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VARIETIES OF PERIODIC ATTRACTOR IN CELLULAR AUTOMATA

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ABSTRACT. We apply three alternate definitions of "attractor" to cellular automata. Examples are given to show that using the different definitions can give different answers to the question "Does this cellular automaton have a periodic attractor?" The three definitions are the topological notion of attractor as used by C. Conley, a more measure-theoretic version given by J. Milnor, and a variant of Milnor's definition that is based on the concept of the "center of attraction" of an orbit. Restrictions on the types of periodic orbits that can be periodic attractors for cellular automata are described. With any of these definitions, a cellular automaton has at most one periodic attractor.

Additionally, if Conley's definition is used, then a periodic attractor must be a fixed point. Using Milnor's definition, each point on a periodic attractor must be fixed by all shifts, so the number of symbols used is an upper bound on the period; whether the actual upper bound is 1 is unknown. With the third definition this restriction is removed, and examples are given of one-dimensional cellular automata on three symbols that have finite "attractors" of arbitrarily large size (with the third definition, a finite attractor is not necessarily a single periodic orbit).

The purpose of this paper is to describe the types of periodic orbits that can be "attractors" of a cellular automaton. We will consider three different definitions of "attractor": one that is based upon topological dynamics, a second that is based upon a mixture of topological and measure theoretic dynamics, and a third that is even more measure theoretic in nature. Examples will be given to show that these three definitions can lead to different answers to the question of the existence of a periodic "attractor" for cellular automata. We begin with an informal discussion of some of the results. Details and precise definitions are given in later sections.

A cellular automaton is a type of endomorphism of a certain function space. Let $\Sigma = \{x : \mathbf{Z}^m \to S\}$, where $\mathbf{Z}^m$ is the integer lattice in $\mathbf{R}^m$ and S is a finite set, called the symbol set. A shift on Σ is a map $\sigma_t : \Sigma \to \Sigma$ of the form $(\sigma_t x)(n) = x(n+t)$ for some $t \in \mathbf{Z}^m$. A *cellular automaton* is a continuous map $f : \Sigma \to \Sigma$ that commutes with all of these shifts. Cellular automata have received much attention in recent years. One reason for this attention is the fact that, especially for $m = 1$ or 2, cellular automata are readily accessible

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to computer experimentation [15, 4, 6]. Such experiments have shown that certain automata seem to have the following property: most of the forward iterates of a typical initial point are close to a particular periodic orbit, but this "eventual periodicity" is occasionally interrupted by some short-term anomalous behavior; moreover these interruptions appear to occur less and less frequently as the number of observed iterates increases. An obvious question arises: in what sense—if any—is the periodic orbit an "attractor?"

We will consider three variants of the notion of "attractor." The first is the definition used by C. Conley [1] and others: a periodic orbit γ is a *topological attractor* for f is there is a closed neighborhood U of γ with the properties that $f(U)$ is contained in the interior of U and the intersection of all the forward iterates of U by f is γ. This notion of attractor was used to study cellular automata in [10]. One result contained in [10] is that if γ is a periodic topological attractor of a cellular automaton f then

(i) γ must be a fixed point of f, $\gamma = \{p\}$.

(ii) this fixed point p, when thought of as a map from $\mathbf{Z}^m$ to S, is a constant map.

(iii) $\{p\}$ is the only periodic topological attractor of f.

(iv) the omega-limit set of x is equal to $\{p\}$ for an open and dense set of points $x \in \Sigma$.

In short, the existence of a periodic topological attractor puts severe restrictions on the dynamics of f. A heuristic explanation of this is given below, following 2.2.

A second, and less restrictive, definition of "attractor" is due to J. Milnor [12]. His definition uses a probability measure μ on Σ. The probability measures that we will consider are the Bernoulli product measures (which are defined in the next section). We will call a periodic orbit γ a μ-attractor if there is a set $\mathscr{R}$ of positive μ-measure with the property that the omega limit set of x is equal to γ for all $x \in \mathscr{R}$. We will establish the following result:

Theorem A. *If a cellular automaton f has a periodic μ-attractor γ then the points of γ must all be fixed by every shift of the underlying lattice (i.e., each of these points is a constant map $\mathbf{Z}^m \to S$). Also, γ is the only μ-attractor of f, and $\omega(x) = \gamma$ for μ-almost all points $x \in \Sigma$.*

The proof of Theorem A does not give any restriction on the period of γ. However the only examples known to the author have period 1.

Question. Must a periodic μ-attractor of a cellular automaton be a fixed point?

A particular example of a cellular automaton where numerical experimentation has indicated the possible existence of a periodic "attractor" is Wolfram's "elementary rule number 110" [15]. In this example there is a pair of period 7 orbits that appear to be in the omega limit set of almost every initial condition. (The shift map interchanges the two orbits.) That is, if an initial point in Σ is chosen randomly and the sequence of its iterates under the cellular automaton is generated, then typically one observes that the sequence of iterates comes arbitrarily close to the period 7 orbits. However, the sequence of iterates does not stay close to the period 7 orbits—occasionally the iterates move away. In experiments these excursions away from the periodic orbits seem to become more and more rare as the number of iterates being observed grows. However Theorem A shows that neither period 7 orbit is a μ-attractor for any allowable choice of μ. (Also, in a compact space an omega limit set cannot be composed of two distinct periodic orbits, so the union of the two orbits is not a μ-attractor either.) In other words, for a typical initial condition the "excursions" away from the periodic orbits keep occurring forever. (An example of a two-dimensional cellular automaton with somewhat similar behavior is described in [7].) If it is in fact true that the frequency of these excursions tends toward 0, then one might consider these period 7 orbits to be an "attractor" in a weaker sense.

The third, and weakest, notion of "attractor" that we will use is that of a *μ-minimal center of attraction.* This notion is due to H. Hilmy [see 9 or 13]; it is similar to the definition of a μ-attractor, the difference being that the omega limit set is replaced by a subset. If f is a cellular automaton and $x \in \Sigma$, define $\mathrm{Cent}(x)$ to be the smallest closed subset F of Σ with the property that if U is a neighborhood of F then the proportion of the points $x, f(x), f^2(x), \ldots, f^n(x)$ that are contained in U tends to 1 as $n \to \infty$. A set $A \subset \Sigma$ is a μ-minimal center of attraction (μMCA) if there is a set $\mathscr{C}$ of positive μ-measure with the property that $\mathrm{Cent}(x) = A$ for all x in $\mathscr{C}$.

Theorem B. *If a cellular automaton has a μMCA A, then A is invariant under all shifts and is the only μ-center of attraction. Moreover,* $\mathrm{Cent}(x) = A$ *for μ-almost all points $x \in \Sigma$.*

Note that any periodic μ-center of attraction is necessarily a μMCA. One important difference between $\mathrm{Cent}(x)$ and $\omega(x)$ is that $\mathrm{Cent}(x)$ can be a finite disjoint union of periodic orbits, so that a finite μ-center of attraction is not necessarily minimal. However it is true any finite μ-center of attraction contains a μMCA.

To contrast Theorem B with Theorem A we will give examples that show the following: there are periodic μMCA for cellular automata with arbitrarily high periods; a finite μMCA may consist of several distinct periodic orbits; and a finite μMCA is not necessarily pointwise fixed by the shifts. Unfortunately we have not been able to determine if the two period 7 orbits of Wolfram's automaton 110 are a μMCA for any measure μ.

Certain relationships between the three notions of attractor are fairly obvious; in particular the following proposition is clear.

Proposition C. *Suppose that f is a cellular automaton with a periodic orbit γ.*
(1) *If γ is a topological attractor then it is a μ-attractor.*
(2) *If γ is a μ-attractor then it is a μMCA.*

The converses to the two assertions of Proposition C are both false; a counterexample to the converse of (1) is Example 4A of [11], and §4 below contains counterexamples to the converse of (2). The relationships between nonperiodic topological attractors, μ-attractors, and μMCA's is less clear; some examples are contained below and in [11]. [6] contains a description of examples of cellular automata where there apparently are infinite μMCA's that are not μ-attractors.

Finally, we will establish the same result as Theorem A but with a slightly weaker hypothesis; this hypothesis is related to a dichotomy for one-dimensional cellular automata that has been discovered by R. Gilman [4, 5].

Theorem D. *Let f be a cellular automaton. Suppose that there is a set P in Σ satisfying $\mu(P) = 1$ and with the property that if x is in P and if B is any finite subset of $\mathbf{Z}^m$, then the restriction of $(f^n x)$ to B is eventually periodic. If f has a minimal μ-attractor A_μ, then A_μ is a periodic orbit, and so A_μ is as described in Theorem* A.

In fact the hypothesis of a minimal μ-attractor in Theorem D can be weakened to the hypothesis that there is a μMCA for f.

The paper is organized as follows: §1 contains general background on cellular automata; §§2 and 3 contain material on the various notions of attractor that we are using, as well as the proofs of Theorems A and B. §4 is devoted to examples illustrating the differences between periodic μ-attractors and periodic μMCA's. §5 contains the proof of Theorem D, and §6 is a brief description of another example.

1. Cellular automata

Let $\Sigma(m, S)$ denote the set of maps from $\mathbf{Z}^m$ to S,

$$\Sigma(m, S) = \{x : \mathbf{Z}^m \to S\};$$

here $\mathbf{Z}^m$ is the integer lattice in $\mathbf{R}^m$ and S is a nonempty finite set, called the *symbol set*. We will usually abbreviate $\Sigma(m, S)$ to Σ. A metric is defined on Σ by $d(x, y) = 2^{-i}$, where $i = \inf\{\|t\| : t \in \mathbf{Z}^m \text{ and } x(t) \neq y(t)\}$, and $\|t\| = \|(t_1, t_2, \dots, t_m)\| = \max|t_j|$. Σ is compact in the topology induced by d.

