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A TRANSITIVE HOMEOMORPHISM ON THE PSEUDOARC WHICH IS SEMICONJUGATE TO THE TENT MAP

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ABSTRACT. A powerful theorem and construction of Wayne Lewis are used to build two homeomorphisms on the pseudoarc, each of which is semiconjugate to the tent map on the unit interval. The first homeomorphism is transitive, thus answering a question of Marcy Barge as to whether such homeomorphisms exist. The second homeomorphism admits wandering points. Also, it is proven that any homeomorphism on the pseudoarc that is semiconjugate to the tent map and is irreducible with respect to the semiconjugacy must either be transitive or admit wandering points.

A *continuum* is a compact connected metric space. A continuum X is *indecomposable* if every proper subcontinuum of X is nowhere dense in X, and X is *hereditarily indecomposable* if each subcontinuum of X is indecomposable. A continuum X is *chainable* if for each $\varepsilon > 0$ there is a chain $C = \{c_0, \dots, c_n\}$ of open sets of diameter less than ε that covers X. (C is a *chain* means that $c_i \cap c_j \neq \emptyset$ iff $|i - j| \leq 1$.) A continuum X is *homogeneous* if for each $x, y \in X$, there is a space homeomorphism h such that $h(x) = y$. A pseudoarc, which is a nonseparating plane continuum, can be characterized as a homogeneous chainable continuum. It can also be characterized as a hereditarily indecomposable, chainable continuum. Pseudoarcs, although chainable, contain no continuous nontrivial images of arcs. In fact, every nondegenerate subcontinuum of a pseudoarc is itself a pseudoarc. Another extraordinary fact about this continuum is that most continua (in the sense that they form a dense G_δ-set in the space of all continua (Hausdorff metric)) in the plane, or in $\mathbb{R}^n$, $n \geq 2$, or in the Hilbert cube are pseudoarcs. Here we will show that this continuum admits a transitive homeomorphism, thus answering a question of Marcy Barge. In addition, this homeomorphism has the property that it is semiconjugate to the tent map on the unit interval. For more information on pseudoarcs and indecomposable continua, see [Bi1]–[Bi4], [L1]–[L2], [K], [KM], [Kr], and [OT].

Exotic continua have been making their appearance in dynamical systems ever since the early part of this century with the work of C. Carathéodory [C],

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G. D. Birkhoff [Bk], M. Charpentier [Ch], and M. L. Cartwright and J. E. Littlewood [CL1]–[CL2]; all actually encountered indecomposable continua that were playing important roles in determining the behaviors of the systems involved. If X is a compact metric space, $F: X \to X$ is continuous, and A is a closed subset of X such that $F(A) = A$, then A is an *attractor* for F if there is some open set u such that $u \supseteq \overline{F(u)}$ and $\bigcap_{n=1}^{\infty} F^n(u) = A$. Probably the most famous example of an indecomposable continuum arising as an attractor in a dynamical system is the invariant continuum in Smale's horseshoe map on the disc [S]. The attractor for that map is a Knaster continuum, and the dynamics of the map are chaotic on a certain invariant Cantor set contained in the continuum. Knaster continua are also indecomposable chainable continua, but unlike pseudoarcs, they have dense arc-components. Further, perhaps it is worth noting that all chainable continua are nonseparating plane continua which have the fixed point property [Ha].

Recently, evidence has accumulated more dramatically than ever that, as Marcy Barge says, complicated dynamics induce complicated topology. Much of the recent work of Barge and his various coauthors has demonstrated a real connection between indecomposability in invariant continua and complex behavior in dynamical systems in the plane. (See [BM1]–[BM4], [B1]–[B4], and [BG].) Recently, Barge and Gillette [BG] obtained two theorems that apply to solutions of the forced van der Pol equations. Cartwright and Littlewood had investigated these equations in the 1940s and 1950s [CL1]–[CL2], and found that, at certain parameter values, an associated Poincaré homeomorphism admits a certain invariant plane separating continuum. They conjectured that this continuum contains an indecomposable continuum. It follows from Barge and Gillette's work that it *is* an indecomposable continuum.

A continuum X is *circularly chainable* if for each $\varepsilon > 0$, there is a circular chain cover $C = \{c_0, \dots, c_n\}$ of open sets of diameter less than ε. ($C = \{c_0, \dots, c_n\}$ is a *circular chain* means that $c_i \cap c_j \neq \emptyset$ iff $|i - j| \leq 1$ or $i = 0$, $j = n$.) In 1982, Michael Handel [H] gave an example of an area preserving C^∞ diffeomorphism H of the plane that has as its invariant set a hereditarily indecomposable, circularly chainable continuum known as a pseudocircle. His diffeomorphism H is minimal on the invariant pseudocircle P_c. (H is *minimal* on P_c means that if $x \in P_c$, $\{H^n(x) | n \in \mathbb{Z}\}$ is dense in P_c.) Further, he gave a C^∞ diffeomorphism H' of the plane that has the pseudocircle as an attractor and on which it is minimal. (Since it is circularly chainable, the pseudocircle does separate the plane.)

Further, Marcy Barge [B3] has shown that the pseudoarc can be a global attractor in a smooth dynamical system on the plane. If F is a homeomorphism on a compact metric space X, then F is *chaotic* (in the sense of R. Devaney [D]) if it is transitive, has sensitive dependence on initial conditions, and has a dense set of periodic points. (F is *transitive* means that there is a point $x \in X$ such that $O_F(x) = \{F^n(x) | n \in \mathbb{Z}\}$ is dense in X, and F has *sensitive* *dependence on initial conditions* means that there is some $\delta > 0$ such that if $x \in X$ and u is an open set containing x, then there are some y in u and positive integer n such that $d(F^n(x), F^n(y)) > \delta$.) In Barge's construction the pseudoarc is not a chaotic attractor, as it is not transitive, does not have sensitive dependence on initial conditions, and does not have a dense set of periodic points. In Handel's construction the pseudocircle is a chaotic attractor. Can one obtain the pseudoarc as a chaotic attractor for a "nice" plane homeomorphism? No one knows as of yet, but the homeomorphicm constructed here on the pseudoarc is transitive and also has sensitive dependence on initial conditions, and thus perhaps represents a first step in answering that question.

If X is a compact metric space and $T: X \to X$ is continuous, then a point $x \in X$ is a *wandering point* for T if there is an open set u in X such that $x \in u$ and $\{T^{-n}(u) | n$ is a nonnegative integer$\}$ is a disjoint collection of open sets. Let $\Omega(T) = \{x \in X | x$ is not a wandering point for $T\}$. Then $\Omega(T)$, which is known as the nonwandering set of T, is a closed subset of X and its complement, the set of wandering points of X, is open in X.

If $h: X \to X$ is continuous and $f: Y \to Y$ is continuous, then h is *semiconjugate* to f if there is a continuous map $\theta: X \to Y$ such that θ is surjective and $\theta h = f\theta$. (This terminology and notation is from Peter Walters' book [W], which also contains more information on these ideas for the interested reader.)

We will use the following theorem, which appears on page 127 of Peter Walter's book. (In particular, the equivalence of statement (ii) and topological transitivity will be used.)

Theorem A. *The following are equivalent for a homeomorphism $T: X \to X$ of a compact metric space.*

(i) *T is topologically transitive.*
(ii) *Whenever E is a closed subset of X and $T(E) = E$ then either $E = X$ or E is nowhere dense (or, equivalently, whenever U is an open subset of X with $T(U) = U$ then $U = \varnothing$ or U is dense).*
(iii) *Whenever U and V are nonempty open sets then there exists $n \in \mathbb{Z}$ with $T^n(U) \cap V \neq \varnothing$.*
(iv) *$\{x \in X | \overline{O_T(x)} = X\}$ is a dense G_δ-set.*

For us, P denotes a pseudoarc, $I = [0, 1]$, $\mathbb{Z}$ is the integers, and $\mathbb{N}$ is the positive integers. If X is a compact metric space, $H(X)$ denotes its group of self-homeomorphisms. All spaces are compact metric. If $E \subseteq X$, then E^0 denotes the interior of E in X, and ∂E denotes the boundary of E in X.

Our main tool is a theorem due to Wayne Lewis [L1], which is stated below. First we will use Lewis' theorem directly, but in order to get the examples desired, we will actually have to get into the construction in Lewis' proof and do some modifying.

Theorem B. *If $f: X \to X$ is a map of the chainable continuum X into itself, there exist a homeomorphism $h: P \to P$ and a continuous surjection $\phi: P \to X$*

such that $f\phi = \phi h$. *(If* f *is onto, the homeomorphism may be taken to be onto.)*

A background theorem we will need is the following classical, important result of R. H. Bing [Bi1]:

Theorem C. *If* M *is a pseudoarc and* G *is an upper semicontinuous collection of proper subcontinua of* M *filling* M, *the resulting decomposition space* M/G *is topologically equivalent to* M.

Also, we will need the following fact, which is surely known. Its proof is straightforward and will be omitted.

