

LINEAR SERIES WITH AN N -FOLD POINT ON A GENERAL CURVE

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ABSTRACT. A linear series $(V, \mathcal{L})$ on a curve X has an N -fold point along a divisor D of degree N if $\dim(V \cap H^0(X, \mathcal{L}(-D))) \geq \dim V - 1$. The dimensions of the families of linear series with an N -fold point are determined for general curves.

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We work over the field of complex numbers $\mathbb{C}$.

Let X be a smooth projective curve. A g_d^r on X is a linear series of dimension r and degree d on X , i.e., a pair $(V, \mathcal{L})$ consisting of a line bundle $\mathcal{L}$ of degree d and an $r+1$ dimensional subspace $V \subset H^0(X, \mathcal{L})$. The g_d^r 's on X are parameterized by a projective scheme $G_d^r(X)$. If X is general in moduli, then $\dim G_d^r(X) = \rho(g, r, d) = g - (r+1)(g+r-d)$ [ACGH].

Definition. We say that a $g_d^r(V, \mathcal{L})$ has an N -fold point along a divisor D of degree $N \geq 2$ in X if $\dim(V \cap H^0(X, \mathcal{L}(-D))) \geq r$.

If $\pi: X \rightarrow V$ is a flat proper irreducible family of smooth curves, there exists a scheme $G_d^r(X/V)$ which is projective over V whose fiber over each $v \in V$ is $G_d^r(X_v)$ where $X_v = \pi^{-1}(v)$ [EH-2]. It is easy to see that, by considering the construction of $G_d^r(X \times_V X^N/X^N)$ as the degeneracy locus of a vector bundle map, there is a closed subscheme $N(X/V) \subset G_d^r(X \times_V X^N/X^N)$ such that the fiber over each point $(P_1, \dots, P_N) \in X_V^N$ consists of the g_d^r 's on X_V with an N -fold point along $\sum P_j$. We will write $N(X)$ if V is a point.

Let $\mathcal{M}_g$ denote the moduli space of smooth curves of genus $g \geq 3$, and let U be the open subset of $\mathcal{M}_g$ corresponding to curves without nontrivial automorphisms. Let $Z \rightarrow U$ be the universal curve over U . We will prove the following theorem.

Theorem. *With $g \geq N \geq 2$, $g \geq 3$, $r \geq 2$ and the notation as above, $\dim N(Z_u) \leq \rho(g, r, d) - r(N-1) + N$ when u is a sufficiently general point of U .*

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This extends a result of Marc Coppens who proved the case when $N = 2$ [C]. It is easy to see that $\rho(g, r, d) - r(N - 1) + N$ is a lower bound for the dimension of $N(Z_u)$ when $N(Z_u) \neq \emptyset$. If $\rho(g, r, d) - r(N - 1) + N \geq 0$ and $\rho(g, r - 1, d - N) \geq 0$ then $N(Z_u) \neq \emptyset$ [S].

Let X be a connected complete curve. We say that X is of compact type if and only singularities of X are ordinary double points, and the dual graph is a tree. We say a connected closed subcurve $Y \subset X$ is a tail if it meets $\overline{X - Y}$ at at most one point. We say that a curve X of genus g is of special type if: it is of compact type; each irreducible component is a nonsingular rational or nonsingular elliptic curve; and each irreducible elliptic component is a tail.

We say that a sequence $\mathbf{a} = (a_0, a_1, \dots, a_r)$ is of type (r, d) if $0 \leq a_0 < a_1 < \dots < a_r \leq d$. If X is a smooth curve containing a point P , and $L = (V, \mathcal{L})$ is a g'_d on X , then the orders of vanishing of the sections of V determine a sequence $\mathbf{a}$ of type (r, d) . We call $\mathbf{a}$ the vanishing sequence of L at P . We denote by $W(\mathbf{a}) = \sum(a_i - i)$ the weight of the sequence $\mathbf{a}$. If $\mathbf{b} = (b_0, \dots, b_r)$ is a sequence of type (r, d) , and a $g'_d L$ has vanishing sequence $\mathbf{a} = (a_0, \dots, a_r)$ at P , we say that L satisfies the vanishing condition $\mathbf{b}$ at P if $a_i \geq b_i$ for $i = 0, \dots, r$.