It will be useful to have a notation for the elements of Σ that are constant mappings. For each $s \in S$ let $s^* \in \Sigma$ be defined by $s^*(t) = s$ for all $t \in \mathbf{Z}^m$.

Definition. (a) A finite, nonempty subset B of $\mathbf{Z}^m$ will be called a *block*.

(b) A map $f : \Sigma \to \Sigma$ is a *cellular automaton* if there is a block B such that for each $t \in \mathbf{Z}^m$ the value of $(fx)(t)$ is completely determined by the finite ordered set $\{x(t + b_j) | b_j \in B\}$.

(c) When f is a cellular automaton acting on $\Sigma(m, S)$ we will say that f is m-dimensional.

There is an equivalent definition due to Curtis, Hedlund, and Lyndon [8]: a cellular automaton is any continuous map $f : \Sigma \to \Sigma$ that commutes with all shifts of the lattice. (For t in $\mathbf{Z}^m$, the shift $\sigma_t : \Sigma \to \Sigma$ is defined by

$(\sigma_t x)(s) = x(t+s)$ for all x in Σ and all s in $\mathbf{Z}^m$.) Thus a map $f : \Sigma \to \Sigma$ is a cellular automaton if and only if f is continuous and $f \circ \sigma_t = \sigma_t \circ f$ for all t in $\mathbf{Z}^m$.

This second definition of cellular automaton is the one that will be most useful in what follows. The fact that cellular automata commute with the shifts means that the ergodic properties of the shifts impose restrictions on the possible dynamics of a cellular automaton. To describe these ergodic properties we first need to define the probability measures on Σ that will be used. These measures are the *Bernoulli product measures.* Bernoulli measures are defined as follows (see [2] for more details): let $S = \{s_1, \dots, s_r\}$ denote the symbol set, and suppose that $p_1, \dots, p_r$ are strictly positive numbers whose sum is one. Given a lattice point t, let $C(t, s_i)$ denote the set $\{x \in \Sigma | x(t) = s_i\}$ and define $\mu_0(C(t, s_i)) = p_i$. If $t_1, \dots, t_k$ are distinct lattice points, define $\mu_0\{\bigcap C(t_j, s_{i(j)})\} = \prod p_{i(j)}$. μ_0 extends to a Borel probability measure μ on Σ called the *Bernoulli product measure with weights* $p_1, \dots, p_r$.

A Bernoulli measure μ is invariant under all shifts, and for each nonzero lattice point t, the measure-theoretic dynamical system (Σ, σ_t, μ) is ergodic: if Y is a Borel subset of Σ that is invariant under σ_t $(t \neq 0)$, then $\mu(Y)$ is either 0 or 1.

The following result will be used in the proofs of Theorems A and B; it is taken from [11]. Call a collection of measurable sets *μ-nearly disjoint* if the intersection of any two of the sets has measure 0.

1.1 **Proposition.** *Suppose that $\mathfrak{B}$ is a μ-nearly disjoint collection of subsets of Σ, each of which has positive measure. Let $\sigma = \sigma_t$ for some nonzero $t \in \mathbf{Z}^m$. If $\mathfrak{B}$ is σ-invariant $(B \in \mathfrak{B} \Rightarrow \sigma(B) \in \mathfrak{B})$ then either $\mathfrak{B}$ is empty or else $\mathfrak{B}$ consists of a single set.*

Proof. It is well known that the measure-theoretic dynamical system (Σ, σ, μ) is strongly mixing [2]; it follows that if B, B' are in $\mathfrak{B}$, then $\sigma^n(B) \cap B'$ has positive measure for all sufficiently large n. The assumptions on $\mathfrak{B}$ now imply that $\sigma^n(B) = B'$ for all large n; in particular $\sigma^{n+1}(B) = \sigma^n(B)$ so that $\sigma(B) = B$. By ergodicity $\mu(B) = 1$, and so the "near disjointness" of $\mathfrak{B}$ implies that B is the only element of $\mathfrak{B}$. □

2. Topological attractors and μ-attractors

This section contains background on two of the types of "attractor" that we will be considering. Conley's definition of a *topological attractor* was given in the introduction. This topological notion of attractor was used to study cellular automata in [10]. All of the results alluded to in the introduction that concern topological attractors can be found in [10]. In [12] J. Milnor gave a more general definition of "attractor;" to minimize confusion we will call one of Milnor's attractors a *μ-attractor.* A comparison of the notions of topological attractor and μ-attractor in the context of cellular automata is contained in [11]. The definition of a *μ-minimal center of attraction*, (or μMCA for short) is a variant of the definition of μ-attractor. It can be found in §4.

This section is largely descriptive; many of the results are taken from [11 and 12], where more detailed proofs can be found. We begin with the definition of a μ-attractor, as found in [12]. Let $\omega(x)$ denote the *omega limit set* of x, that is, the smallest closed subset Y of Σ satisfying $\mathrm{dist}(f^n(x), Y) \to 0$ as $n \to \infty$. Equivalently, $\omega(x) = \{y \in \Sigma | f^{n_i}(x) \to y$ for some sequence of integers n_i with $n_i \to \infty\}$.

2.1 **Definition.** If A is a subset of S let $\rho(A)$ denote the set of points whose omega limit sets are contained in A. $\rho(A)$ is called the *realm* of A. If A is closed then the set $\rho(A)$ is automatically measurable.

When we need to emphasize the dependence of $\omega(x)$ or $\rho(A)$ on f we will write $\omega(x; f)$ for $\omega(x)$ and $\rho(A; f)$ for $\rho(A)$.

2.2 **Definition** [12]. A closed subset A of Σ is an *μ-attractor* for f if
(a) $\mu(\rho(A)) > 0$ and
(b) $\mu(\rho(B)) < \mu(\rho(A))$ for any proper closed subset B of A.

A μ-attractor A is *minimal* if $\mu(\rho(B)) = 0$ for any proper closed subset B of A.

Remarks. (1) The collection of μ-attractors for f can vary as μ varies; an example illustrating this is contained in §4C of [11].

(2) It follows from 2.2(b) that μ-attractors are invariant sets: $f(A) = A$ for any μ-attractor A.

(3) If γ is a periodic μ-attractor, then 2.2 does not imply that γ is stable: points near γ may be taken far from γ under iteration by f. In this sense 2.2 is quite different from the usual description of a "periodic attractor" in a smooth dynamical system. To see why this lack of a requirement of stability is reasonable in discussing cellular automata consider the case of a fixed point. In smooth dynamical systems stability of a fixed point is usually the consequence of local linearizability: if the derivative at the fixed point is a contraction, then the nonlinear map is a local contraction and so the periodic orbit is stable. For a cellular automaton f the phase space Σ is a Cantor set, and, except for very trivial examples f is not locally a contraction anywhere in Σ. (The reason for this is evident if one considers the definition of a cellular automaton as a block map: if $x \in \Sigma$ and the values of $x(t)$ are specified for some finite number of lattice points t, then in general the values of $(fx)(t)$ are determined for fewer values of t.) Example 4A of [11] contains an example of a cellular automaton with a fixed point that is a μ-attractor for every Bernoulli measure μ, but this fixed point is not stable.

The next two lemmas concern the way in which μ-attractors can be detected and decomposed.

2.3 **Lemma.** *If K is a closed set with $\mu(\rho(K)) > 0$ then there is a μ-attractor A in K with $\mu(\rho(A)) = \mu(\rho(K))$.*

Proof (taken from Lemmas 1 and 2 of [12]). Begin by choosing a countable basis for K in the relative topology. Given one of the sets from this basis, say that it is "rarely visited" if the set of $x \in \rho(K)$ such that $\omega(x)$ meets the basic set has measure 0. Let U be the union of all the sets in the basis that are rarely visited. Then $A = K - U$ is the desired μ-attractor. □

Corollary. *Suppose that $F_1, F_2, \ldots$ is an increasing sequence of closed sets, each of which is mapped into itself by f. Let $K = \bigcup F_j$. If $\mu(K) > 0$, and if $\varepsilon > 0$, then there is an μ-attractor A in K with $\mu(\rho(A)) > \mu(K) - \varepsilon$.*

Proof. The measure μ is regular, so $\mu(K) = \lim \mu(F_j)$. Pick j so that $\mu(F_j) > \mu(K) - \varepsilon$. Since F_j is closed and forward invariant, $F_j \subset \rho(F_j)$ and so 2.3 shows that there is a μ-attractor A in F_j with $\mu(\rho(A)) = \mu(\rho(F_j)) \geq \mu(F_j) > \mu(K) - \varepsilon$. □

Note that if K is an open set that is not also closed, and if f is the identity, then K satisfies the hypotheses of the corollary but there is no μ-attractor $K \subset A$ with $\mu(\rho(A)) = \mu(\rho(K))$.

2.4 **Lemma.** *Suppose that $k \geq 1$, that $A_1, \ldots, A_k$ are closed, nonempty, pairwise disjoint sets that form a cycle, i.e. $f(A_k) = A_1$ and $f(A_j) = A_{j+1}$ for $1 \leq j < k$. Let $A = \bigcup A_j$. then $\rho(A; f) = \bigcup \rho(A_j; f^k)$.*

Proof. Clearly the left-hand side contains the right. To obtain the opposite inclusion, choose a collection of pairwise disjoint closed neighborhoods U_j of A_j with

$$(*) \qquad \text{if } y \in U_j \text{ and } f^k(y) \in U_i \text{ then } i = j.$$

This is possible since each A_j is a closed invariant set of f^k. Let $U = \bigcup U_j$. If $x \in \rho(A; f)$ then $f^n(x) \in U$ for all large n. By $(*)$ there is a j such that $f^{nk}(x) \in U_j$ for all large n. It follows that $\omega(x; f^k) \subset \omega(x; f) \cap U_j \subset A \cap U_j = A_j$, so $x \in \rho(A_j; f^k)$. □

The next three results concern the structure of minimal μ-attractors.