Lemma D. *If* P *is a pseudoarc,* $h \in H(P)$, *and* G *is an upper semicontinuous collection of proper subcontinua of* P *filling* P *such that if* $A \in G$, *then* $h(A) \in G$, *then* P/G *is homeomorphic to* P *and* $\hat{h} \in H(P/G)$, *where* $\hat{h}$ *is the homeomorphism on* P/G *induced by* h *(i.e.,* $\hat{h}(A)$ *is defined as the set* $h(A)$ *for* $A \in G$).

Consider $f\colon I \to I$ defined by

$$f(x) = \begin{cases} 2x, & x \in [0, 1/2], \\ 2 - 2x, & x \in [1/2, 1]. \end{cases}$$

The map f is called the tent map and, if for $x \in I$, $O_f^+(x) = \{f^n(x) | n \in \mathbb{N} \cup \{0\}\}$, then $\{x \in I | O_f^+(x)$ is dense in $I\}$ is a dense G_δ-set in I. (See [D, p. 52] and [W, p. 127].)

Now apply Lewis' theorem to obtain $\Theta\colon P \to I$, a continuous surjection, and $h \in H(P)$ such that $\Theta h = f\Theta$. If h has the additional properties that (1) if P' is a nondegenerate subcontinuum of P, then $\Theta(P')$ is a nondegenerate interval in I, and (2) if P' is a proper subcontinuum of P, then $\Theta(P') \neq I$ or $h(P') \neq P'$, we will say that h is *irreducible with respect to the semiconjugacy.* The next two lemmas demonstrate that if $h \in H(P)$ is semiconjugate to the tent map, then h induces a homeomorphism on P semiconjugate to the tent map which is irreducible.

Lemma 1. *Suppose that* $f\colon I \to I$ *is a continuous surjection,* $\Theta\colon P \to I$ *is a continuous surjection,* $h \in H(P)$, *and* $f\Theta = \Theta h$. *Let* $M = \{\Theta^{-1}(a) | a \in I\}$, $R = \{K | K$ *is a component of some* $\Theta^{-1}(a) \in M\}$. *Then* R *is an upper semicontinuous decomposition of* P *and* $P/R \cong P$. *Further,* $\hat{h}\colon P/R \to P/R$ *defined by* $\hat{h}(K) = h(K)$ *is a homeomorphism in* $H(P/R)$, *and* $\widehat{\Theta}\colon P/R \to I$ *defined by* $\widehat{\Theta}(K) = a$, *where* $K \subseteq \Theta^{-1}(a)$ *is continuous and onto and* $\widehat{\Theta}\hat{h} = f\widehat{\Theta}$.

Proof. Since M is an upper semicontinuous decomposition of P and $R \cap \Theta^{-1}(a)$ is an upper semicontinuous decomposition of $\Theta^{-1}(a)$ for $a \in I$, R is an upper semicontinuous decomposition of P into points and pseudoarcs. From Theorem C, it follows that P/R is homeomorphic to P. For $K \in P/R$, define $\widehat{\Theta}(K) = \Theta(K)$. Since $\Theta(K)$ is degenerate by definition, $\widehat{\Theta}$ is well-defined and onto. It is also continuous, for if K_1, K_2, ... converges to K

in P/R, then for $i \in \mathbb{N}$, $K_i \subseteq \Theta^{-1}(a_i)$ for some $a_i \in I$, and $K \subseteq \Theta^{-1}(a)$ for some $a \in I$. In the quotient topology in P/M, $\Theta^{-1}(a_1)$, $\Theta^{-1}(a_2), \ldots$ converges to $\Theta^{-1}(a)$, and $a_1, a_2, \ldots$ converges to a. That $\hat{h} \in H(P/R)$ follows from our Lemma D.

Suppose $K \in R$ and $x \in K$. Then $K \subseteq \Theta^{-1}(a)$ for some $a \in I$ and $\Theta h(x) = f\Theta(x)$, and $\Theta(x) = a$ imply $\widehat{\Theta}(K) = a$ and $f\widehat{\Theta}(K) = f(a)$. Also, $\widehat{\Theta}\hat{h}(K) = \widehat{\Theta}(h(K))$, $h(x) \in h(K)$ and $\Theta h(x) = f\Theta(x) = f(a)$ imply $f(a) = f\widehat{\Theta}(K) = \widehat{\Theta}\hat{h}(K)$. □

Lemma 2. *Suppose $h \in H(P)$, $\Theta: P \to I$ is a continuous surjection, $f: I \to I$ is a continuous surjection, and $f\Theta = \Theta h$. Then there is a subcontinuum P' of P such that*

(1) $\Theta(P') = I$,

(2) $h(P') = P'$, *and*

(3) *if P'' is a proper subcontinuum of P', then either $\Theta(P'') \neq I$ or $h(P'') \neq P''$.*

Proof. Let $\mathbf{C} = \{\widehat{P} | \widehat{P}$ is a subcontinuum of P, $\Theta(\widehat{P}) = I$, and $h(\widehat{P}) = \widehat{P}\}$. Note that $P \in \mathbf{C}$, so $\mathbf{C} \neq \varnothing$. Consider a maximal monotonic subcollection $\widehat{\mathbf{C}}$ of $\mathbf{C}$ that contains P, and let $P' = \bigcap \widehat{\mathbf{C}}$. Now $P' \neq \varnothing$ and, further, $P' \in \widehat{\mathbf{C}}$ but no proper subcontinuum of P' is in $\widehat{\mathbf{C}}$. □

If X is an indecomposable continuum and $p \in X$, then the *composant* C of p in X is the union of all proper subcontinua in X that contain p. Composants of indecomposable continua are always first category connected σ-compact subsets. Two different composants do not intersect and each indecomposable continuum has $\mathbf{c}$ of them.

Theorem 3. *Suppose h is a homeomorphism on the pseudoarc that is semiconjugate to the tent map f on I, and that Θ denotes a continuous surjection from P to I such that $f\Theta = \Theta h$. Suppose h is irreducible with respect to this semiconjugacy. If $p_0 \in P$ such that $h(p_0) = p_0$ (there must be such a point, since P has the fixed point property), and P_0 is a nondegenerate continuum containing p_0, then $\bigcup_{n\in\mathbb{Z}} h^n(P_0)$ is the composant of P that contains p_0.*

Proof. Since $p_0 \in \bigcap_{n\in\mathbb{Z}} h^n(P_0)$, $B = \bigcup_{n\in\mathbb{Z}} h^n(P_0)$ is connected and is a subset of the composant of P containing p_0. If $\overline{B} \neq P$, $\overline{B}$ is nowhere dense in P and either $\Theta(\overline{B}) \neq I$ or $h(\overline{B}) \neq \overline{B}$. But $h(B) = B$, so $h(\overline{B}) = \overline{B}$ and it must be the case that $\Theta(\overline{B}) \neq I$. However, $\Theta(\overline{B})$ is a nondegenerate interval, so $\Theta(\overline{B}) = [a, b]$ where either $a > 0$ or $b < 1$.

Also, $\Theta(P_0)$ is a nondegenerate interval, and there is $p \in P_0$ such that $\Theta(p) = \alpha$ where $\alpha \in A = \{\tilde{a} \in I | O_f^+(\tilde{a})$ is dense in $I\}$. Hence, there is $n \in \mathbb{N}$ such that $f^n(\alpha) \notin [a, b] = \Theta(\overline{B})$. But $p \in P_0$, and $h^n(p) \in h^n(P_0) \subseteq B$, so $\Theta h^n(p) \in \Theta(B) \subseteq [a, b]$, which is a contradiction.

Then B is dense in P and $\overline{B} = P$. If $B \neq C$, the composant of P containing p_0, there is $q \in C - B$ and there is a proper subcontinuum K such that p_0 and q are in K. Since $p_0 \in K \cap h^n(P_0)$ for $n \in \mathbb{Z}$, $h^n(P_0) \subseteq K$ for each n, for otherwise $K \subset B$. (Recall that if two continua in the pseudoarc intersect, then one contains the other.) But this will not work either, for now $B \subset K$ with B dense in P and K nowhere dense in P. □

Theorem 4. *Suppose h is a homeomorphism on the pseudoarc P that is semiconjugate to the tent map on I, and that Θ denotes a continuous surjection from P to I such that $f\Theta = \Theta h$ with h irreducible with respect to this semiconjugacy. Then either the homeomorphism h admits a wandering point or h is transitive.*

Proof. Suppose h does not admit wandering points and h is not transitive. Theorem A implies that there must exist a closed set E, with nonempty interior, such that $h(E) = E$, and $E \neq X$. Without loss of generality, let us assume that $\overline{E^0} = E$. Now h must admit a fixed point p_0, and either $p_0 \in E$ or $p_0 \in \overline{P - E}$. Let us assume that $p_0 \in E$. There is a sequence $x_1, x_2, \ldots$ converging to p_0 such that $x_j \in E^0$. For each pair i, j of positive integers there is an open set u_{ij} such that $x_i \in u_{ij} \subseteq S_{2^{-j}}(x_i) \cap E^0$, where $S_{2^{-j}}(x_i)$ denotes the 2^{-j} neighborhood about x_i. Further, there is an open set v_{ij} such that $x_i \in v_{ij} \subseteq \overline{v_{ij}} \subseteq u_{ij}$, and there is $\varepsilon_{ij} > 0$ such that if Q is a component of $\overline{u_{ij}}$ which intersects v_{ij}, then $\Theta(Q)$ has diameter greater than ε_{ij}. Because of the way f expands nondegenerate intervals, there is some positive integer n'_{ij} such that if $n \geq n'_{ij}$, $f^n\Theta(Q) = [0, 1]$. Since x_i is not a wandering point, it follows that for infinitely many n, $h^n(v_{ij}) \cap v_{ij} \neq \emptyset$.

