude that T'_n contains no good pair of Klein bottles (in contrast to T'_1). In particular, there has to be an index j such that T'_j, but not T'_{j+1}, contains a good pair. Specifically, there is a torus T_0 in M and Klein bottles K_1, K_2 in T'_j such that $T_0 \cap K_i$, $i = 1$ and 2, is a nonseparating curve. At this point observe that M cannot be a Seifert fibre space with a Klein bottle as orbit surface (the rank of the first integral homology group subdominates the Heegaard genus). In particular, at least one component of $(M-U(K_1 \cup T_0 \cup K_2))^-$ contains no essential disc (T_0 is incompressible and M is irreducible). It follows that ∂C_j can be pushed out of $T_0 \cap (K_1 \cup K_2)$, using an isotopy in $K_1 \cup K_2$. By our choice of the index j, it follows furthermore that the compression-annulus C_j has to join K_1 with K_2. Thus, altogether, we have verified that C_j is an annulus, $C_j \cap (K_1 \cup K_2) = \partial C_j$, joining a nonseparatnig curve in K_1 with a nonseparating curve in K_2. In this case, another simple counting-argument (involving the directions of the normal vectors) shows that some component of T'_{j+1} has to be a nonseparating torus. But this contradicts our minimal condition on $T \cap V_1$.

Case 2. *T is separating.*

Let $\mathscr{D}$ be a system of discs in V_2 which splits V_2 into 3-balls and which, in addition, intersects T in curves whose number is as small as possible. Consider the arc-system $T \cap \mathscr{D}$ and let b be an outermost arc, i.e. an arc which separates a disc D_0 from $\mathscr{D}$ with $D_0 \cap T = b$. By Lemma 2.1, it follows that ∂b has to lie in one component B_0 of $T \cap V_1$ (otherwise $T \cap V_1$ could be reduced by some isotopy). Let E be the regular neighborhood of B_0 in V_1, and set $b_0 := D_0 \cap \partial \mathscr{D}$ and $b'_0 := (b_0 - E)'$.

Note that $B_0 \cup b_0$ is not contractible. To see this, simply observe that $B_0 \cup b_0$ can be isotoped across D_0 and into $B_0 \cup b$ and then the claim follows from the fact that T is incompressible in M and that b is essential in T (that b is essential in T follows from Lemma 3.4 since $T \cap V_1$ is minimal and so, by Proposition 2.3, T is strictly normal).

Let W be that component of $(V_1 - E)^-$ which contains b'_0. Now, B_0 may or may not be separating in V_1. But V_1 is a handlebody of genus two, and so, in any case, W is a solid torus and every essential disc in V_1 contained in $W - b_0$ can be properly isotoped in V_1 into E since $B_0 \cup b_0$ is not contractible and since b_0 meets T from one side only (T is separating). Thus we may suppose that $T \cap W = \emptyset$.

Define $N := (M - W)^-$. Then N is a one-relator 3-manifold, in the sense of [Joh 2], with $\partial N = \partial W$ being a torus. In particular, there is an arc t in N such that $(N - U(t))^-$ is a handlebody.

Since $T \cap W = \emptyset$, we have that $T \subset N$. The torus T splits t into a collection of arcs, and let t_1, t_2 be those two arcs from this collection which contain end-points of t. Then t_1 is actually an essential arc in some essential annulus A_1 joining T with ∂N with $A_1 \cap t = t_1$ (see again our construction in 4.3, Case 1). Moreover, either t_2 or $(t - t_2)^-$ lies in such an annulus A_2 as well. We suppose t_2 lies in such an annulus (the other case being similar).

Let F_1, F_2 be the two components of $(\partial N - U(A_1 \cup A_2))^-$, let B_1, B_2 be the two components of $(T - U(A_1 \cup A_2))^-$, and let B'_i, $i = 1, 2$, denote the union of B_i with the two annuli from $(\partial U(A_1 \cup A_2) - (T \cup \partial N))^-$ meeting B_i. Let the indices be chosen so that $B'_1 \cup F_1$ and $B'_2 \cup F_2$ form two disjoint tori.

These tori intersect t in strictly less points than T, and so both of them have to be compressible, by our minimality condition on $T \cap V_1$. More precisely, they bound solid tori, say X_1, X_2, for otherwise they lie in 3-balls (M is irreducible) which is impossible since A_1, A_2 are essential and since T is incompressible.

Suppose B_1' is ∂-parallel in N and $(t \cap T) = 2$, then w.l.o.g. X_1 is a parallelity-region for B_1' (otherwise X_2 is a parallelity-region for B_2'), and so $B_1' \cap t = \emptyset$ (by our minimality condition on $T \cap V_1$). Then sliding the arc t_1 (fixing the end-point $t_1 \cap T$) across this parallelity region and across the arc t_2, we obtain from t a graph which, after another general position isotopy, intersects T in strictly fewer points than t. The union of W with a regular neighborhood of this graph is a handlebody of genus two which is ambient isotopic to V_1. Thus T is ambient isotopic to some torus which intersects V_1 in strictly fewer discs than T. But this is a contradiction to our minimality condition on $T \cap V_1$. In particular, the winding numbers of the annuli A_1, A_2 with respect to the solid tori X_1, X_2 have to be strictly larger than one.