In §2 we show the existence of a family of smooth curves $X_{T-\{0\}} \rightarrow T - \{0\}$ which specialize to a curve X_0 of special type and a family A' of g'_d 's with N -fold points on $X_{T-\{0\}} \rightarrow T - \{0\}$. The N -fold points of A' specialize to a genus M tail of X_0 . The g'_d 's of A' also satisfy a vanishing condition $\mathbf{a}$ at a point. The relative dimension of A' over $T - \{0\}$ is 0, and the codimension of A' in $N(X_{T-\{0\}}/T - \{0\})$ is $\leq N - M + W(\mathbf{a})$.

In §3 we use the theory of limit linear series as developed by Eisenbud and Harris [EH-2] to show that the crude limit linear series on X_0 induced by A' forces $\rho(g, r, d) - r(N - 1) + M - W(\mathbf{a}) \geq 0$.

The products involving $\mathbf{P}^1$ in the proof of Lemma 1 are taken over $\text{Spec } \mathbb{C}$. All other products are fibered over $\overline{\mathcal{M}}_g$ unless specified otherwise.

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We will make use of Knudsen's results concerning stable n -pointed curves [K1, K2]. A stable n -pointed curve is a connected projective curve X and n distinct nonsingular points $P_1, \dots, P_n$ of X such that: the only singularities of X are ordinary double points; and on every smooth rational component $Y \subset X$, $\#\{P_i \mid P_i \in Y\} + \#\{Y \cap \overline{X - Y}\} \geq 3$. For each g and n , there exists a coarse moduli space for n -pointed stable curves of genus g which we denote by $\overline{\mathcal{M}}_{g,n}$. For each g and n , $\overline{\mathcal{M}}_{g,n}$ is a projective variety.

The functors of relative stable n -pointed curves and relative stable $n - 1$ -pointed curves with an additional section are isomorphic. For each stable n -

pointed curve $(X, P_1, \dots, P_n)$ there is a curve X_c and morphism $c: X \rightarrow X_c$ such that: $(X_c, c(P_1), \dots, c(P_{n-1}))$ is a stable $n-1$ -pointed curve; and either c is an isomorphism, or P_n lies on a rational component $Y \subset X$ whose image in X_c is a point and $c|_{X-Y}$ is an isomorphism of $X-Y$ with $X_c - c(Y)$. When we wish to consider $\overline{\mathcal{M}}_{g,n}$ as coarsely representing the functor of stable $n-1$ -pointed curves with an additional section, we will write $\overline{\mathcal{M}}_{g,n} \simeq \overline{\mathcal{Z}}_{g,n-1}$. When $n=0$ we will write $\overline{\mathcal{M}}_g$ instead of $\overline{\mathcal{M}}_{g,0}$ and $\overline{\mathcal{Z}}_g$ instead of $\overline{\mathcal{Z}}_{g,0}$.

For each $m < n$ and each subset $\{i_1 < \dots < i_m\}$ of $\{1, \dots, n\}$, we have a *contraction* morphism $\pi: \overline{\mathcal{M}}_{g,n} \rightarrow \overline{\mathcal{M}}_{g,m}$ obtained by forgetting the points not indexed by $\{i_1 < \dots < i_m\}$ and collapsing certain rational subcurves, if necessary.

There is a natural *clutching* morphism

$$\gamma: \overline{\mathcal{M}}_{g_1, n+1} \times_{\mathbf{C}} \overline{\mathcal{M}}_{g_2, m+1} \rightarrow \overline{\mathcal{M}}_{g_1+g_2, n+m}.$$

If $(X, P_1, \dots, P_{n+1})$ and $(Y, Q_1, \dots, Q_{m+1})$ correspond to a point $(x, y) \in \overline{\mathcal{M}}_{g_1, n+1} \times_{\mathbf{C}} \overline{\mathcal{M}}_{g_2, m+1}$, then $\gamma(x, y)$ corresponds to the $n+m$ -pointed curve obtained by joining X and Y at the points P_{n+1} and Q_{m+1} .

Fix $g \geq 3$, $2 \leq N \leq g$, $r \geq 2$, and d . As before, we let $U \subset \overline{\mathcal{M}}_g$ be the open subset corresponding to smooth curves without nontrivial automorphisms, and we let $\pi: Z \rightarrow U$ be the universal curve over U . Let $H \subset N(Z/U)$ be a component of the scheme of g_d^r 's with N -fold points whose dimension is maximal with respect to the property that $\pi(H) = U$.