Choose an $n > n'_{ij}$ such that $v_{ij} \cap h^n(v_{ij}) \neq \emptyset$. If $y \in h^n(v_{ij}) \cap v_{ij}$, then $y = h^n(y')$ for some $y' \in v_{ij}$, and if C' denotes the component of $\overline{u_{ij}}$ that contains y', $y \in h^n(C') \subseteq h^n(\overline{u_{ij}})$. Thus $C' \cap v_{ij} \neq \emptyset$ and $h^n(C') \cap v_{ij} \neq \emptyset$. Because of the way f expands nondegenerate intervals, and the fact that $\Theta(C')$ contains an interval of length at least ε_{ij}, we may assume that $\Theta(h^n(C')) = [0, 1]$ for sufficiently large n. Let one such $h^n(C')$ be denoted by D_{ij}. Then, in the Vietoris topology, there is a continuum D_i that contains x_i, and is a limit continuum of D_{i1}, $D_{i2}, \ldots$. Thus, $\Theta(D_i) = [0, 1]$ and $D_i \subseteq E$. But then, there is a continuum D containing p_0 such that D is a limit continuum of the sequence $D_1, D_2, \ldots$ of continua, $\Theta(D) = [0, 1]$, $D \subseteq E$. Since D cannot be degenerate, $\bigcup_{n \in \mathbb{Z}} h^n(D)$ is dense in P (Theorem 3), and $\bigcup_{n \in \mathbb{Z}} h^n(D) \subseteq E$, which is a contradiction. □

What we would like to be able to do is to conclude that a homeomorphism $h \in H(P)$ semiconjugate to the tent map must be transitive. But we cannot, and in fact it need not be as we shall see later. However, this author does not know whether or not a homeomorphism $h \in H(P)$ semiconjugate to the tent map and irreducible with respect to the semiconjugacy has to be transitive. In other words, in this situation perhaps it is not possible for a homeomorphism on the pseudoarc to have wandering points.

If we go into Lewis' construction and do some modifying, we can construct a homeomorphism on P which is transitive, semiconjugate to the tent map, and irreducible with respect to the semiconjugacy. After doing that, we will construct another homeomorphism on P semiconjugate to the tent map, but this one will have wandering points. (It is probably not irreducible with respect to the semiconjugacy.) Before we can do these things, we need to express the tent map in terms of chains. Also, we need some background and notatation.

A chain $C = \{c_0, \dots, c_n\}$ is *taut* whenever $c_i \cap c_j \neq \varnothing$ if and only if $\overline{c}_i \cap \overline{c}_j \neq \varnothing$. A chain covers a set A *essentially* if there is a continuum Q contained in A such that each link contains a point of Q not in the closure of any other link. An open set o in a space X is *regular* if $\overline{o}^0 = o$. A chain is *regular* if its links are regular. In the discussion that follows we will assume that our open chain covers are regular, taut, and essential.

If B is a collection of sets, then B^* denotes the union of the sets in B. If C is a collection of sets, then C is an *amalgamation* of B if $B^* = C^*$ and each set in C is the union of some sets in B. If the closure of each set in B is a subset of a set in C, then B is said to *closure refine* C. The chain C *properly covers* the chain B if B closure refines C and B does not refine any proper subchain of C.

If $C = \{c_0, \dots, c_n\}$ is an open chain in a space X (which does not necessarily cover X), then for $c \in C$,

$$i(c, C) = \{y \in c \mid y \notin \overline{c}' \text{ for } c' \in C - \{c\}\}.$$

If $m, n \in \mathbb{N} \cup \{0\}$, $m < n$, let $I[m, n] = \{m, m+1, \dots, n\}$. A surjection $f\colon I[m', n'] \to I[m, n]$ is called a (*light*) *pattern* provided $|f(i+1) - f(i)| \leq 1$ ($|f(i+1) - f(i)| = 1$, respectively) for $i = m', \dots, n'-1$. If $V = \{V_{m'}, V_{m'+1}, \dots, V_{n'}\}$ and $U = \{U_m, U_{m+1}, \dots, U_n\}$ are chain covers of the compactum X, and $f\colon I[m', n'] \to I[m, n]$ is a pattern, we will say that V *follows the pattern* f *in* U provided $V_i \subseteq U_{f(i)}$ for each $i \in I[m', n']$. We call f a *pattern on* U.

If the chain $C = \{c_0, \dots, c_m\}$ refines the chain $D = \{d_0, \dots, d_n\}$, then C is *crooked* in D provided that for every p, s, i, j, where $j > i+2$, $c_p \subseteq d_i$, and $c_s \subseteq d_j$, there exist q, r with $c_q \subseteq d_{j-1}$, $c_r \subseteq d_{i+1}$, and either $p < q < r < s$, or $p > q > r > s$.

The following fact is used in the constructions that are coming. (This lemma appeared in this form in Lewis' paper [L2].)

Lemma E. *If X is a nondegenerate chainable continuum such that for each chain C covering X and $\varepsilon > 0$, there is a chain D of mesh less than ε which covers X and is crooked in C, then X is a pseudorac.*

In the constructions that follow, we have an extensive need to indicate sequences of chains, specific subchains of chains, links of chains, and patterns chains follow in other chains. With that in mind, we will make the following notational conventions: chains will be denoted with uppercase letters, and possibly additional symbols, and links of chains with the associated lowercase letters, associated symbols, and link numbers. So, for example,

$$\begin{aligned} C_1 &\equiv \{c(1, 0), \dots, c(1, m)\} \equiv C_1[0, m], \\ \widetilde{D}_2 &\equiv \{\tilde{d}(2, k), \dots, \tilde{d}(2, l)\} \equiv \widetilde{D}_2[k, l], \\ F &\equiv \{f(1), \dots, f(n)\} \equiv \{f_1, \dots, f_n\} \equiv F[1, n]. \end{aligned}$$

If A and B are chains, then $A \vee B = \{a \cap b | a \in A, b \in B$ and $i(a, A) \cap i(b, B) \neq \varnothing\}$.

Please note that what follows is largely just Lewis' construction applied to this particular function, the tent map. We have relaxed some of his requirements and sacrificed some of his efficiency (for example, we will gain control over mesh size only slowly), but also have put in more details than Lewis himself did in the hope that the reader will more easily understand the construction and then the modifications necessary to achieve our ends.

1. **The tent map in terms of chains.** For convenience, we will use chains whose links are closed neighborhoods of I, rather than open sets. For $i \geq 1$, let

$$\begin{aligned} C_i &= \left\{ \left[\frac{j}{2^{i+2}}, \frac{j+1}{2^{i+2}}\right] \middle| 0 \leq j < 2^{i+2} \right\} = \{c(i, j) | 0 \leq j < 2^{i+2}\}, \\ A_i &= \{c(i, j) \cup c(i, 2^{i+2} - j - 1) | 0 \leq j < 2^{i+1}\} = A_i[0, 2^{i+1} - 1], \\ B_i &= \{c(i, 2j) \cup c(i, 2j+1) | 0 \leq j < 2^{i+1}\} = B_i[0, 2^{i+1} - 1]. \end{aligned}$$

It is indeed the case that $B_{i+1} = C_i$, but these chains will be playing different roles in our construction, and we need them both.

Define $[[\ ]]: \mathbb{R} \to \mathbb{Z}$ by $[[r]] =$ greatest integer $\leq r$. For $i \geq 1$, define $\beta_i: I[0, 2^{i+2} - 1] \to I[0, 2^{i+1} - 1]$ by

$$\beta_i(j) = \begin{cases} j & \text{for } j \in I[0, 2^{i+1} - 1], \\ 2^{i+2} - j - 1 & \text{for } j \in I[2^{i+1}, 2^{i+2} - 1], \end{cases}$$

and for $i \geq 0$ define $\alpha_i: I[0, 2^{i+2} - 1] \to I[0, 2^{i+1} - 1]$ by $\alpha_i(j) = [[j/2]]$ for $0 \leq j < 2^{i+2}$. For each i, α_i and β_i are patterns; and C_i follows α_i in B_i, A_{i+1} follows α_i in A_i, B_{i+1} follows α_i in B_i, and C_i follows β_i in A_i. (See Figure 1.)