2.5 **Lemma.** *If A is a minimal μ-attractor, then $\omega(x) = A$ for μ-almost all x in $\rho(A)$.*

Proof. See Lemma 3 of [12]. The idea is that if the conclusion of the lemma were false then there would be a proper closed subset K of A with $\mu(\rho(K)) > 0$, so that 2.3 would contradict the minimality of A. □

Corollary. *If A, A^* are distinct minimal μ-attractors, then $\rho(A) \cap \rho(A^*)$ has measure 0. (In the terminology of* 1.1, *the collection of realms of minimal μ-attractors is nearly disjoint.)*

2.6 **Lemma.** *Let t be any lattice point. A is a (minimal) μ-attractor for f if and only if $\sigma_t(A)$ is. Moreover, $\rho(\sigma_t(A)) = \sigma_t(\rho(A))$, so that $\mu(\rho(\sigma_t(A))) = \mu(\rho(A))$.*

Proof. This follows easily from the fact that f commutes with σ_t.

2.7 **Proposition.** *If a cellular automaton has a minimal μ-attractor A, then A is the only μ-attractor and its realm has measure* 1.

Proof. Let $\mathfrak{B} = \{\rho(M)|M$ is a minimal μ-attractor$\}$. By 1.1 $\mathfrak{B} = \{\rho(A)\}$ and $\rho(A) = 1$. If A' is a μ-attractor with $A' \neq A$ then the combination of $\rho(A) = 1$ and 2.5 would imply the existence of a point $x \in \rho(A')$ with $\omega(x) = A$, so that $A \subset A'$. Then $\rho(A') = \rho(A) = 1$, and we have contradicted condition 2.2(b) in the definition of a μ-attractor. □

2.8 *Remark.* If a periodic orbit is a μ-attractor, it is automatically minimal, so it is the only μ-attractor and its realm has full measure.

We will show that if there is a periodic μ-attractor then each of its points is fixed by all shifts.

2.9 **Theorem.** *Suppose the γ is a periodic orbit for a cellular automaton f. If there is a Bernoulli measure μ such that γ is a μ-attractor, then γ is pointwise fixed by all shifts (in other words, if $q \in \gamma$ is thought of as as map $q : \mathbf{Z}^m \to S$, then it is a constant map: $q(r) = q(t)$ for all r, t in $\mathbf{Z}^m$).*

Proof. Let $\gamma = \{q_0, \dots, q_{k-1}\}$, so that each q_j is a fixed point of f^k. By 2.4 and 2.8 $1 = \mu(\rho(\gamma; f)) = \Sigma\mu(\rho(q_j, f^k))$, so there is some value $j = j'$ such that $\rho(q_{j'}; f^k)$ has nonzero measure; to simplify the notation assume that $j' = 0$. Now $\{q_0\}$ is a fixed μ-attractor for the cellular automaton f^k, so 2.8 and 2.7 show that it is the only μ-attractor of f^k. This uniqueness combines with 2.6 to show that q_0 is fixed by all shifts. Finally, since the shifts commute with f and since the remaining q_j are forward iterates of q_0, each q_j is also fixed by all shifts: $\sigma_t(q_j) = \sigma_t \circ f^j(q_0) = f^j \circ \sigma_t(q_0) = f^j(q_0) = q_j$. □

Theorem A is the combination of 2.7–2.9.

3. μ-MINIMAL CENTERS OF ATTRACTION

If f is a cellular automaton, $x \in \Sigma$, and E is a subset of Σ, let $P_n(x; E)$ denote the proportion of the first n iterates of x that lie in E:

$$P_n(x; E) = \frac{1}{n} \cdot \left(\sum_{j=0}^{n-1} \chi_E(f^j x) \right)$$

(χ_E is the characteristic function of E).

3.1 **Definition.** If $x \in \Sigma$ and f is a cellular automaton, say that a closed nonempty subset Y of Σ is a *center* for x if

$$\liminf_{n\to\infty} P_n(x; U) = 1$$

for every neighborhood U of Y.

It is not hard to check that the collection of centers for x has the finite intersection property, so that the intersection of all of the centers for x is nonempty. Call this intersection the *minimal center for* x, and denote it by $\mathrm{Cent}(x)$.

Remarks. $\mathrm{Cent}(x)$ is compact and nonempty, $\mathrm{Cent}(x)$ is a subset of the omega limit set of x, and $f(\mathrm{Cent}(x)) = \mathrm{Cent}(x)$.

3.2 **Lemma.** *$y \in \mathrm{Cent}(x)$ if and only if $\limsup_{n\to\infty} P_n(x; V) > 0$ for every neighborhood V of y.*

Proof. Suppose $y \in \mathrm{Cent}(x)$ but that $\lim_{n\to\infty} P_n(x; V) = 0$ for some neighborhood V of Y. Then $K = \mathrm{Cent}(x) - V$ is closed, nonempty, and strictly smaller than $\mathrm{Cent}(x)$. Moreover, K is a center for x; if U is any neighborhood of K, then $U \cup V$ is a neighborhood of $\mathrm{Cent}(x)$ so that $P_n(x; U \cup V) \to 1$ as $n \to \infty$. But $P_n(x; U \cup V) \leq P_n(x; U) + P_n(x; V)$ and $P_n(x; V) \to 0$, so $P_n(x; U) \to 1$. This is a contradiction, since $\mathrm{Cent}(x)$ is the smallest center for x. Conversely, suppose that y is not in $\mathrm{Cent}(x)$. Then there are disjoint neighborhoods U of $\mathrm{Cent}(x)$ and V of y. The fact that these neighborhoods are disjoint means that $P_n(x; U \cup V) = P_n(x; U) + P_n(x; V)$ and so $P_n(x; V) \leq 1 - P_n(x; U)$. The right side of this inequality tends to 0 as $n \to \infty$, and so $\lim_{n\to\infty} P_n(x; V) = 0$. □

Definition. If T is a closed subset of Σ, let $\psi(T) = \{x \mid \mathrm{Cent}(x) \subset T\}$.

3.3 **Lemma.** *For $T \subset \Sigma$ closed, $\psi(T)$ is a Borel set.*

Proof. Let U be an open neighborhood of T. Define the sets

$$U(n, \varepsilon) = \{x \mid P_n(x; U) > 1 - \varepsilon\},$$
$$U(\varepsilon) = \bigcup_{k\geq 1} \bigcap_{n\geq k} U(n, \varepsilon).$$

Clearly $\psi(T) \subset U(\varepsilon)$ for any $\varepsilon > 0$, so that $\psi(T) \subset U' = \bigcap U(1/m)$. Obviously U' is a Borel set. Using compactness we can find a nested sequence of open sets, $U_1 \supset U_2 \supset \cdots$, with the property that any neighborhood of T contains one of the sets U_j. Let $Z = \bigcap U_j'$; the previous remarks show that Z is a Borel set and that $\psi(T)$ is contained in Z. To finish we show that $\psi(T)$ contains Z. Suppose $x \in Z$ and that V is some neighborhood of T. For large enough j we have $U_j \subset V$. Since $x \in U_j'$ we know that if $\varepsilon > 0$ then $P_n(x; V) \geq P_n(x; U_j) > 1 - \varepsilon$ for all sufficiently large n so that $\liminf P_n(x; V) = 1$ and $\mathrm{Cent}(x) \subset V$. Since V was an arbitrary neighborhood of T, we see that $\mathrm{Cent}(x) \subset T$, as desired. □

We will use the notations $\mathrm{Cent}(x; f) = \mathrm{Cent}(x)$, $\psi(x; f) = \psi(x)$ whenever we need to indicate the dependence of these sets upon f.

3.4 **Definition.** A closed, nonempty subset Y of Σ is a *μ-center of attraction* of f if

(a) $\mu(\psi(Y)) > 0$ and

(b) $\mu(\psi(T)) < \mu(\psi(Y))$ for any proper closed subset T of Y.

A μ-center of attraction Y is *minimal* if $\mu(\psi(T)) = 0$ for each proper closed subset T of Y. We will abbreviate "μ-minimal center of attraction" to "μMCA." Just as in the case of μ-attractors, any μ-center of attraction of f is invariant, $f(Y) = Y$. In fact most of the results concerning μ-attractors in [12 and 11] can be established for μ-centers of attraction by using the same arguments. Several of these results are listed in the next lemma; the proofs of all but the first are left to the reader.