We have a natural morphism $\alpha: H \rightarrow (\overline{\mathcal{Z}}_g)^N$. Let B be the closure of $\alpha(H)$ in $(\overline{\mathcal{Z}}_g)^N$, and let $\beta: B \rightarrow \overline{\mathcal{M}}_g$ be the natural morphism. Note that $\beta(B) = \overline{\mathcal{M}}_g$, because $\pi(H) = U$. Let $M = N - \min\{\dim \beta^{-1}(x) \mid x \in \overline{\mathcal{M}}_g\}$.

Note that $0 \leq M \leq N$. If $M = 0$, then the theorem holds, because the g_d^r 's on a general curve with an N -fold point along a divisor whose support is a general point have codimension $r(N-1)$ by Theorem 4.5 of [EH-2]. We will henceforth assume $M \geq 1$.

Lemma 1. *There exists a point $b \in B$ corresponding to a curve X_b which is of special type and points $P_1, \dots, P_N \in X_b$ which lie on a tail of genus $\leq M$ or $P_1 = P_2 = \dots = P_N$ is a nonsingular point on a rational component of X_b .*

Proof. All products involving $\mathbf{P}^1$ in this proof are fibered over $\text{Spec } \mathbf{C}$. All other products are fibered over $\overline{\mathcal{M}}_g$.

There is a set-theoretic map on closed points $\delta: \overline{\mathcal{M}}_{0,g} \times (\overline{\mathcal{Z}}_g)^N \rightarrow (\overline{\mathcal{Z}}_{0,g})^N$ which we will now describe. A point $w \in \overline{\mathcal{M}}_{0,g} \times (\overline{\mathcal{Z}}_g)^N$ corresponds to a g -pointed rational curve $(X, Q_1, \dots, Q_g)$ and N points $P_1, \dots, P_N$ on the stable genus g curve $\tilde{X}_{Q_1, \dots, Q_g}$ obtained by attaching a fixed elliptic curve E to X at each point $Q_1, \dots, Q_g$. Let $\eta: \tilde{X}_{Q_1, \dots, Q_g} \rightarrow X$ be the natural map which collapses the elliptic tails. The point $\delta(w) \in (\overline{\mathcal{Z}}_{0,g})^N$ corresponds to $(X, Q_1, \dots, Q_g)$ and the N points $\eta(P_1), \dots, \eta(P_N)$.

For each $I = (i_1, \dots, i_N)$ such that $0 \leq i_j \leq g$ for $j = 1, \dots, N$ there is a scheme D_I which parameterizes g -pointed rational curves $(X, Q_1, \dots, Q_g)$ and points $P_1, \dots, P_N$ on $\tilde{X}_{Q_1, \dots, Q_g}$ such that each P_j lies on the elliptic tail attached to X at the point Q_{i_j} if $i_j > 0$, or P_{i_j} lies on X if $i_j = 0$. Each D_I is easily shown to exist by considering contraction and clutching morphisms.

Furthermore, there are morphisms $\phi_I: D_I \rightarrow (\overline{\mathcal{Z}}_{0,g})^N$ and $\Psi_I: D_I \rightarrow \overline{\mathcal{M}}_{0,g} \times (\overline{\mathcal{Z}}_g)^N$ such that $\delta \circ \Psi_I = \phi_I$. It follows that $\delta(p_Z^{-1}(B))$ is a closed subset of $(\overline{\mathcal{Z}}_{0,g})^N$ where p_Z is the projection of $\overline{\mathcal{M}}_{0,g} \times (\overline{\mathcal{Z}}_g)^N$ to $(\overline{\mathcal{Z}}_g)^N$.

There are morphisms

$$\zeta_N: (\overline{\mathcal{Z}}_{0,g})^N \rightarrow (\overline{\mathcal{Z}}_{0,3})^N \simeq (\mathbf{P}^1)^N \quad \text{and} \quad \zeta_g: (\overline{\mathcal{Z}}_{0,g})^g \rightarrow (\overline{\mathcal{Z}}_{0,3})^g \simeq (\mathbf{P}^1)^g$$