For $x \in I$, there is an infinite sequence of integers $j(x, 1)$, $j(x, 2), \dots$ such that $x \in a(i, j(x, i))$ and $\alpha_i(j(x, i+1)) = j(x, i)$ for $i \in \mathbb{N}$. If we define $f(x) = \bigcap_i b(i, j(x, i))$, then $f: I \to I$ is the tent map.

2. **Lifting the tent map to a pseudoarc homeomorphism.** For $k, i \in \mathbb{N}$, we will be choosing $\widehat{F}_i$ to be an open taut chain in the plane such that

(1) $\widehat{F}_i = \{\hat{f}(i, j) | 0 \leq j < 2^{i+3}\}$,
(2) if $i > k$, $\widehat{F}_i$ closure refines $\widehat{F}_k$, and

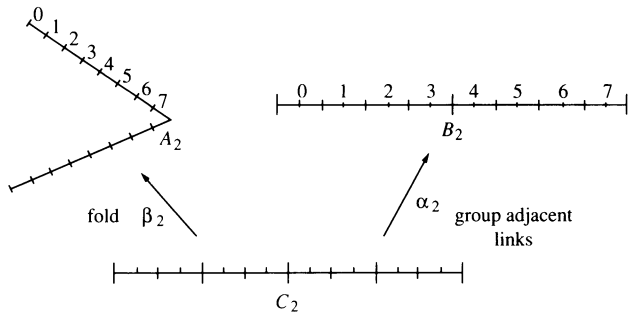

FIGURE 1

(3) $\widehat{F}_{i+1}$ follows α_{i+2} in $\widehat{F}_i$, with $\bigcap_{i=1}^{\infty} \hat{f}(i, j_i) \neq \emptyset$, closed, and nowhere dense in $\mathbb{R}^2$ whenever $j_1, j_2, \ldots$ is an infinite sequence such that $\alpha_{i+2}(j_{i+1}) = j_i$.

Then we define for $i \in \mathbb{N}$

(4)

$$\text{the chain } F_i = \{\hat{f}(i, j) \cup \hat{f}(i, 2^{i+3} - j - 1) | 0 \leq j < 2^{i+2}\} = F_i[0, 2^{i+2} - 1],$$

(5) the chain $E_i = \{f(i, 2j) \cup f(i, 2j+1) | 0 \leq j < 2^{i+1}\} = E_i[0, 2^{i+1} - 1]$,

(6) the chain $\widetilde{E}_i = \{\hat{f}(i, 2j) \cup \hat{f}(i, 2j+1) | 0 \leq j < 2^{i+2}\} = \widetilde{E}_i[0, 2^{i+2} - 1]$, and

(7)

$$\text{the chain } D_i = \{f(i, j) \cup f(i, 2^{i+2} - j - 1) | 0 \leq j < 2^{i+1}\} = D_i[0, 2^{i+1} - 1].$$

Initially we will only be able to choose $\widehat{F}_1$ (which then determines F_1, $\widehat{E}_1$, E_1, and D_1), and can only choose $\widehat{F}_2$ after having constructed some other chains which are needed to start building both our pseudoarc and our pseudoarc homeomorphism. We will have to alternate back and forth all the way down choosing step-by-step our sequences of chains.

The F_i chains for the pseudoarc and pseudoarc homeomorphism construction correspond to the C_i chains in the tent map construction, while the D_i (respectively, E_i) chains correspond to the A_i (respectively, B_i) chains. Please refer to Figures 1–4 as an aid in understanding the somewhat complicated construction we have started and will be continuing. (For obvious reasons, the figures are simplified. In particular, crookedness is only indicated.)

If $\varepsilon > 0$, x is a point in $\mathbb{R}^2$, $S_\varepsilon(x)$ will denote the ε-neighborhood of x with respect to the usual plane metric. Further, if $A \subseteq \mathbb{R}^2$, $S_\varepsilon(A) = \{y \in \mathbb{R}^2 | d(y, z) < \varepsilon \text{ for some } z \in A\}$.

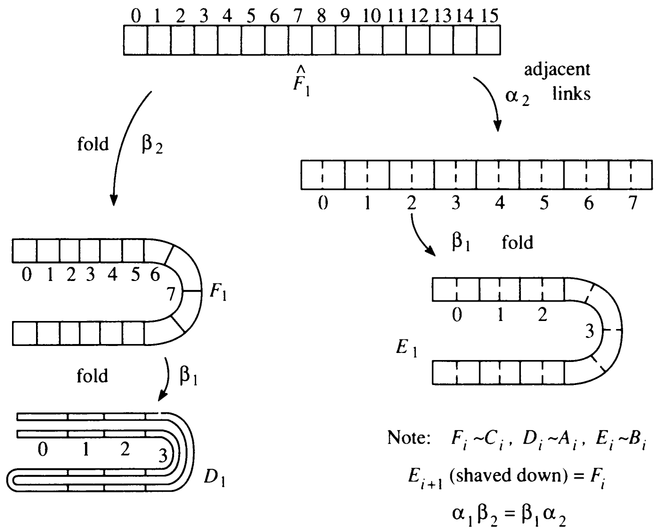

FIGURE 2

Note that D_{i+1} follows α_i in D_i, as does E_{i+1} in E_i, while F_{i+1} and $\widetilde{E}_{i+1}$ follow α_{i+1} in F_i and $\widetilde{E}_i$, respectively, and $\widehat{F}_{i+1}$ follows α_{i+2} in $\widehat{F}_i$. Also, F_i follows β_i in D_i, and $\widetilde{E}_i$ follows β_i in E_i, and E_{i+1} closure refines F_i. Define $\delta_i\colon I[0, 2^{i+2}-1] \to I[0, 2^{i+2}-1]$ by $\delta_i(j) = j$, so that E_{i+1} follows δ_i in F_i. (Please see Figure 2.)

Choose a chain G_1 with the following properties. (See Figure 3.)

(8) G_1 refines and is crooked in F_1, and $G_1 = G_1[0, n_1]$.

(9) G_1 follows the pattern η_1 in F_1 with η_1 light.

(10) $G_1^* \cap i(\hat{f}(1, 0), \widehat{F}_1) \neq \varnothing$ and $G_1^* \cap i(\hat{f}(1, 2^4-1), \widehat{F}_1) \neq \varnothing$.

(11) If M_1 is an arc which is a nerve for G_1, $G_1^* \subseteq S_{2^{-4}}(M_1)$.

(12) If $f \in F_1$, $f \cap G_1^*$ is a union of links of G_1.

There is a chain H_1 such that

(13) $H_1 = H_1[0, n_1]$ follows η_1 in $\widetilde{E}_1$ (which follows β_1 in E_1),

(14) $H_1^* \cap i(\hat{f}(1, 0), \widehat{F}_1) \neq \varnothing$ and $H_1^* \cap i(\hat{f}(1, 2^4-1), \widehat{F}_1) \neq \varnothing$,

(15) if $\tilde{e} \in \widetilde{E}_1$, $\tilde{e} \cap H_1^*$ is a union of links of H_1, $\overline{H}_1^* \subseteq G_1^*$, and

(16) if $h(1, s) \in H_1$, then $h(1, s)$ intersects any link of $\widetilde{E}_1$ adjacent to $\tilde{e}(1, \eta_1(s))$. (See Figure 3.)

(To construct H_1, think of sticking an arc (which will *not* be a nerve for H_1), through G_1^*, so that the arc has wiggles both because of the way G_1 sits in F_1 and the way we want H_1 to sit in $\widetilde{E}_1$. Also, when the arc has to turn around because of the H_1 in $\widetilde{E}_1$ pattern (η_1), make sure that it goes almost all the way to both of the adjacent link intersection sets (if not in an end link). This is because the F_1 links cut the $\widetilde{E}_1$ links, and that cut has already determined somewhat the surjection Θ.