3.5 **Lemma.** (a) *If Z is a closed set with $\mu(\psi(Z)) > 0$, then there is a μ-center of attraction Y in Z with $\mu(\psi(Y)) = \mu(\psi(Z))$.*

(b) *If Y is a μMCA then* $\text{Cent}(x) = Y$ *for μ-almost all x in $\psi(Y)$.*

(c) *If Y_1 and Y_2 are μMCA's and $\mu(\psi(Y_1 \cap Y_2)) > 0$, then $Y_1 = Y_2$.*

(d) *Let t be any lattice point. Y is a μ-center of attraction or a μMCA if and only if $\sigma_t(Y)$ is. Moreover, $\psi(\sigma_t(Y)) = \sigma_t(\psi(Y))$, so that $\mu(\psi(\sigma_t(Y))) = \mu(\psi(Y))$.*

Proof of (a). (Compare with 2.3.) Given a countable basis $\{U_i\}$ for Z in the relative topology, let U be the union of the sets U_i that satisfy

$$\mu(\{x \in \psi(Z) | \text{Cent}(x) \cap U_i \neq \varnothing\}) = 0.$$

Let $Y = Z - U$. It follows from the definition of U that $\mu(\psi(Y)) = \mu(\psi(Z)) > 0$. Now suppose that K is a proper closed subset of Y. Then $Z - K$ is open in the relative topology, so one of the basic open sets U_j is contained in $Z - K$ and meets Y. Consequently U_j is not one of the sets comprising U, so $\mu(\{x \in \psi(Z) | \text{Cent}(x) \cap U_j \neq \varnothing\}) > 0$ which means that $\text{Cent}(x)$ is not in K for any x in this last set. It follows that $\mu(\psi(K))$ is strictly smaller than $\mu(\psi(Y))$, and we conclude that Y is a μ-center of attraction. □

3.6 **Theorem B.** *If a cellular automaton has a μMCA Y, then Y is the only μ-center of attraction, $\psi(Y)$ has full measure, and Y is left invariant by all shifts.*

Proof. Let $\mathfrak{B} = \{\psi(T) | T \text{ is a } \mu MCA\}$. By 1.1 and 3.5 $\mathfrak{B} = \{\psi(Y)\}$ and $\mu(\psi(Y)) = 1$. If $T \neq Y$ is a μ-center of attraction, then 3.5(b) and the fact that $\psi(Y)$ has full measure lead to the conclusion that $Y \subset T$. But then T fails to satisfy 3.4(b) so it could not be a μ-center of attraction. □

3.7 *Remark.* There are two basic results about μ-attractors that do not carry over to μMCA's:

(1) A μMCA need not be chain recurrent; in fact a μMCA can be composed of finitely many pairwise disjoint closed invariant sets.

(2) The analogue of 2.4 is false: when $A_1, A_2, \dots, A_k$ is a cycle of closed, pairwise disjoint, f^k-invariant sets, the equation $\psi(\bigcup A_j; f) = \bigcup \psi(A_j; f^k)$ may not be true.

Examples of cellular automata that illustrate 3.7 are given in the next section; however simple examples other than cellular automata are easy to construct. Suppose φ is a flow on $\mathbf{R}^2$ with the following properties:

(a) φ is symmetric about the origin;

(b) φ has three equilibria, (0, 0), $A_1 = (0, 1)$ and $A_2 = (0, -1)$.

(c) the unit circle is φ-invariant, and is equal to $\omega(z)$ for every $z \neq (0, 0)$ that is not on the unit circle.

(For instance, φ could be the flow of the differential equation in polar coordinates: $r' = r(1-r)$, $\theta' = (r-1)^2 + \cos^2(\theta)$.) Let f be the time-one map of this flow. It is easy to see that $\mathrm{Cent}(z; f) = \{A_1, A_2\}$ for every $z \neq (0, 0)$ that is not on the unit circle, so that f demonstrates 3.7(1) (using Lebesgue measure λ). Now let τ be the involution $\tau(x, y) = (-x, -y)$ and let $g = f \circ \tau$, so that $\{A_1, A_2\}$ is a period two orbit of g. Once again $\mathrm{Cent}(z; g) = \mathrm{Cent}(z; g^2) = A$ for all $z \neq (0, 0)$ that are not on the unit circle. Thus $\psi(A; g) = \mathbf{R}^2 - (0, 0)$ while $\psi(A_1; g^2) \cup \psi(A_2; g^2)$ is only the unit circle, and so g demonstrates 3.7(2).

4. Examples of μMCA's

In this section we will give examples that show that the gap between the conclusions of Theorems A and B is necessary. We will show

(1) a cellular automaton can have a periodic μMCA that is not a μ-attractor: for each $N > 2$ there is a cellular automaton f and a measure μ such that f has a periodic orbit of period N which is a μMCA but not a μ-attractor.

(2) a finite μMCA need not be a single periodic orbit.

(3) the points of a finite μMCA need not be fixed by the shift: given any N there is a one-dimensional cellular automaton with a finite μMCA that contains points whose least period under σ is N.

The construction of a cellular automaton with a period N μMCA that is not a μ-attractor depends on whether N is even or odd. We begin with the simplest case, $N = 3$. The cellular automaton f will be one-dimensional, and the symbol set will be $\{0, 1, 2\}$. f is defined in terms of random walks on the integers. Associate to each $x \in \Sigma$ a mapping $W_x : \mathbf{Z} \to \mathbf{Z}$ given as follows. For each $n \in \mathbf{Z}$ let $\Delta_x(n)$ be the integer defined by the two conditions

$$\Delta_x(n) \in \{-1, 0, 1\} \quad \text{and} \quad \Delta_x(n) = x(n+1) - x(n) \pmod 3.$$

W_x is defined inductively by

$$\begin{aligned} &W_x(0) = x(0), \\ &W_x(n+1) = W_x(n) + \Delta_x(n) \quad \text{if } n \geq 0, \\ &W_x(n) = W_x(n+1) - \Delta_x(n) \quad \text{if } n < 0. \end{aligned}$$

4.1 *Remark.* Note that interchanging the values of $x(n)$ and $x(n+1)$ has the effect of multiplying $\Delta_x(n)$ by -1. It follows that if μ is any Bernoulli measure then $\int \Delta_x(0)\, d\mu(x) = 0$. This equality will be needed later on.

W_x has a natural interpretation as a random walk on a vertical number line: the walker begins at height $x(0)$, and at the nth step the walker either stands still, moves up one unit, or moves down one unit, depending upon the value of $\Delta_x(n)$.

It is important to note that the bisequence x can be reconstructed from the graph of W_x. If $\varphi : \mathbf{Z} \to \mathbf{Z}$ is any map, let $J(\varphi) \in \Sigma$ be defined by $(J\varphi)(n) = \varphi(n) \pmod 3$. It is easy to verify that $J(W_x) = x$ for all x. It will be useful to make a slight extension. Consider the set $\mathscr{M}$ of all maps $\varphi : \mathbf{Z} \to \mathbf{Z}$ with the property that $|\varphi(n+1) - \varphi(n)| \leq 1$ for all n. We will say that φ, ψ in $\mathscr{M}$ are equivalent, $\varphi \sim \psi$, if $J(\varphi) = J(\psi)$. Geometrically speaking, φ and ψ are equivalent if the graph of one is a vertical translation of the graph of the other, and the amount of translation is a multiple of three.

4.2 **Lemma.** *For each* $\varphi \in \mathscr{M}$, $W \circ J(\varphi) \sim \varphi$.

Proof. By the definition of equivalence, it suffices to check that $J(W_x) = x$ for every $x \in \Sigma$. This is just the definition of J. □

Now consider the map $\Gamma : \mathscr{M} \to \mathscr{M}$ given by

$$(\Gamma\varphi)(n) = 1 + \max\{\varphi(n), \varphi(n+1)\}.$$

The graph of $\Gamma\varphi$ is obtained by first moving the entire graph of φ up one unit, and then moving each increasing segment on the graph one unit to the left, filling in valleys on the left and leaving plateaus on the right. It is clear from the definition of $\mathscr{M}$ that $\Gamma\varphi$ is an element of $\mathscr{M}$ whenever φ is. Moreover, if the graph of φ is a vertical translate of the graph of ψ then the same is true of $\Gamma(\varphi)$ and $\Gamma(\psi)$, and the vertical displacement between the graphs is the same. In particular, if $\varphi \sim \psi$, then $\Gamma(\varphi) \sim \Gamma(\psi)$. Γ induces a map $f : \Sigma \to \Sigma$ by the formula

$$W_{f(x)} \sim \Gamma(W_x)$$

i.e., by $f(x) = J \circ \Gamma(W_x)$. This map f is a cellular automaton: it is clear that f is continuous, and shift invariance follows by noting that the graph of $W_{\sigma(x)}$ is obtained from the graph of W_x by shifting the graph one unit horizontally, and possibly shifting three units vertically. The lemma, combined with the observation that Γ preserves equivalence, shows that $f^k = J \circ \Gamma^k \circ W$ for all $k \geq 0$.

Let 0^*, 1^*, 2^*, denote the three constant bisequences in Σ ($0^*(n) = 0$ for all n, etc.). Clearly $\gamma = \{0^*, 1^*, 2^*\}$ is a periodic orbit of f; we will show that γ is a μMCA but is not a μ-attractor. The following lemma is clear.

4.3 **Lemma.** *For any* $\varphi \in \mathscr{M}$ *and any* $k \geq 0$, *the value of* $(\Gamma^k\varphi)(n)$ *is equal to* $k +$ (*the maximum value of* φ *on the interval* $[n, n+k]$).

4.4 **Proposition.** (a) $\omega(x) = \gamma$ *if and only if the function* W_x *is bounded above on* $[0, \infty)$ *and achieves its maximum value infinitely often in* $[0, \infty)$.

(b) *Let* $MV(k, x)$ *denote maximum value of* $W_x(j) - W_x(0)$ *for* $0 \le j \le k$. *Then* $\mathrm{Cent}(x) = \gamma$ *if and only if* $MV(k, x)/k \to 0$ *as* $k \to \infty$.