which correspond to forgetting about the last $g - 3$ points on a g -pointed rational curve and designating the first three points as 0, 1 and ∞ . The morphisms ζ_N and ζ_g are products of contraction morphisms. There is a morphism $\varepsilon: \overline{\mathcal{M}}_{0,g} \rightarrow (\overline{\mathcal{Z}}_{0,g})^g$ which corresponds to associating the g -pointed rational curve $(X, Q_1, \dots, Q_g)$ to itself and the points $Q_1, \dots, Q_g$ on X . The morphism ε exists because of the functorial properties $\overline{\mathcal{M}}_{0,g}$ and $\overline{\mathcal{Z}}_{0,g}$. We get a set-theoretic map

$$f = (\zeta_g \circ \varepsilon \circ p_M) \times (\zeta_N \circ \delta): \overline{\mathcal{M}}_{0,g} \times (\overline{\mathcal{Z}}_g)^N \rightarrow (\mathbf{P}^1)^g \times (\mathbf{P}^1)^N$$

where p_M is the projection of $\overline{\mathcal{M}}_{0,g} \times (\overline{\mathcal{Z}}_g)^N$ onto $\overline{\mathcal{M}}_{0,g}$. Furthermore, $f(p_Z^{-1}(B))$ is closed in $(\mathbf{P}^1)^g \times (\mathbf{P}^1)^N$.

We will prove the lemma by showing that there exists a point

$$(Q_1, \dots, Q_g, P_1, \dots, P_N) \in f(p_Z^{-1}(B)) \subset (\mathbf{P}^1)^g \times (\mathbf{P}^1)^N$$

such that either: (i) $P_1 = \dots = P_N$ and at most M of the Q_i are equal to P_1 ; or (ii) there exists a point $Q \in \mathbf{P}^1$ such that all of the P_i are distinct from Q and at least $g - M$ of the Q_i are equal to Q . Suppose that $(X, Q_1, \dots, Q_g)$ is a stable g -pointed rational curve. Let $\tilde{X}_{Q_1, \dots, Q_g}$ be the curve obtained by attaching the elliptic curve E at each Q_i , and let $\eta: \tilde{X}_{Q_1, \dots, Q_g} \rightarrow X$ be the map which collapses the elliptic tails as before. The map $\overline{\mathcal{Z}}_{0,g} \rightarrow \overline{\mathcal{Z}}_{0,3} \simeq \mathbf{P}^1$ corresponding to forgetting about the last $g - 3$ points of a g -pointed rational curve and designating the first three points as 0, 1 and ∞ induces a morphism $X \rightarrow \mathbf{P}^1$ and hence a morphism $\tilde{X}_{Q_1, \dots, Q_g} \rightarrow \mathbf{P}^1$. Note that if P is a point in $\mathbf{P}^1$ and k of the points $Q_1, \dots, Q_g$ map to P , then the pre-image of P in $\tilde{X}_{Q_1, \dots, Q_g}$ is a tail of genus k . hence condition (i) above implies that there exists a point in B corresponding to a curve and N points which lie on a tail of genus $\leq M$. Note that the preimage of $\mathbf{P}^1 - \{P\}$ in $\tilde{X}_{Q_1, \dots, Q_g}$ is a tail of genus $g - k$. Thus condition (ii) will also imply the lemma.

Let $D_k = \{(P_1, \dots, P_N) \in (\mathbf{P}^1)^N \mid \text{at least } k \text{ points coincide}\}$. We will use the fact that if Y is a closed subset of $(\mathbf{P}^1)^N$ and $\dim(Y) \geq k-1$, then $D_k \cap Y \neq \emptyset$. This fact follows from: the diagonal is ample in $\mathbf{P}^1 \times \mathbf{P}^1$; thus $\dim(D_k \cap Y) \geq 1$ implies $\dim(D_{k+1} \cap Y) \neq \emptyset$; and $\text{codim } D_k = k-1$.

Let p_g and p_N denote the projections of $(\mathbf{P}^1)^g \times (\mathbf{P}^1)^N$ to $(\mathbf{P}^1)^g$ and $(\mathbf{P}^1)^N$, respectively.

The dimension of the fiber of $f(p_Z^{-1}(B))$ over each point of $(0, 1, \infty) \times (\mathbf{P}^1)^{g-3}$ is $\geq N-M$, because the dimension of the fiber of B over each point of $\overline{\mathcal{M}}_g$ is $\geq N-M$. Hence

$$p_g(f(p_Z^{-1}(B)) \cap (\mathbf{P}^1)^g \times D_{N-M+1}) = (0, 1, \infty) \times (\mathbf{P}^1)^{g-3}.$$

Let $\kappa = \max\{k \mid (\mathbf{P}^1)^g \times D_k \cap f(p_Z^{-1}(B)) \neq \emptyset\}$. We will prove the lemma by showing that condition (i) holds if $\kappa = N$ and condition (ii) holds if $\kappa \leq N-1$.