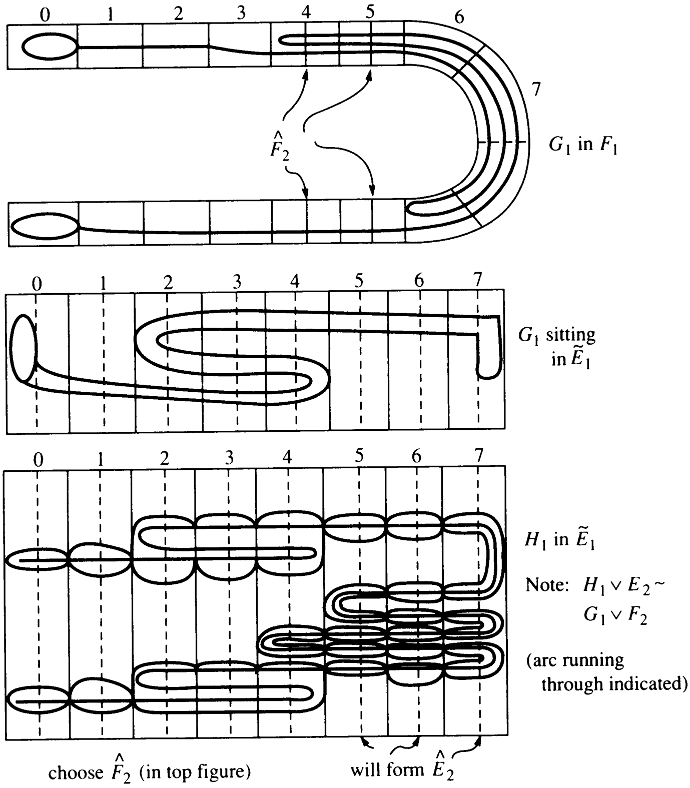

FIGURE 3

Now it is time to choose $\hat{F}_2$. Roughly, all we will be doing is splitting the links of $\hat{F}_1$ by splitting the links of H_1 and G_1 (the α-pattern). But H_1 is *already* split by $\hat{F}_1$, although the links of the resulting chain $\tilde{F}_1 = \{f \cap H_1^* | f \in \hat{F}_1\}$ need to be trimmed down some. Choose $\hat{F}_2$ so that $\hat{F}_2$ closure refines $\tilde{F}_1$, follows α_4 in $\tilde{F}_1$, and if $\tilde{g}$ is in $\tilde{G}_1 = \{g \cap \hat{F}_2^* | g \in G_1\}$, then $\tilde{g}$ is split into exactly two *important* pieces by $\hat{F}_2$. (See Figure 3. Turnaround links are actually split into three pieces; other links are split into two pieces.) Once $\hat{F}_2$ is chosen, F_2, $\tilde{E}_2$, E_2, and D_2 are all automatically determined and we are ready to proceed. Also, (if the fattened up arc that is H_1^* is skinny enough) we may assume that $H_1 \vee \tilde{E}_2$ is a chain, it refines both H_1 and $\tilde{E}_2$, and the chain $G_1 \vee F_2$ has the same number of links as $H_1 \vee \tilde{E}_2$ does. Moreover, it is associated with $H_1 \vee \tilde{E}_2$ in a very nice way, so that we will be able to set chains up as follows. (Note that the pieces into which $\hat{F}_2$ splits each link of G_1 are the links of $G_1 \vee F_2$.)

There is a chain H_2 in H_1^* such that

(17) H_2 refines and is crooked in both H_1 and $\tilde{E}_2$;
(18) if M_2 is an arc which is a nerve for H_2, $H_2^* \subseteq S_{2^{-5}}(M_2)$;
(19) if $\tilde{e} \in \tilde{E}_2$, $\tilde{e} \cap H_2^*$ is a union of links of H_2;
(20) H_2 follows the pattern η_2 in $\tilde{E}_2$ and the pattern ξ_1 in H_1 with η_2 a light pattern;

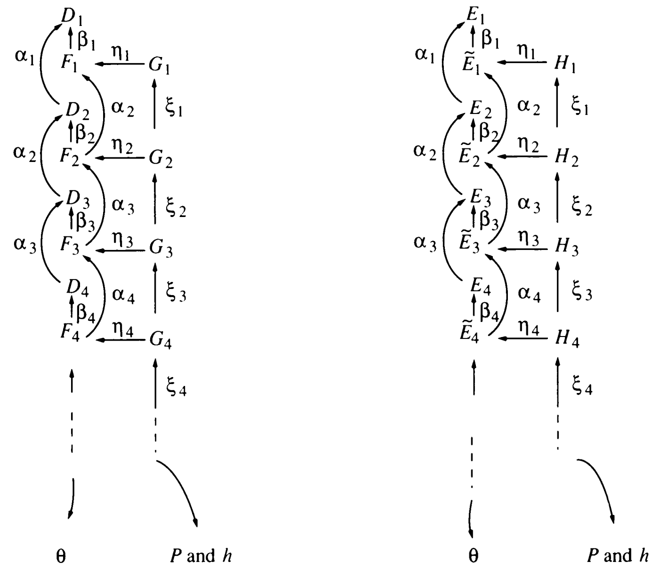

FIGURE 4

(21) $H_2^* \cap i(\hat{f}(2, 0), \widehat{F}_2) \neq \varnothing$, and $H_2^* \cap i(\hat{f}(2, 2^5 - 1), \widehat{F}_2) \neq \varnothing$; and

(22) if $h(2, s) \in H_2$, then $h(2, s)$ intersects any link of $\widetilde{E}_2$ adjacent to $\tilde{e}(1, \eta_1(s))$.

Then it is possible to put a chain G_2 in H_2^* so that

(23) $G_2 = G_2[0, n_2]$ refines G_1 and F_2;

(24) G_2 follows η_2 in F_2 and ξ_1 in G_1;

(25) $G_2^* \cap i(\hat{f}(2, 0), \widehat{F}_2) \neq \varnothing$ and $G_2^* \cap i(\hat{f}(2, 2^5 - 1), \widehat{F}_2) \neq \varnothing$; and

(26) if $f \in F_2$, $f \cap G_2^*$ is a union of links of G_2.

Having G_2, we now choose $\widehat{F}_3$ (and thus, F_3, $\widetilde{E}_3$, E_3, and D_3), consider the collection $G_2 \vee F_3$, and find a refining chain G_3, etc. (See Figure 4.) Continue this process, constructing the sequences of chains $G_1, G_2, \ldots$ and $H_1, H_2, \ldots$ such that

(27) G_i follows η_i in F_i, G_{i+1} follows ξ_i in G_i with G_{i+1} crooked in both F_{i+1} and G_i;

(28) H_i follows $\beta_i \eta_i$ in E_i, H_{i+1} follows ξ_i in H_i; and, additionally,

(29) $\overline{G_{i+1}^*} \subseteq G_i^*$, $\overline{H_{i+1}^*} \subseteq H_i^*$; $\overline{G_{i+1}^*} \subseteq H_i^*$, $\overline{H_{i+1}^*} \subseteq G_i^*$; and

(30) $\lim_i \operatorname{mesh} G_i = \lim_i \operatorname{mesh} H_i = 0$.

Let $P = \bigcap_{i=1}^{\infty} G_i^* = \bigcap_{i=1}^{\infty} H_i^*$. It follows from Lemma E that P is a pseudoarc. Define $h \in H(P)$ by $h(x) = \bigcap_{i=1}^{\infty} h(i, j(x, i))$, where $j(x, 1)$, $j(x, 2)$, $\ldots$ is an infinite sequence of integers such that for each i

(31) $x \in g(i, j(x, i))$, and

(32) $\xi_i(j(x, i+1)) = j(x, i)$.

Define $\Theta: P \to I$ by $\Theta(x) = \bigcap_{i=1}^{\infty} c(i, \eta_i(j(x, i)))$.

Let us verify that $f\Theta(x) = \Theta h(x)$. First, we refer the reader to Figure 4, and emphasize that the diagrams in those sequences of chains and patterns commute, and that E_{i+1} follows δ_i in F_i, where $\delta_i(j) = j$. Then

$$\begin{aligned} f\Theta(x) &= f\left(\bigcap_{i=1}^{\infty} c(i, \eta_i j(x, i))\right) = f\left(\bigcap_{i=1}^{\infty} a(i, \beta_i\eta_i j(x, i))\right) \\ &= \bigcap_{i=1}^{\infty} b(i, \beta_i\eta_i j(x, i)) = \bigcap_{i=1}^{\infty} b(i, \beta_i\eta_i\xi_i j(x, i+1)) \\ &= \bigcap_{i=1}^{\infty} b(i, \beta_i\alpha_{i+1}\eta_{i+1}(j(x, i+1))) \\ &= \bigcap_{i=1}^{\infty} b(i, \alpha_i\beta_{i+1}\eta_{i+1}(j(x, i+1))) \end{aligned}$$

and

$$\begin{aligned} \Theta h(x) &= \Theta\left(\bigcap_{i=1}^{\infty} h(i, j(x, i))\right) = \Theta\left(\bigcap_{i=1}^{\infty} e(i, \beta_i\eta_i j(x, i))\right) \\ &= \Theta\left(\bigcap_{i=1}^{\infty} e(i+1, \beta_{i+1}\eta_{i+1} j(x, i+1))\right) \\ &= \Theta\left(\bigcap_{i=1}^{\infty} f(i, \delta_i\beta_{i+1}\eta_{i+1} j(x, i+1))\right) \\ &= \bigcap_{i=1}^{\infty} c(i, \beta_{i+1}\eta_{i+1} j(x, i+1)) \\ &= \bigcap_{i=1}^{\infty} b(i, \alpha_i\beta_{i+1}\eta_{i+1} j(x, i+1)). \end{aligned}$$

Thus, $f\Theta(x) = \Theta h(x)$.