Proof. (a) $\omega(x) = \gamma$ if and only if for each interval $I_m = [-m, m]$ the restriction of $f^k(x)$ to I_m takes on only one value for each sufficiently large integer k. This is the same as saying that the part of the graph of $W_{f^k(x)}$ that lies over I_m is horizontal for all sufficiently large k. As noted above, this graph is equivalent to that of $\Gamma^k W_x$ (i.e., the first graph is a vertical translate of the second with the vertical distance between the graphs divisible by 3). Using the lemma it is easy to see that the graphs of $\Gamma^k W_x$ over I_m can be horizontal for all large k if and only if W_x achieves its maximum on $[0, \infty)$ and does so at some point $n \ge m$. □

Proof of (b). If W_x is bounded above on $[0, \infty)$ then (b) follows from (a), so we will assume that W_x is not bounded above on $[0, \infty)$. Saying that $\mathrm{Cent}(x) = \gamma$ is equivalent to saying that for each $m \ge 1$ the proportion of the integers j in $[0, k]$ satisfying

(∗) the graph of the restriction of $\Gamma^j W_x$ to I_m is horizontal

tends to 1 as $k \to \infty$. Using the lemma, we see that (∗) fails if and only if the maximum value of W_x on $[-m, m + j]$ is larger than the maximum value of W_x on $[-m, -m + j]$. Since we are assuming that $\limsup_{n\to\infty} W_x(n) = \infty$, as long as j is large this will be true if and only if $MV(j+m, x) > MV(j-m, x)$. Let N_k denote the number of values of j in $[0, k]$ for which this happens; the preceding comments can be rephrased by saying that $\mathrm{Cent}(x) = \gamma$ is equivalent to $\lim_{k\to\infty}(N_k)/k = 0$. $MV(k, x)$ is equal to the number of integers i in $[0, k]$ such that $W_x(i) > W_x(j)$ for all $0 \le j < i$. Each such integer i contributes to at most $2m$ to the quantity N_k, and any new maximum that is encountered at position i with $m \le i \le k$ contributes at least 1 to N_k. Thus

$$2m \cdot MV(k, x) \ge N_k \ge MV(k, x) - m.$$

In summary, $\lim_{k\to\infty}(N_k)/k = 0$ if and only if $\lim MV(k, x)/k = 0$, and we are done. □

4.5 **Proposition.** *Let* μ *be any Bernoulli measure on* $\{0, 1, 2\}$. *Then the following are true* μ*-almost everywhere.*

(a) W_x *has no finite upper bound on* $[0, \infty)$.

(b) $MV(k, x)/k \to 0$ *as* $k \to \infty$.

Proof. In the situation where μ is the balanced Bernoulli measure (the measure that gives equal weight to each of the three symbols) (a) is a standard result concerning unbiased one-dimensional random walks; see [3]. For more general μ we use the following argument. Recall that the graph of $W_{\sigma(x)}$ is obtained by translating the graph of W_x horizontally one unit and vertically by 0 or 3 units. From this it is clear that the set of x such that W_x is bounded above is shift-invariant. Let Q denote the complementary set,

$$Q = \{x | W_x \text{ is not bounded above on } \mathbf{Z}\}.$$

The ergodic theorem implies that Q either has measure 0 or measure 1. We finish the proof of (a), assuming that Q has measure 1: consider the involution $\tau : \Sigma \to \Sigma$ defined by $(\tau x)(n) = x(-n)$. τ is measure preserving, so $Q' = Q \cap \tau(Q)$ also has measure 1. The effect of τ on the random walk is just to reflect its graph across the vertical axis, $W_{\tau(x)}(n) = W_x(-n)$. (a) holds for all $x \in Q'$.

Now we show that Q has measure 1. Note that if $x \in \Sigma$ and if z is obtained from x by reversing the sequence $x(0), x(1), \ldots, x(n)$, then $W_z(n) - W_z(0) = -[W_x(n) - W_x(0)]$ (z is defined by $z(j) = x(n-j)$ for $0 \leq j \leq n$). Since μ is a product measure,

$$\mu\{x | W_x(n) - W_x(0) = K\} = \mu\{x | W_x(n) - W_x(0) = -K\}$$

for each integer K. Consequently the set of x for which W_x is bounded above has the same measure as the set of x for which W_x is bounded below. In other words, if Q has measure 0 then the set of x for which W_x is bounded has measure 1. The following lemma shows that this is not the case, and so finishes the proof of (a).

Lemma 4.6. *The set of x such that W_x is bounded has measure* 0.

Proof. Almost every bisequence $x \in \Sigma$ has the property that every possible finite string of symbols occurs somewhere in x. In particular, every such x contains long runs of the form $012012\ldots012$, say of length $3N$. The graph of W_x over the interval corresponding to such a run is a line segment with slope 1. Since this occurs for every value of N, W_x is not bounded for any such x. □

To prove (b) let M_k denote the maximum value of W_x on $[0, k]$. Since $MV(k, x) = M_k - W_x(0)$, it is enough to show that M_k/k tends to 0 with probability 1.

Define $a_0 = b_0 = 0$, and for $k > 0$ define

$$a_k = W_x(k)/k, \qquad b_k = M_k/k.$$

We claim that if $\lim a_k = 0$ then $\lim b_k = 0$ as well. This is not too hard to see: for each k there is a value $j = j(k)$ in the interval $[0, k]$ such that $M_k = M_j = W_x(j)$, so that

$$(*) \qquad a_j = b_j \geq b_k.$$

Now there are two cases: if the sequence M_k is unbounded, then the integers $j(k)$ go to ∞ as k does, and so $(*)$ establishes the claim; the remaining case is that the sequence M_k is eventually constant, in which case it is obvious that b_k tends to 0.

The proof is finished by showing that the sequence a_k (which depends on $x \in \Sigma$) tends to 0 with probability 1. Consider the function Δ used to define Γ; set $T(x) = \Delta_x(0)$. Note that $T(\sigma^j x) = \Delta_x(j)$, and recall from 4.1 that $\int T\,d\mu = 0$ for any Bernoulli measure μ. The ergodic theorem shows that for μ-almost every $x \in \Sigma$,

$$0 = \lim_{k\to\infty} \frac{1}{k} \sum_{j=0}^{k-1} T(\sigma^j x) = \lim_{k\to\infty} \frac{1}{k} \sum_{j=0}^{k-1} \Delta_x(j)$$

$$= \lim_{k\to\infty} \frac{1}{k}(W_x(k) - W_x(0)) = \lim W_x(k)/k. \quad \square$$

The combination of 4.4 and 4.5 establishes the following result.

4.7 **Theorem.** *Let μ be any Bernoulli measure on* $\{0, 1, 2\}$.
(a) *γ is not a μ-attractor for f.*
(b) *γ is the μ-minimal center of attraction for f.*

Remark. The block-map definition of the automaton f is

$$(fx)(n) = (1 + x_{n+1}) \bmod 3 \quad \text{if } x(n+1) = x(n) + 1 \pmod 3,$$
$$(fx)(n) = (1 + x_n) \bmod 3 \qquad \text{otherwise}.$$

The remainder of this section contains various modifications of the previous example. It will often be convenient to restrict to the case of a *balanced Bernoulli measure*, which we will denote by ν. (ν is the measure that gives equal weight to each of the symbols in S.) We begin by giving an example of a cellular automaton that has a finite νMCA that consists of more than one orbit.

Example 4.8. Given $\varphi : \mathbf{Z} \to \mathbf{Z}$ define a new map $\Gamma_0\varphi$ by $(\Gamma_0)(n) = \max\{\varphi(n), \varphi(n+1)\} = (\Gamma\varphi)(n) - 1$; use Γ_0 in place of Γ in the previous construction to define a cellular automaton f_0. Each point of γ is fixed by f_0. f and f_0 commute, and their third iterates are equal, so it is clear that $\mathrm{Cent}(x; f_0) \subset \gamma$ for μ-almost all x. In order to conclude that γ is the νMCA of f_0 we need to show that $\nu(\psi(X; f_0)) = 0$ for each of the 6 proper subsets X of γ. Since ν is the balanced measure, this is easy to accomplish. Any permutation of the symbols induces a measure preserving automorphism on Σ. Let π be the automorphism induced by the permutation $s \to s+1 \pmod 3$. Note that the graph of $W_{\pi(x)}$ is obtained by vertically translating the graph of W_x either one unit up or two units down. Thus π maps $\psi(s^*; f_0)$ to $\psi((s+1)^*; f_0)$. It follows that $\nu(\psi(0^*; f_0)) = \nu(\psi(1^*; f_0)) = \nu(\psi(2^*; f_0))$. By Theorem B at most one of these sets has positive measure, so they must all have measure 0.

The same argument shows that if X is one of the two-element subsets of γ and $\psi(X)$ has positive measure, then the same is true for all two-element subsets. All of these sets are shift invariant, so the assumption of positive measure implies that they have measure 1, and so their intersection has measure

1. This is absurd, since the intersection is empty. Thus $\psi(X; f_0)$ has measure 0 for all proper subsets of γ, so that γ is the νMCA for f_0.

It is an easy exercise in uniform continuity to show that whenever g is a cellular automaton and $\mathrm{Cent}(x; g)$ consists of a finite number of fixed points, then $\mathrm{Cent}(x; g^j) = \mathrm{Cent}(x; g)$ for each $j \geq 1$. Since $f^3 = f_0^3$ this shows that γ is also the νMCA for f^3. Thus f is an example of a cellular automaton illustrating 3.7(2), since $\psi(\gamma; f)$ has full measure while $\bigcup_{0\leq j\leq 2}\psi(j^*; f^3) = \bigcup_{0\leq j\leq 2}\psi(j^*; f_0^3) = \bigcup_{0\leq j\leq 2}\psi(j^*; f_0)$ has measure 0.