Suppose $\kappa = N$. Then $p_g(f(p_Z^{-1}(B)) \cap (\mathbf{P}^1)^g \times D_N)$ has codimension $\leq N - (N-M+1) = M-1$ in $(0, 1, \infty) \times (\mathbf{P}^1)^{g-3}$. Hence

$$\dim(p_g(f(p_Z^{-1}(B)) \cap (\mathbf{P}^1)^g \times D_N)) \geq g-2-M.$$

Let $E_{M+1} = \{(Q_1, \dots, Q_g) \in (0, 1, \infty) \times (\mathbf{P}^1)^{g-3} \mid \text{at least } M+1 \text{ of the points } Q_1, \dots, Q_g \text{ coincide}\}$. We have

$$\dim(E_{M+1}) = g-3-M < \dim p_g(f(p_Z^{-1}(B)) \cap (\mathbf{P}^1)^g \times D_N).$$

Thus we can find $(Q_1, \dots, Q_g, P_1, \dots, P_N) \in f(p_Z^{-1}(B))$ so that $P_1 = \dots = P_N$ and at most M of the Q_i 's are equal to P_1 .

Suppose $\kappa \leq N-1$. Let W be a component of $f(p_Z^{-1}(B)) \cap (\mathbf{P}^1)^g \times D_\kappa$. Then $p_N(W)$ is a point $(P_1, \dots, P_N) \in (\mathbf{P}^1)^N$, because otherwise $(\mathbf{P}^1)^g \times D_{\kappa+1} \cap f(p_Z^{-1}(B)) \neq \emptyset$. Choose a point $R \in \mathbf{P}^1$ so that $R \neq P_i$ for $i = 1, \dots, N$. If $\kappa = N-1$, we choose $R \in \{0, 1, \infty\}$. Now W has codimension $\leq \kappa - (N-M+1)$ in $f(p_Z^{-1}(B)) \cap (\mathbf{P}^1)^g \times D_{N-M+1}$ so $p_g(W)$ has codimension $\leq \kappa - (N-M+1)$ in $(0, 1, \infty) \times (\mathbf{P}^1)^{g-3}$. Thus $\dim p_g(W) \geq g-M$ if $\kappa \leq N-2$, and $\dim p_g(W) \geq g-M-1$ if $\kappa = N-1$. Since $p_g(W)$ is closed in $(0, 1, \infty) \times (\mathbf{P}^1)^{g-3}$ there are $\dim(p_g(W))$ factors of $(\mathbf{P}^1)^{g-3}$ so that the projection of $p_g(W)$ to the product of these factors is onto. Thus there is a point $(Q_1, \dots, Q_g) \in p_g(W) \subset (0, 1, \infty) \times (\mathbf{P}^1)^{g-3}$ so that at least $g-M$ of the Q_i 's are equal to R . Thus condition (ii) holds and the lemma follows.

We can find a smooth curve T containing a point 0 and a morphism $\phi: T \rightarrow B \subset (\overline{\mathcal{Z}}_g)^N$ such that $\phi(0)$ is the point $b \in B$ described in Lemma 1, and the induced map $T \rightarrow \overline{\mathcal{M}}_g$ sends $T - \{0\}$ to the subset of U where the fibers of $\beta: B \rightarrow \overline{\mathcal{M}}_g$ have dimension $N-M$.

After replacing T with a base extension, if necessary, there is a family of 0-pointed stable curves $X \rightarrow T$ which corresponds to the morphism $T \rightarrow \overline{\mathcal{M}}_g$.