Also, with careful choices of $\widehat{F}_i$ and G_i^* or H_i^* at each level, we may assume that the preceding construction yields the pseudoarc P and surjection Θ such that if P' is a nondegenerate subcontinuum of P, then $\Theta(P')$ is also nondegenerate. □

3. **A transitive homeomorphism on the pseudoarc.** In order to ensure that our homeomorphism h will be transitive, we need to put some additional requirements on the chains in our sequences $G_1, G_2, \dots$ and $H_1, H_2, \dots$ while retaining all those already noted.

Choose G_1 so that the first and last links of G_1 are in $\hat{f}(1, 0)$, no other links of G_1 are in $\hat{f}(1, 0)$, and these first and last links both intersect $i(\hat{f}(1, 0), \widehat{F}_1)$. Then choose H_1 so that $h(1, 0) \cap i(g(1, 0), G_1) \neq \varnothing$, $h(1, n_1) \cap i(g(1, 0), G_1) \neq \varnothing$, and then continue: for each i, $g(i, 0) \cup g(i, n_i) \subseteq \hat{f}(i, 0) \cap g(i-1, 0)$ and these are the only links of G_i in $\hat{f}(i, 0) \cap g(i-1, 0)$. Then choose H_i

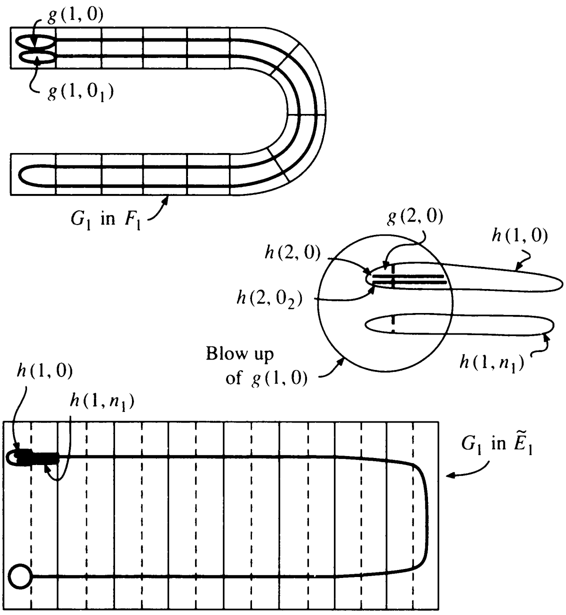

FIGURE 5. For h transitive

so that $h(i, 0) \cap i(g(i, 0), G_i) \neq \emptyset$, $h(i, n_i) \cap i(g(i, 0), G_i) \neq \emptyset$. (This construction is illustrated in Figure 5.)

Also, note that by construction, if $\widehat{G}$ is a proper subchain of G_1, then $H_1^* \not\subset \widehat{G}^*$; if $\widehat{H}$ is a proper subchain of H_1, then $H_2^* \not\subset \widehat{H}^*$, etc. We obtain then our homeomorphism h and our surjection Θ, with Θ having the property that if P' is a nondegenerate subcontinuum of P, then $\Theta(P')$ is nondegenerate. But it is also the case that if P' is a proper subcontinuum of P, then $h(P') \neq P'$ or $\Theta(P') \neq [0, 1]$, as we now prove.

Suppose P' is a subcontinuum of P, $h(P') = P'$, and $\Theta(P') = [0, 1]$. Since $\Theta(P') = [0, 1]$, if $i \in \mathbb{N}$, either

(1) $\hat{f}(i, j) \cap P' \neq \emptyset$ for $j \in I[0, 2^{i+2} - 1]$, or
(2) $\hat{f}(i, j) \cap P' \neq \emptyset$ for $j \in I[2^{i+2}, 2^{i+3} - 1]$.

Then, in either case, $h(P') \cap \tilde{e}(i, j) \neq \emptyset$ for $j \in I[0, 2^{i+2} - 1]$, and $P' \cap \tilde{e}(i, j) \neq \emptyset$ for $j \in I[0, 2^{i+2} - 1]$. It follows that $P' \cap \hat{f}(i, j) \neq \emptyset$ for $i \in \mathbb{N}$, $j \in I[0, 2^{i+3} - 1]$.

If $P' \cap g(1, 0) = \emptyset$, then $P' \cap g(1, n_1) \neq \emptyset$ and $(\Theta^{-1}(0) \cap P') \cap g(1, n_1) \neq \emptyset$. Hence, $h(\Theta^{-1}(0) \cap P') \cap h(1, n_1) \neq \emptyset$ and, in fact, $h(\Theta^{-1}(0) \cap P') \cap g(1, 0) \neq \emptyset$. But this cannot be, for $h(\Theta^{-1}(0) \cap P') \subseteq h(P') = P'$. Then $P' \cap g(1, 0) \neq \emptyset$ and $P' \cap g(1, 0) \cap \Theta^{-1}(0) \neq \emptyset$.

Further, $P' \cap g(2, 0) \cap \Theta^{-1}(0) \neq \emptyset$, for

(3) $g(1, 0) \cap P' \cap \Theta^{-1}(0) = (g(2, 0) \cup g(2, n_2)) \cap P' \cap \Theta^{-1}(0)$, and
(4) $h(g(2, n_2) \cap P' \cap \Theta^{-1}(0)) \subseteq g(2, 0) \cap P' \cap \Theta^{-1}(0)$.

Thus, $g(2, 0) \cap P' \cap \Theta^{-1}(0) \neq \emptyset$, and by induction, $g(i, 0) \cap P' \cap \Theta^{-1}(0) \neq \emptyset$ for $i \in \mathbb{N}$.

Suppose that for each i, $P' \cap g(i, n_i) = \emptyset$. It follows that $P' \cap \Theta^{-1}(0) \cap \hat{f}(1, 0) = \{p_0\} = \bigcap_{i=1}^{\infty} g(i, 0)$. However, this is impossible, for $P' \cap \Theta^{-1}(1)$ separates P', and is thus uncountable. It follows that $h(P' \cap \Theta^{-1}(1)) \subseteq P' \cap \Theta^{-1}(0) \subseteq (\hat{f}(1, 0) \cup \hat{f}(1, 15))$, $h(P' \cap \Theta^{-1}(1))$ is uncountable, and so is $h^2(P' \cap \Theta^{-1}(1))$, which is in $\hat{f}(1, 0)$. Therefore, so is $g(i, 0)$ for each i. Then $P' \cap g(i, n_i) \neq \emptyset$ for some i. A similar argument gives that $P' \cap g(i, n_i) \neq \emptyset$ for infinitely many i. But then $P' = P$. Therefore, h is irreducible with respect to the semiconjugacy.

Now $\Theta^{-1}(0) = \bigcap_{i=1}^{\infty} f(i, 0)$, and if $x \in \Theta^{-1}(0)$, then $h(x), h^2(x), \ldots$ converges to $\bigcap_{i=1}^{\infty} g(i, 0) = \{p_0\}$. (This is because the links $h(1, 0)$, $h(1, n_1)$ both intersect $i(g(1, 0), G_1)$, so $h(\Theta^{-1}(0)) \subseteq g(1, 0) \cup g(1, n_1)$, $h^2(\Theta^{-1}(0)) \subseteq g(2, 0) \cup g(2, n_2) \subseteq g(1, 0)$, etc.)

Further, if $x \in \Theta^{-1}(p/2^q)$ for some $q \in \mathbb{N}$, $0 \leq p \leq 2^q$, then eventually $h^n(x) \in \Theta^{-1}(0)$ and $\{h^n(x)\}_{n \in \mathbb{N}}$ also converges to p_0.

We now show that h admits no wandering points, for suppose it does. Then there is a nonempty open set o in P such that the collection $\{h^n(o) | n \in \mathbb{Z}\} = E$ is mutually disjoint. But o contains some x in some $\Theta^{-1}(p/2^q)$, and that point x is in some nondegenerate continuum K_x in o.

Now E^* is invariant under h and we may assume that $\overline{E^*} \neq P$, but for some N_x, $\Theta(h^n(K_x)) = [0, 1]$ for $n \geq N_x$, $h^n(K_x) \subseteq E^*$, and $\lim_{n \to \infty} h^n(K_x)$ contains a nondegenerate continuum $K \subseteq \overline{E^*}$, which contains p_0. By Theorem 3, this cannot happen. Then h is transitive. □

4. A homeomorphism on the pseudoarc which is semiconjugate to the tent map and admits wandering points. Again, in order to construct h so that it has wandering points and is semiconjugate to the tent map, we must put some additional requirements on the chains in our sequences $G_1, G_2, \ldots$ and $H_1, H_2, \ldots$ while retaining those in §2. Our aim is to construct h so that, *roughly*, $h(i+1, n_{i+1}) \approx g(i, n_i)$ and therefore,

$$h^{-1}(h(i, n_i)) \approx g(i, n_i) \approx h(i+1, n_{i+1}).$$

This will mean that $h(1, n_1)$ contains a wandering point.

For $i \in \mathbb{N}$, let

$$F_i^{\#} = (\widehat{F}_i - \{\hat{f}(i, 2^{i+2} - 1), \hat{f}(i, 2^{i+2})\}) \cup \{\hat{f}(i, 2^{i+2} - 1) \cup \hat{f}(i, 2^{i+2})\}.$$

The reader should refer to Figure 6.