4.9 **Example.** Next we indicate how to use f to create a cellular automaton g with the property that g has a finite νMCA and some of the points of this νMCA are not fixed by the shift. The idea is fairly simple: consider the product $\Sigma\times\Sigma$ with the measure $\nu\times\nu$. The bijection $\tau : (\Sigma, \mu) \to (\Sigma\times\Sigma, \nu\times\nu)$ given by $\tau(x) = (y, z)$ where

$$y(n) = x(2n) \quad \text{and} \quad z(n) = x(2n+1)$$

is a measure-theoretic isomorphism. Let $g : \Sigma \to \Sigma$ be defined by

$$g = \tau^{-1} \circ (f \times f) \circ \tau .$$

Note that g is a cellular automaton: the fact that f was used on each factor in $\Sigma \times \Sigma$ ensures that g commutes with σ. Let γ be as above and consider $A = \tau^{-1}(\gamma\times\gamma)$. The nine elements of A are the points of the form $\{x | x(n+2) = x(n) \text{ for all } n\}$; each of these points has period 3 under g and period 1 or 2 under σ.

It suffices to show that the $(\nu\times\nu)$-MCA of $f\times f$ is $\gamma\times\gamma$. It is clear that $(\nu\times\nu)(\psi(\gamma\times\gamma, f\times f)) = 1$; it remains to show that $(\nu\times\nu)(\psi(X)) = 0$ for every proper invariant subset of $\gamma\times\gamma$. For $k = 0, 1, 2$ let $\mathscr{O}_k$ denote the $f\times f$ orbit of the point $(0^*, k^*)$. Each $\mathscr{O}_k$ is a period three orbit of $f\times f$, and $\gamma\times\gamma$ is composed of these three orbits. We need to show that $(\nu\times\nu)(X) = 0$ whenever X is composed either of one or of two of the $\mathscr{O}_k$. The argument is like the one used in Example 4.8. Let $\pi : \Sigma \to \Sigma$ be as in 4.8. If $(x, y) \in \psi(\mathscr{O}_0)$ then with high probability the point $(f^k(x), f^k(y))$ is close to $\mathscr{O}_0 = \{(0^*, 0^*), (1^*, 1^*), (2^*, 2^*)\}$; when this occurs $(f^k(x), f^k(\pi(y)))$ is close to $\mathscr{O}_1$. It follows as in 4.8 that $\psi(\mathscr{O}_0)$ has measure 0. In a similar way one shows that $\psi(X)$ has measure 0 for all proper subsets of $\gamma\times\gamma$, so that $\gamma\times\gamma$ is the $(\nu\times\nu)$-MCA for $f\times f$.

Remark 4.10. By splitting Σ into N factors instead of into 2 we can construct a cellular automaton with a finite νMCA that contains some points whose least period under σ is N.

4.11 *Other odd periods.* Theorem 4.7 holds with essentially the same proof for many other cellular automata; one can use the transformation Γ with a variety of random walks to define cellular automata with various properties for which

4.7 holds. In particular for each odd integer $K \geq 3$ there is a cellular automaton on K symbols with a periodic orbit of period K that is a μMCA but not a μ-attractor, for every Bernoulli measure μ. The construction is a direct generalization of the case $K = 3$. Suppose $K = 2N + 1 \geq 3$ and take the symbol set S to be $\{1, \dots, 2N+1\}$. Let $\Delta : S \times S \to \mathbf{Z}$ be the map satisfying

4.11(i) $-N \leq \Delta(s, t) \leq N$ for all s, $t \in S$,

4.11(ii) $\Delta(s, t)$ is equivalent to $t - s$ modulo K.

These conditions determine Δ; note that

4.11(iii) $\Delta(s, t) = -\Delta(t, s)$ for all s and t,

4.11(iv) $\Delta(s+1, t+1) = \Delta(s, t)$ for all s, t (addition $\bmod K$).

Use Δ to associate a random walk W_x to each $x \in \Sigma$, just as before: $W_x(0) = x(0)$ and $W_x(n+1) = W_x(n) + \Delta(x(n+1), x(n))$. 4.11(iii) ensures that the ergodicity condition 4.1 will hold. W maps Σ into

$$\mathscr{M} = \{\varphi : \mathbf{Z} \to \mathbf{Z} | -N \leq \varphi(j+1) - \varphi(j) \leq N \text{ for all } j\}.$$

Let Γ be as before, and note that $\mathscr{M}$ is taken into itself by Γ. Define $J : \mathscr{M} \to \Sigma$ by the formula $(J\varphi)(n) = \varphi(n) \pmod K$.

Lemma. *$J(W_x) = x$ for each $x \in \Sigma$.*

Proof. Recall that $W_x(0) = x(0)$ and that for $n \geq 0$,

$$W_x(n+1) = x(0) + \sum_{0 \leq j \leq n} \Delta(x(j), x(j+1)),$$

so that

$$\begin{aligned} J \circ W_x(n+1) &= x(0) + \sum_{0 \leq j \leq n} \Delta(x(j), x(j+1)) && \pmod K, \\ &= x(0) + \sum_{0 \leq j \leq n} [x(j+1) - x(j)] && \pmod K, \\ &= x(n+1) && \pmod K. \end{aligned}$$

The argument for $n < 0$ is similar. □

At this stage we once again define the cellular automaton f by the formula $W_{f(x)} = J \circ \Gamma(W_x)$. The lemma and 4.11(iv) ensure that $f^k = J \circ \Gamma^k \circ W$ for each $k \geq 1$. The rest of the argument is as before.

4.12 *Even periods.* When K is even, the previous construction breaks down. One can use conditions 4.11(i), (ii), (iii) to obtain a map $\Delta : \{1, \dots, 2N\}^2 \to \{-N, \dots, N\}$ (although these conditions do not completely determine Δ). Moreover it is still possible to define a cellular automaton f by the equation $f = J \circ \Gamma \circ W$. The difficulty is that this Δ cannot satisfy 4.11(iv). Because of this it is no longer true that $f^k = J \circ \Gamma^k \circ W$. However, by restricting our attention to the balanced measure ν, we can find periodic νMCA's of all periods larger than 2. Suppose $S = \{1, \dots, K\}$ where $K = 2N \geq 4$. Define

$\Delta : S \times S \to \mathbf{Z}$ by

4.12(i) $\Delta(s, t) = t - s \pmod{2N}$;

4.12(ii) $\Delta(s, t) \in \{-N+1, \dots, N\} \cup \{-3N+1\}$;

4.12(iii) when $t - s = 1 - N \pmod{2N}$ then $\Delta(s, t) = 1 - N$ if s is even and $\Delta(s, t) = 1 - 3N$ if s is odd.

Using this Δ with W, J and Γ as before we get a cellular automaton f satisfying $f^k = J \circ \Gamma^k \circ W$ for all $k \geq 1$. However, since Δ is no longer antisymmetric, different arguments are required to establish 4.5 and to show the result of 4.1, namely that $\int \Delta(x(0), x(1))\, d\nu = 0$. This last equality is a simple calculation, based on the fact that ν is the balanced measure:

$$\int \Delta(x(0), x(1))\, d\nu = \sum_{s, t \in S} \Delta(s, t) p_s p_t = \frac{1}{K^2} \cdot \sum_s \sum_t \Delta(s, t)$$

and the double sum is 0 since $\sum_t \Delta(s, t) = N$ when s is even and $= -N$ when s is odd.

In the proof of 4.5 the antisymmetry of Δ was used to show that W_x is not bounded above for almost all x. We get the same conclusion with the altered definition of Δ as follows. Use Δ to define a Markov chain on $\mathbf{Z}$: the transition probability $P(i, j) = 1/K$ if there is an $x \in \Sigma$ and $n \in \mathbf{Z}$ with $W_x(n) = i$ and $W_x(n+1) = j$. Choose a positive integer m and approximate this infinite Markov chain with a finite one whose states are $[-mK, mK + K - 1]$, with the endpoints as absorbing states. The interior states are all clearly transitive, so with probability one, as the process evolves from any initial configuration it approaches a limiting configuration that is concentrated at the two endpoints [3]. If the initial distribution is the one with equal weights at $0, 1, \dots, K-1$ and weight 0 elsewhere, then the two limiting probabilities of being absorbed at an endpoint are each equal to 1/2. To see this note that the initial probability of being at an odd-numbered state is 1/2, and that this property is preserved as the chain evolves. In particular, the limiting probability of being absorbed by the odd endpoint $mK + K - 1$ is 1/2. It follows that for the original infinite Markov chain the sets $\{x \in \Sigma \mid \sup(W_x) > L\}$ have measure at least 1/2 for every L, so that the set of x for which W_x is bounded above cannot have measure 1; since this last set is shift invariant, it must therefore have measure 0. The remainder of the proof of 4.5 is as before.

When $K = 2$, the argument breaks down in several places. Basically the difficulty is that with only two symbols it is not possible to define the walk W_x so that at each point there is a possibility of moving up or down or staying level.

Problem. For some Bernoulli measure μ, find an example of a cellular automaton f that has a period two μMCA that is not a μ-attractor.

4.13 *Other variations.*

4.13(i) Given an integer p define

$$(\Gamma_p \varphi)(k) = p + \max\{\varphi(k), \varphi(k+1)\}.$$

Exactly as above the formula $f_p = J \circ \Gamma_p \circ W$ defines a cellular automaton with the property that the set $C = \{s^* | s \in S\}$ of constant bisequences is a μ-center of attraction but is not a μ-attractor for f_p. Each point of C is a periodic point of f; if K and p are relatively prime then C is a single periodic orbit, and it is clear that C is the μMCA of f. When K and p are not relatively prime C consists of several periodic orbits; in this case to show that C is the μMCA one must show as in 4.9 that no proper invariant subset of C is a μ-center of attraction. If $\mu = \nu$ (the balanced measure) this is done just as above. Taking $p = 0$ gives a cellular automaton with a νMCA consisting of K fixed points.