Note that if T is replaced with a base extension and the singularities of X are resolved by blowing up, the curve X_0 will change by inserting chains of rational curves at the nodes of X_0 . Thus we may assume (by replacing T with a base extension and blowing up the singularities of X if necessary) that there is a family of curves $X \rightarrow T$ which extends $Z \times_U (T - \{0\})$ and has the following properties: (1) X_0 is of special type; (2) there is a tail Y of X_0 of genus $\leq M$ so that the sections $s_i: T \rightarrow X$ induces by the map $\phi: T \rightarrow B \subset (\overline{Z}_g)^N$ are such that each $s_i(0)$ is smooth point of X_0 which lies in Y (if $P_1 = P_2 = \dots = P_N$ is a point on a rational component of X_b then, after blowing up, there will exist s_i such that the $s_i(0)$ lie on a tail of genus 0 in X_0); (3) there is a section $s: T \rightarrow X$ such that $s(0)$ is a smooth point of X_0 which lies in a rational component of $X_0 - Y$; and (4) X is smooth. Note that these properties are unchanged if T is replaced with a base extension which sends one point to 0 and the singularities of the new X are resolved by blowing up.

Theorem 2. *There is a sequence $\mathbf{a}$ of type (r, d) and a closed subscheme A of $H \times_B (T - \{0\})$ such that: A consists of all linear series in $H \times_B (T - \{0\})$ which satisfy vanishing condition $\mathbf{a}$ along $s(T - \{0\})$; the fibers A_t are nonempty for each $t \in T - \{0\}$; and $\dim A_t = 0$ for all but finitely many $t \in T - \{0\}$.*

Proof. Note that $H \times_B (T - \{0\})$ is proper over $T - \{0\}$, and the subset of linear series satisfying a particular vanishing condition along $s(T - \{0\})$ is closed. Since there are only finitely many sequences of type (r, d) , the lemma is a consequence of the following.

Lemma 2a. *Let X be a smooth curve. Let A be a closed subset of $G_d^r(X)$, and let $\mathbf{a}$ be a sequence of type (r, d) . If $\dim A \geq 1$ and every linear series in A satisfies vanishing condition $\mathbf{a}$ at a point $P \in X$, then there exists a sequence $\mathbf{a}'$ with $a'_i \geq a_i$ for $i = 0, \dots, r$ and $a'_k > a_k$ for some k such that A contains a linear series which satisfies vanishing condition $\mathbf{a}'$ at P .*

Proof. We may assume A is irreducible. Consider the natural map $\Phi: A \rightarrow \text{Pic}^d(X)$. Suppose for some $x \in \text{Pic}^d(X)$, the fiber $\Phi^{-1}(x)$ has dimension ≥ 1 , and let $\mathcal{L}$ be the line bundle corresponding to x . The set of $(r+1)$ dimensional vector spaces of $H^0(X, \mathcal{L})$ which have vanishing sequence $\mathbf{a}$ at P is a Schubert variety, and hence is affine. It follows that A contains a g_d^r which does not have vanishing sequence $\mathbf{a}$ at P , because $\Phi^{-1}(x)$ is closed and $\dim \Phi^{-1}(x) \geq 1$. So we may assume $\Phi(A)$ contains a closed curve in $\text{Pic}^d(X)$.

If the lemma were false, then there would exist a family F of $g_{d-a_r}^0$'s obtained from A by taking $(H^0(Y, \mathcal{L}(-a_r P)) \cap V, \mathcal{L}(-a_r P))$ for every $(V, \mathcal{L})$ of A . But a family of $g_{d-a_r}^0$'s is a family of divisors of degree $d - a_r$. Since $\dim(F) \geq 1$, it must contain a divisor with P in its support. But the linear series in A associated with this divisor satisfies vanishing condition $(a_0, \dots, a_{r-1}, a_r + 1)$.

If T is replaced with an appropriate base extension, and the singularities of X blown up, we may also assume that $A \rightarrow T - \{0\}$ gives an isomorphism $A' \rightarrow T - \{0\}$ for some component A' of A .

The codimension of $H \times_B T$ in $H \times_U T$ is $N - M$, and A' has codimension $\leq W(\mathbf{a})$ in $H \times_B T$. Thus the theorem will follow when we show that $W(\mathbf{a}) + N - M \leq \rho(g, r, d) - r(N - 1) + N$.