Choose G_1 so that

(1) $g(1, n_1) \subseteq \hat{f}(1, 7) \subseteq f(1, 7)$,

(2) $G_1[a_1, n_1]$ is a minimal final subchain of G_1 properly covered by $\widehat{F}_1[0, 7]$,

(3) $G_1[a_1, b_1]$ is a minimal subchain properly covered by $\widehat{F}_1[0, 3]$,

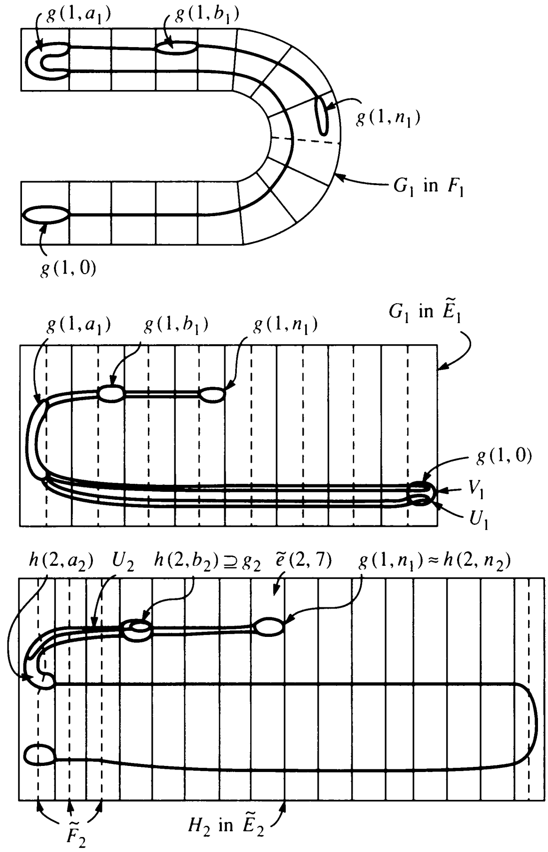

FIGURE 6. For h with wandering points

(4) $G_1[0, a_1]$ is a minimal initial subchain properly covered by $F_1^{\#}$, and

(5) each link of G_1 is connected. (Using $F_1^{\#}$ rather than $\widehat{F}_1$ ensures that G_1 will follow a light pattern in F_1. In turn, this helps simplify and avoid problems with the link "splittings" that will come in subsequent steps of the basic construction.)

Next, find two connected open sets U_1 and V_1 in $G_1[0, a_1 - 1]^*$ such that

(6) $G_1[0, a_1 - 1]$ essentially covers U_1 and V_1, and

(7) $\overline{U_1} \cap \overline{V_1} = \varnothing$.

Choose chains H_1^1, H_1^2, and H_1^3 as follows:

(8) $H_1^1 = H_1^1[0, a_1]$ follows $\eta_1|I[0, a_1]$ in $\widetilde{E}_1$ with $\overline{H_1^{1*}} \subseteq U_1 \cup g(1, a_1)$.

(9) $H_1^2 = H_1^2[a_1, n_1]$ follows $\eta_1|I[a_1, n_1]$ in $\widetilde{E}_1$ and $\overline{H_1^{2*}} \subseteq V_1$.

(10) $H_1^3 = H_3[a_1, b_1]$ follows $\eta_1|I[a_1, b_1]$ in $\widetilde{E}_1$ and $\overline{H_1^{3*}} \subseteq G_1[a_1 + 1, n_1]^*$.

(11) There exists an open set O_1 whose closure is contained in $g(1, a_1)$ and that intersects $i(\underline{g(1, a_1)}, G_1)$ such that $H_1^{3*} \cup O_1 \cup H_1^{2*} \cup H_1^{1*}$ is connected, but $\overline{O_1} \cap \overline{h^1(1, 0)} = \varnothing$.

(12) The only link of $H_1^1 \cup H_1^2 \cup H_1^3$ that intersects $i(g(1, n_1), G_1)$ is $h^3(1, b_1)$. (Actually, this is redundant.)

Define $H_1[0, n_1]$ by

$$h(1, i) = \begin{cases} h^1(1, i) & \text{for } i \in I[0, a_1 - 1], \\ O_1 \cup h^1(1, a_1) \cup h^2(1, a_1) \cup h^3(1, a_1) & \text{for } i = a_1, \\ h^2(1, i) \cup h^3(1, i) & \text{for } i \in I[a_1 + 1, b_1], \\ h^2(1, i) & \text{for } i \in I[b_1 + 1, n_1]. \end{cases}$$

Then H_1 follows η_1 in $\widetilde{E}_1$ and it is time to choose $H_2 = H_2[0, n_2]$. Recall that H_2 actually refines $H_1 \vee \widetilde{E}_2$, and choose H_2 with the following properties:

(13) The final link $h(2, n_2) = g(1, n_1) \cap H_2^* \cap \tilde{e}(2, 7)$, and no other link of H_2 intersects $i(g(1, n_1), G_1)$.

(14) $H_2[a_2, n_2]$ is a minimal final subchain of H_2 properly covered by $\widetilde{E}_2[0, 7]$. (It follows that $H_2[a_2, n_2]^* \subseteq H_1[a_1, n_1]^*$.)

(15) $H_2[a_2, b_2]$ is a minimal subchain properly covered by $\widetilde{E}_2[0, 3]$.

(16) Both $h(2, 0)$ and $h(2, a_2)$ intersect $i(\hat{f}(2, 0), \widehat{F}_2)$.

(17) There exists a connected open set g_2 such that $g_2 \subseteq i(h(2, b_2), H_2) \cap \hat{f}(2, 7)$.

(18) There exists a connected open set U_2 essentially covered by $H_2[a_2, b_2]$ such that $g_2 \subseteq U_2$.

(19) There exists a connected open set V_2 essentially covered by $H_2[0, n_2]$ with $\overline{U_2} \cap \overline{V_2} = \varnothing$.

(20) Both U_2 and V_2 intersect $h(2, a_2) \cap \hat{f}(2, 0)$.

Choose chains G_2^1 and G_2^2 as follows:

(21) $G_2^1 = G_2^1[0, a_2]$ follows $\eta_2 | I[0, a_2]$ in $F_2[0, a_2]$, follows $\xi_1 | I[0, a_2]$ in G_1, and $\overline{G_2^{1*}} \subseteq V_2$.

(22) $G_2^2 = G_2^2[a_2, n_2]$ follows $\eta_2 | I[a_2, n_2]$ in F_2, $\xi_1 | I[0, a_2]$ in G_1, $G_2^{2*} \subseteq U_2$, and $g_2 = g(2, n_2)$.

Define G_2 by

$$g(2, i) = \begin{cases} g^1(2, i) & \text{for } i \in I[0, a_2 - 1], \\ h(2, a_2) \cap \hat{f}(2, 0) & \text{for } i = a_2, \\ g^2(2, i) & \text{for } i \in I[a_2 + 1, n_2]. \end{cases}$$

Suppose $G_2[a_2, b_2]$ is the minimal subchain of $G_2[a_2, n_2]$ properly covered by $\widehat{F}_2[0, 3]$, and choose the chain G_3 with the following properties:

(23) $G_3[a_3, n_3]$ is a minimal final subchain of G_3 properly covered by $\widehat{F}_3[0, 7]$ with $g(3, a_3) \subseteq g(2, a_2)$ and $g(3, n_3) \subseteq g(2, b_2)$.

(24) $G_3[a_3, b_3]$ is a minimal subchain properly covered by $\widehat{F}_3[0, 3]$.

(25) $G_3[0, a_3]$ is properly covered by $F_3^{\#} \vee G_2$.

(26) Each link of G_3 is connected.

(27) $G_3[a'_3, a_3]$ is the *only* minimal subchain of G_3 whose first and last links are contained in $g(2, a_2) \cap \hat{f}(3, 0)$, and such that $G_3[a'_3, a_3]^*$ intersects $g(2, n_2)$.
(Note that even *with* the requirements of G_3 being crooked in both F_3 and G_2, this is possible. However, it is *only* possible because $g(2, n_2)$ is an end link of G_2.)

(28) The first link $g(3, 0)$ is contained in the last link of $\widehat{F}_3$.

Choose chains H_3^1, H_3^2, and H_3^3 as follows:

(29) $H_3^1 = H_3^1[0, a_3]$ follows $\eta_3|I[0, a_3]$ in $\widetilde{E}_3$ and $\xi_2|I[0, a_3]$ in H_2 with $\overline{H_3^{1*}} \subseteq G_3[0, a'_3]^*$.

(30) $H_3^2 = H_3^2[a_3, n_3]$ follows $\eta_3|I[a_3, n_3]$ in $\widetilde{E}_3$ and $\xi_2|I[a_3, n_3]$ in H_2 with $\overline{H_3^{2*}} \subseteq G_3[a'_3, a_3]^*$.