4.13(ii) Further examples can be generated by applying the construction of 4.9 to the cellular automata in 4.11 and 4.12.

4.13(iii) There are still other cellular automata for which the conclusion of 4.7 appears to hold true. For instance, let g be the automaton acting on the three-shift which is given by

$$(gx)(n) = 1 + (\text{the minimum of } x(n-1), x(n), x(n+1)),$$

where the addition is computed modulo 3. In other words,

$$\begin{aligned} (gx)(n) &= 1 \quad \text{if any of } x(n-1), x(n), x(n+1) \text{ are } 0, \\ (gx)(n) &= 0 \quad \text{if all of } x(n-1), x(n), x(n+1) \text{ are } 2, \\ (gx)(n) &= 2 \quad \text{otherwise}. \end{aligned}$$

The idea is that there is a connection between g and the map f of 4.7. The details are cumbersome; a brief outline of the idea is given in §6.

5. Theorem D

In this section we will give a result that is slightly stronger than Theorem A: if there is a minimal μ-attractor and if the sequence of iterates of almost every point is eventually periodic on each block, then the μ-attractor is a single periodic orbit which is pointwise fixed by all shifts. Before doing this we will give some background concerning eventual periodicity on blocks. Much of this material is either contained in or was suggested by R. Gilman's papers [4, 5]. As before, let B denote a block, i.e., a finite, nonempty subset of $\mathbf{Z}^m$.

5.1 **Definition.** Suppose $x \in \Sigma$ and that f is a cellular automaton.

(a) $P_B(f) = \{x | f^i(x) \text{ is eventually periodic on } B\}$.

(b) $P(f) = \bigcap_B P_B(f)$.

5.2 *Remarks.* (a) $\sigma_t(P_B(f)) = P_{B'}(f)$ where $B' = \{b-t | b \in B\}$, so $\sigma_t(P(f)) = P(f)$.

(b) by (a) and ergodicity the measure of $P(f)$ is either 0 or 1.

(c) if f has a periodic μ-attractor, then $\mu(P(f)) = 1$.

5.3 **Lemma.** *If $\mu(P(f)) = 1$, then for any $\tau < 1$ there is a closed set Y in Σ with*

(a) $f(Y) \subset Y$,

(b) $\mu(Y) > \tau$,

(c) *the restriction of* f *to* Y *is equicontinuous (i.e., the family of maps* $\{f^j | j \geq 0\}$ *is equicontinuous on* Y).

Proof. See Proposition 4 of [5]. □

5.4 **Definition.** For each $m = 1, 2, \ldots$ use 5.3 to select a closed, forward invariant set Y_m with $\mu(Y_m) > (m-1)/m$. Let Z denote the union of all of the sets Y_m. Note that Z is forward invariant and has full measure.

Remark. Define a subset $E(f)$ of Σ as follows: x is in $E(f)$ if and only if for each block B, the set

$$\{y | (f^i y)(b) = (f^i x)(b) \text{ for all } i \geq 0 \text{ and all } b \in B\}$$

has strictly positive measure. If follows from 5.3 that if $\mu(P(f)) = 1$ then $E(f)$ also has full measure. In [4, 5] R. Gilman proves a converse of this fact for one-dimensional cellular automata: if the set $E(f)$ has full measure then so does $P(f)$. His techniques do not extend to higher dimensional automata, and it is unknown whether $\mu(E(f)) = 1$ implies $\mu(P(f)) = 1$ when f is a cellular automaton of dimension greater than one.

5.5 **Theorem D.** *Suppose that* $\mu(P(f)) = 1$ *and that* f *has a minimal* μ-*attractor* A. *Then* A *is a periodic orbit, and each point of* A *is fixed by all shifts.*

Proof. In view of Theorem A it is enough to show that A is a periodic orbit. Let B be the block $\{0\}$. Since $P_B(f)$ and $\rho(A; f)$ both have measure 1, there is a point q with $\omega(q) = A$ and with $(f^n q)(0)$ eventually periodic, say of least period k. Let $\delta_1, \ldots, \delta_k$ be the symbols such that $(f^{nk+i} q)(0) = \delta_i$ for all large n and each i satisfying $1 \leq i \leq k$. Let $D^1 = \{D^1(n) | 1 \leq n < \infty\}$ denote the periodic symbol sequence of period k defined by

$$D^1 = \delta_1, \delta_2, \ldots, \delta_k, \delta_1, \ldots.$$

For $j = 2, 3, \ldots, k$ let D^j denote the sequence obtained from D^1 by deleting the first $j-1$ elements of D^1, so

$$D^j = \delta_j, \delta_{j+1}, \ldots, \delta_k, \delta_1, \ldots.$$

Since k is the least period of D^1 all of the sequences $D^1, \ldots, D^k$ are distinct.

5.6 **Lemma.** *If* $y \in A$ *then there is a uniquely determined integer* $j \in \{1, 2, \ldots, k\}$ *such that* $y \in A_j$, *where*

$$A_j = \{x \in A | (f^i x)(0) = D^j(i) \textit{ for all } i \geq 0\}.$$

Proof. Let q be as above and suppose that y is a point of A. $\omega(q)$ contains y, so there is a sequence of integers $n_m \to \infty$ with $f^{n_m}(q) \to y$. In fact there is such a sequence with the additional property that all of the integers n_m are equivalent modulo k; let j denote the integer in the range $1, \ldots, k$ such that $n_m = j \pmod{k}$ for every m. Set $q_m = f^{n_m}(q)$; it follows that $q_m(0) = \delta_j$ for all large m. Since $q_m \to y$ we see that $y(0) = \delta_j$. Moreover, the continuity of f implies that $\lim_{m\to\infty} f^i(q_m) = f^i(y)$ for any $i \geq 0$ so that

$$(*) \qquad (f^i y)(0) = \lim_{m\to\infty} (f^i q_m)(0) = \delta_{(j+1) \bmod k}.$$

In other words the sequence $(f^i y)(0)$ is the periodic sequence D^j. □

5.7 **Corollary.** *Let Z be the set of* 5.4; *if x is any point in $Z \cap \rho(A; f)$ then there is an integer j, $1 \leq j \leq k$, such that the sequence $(f^i x)(0)$ and $D^j(i)$ agree for all sufficiently large i.*

Proof. Given such an x, there is a set Y as in 5.3 containing x. Since $\omega(p) = A$ for almost all points p, the fact that Y has positive measure, is closed, and is forward invariant implies that Y contains A. Let $\varepsilon > 0$ be chosen so that any two points y, z of Σ that are within ε of each other must satisfy $y(0) = z(0)$. Now use the equicontinuity of 5.3 to select $\delta > 0$ small enough that if y, z are in Y and are within δ of each other, then $f^n(y)$ and $f^n(z)$ are within ε of each other for all $n \geq 0$. There is an integer M such that $f^M(x)$ is within δ of some point $y \in A$. By the lemma $y \in A_j$ for some j and so the choice of δ shows that $\{(f^{M+i}x)(0) | i \geq 0\} = D^j$. □

The sets A_j defined in 5.6 form a closed, pairwise disjoint cover of A. Additionally, $f(A_j) = A_{j+1 \pmod{k}}$, so that 2.4 implies that $\rho(A; f) = \bigcup \rho(A_j; f^k)$. We conclude

$$(**) \qquad 1 = \sum \mu(\rho(A_j; f^k)).$$

Next we would like to show that there is a single value of j, say $j = j'$ with

$$(***) \qquad 1 = \mu(\rho(A_{j'}; f^k)).$$

This is an immediate consequence of $(**)$ and the following lemma.

5.8 **Lemma.** *For each j, $\mu[\rho(A_j; f^k)]$ is either 0 or 1.*

Proof. Let $r_j = \mu[\rho(A_j; f^k)]$. The idea is to exploit 2.3 to show that if the lemma is false, then there is no minimal μ-attractor. For each $M \geq 0$ and each $j = 1, 2, \ldots, k$ consider the set

$$X_{j,M} = \{x \in \Sigma | (f^i x)(0) \text{ is equal to } D^j(i) \text{ for } i \geq M\},$$

and define $X_j = \bigcup_M X_{j,M}$. Note that

5.9(a). each set $X_{j,M}$ is closed, and

5.9(b). $f(X_{j,M}) \subset X_{i,M-1} \subset X_{i,M}$ where $i \equiv j+1 \pmod{k}$.

It follows from 5.9(b) that $f^k(X_{j,M}) \subset X_{j,M}$ for each j; in particular $f^k(X_j) \subset X_j$, so that $X_j \subset \rho(X_j; f^k)$. The sequences D^j are distinct, so the sets X_j

are pairwise disjoint; similarly the sets $\rho(X_j; f^k)$ are pairwise disjoint. By 5.7 almost every point of $\rho(A_j; f^k)$ is contained in X_j, so we have $r_j \le \mu(X_j) \le \mu(\rho(X_j; f^k))$. Applying $(**)$ gives

$$1 = \sum r_j \le \sum \mu(X_j) \le \sum \mu(\rho(X_j; f^k)) = \mu(\bigcup \rho(X_j; f^k)) \le 1, \text{ so}$$

5.9(c) $\mu(\rho(X_j; f^k)) = \mu(X_j) = r_j$ for each j.