Suppose B_1', say, meets t. Suppose also that B_1' and B_2' are not ∂-parallel in N. Then observe that B_1' has to intersect t in at least two points since B_1' is separating (see construction of X_1). Thus, by Lemma 4.3, there is an essential annulus-compression which reduces the number of intersections of $t \cap T$ by exactly two. More precisely, the corresponding compression-annulus A' may be chosen (see construction in 4.3, Case 1) so that one boundary curve of A' equals a boundary curve of A_1 (or A_2). In particular, A' lies in the complement of $X_1 \cup X_2$. Moreover, $\partial A'$ splits T into two annuli, and the respective unions of the latter annuli with A' form two tori. By our minimality condition on $T \cap V_1$, both of these tori are compressible, for they intersect t in strictly fewer points than T. It follows (see above) that these tori bound solid tori X_3, X_4 and w.l.o.g. we may suppose the indices are chosen so that $X_3 \cap T \subset X_1$. Observe that the winding number of the annulus A' with respect to the solid torus X_3 as well as X_4 has to be strictly larger than one, for otherwise an appropriate isotopy of T across X_3 resp. X_4 would reduce the intersection $T \cap t$ in contradiction to our minimality condition on $T \cap V_1$. But $X_3 \cup X_4$ equals the complement of $X_1 \cup X_2$ and so the union $X_1 \cup X_3$ as well as the union $X_2 \cup X_4$ is a Seifert fibre space over the disc with two exceptional fibres (the winding numbers of A_1 with respect to X_1 and X_2 are strictly larger than one). In particular, neither $X_1 \cup X_3$ nor $X_2 \cup X_4$ contains an essential disc. But the torus $\partial(X_1 \cup X_3)$ intersects t in strictly fewer points than T. Thus, by our minimality condition on $T \cap V_1$, there has to be a compression-disc for $\partial(X_1 \cup X_3)$. By what has been seen before, the disc has to lie in W and so its boundary has to lie in the annulus $X_1 \cap W$. It follows that $A_1 \cap \partial N$ bounds a disc in W which, however, is impossible since $A_1 \cap T$ is essential in T and since T is incompressible. Finally, observe that we get the same contradiction of B_1', say, is ∂-parallel in N (in this case replace X_1 by $X_1 \cup W$ in the above argument).

By what we have seen so far, T has to intersect t in exactly two points. In order to continue, consider the curve $A_1 \cap \partial N$. By construction, this curve lies in the boundary of the solid torus W, and the winding number w of this curve with respect to W is greater than or equal to one (T is incompressible).

If the winding number w is greater than one, then $X_1 \cup W$ is a Seifert fibre space over the disc with two exceptional fibres, and the torus

$$T' := \partial(X_1 \cup U(W))$$

intersects t in two points. But $(N \cap U(W) - X_1)^-$ is a parallelity region and sliding t_1 (fixing the end-point $t_1 \cap T$) across this region and across the arc t_2, we obtain from t a graph which, after a small general position isotopy, intersects T' in strictly fewer points. Thus, by our minimal condition on $T \cap V_1$, the torus T' has to be compressible (see above). But this in turn is also impossible, for $X_1 \cup W$ contains no essential disc (since it is a Seifert fibre space different from a solid torus) and $(N - X_1 \cup W)^-$ contains no essential disc (since X_2 is a solid torus and since T is incompressible).

If the winding number w is one, then $W \cup X_1$ is a solid torus. In fact, it is easily verified that $(W \cup X_1) \cup U(t)$ as well as its complement in M is a handlebody (every incompressible surface splits a handlebody into handlebodies). Now, sliding t_1 along B_1' and across t_2 (fixing the end-point $t_1 \cap T$), we obtain from t a graph t' which, after a general position isotopy, intersects T in exactly one point. But $V_1' := W \cup X_1 \cup U(t')$ as well as $V_2' := (M - V_1')^-$ is a handlebody, and so (M, V_1', V_2') is a genus two Heegaard-splitting such that $T \cap V_1'$ consists of strictly fewer discs than $T \cap V_1$ (namely one disc as opposed to two discs). This is a (formal) contradiction to our minimal choice of the torus T and the Heegaard-splitting (M, V_1, V_2).

Thus, in any case, we obtain a contradiction to our assumption, and so the proof of Proposition 6.1 is complete. □

6.2. *Remarks.* (1) By a similar reasoning as in 6.1, one can also prove the following: If (M, V_1, V_2) is a Heegaard-decomposition of any Haken 3-manifold and if S is a separating, incompressible surface in M intersecting V_1 in a minimal number of discs, then either S intersects V_1 in at most $6g - 11$ discs, or S bounds a 3-manifold obtained from a handlebody by attaching a 2-handle.

(2) The phenomenon encountered in the proof of Proposition 6.1 can be made explicit and gives rise to candidates for inequivalent Heegaard-splittings of genus two.

Added in proof. I am grateful to T. Kobayashi for pointing out an oversight occurring in that subcase of Case 2 in 6.1 which is not carried out but proclaimed similar. The proof remans valid if we add to the hypothesis of Theorem 6.1: *M has no torus which intersects V_1 in two discs and which splits M into a simple 3-manifold* [Joh 1] *and a Seifert fibre space over the disc with two exceptional fibres.*

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