(31) $H_3^3 = H_3^3[a_3, b_3]$ follows $\eta_3|I[a_3, b_3]$ in $\widetilde{E}_3$ and $\xi_2|I[a_3, b_3]$ in H_2 with $\overline{H_3^{3*}} \subseteq G_3[a_3 + 1, n_3]^*$.

(32) There exists an open set O_3 whose closure is contained in $g(3, a_3) \cup g(3, a'_3)$ and that intersects $i(g(3, a_3), G_3)$ such that $H_3^{1*} \cup H_3^{2*} \cup H_3^{3*} \cup O_3$ is connected, but $\overline{O_3}$ fails to intersect the closure of any link of any of the H_3^i chains except for those numbered $a_3 - 1$ or $a_3 + 1$.

Define H_3 by

$$h(3, i) = \begin{cases} h'(3, i) & \text{for } i \in I[0, a_3 - 1], \\ 0_3 \cup h'(3, a'_3) \cup h^2(3, a_3) \cup h^3(3, a_3) & \text{for } i = a_3, \\ h^2(3, i) \cup h^3(3, i) & \text{for } i \in I[a_3 + 1, b_3], \\ h^2(3, i) & \text{for } i \in I[b_3 + 1, n_3]. \end{cases}$$

Note that $h(3, n_3) = g(2, n_2) \cap H_3^* \cap \tilde{e}(3, 7)$. Choose H_4 with the following properties:

(33) The final link $h(4, n_4) = g(3, n_3) \cap H_4^* \cap \tilde{e}(4, 7)$.

(34) $H_4[a_4, n_4]$ is a minimal final subchain properly covered by $\widetilde{E}_4[0, 7]$.

(35) $H_4[a_4, b_4]$ is a minimal subchain properly covered by $\widetilde{E}_4[0, 3]$.

(36) Both $h(4, 0)$ and $h(4, a_4)$ intersect $i(\hat{f}(4, 0), \widehat{F}_4)$.

(37) There exists a connected open set g_4 such that $g_4 \subseteq i(h(4, b_4), H_4) \cap \hat{f}(4, 7)$.

(38) There exists a connected open set U_4 essentially covered by $H_4[a_4, b_4]$ such that $g_4 \subseteq U_4$.

(39) There exists a connected open set V_4 essentially covered by $H_4[0, n_4]$ with $\overline{U}_4 \cap \overline{V}_4 = \emptyset$.

(40) Both U_4 and V_4 intersect $h(4, a_4) \cap \hat{f}(4, 0)$.

The construction of H_4 in H_3 is similar to the construction of H_2 in H_1. We proceed, putting G_4 in G_3 in a way similar to the way we put G_2 in G_1. Having G_4, we choose G_5 similar to the way we chose G_3, etc. Thus, we choose G_5, then H_5 and H_6, choose G_6 and G_7, then H_7 and H_8, and continue. Note that we require that for $i \in \mathbb{N}$, $h(i+1, n_{i+1}) = g(i, n_i) \cap H^*_{i+1} \cap \tilde{e}(i+1, 7)$. We require that for $i \in \mathbb{N}$, $h(i+1, n_{i+1}) = g(i, n_i) \cap H^*_{i+1} \cap \tilde{e}(i+1, 7)$.

Having constructed our pseudoarc P and homeomorphism h, we need to show that we do indeed have wandering points.

To that end, define for $i, k \in \mathbb{N}$, $k > i$, and $j \in [0, n_i]$,

$$A_{ijk} = \{g(k, l) | \xi_i \dots \xi_{k-2}\xi_{k-1}(l) = j\}$$

and

$$B_{ijk} = \{h(k, l) | \xi_i \dots \xi_{k-2}\xi_{k-1}(l) = j\}.$$

Then define

$$\hat{g}(i, j) = g(i, j) \cap A^*_{i,j,i+1} \cap A^*_{i,j,i+2} \cap \cdots$$

and

$$\hat{h}(i, j) = h(i, j) \cap B^*_{i,j,i+1} \cap B^*_{i,j,i+2} \cap \cdots .$$

Note that $i(g(i, j), G_i) \cap P \subseteq \hat{g}(i, j)$ and $i(h(i, j), H_i) \cap P \subseteq \hat{h}(i, j)$, so $\hat{g}(i, j)^\circ \neq \varnothing$ and $\hat{h}(i, j)^\circ \neq \varnothing$ (in P). Moreover, each link of $A_{i,j,k}$ is contained in some link of $F^\#_k$. If $k > i+1$, $F^\#_k$ closure refines $F^\#_{k-1}$. Thus, $\overline{A^*_{i,j,k}} \subseteq A^*_{i,j,k-1}$. Hence, $\hat{g}(i, j)$ is closed in P. Similarly, $\hat{h}(i, j)$ is closed in P.

Recall that for $i \in \mathbb{N}$,

(41) $h(i+1, n_{i+1}) = g(i, n_i) \cap H^*_{i+1} \cap \tilde{e}(i+1, 7)$;

(42) $G_i(H_i)$ follows a light pattern in $F^\#_i(\widetilde{E}_i)$; and

(43) the intersection of each link of $F^\#_i(E_i)$ with P is a union of links of $G_i(H_i)$ intersected with P.

Now $A^*_{i,n_i,i+1} \cap P \subseteq h(i+1, n_{i+1})$, for otherwise for some $g(i+1, j)$ which is contained in $g(i, n_i)$, $g(i+1, j) \cap P$ is not contained in $h(i+1, n_{i+1})$. However, $g(i+1, j) \subseteq \hat{f}(i+1, 14)$ or $g(i+1, j) \subseteq \hat{f}(i+1, 15)$. Since $\tilde{e}(i+1, 7) = \hat{f}(i+1, 14) \cup \hat{f}(i+1, 15)$, $g(i+1, j) \subseteq \tilde{e}(i+1, 7)$. It follows that $g(i+1, j) \cap P \subseteq h(i+1, n_{i+1}) = g(i, n_i) \cap H^*_{i+1} \cap \tilde{e}(i+1, 7)$, and that $A^*_{i,n_i,i+1} \cap P \subseteq h(i+1, n_{i+1})$, and $A^*_{i,n_i,i+1} \cap P = h(i+1, n_{i+1}) \cap P$.

Moreover, since

$$P \cap A^*_{i,n_i,k} = g(i, n_i) \cap P \cap (\{\hat{f}(k, l) | l \in I[7 \cdot 2^{k-i}, 8 \cdot 2^{k-i} - 1]\}^*);$$

$$P \cap B^*_{i,n_i,k} = h(i, n_i) \cap P \cap (\{\tilde{e}(k, l) | l \in I[7 \cdot 2^{k-i-1}, 8 \cdot 2^{k-i-1} - 1]\}^*;$$

and

$$\{\tilde{e}(k, l) | l \in I[7 \cdot 2^{k-i-1}, 8 \cdot 2^{k-i-1} - 1]\}^* = \{\hat{f}(k, l') | l' \in I[7 \cdot 2^{k-i}, 8 \cdot 2^{k-i} - 1]\}^*,$$

it follows that for $k > i+1$,

$$A^*_{i,n_i,k} \cap P = B^*_{i+1,n_{i+1},k} \cap P.$$

Finally, we have that $\hat{g}(i, n_i) = \hat{h}(i+1, n_{i+1})$.

If $x \in \hat{h}(j, n_j)$, there is an infinite sequence $k(x, 1)$, $k(x, 2), \ldots$ of integers such that for $i \geq j$, $x \in h(i, k(x, i))$ and $\xi_i(k(x, i+1)) = k(x, i)$, and $h^{-1}(x) = \bigcap_{i=j}^{\infty} g(i, k(x, i))$. It follows that $h^{-1}(\hat{h}(j, n_j)) \subseteq \hat{g}(j, n_j)$, and a similar argument gives that $h(\hat{g}(j, n_j)) \subseteq \hat{h}(j, n_j)$. Thus, $h^{-1}(\hat{h}(j, n_j)) = \hat{g}(j, n_j)$.

Then $h^{-1}(\hat{h}(1, n_1)) = \hat{g}(1, n_1) = \hat{h}(2, n_2)$, so

$$h^{-2}(\hat{h}(1, n_1)) = h^{-1}(\hat{h}(2, n_2)) = \hat{h}(3, n_3),$$

etc. Clearly, the collection $\{\hat{h}(i, n_i) | i \in \mathbb{N}\}$ consists of disjoint sets, so we have our wandering set, for if $\{h^{-n}(\hat{h}(1, n_1)^0) | n \in \mathbb{N}\}$ consists of disjoint sets, so does $\{h^{-n}(\hat{h}(1, n_1)^0) | n \in \mathbb{Z}\}$. □

The homeomorphism with wandering points that was just constructed has the property that if P' is a nondegenerate subcontinuum of P, then $\Theta(P')$ is nondegenerate, but I do not know whether or not it has the property that if P' is a proper subcontinuum of P, then either $\Theta(P') \neq [0, 1]$ or $h(P') \neq P'$.

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