By 5.9(a)–(b) X_j is a countable, increasing union of closed, f^k-invariant sets, so the corollary to 2.3 implies that for each j with $r_j > 0$ there is a μ-attractor M_j of f^k with $M_j \subset X_j$. In fact, for each such j

5.10. $0 < \mu(\rho(M_j; f^k)) \le \mu(\rho(X_j; f^k)) = r_j$.

Now suppose that there is a value $j = j'$ with $0 < r_{j'} < 1$. Using 5.10 and 2.7 we see that $M_{j'}$ is not a minimal μ-attractor of f^k. Therefore there is a μ-attractor N' for f^k with $N' \subset M_{j'} \subset X_{j'}$ and with

$$0 < \mu(\rho(N'; f^k)) < \mu(\rho(M_{j'}; f^k)) \le r_{j'}.$$

Let $N = \bigcup\{f^i(N') | 0 \le i \le k-1\}$. N is closed, and $f^k(N') \subset N'$, so $f(N) \subset N$. Applying 2.4,

$$\begin{aligned}\mu(\rho(N; f)) &= \sum_{i=0}^{k-1} \mu(\rho(f^i(N'); f^k)) \\ &\le \mu(\rho(N'; f^k) + \left[\sum_{j\ne j'} \mu(\rho(X_j; f^k))\right] \\ &= \mu(\rho(N'; f^k)) + \sum r_j - r_{j'} \\ &= 1 - [r_{j'} - \mu(\rho(N'; f^k))].\end{aligned}$$

The quantity in square brackets is positive, so $\mu(\rho(N; f)) < 1$; on the other hand $\mu(\rho(N; f)) > \mu(\rho(N'; f^k)) > 0$. Using 2.3 again, we see that there is an μ-attractor for f whose realm has measure strictly between 0 and 1. But then by 2.7 f has no minimal μ-attractor, contradicting our assumption. Hence r_j is either 0 or 1 for each j. □

Now we can finish the proof of Theorem D. In light of Proposition 2.9 all we need to do is to show that there is a periodic μ-attractor. To simplify notation assume that $j' = 1$ in $(***)$. It follows that the set

$$C_0 = \{x \in \Sigma | (f^{kn}x)(0) = \delta_1 \text{ for all sufficiently large } n\}$$

has measure 1. Define $C = \bigcap C_t$ where $C_t = \sigma_t(C_0)$; note that for each $t \in \mathbf{Z}^m$

C_t has full measure so that $\mu(C) = 1$ as well. However

$$\begin{aligned}\sigma_t(C_0) &= \{\sigma_t(x) | (f^{kn}x)(0) = \delta_1 \text{ for all sufficiently large } n\} \\ &= \{\sigma_t(x) | (\sigma_{-t} f^{kn} \sigma_t x)(0) = \delta_1 \text{ for all sufficiently large } n\} \\ &= \{y | (\sigma_{-t} f^{kn} y)(0) = \delta_1 \text{ for all sufficiently large } n\} \\ &= \{y | (f^{kn} y)(-t) = \delta_1 \text{ for all sufficiently large } n\}.\end{aligned}$$

Consequently if $x \in C$ then for each $t \in \mathbf{Z}^m$ there is an integer $N(t)$ such that $(f^{kn}x)(t) = \delta_1$ for all $n \geq N(t)$. Consider the fixed point q_1 of f^k that is defined by $q_1(t) = \delta_1$ for all t. It is now clear that $\omega(x; f^k) = \{q_1\}$ for all $x \in C$. Since q_1 is fixed by f^k it is a periodic point of f. If $x \in C$ then $\omega(x; f)$ is equal to the orbit of q_1, and so this orbit is a periodic μ-attractor for f. □

Gilman's main result in [5] is that either $\mu(P(f)) = 1$ or else f resembles an expansive map, in the following sense: there is a positive constant ε with the property that for any x in Σ the set of points y satisfying $f^j(x)$ and $f^j(y)$ are at least ε apart for some $j \geq 0$ has full measure. The following corollary shows that despite the strong tendency towards periodicity in examples like 4.9, they are examples of the expansive case in Gilman's theorem.

5.11 **Corollary.** *Suppose the $\mu(P(f)) = 1$ and that f has a minimal μMCA A. Then A is a minimal μ-attractor, and so by the Theorem* A *is a single periodic orbit, each point of which is fixed by all shifts.*

Proof. By 5.3 there is a closed, forward invariant set $Y \subset \Sigma$ such that Y has positive measure and the restriction of f to Y is equicontinuous. By 3.5(b) and the fact that $\mu(\psi(A)) = 1$ we know that almost every point $y \in Y$ satisfies $A = \text{Cent}(y)$. Since Y is closed and invariant we know that $\omega(y) \subset Y$; in particular $A \subset Y$ because $\text{Cent}(y) \subset \omega(y)$. Now we use the equicontinuity: for some large integer n we can be sure that $f^n(y)$ is as close as we like to A. Since y and A are both in Y, equicontinuity implies that all subsequent iterates of y also stay close to A. This shows that $\omega(y) \subset A$. The opposite inclusion is automatic, and we conclude that $\omega(y) = A$ for almost every point of Y. Since Y can be chosen to have measure arbitrarily close to 1, we see that $\omega(x) = A$ for μ-almost all x in Σ, so that A is a minimal μ-attractor. □

6. Example 4.13(iii)

Let g be the cellular automaton defined in 4.13(iii), and let f continue to stand for the automaton of Theorem 4.7. The map on Σ which interchanges the symbols 0 and 2 defines a topological conjugacy between g and h, where

$$(hx)(n) = 2 + (\text{the maximum of } x(n-1), x(n), x(n+1)),$$

TABLE 1

Block found in x	Replacement block in $D(x)$
$2a2$	222
$2ab2$	2222
101	111
1001	1111
2011	2111
2010	2110
2001	2111
1102	1112
0102	0112
1002	1112

(In the first two lines, a and b are arbitrary elements of the symbol set $\{0, 1, 2\}$.)

For example, if $x = \ldots 21202212001200002012\ldots$
then $D(x) = \ldots 22222222111200002222\ldots$

(again the addition is computed modulo 3). In other words,

$$(hx)(n) = 1 \quad \text{if any of } x(n-1), x(n), x(n+1) \text{ are } 2,$$
$$(hx)(n) = 2 \quad \text{if all of } x(n-1), x(n), x(n+1) \text{ are } 0,$$
$$(hx)(n) = 0 \quad \text{otherwise}.$$

There is a connection between the automata h^3 and $\sigma \circ f^2$ which we will describe below (here σ is the shift to the right, $(\sigma x)(n) = x(n-1)$). The ideas leading to Theorem 4.7 came out of an attempt to understand the dynamics of h.

We begin with some technical preliminaries. The first is to define a "damping operator" $D : \Sigma \to \Sigma$. D is another cellular automaton, which acts on a bisequence $x \in \Sigma$ by replacing certain blocks by other blocks of the same size, as indicated in Table 1 (the replacement block in $D(x)$ occurs in the same positions as the block it replaces from x). The idea behind the replacements is that in the computation of $h(x)$ certain symbols have no effect; for instance an isolated 0 is never the maximum of any trio $x(n-1)$, $x(n)$, $x(n+1)$, so that replacing the isolated 0 by a 1 (or sometimes replacing it by a 2) does not change the value of $h(x)$. We refer to D as a damping operator because the graph of the random walk $W_{D(x)}$ typically has fewer up-and-down fluctuations than that of W_x.

6.1 **Lemma.** (a) $h \circ D = h$.
(b) *any block of consecutive* 0's *in* $D(x)$ *has length at least* 3.
(c) *any block of consecutive* 1's *in* $D \circ h(x)$ *has length at least* 3.

Proof. (a) and (b) follow from the definition of D in terms of the ten block replacements given in Table 1; (c) follows from the fact that any block of consecutive 1's in $h(x)$ has length at least 3. □

Next we give a description of the dynamics of h; in view of 6.1 we restrict h to the subshift of finite type $X \subset \Sigma$, where X is the image of the map $D \circ h$.

6.2 **Proposition.** *Suppose that* $x \in X$. *Then* $(h^3 x)(n) = (f^2 \circ \sigma x)(n)$ *for all* n, *with the following exceptions*:

> *anyplace that* x *contains a string of exactly three or four* 1's, *bounded on each side by a* 2, *then* $h^3(x)$ *has* 2's *in place of the* 1's, *while* $f^2 \circ \sigma(x)$ *has either* 212 *or* 2112.

In terms of the random walk description of §4, both automata in 6.2 have the same general behavior: local maxima (plateaus) widen by one unit in each direction, filling in the valleys as they go. The only exception is that a valley coming from one of the blocks 21112 or 21112 is completely filled by one iteration of h^3, while it takes two iterations of $f^2 \circ \sigma$ to fill in such a valley.

6.2 is established by calculating the effect of each of the two automata on all blocks of the form $a^* b^* c^*$, where $\{a, b, c\} = \{0, 1, 2\}$ and the exponent * indicates that the symbol occurs some finite number of consecutive times (this number being ≥ 3 whenever the symbol is 0 or 1). The details are quite tedious and will not be reproduced here.

6.2 can be viewed as evidence that the conclusion of 4.7 should hold for h. Proving this would involve making the estimates needed to establish the analogue of 4.5, namely that

6.3(a). $W_{D \circ h(x)}$ has no finite upper bound, and

6.3(b). $MV^*(k, D \circ h(x))/k \to 0$ as $k \to \infty$

are true μ-almost everywhere, where $MV^*(k, x)$ denotes the maximum of $W_x(j) - W_x(0)$ on $[-k, k]$. Computer experiments provide evidence in support of 6.3, but it has not been rigorously verified.

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