3

The family of curves $X \rightarrow T$ and A' determine a crude limit linear series on X_0 [EH-2]. This crude limit linear series is a collection consisting of a g_d^r for each of the components of X_0 . If C is a component of X_0 , the $g_d^r(V_C, \mathcal{L}_C)$ on C is determined in the following manner. Let R be the local ring of T at 0, and let η be the generic point of $\text{Spec } R$. Then A' determines a unique line bundle $\mathcal{L}^C$ on $X_{\text{Spec } R}$ such that the restriction of $\mathcal{L}^C$ to X_0 has degree 0 on every component of X_0 except C . Also, A' determines a subspace $V \subset H^0(X_\eta, \mathcal{L}^C|_{X_\eta})$. Let $\tilde{V}_C = V \cap H^0(X_{\text{Spec } R}, \mathcal{L}^C)$. Then $V_C = \tilde{V}_C \otimes k(0) \subset H^0(X_0, \mathcal{L}^C|_{X_0})$ and $\mathcal{L}_C = \mathcal{L}^C|_C$ where $k(0)$ the residue field of R .

Let $Q = Y \cap (\overline{X_0 - Y})$ and let Y' be the component of Y containing Q . Let $Y_1, \dots, Y_k$ be the remaining components of Y . In addition to parameterizing a family of g_d^r 's on $X_{T-\{0\}} \rightarrow T - \{0\}$, A' also determines a family of g_{d-N}^{r-1} 's corresponding to the sections which vanish along $\sum s_i(T - \{0\})$.

Let $(W_{Y'}, \eta_{Y'})$ be the g_{d-N}^{r-1} on Y' of the limit g_{d-N}^{r-1} on X_0 . Then

$$\eta^{Y'} = \mathcal{L}^{Y'} \left(- \sum s_i(T) - \sum n_i Y_i \right) \quad \text{where } n_i \geq 0,$$

so $\eta_{Y'} = \eta^{Y'}|_{Y'} = \mathcal{L}_{Y'}(-\sum Q_i)$ where the Q_i are points not equal to Q . Also,

$$W_{Y'} = (\tilde{V}_{Y'} \cap H^0(X_{\text{Spec } R}, \eta^{Y'})) \otimes k(0) \subset V_{Y'} \cap H^0(\eta_{Y'}).$$

So the vanishing sequence $\mathbf{b}$ of $(W_{Y'}, \eta_{Y'})$ at Q is a subsequence of the vanishing sequence $\mathbf{c}$ of $(V_{Y'}, \mathcal{L}_{Y'})$ at Q .

The following lemma is an immediate consequence of Theorem 4.5 of [EH-2] and Theorem 2.3 of [EH-1].

Lemma 3. *If C is a genus g curve of special type and P and Q are two smooth points in rational components of C , then the existence of a crude limit g_d^r on C , which satisfies vanishing conditions $\mathbf{a}$ and $\mathbf{b}$ at points P and Q , respectively, implies that $\rho(g, r, d) - W(\mathbf{a}) - W(\mathbf{b}) \geq 0$.*

Now the crude limit g_{d-N}^{r-1} on X_0 determined by A' restricts to a limit g_{d-N}^{r-1} on the genus M tail Y which satisfies vanishing condition $\mathbf{b}$ at Q . Hence $\rho(M, r - 1, d - N) - W(\mathbf{b}) \geq 0$. Since $\mathbf{b}$ is a subsequence of $\mathbf{c}$ we have $W(\mathbf{c}) \leq W(\mathbf{b}) + d - r$. So $W(\mathbf{c}) \leq \rho(M, r - 1, d - N) + d - r$.

Let F be the component of $\overline{X_0 - Y}$ which contains Q . If c' is the vanishing sequence of V_F at Q , then the definition of crude limit linear series requires

$$\begin{aligned} W(c') &\geq (r+1)(d-r) - W(c) \\ &\geq r(d-r) - \rho(M, r-1, d-N). \end{aligned}$$

Now the crude limit g_d^r on X_0 determined by A' restricts to a crude limit g_d^r on $\overline{X - Y}$ satisfying vanishing conditions c at Q and a at $s(0)$. Thus,

$$\begin{aligned} 0 &\leq \rho(g - M, r, d) - W(a) - W(c') \\ &\leq \rho(g - M, r, d) - W(a) + \rho(M, r-1, d-N) - r(d-r) \\ &= (r+1)(d-r) - r(g-M) - w(a) \\ &\quad + r(d-N-r+1) - (r-1)M - r(d-r) \\ &= (r+1)(d-r) - rg - W(a) - r(N-1) + N - (N-M) \\ &= \rho(g, r, d) - W(a) - r(N-1) + N - (N-M). \end{aligned}$$

Thus $W(a) + N - M \leq \rho(g, r, d) - r(N-1) + N$, and the theorem follows.

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