

SOLVING ORDINARY DIFFERENTIAL EQUATIONS IN TERMS OF SERIES WITH REAL EXPONENTS

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ABSTRACT. We generalize the Newton polygon procedure for algebraic equations to generate solutions of polynomial differential equations of the form $\sum_{i=0}^{\infty} \alpha_i x^{\beta_i}$ where the α_i are complex numbers and the β_i are real numbers with $\beta_0 > \beta_1 > \dots$. Using the differential version of the Newton polygon process, we show that any such a series solution is finitely determined and show how one can enumerate all such solutions of a given polynomial differential equation. We also show that the question of deciding if a system of polynomial differential equations has such a power series solution is undecidable.

When one looks for solutions of differential equations, one is forced to deal with the question: what form can these solutions have? The first natural class of solutions is the set of formal power series. An algorithm to determine whether a system of differential polynomial equations with coefficients in $\mathbb{C}(x)$ has such a solution is given in [DL84]. Even if we only consider algebraic equations, such solutions are not enough; one must consider fractional power series solutions (Puiseux series). In this paper we consider solutions of differential equations of the form $\sum_{i=0}^{\infty} \alpha_i x^{\beta_i}$ where $\alpha_i \in \mathbb{C}$ and $\beta_i \in \mathbb{R}$ with $\beta_0 > \beta_1 > \dots$. In §1, we show (Theorem 1.1) that if such a series satisfies a polynomial differential equation, then the β_i can have no finite limit point. In particular $\sum_{i=0}^{\infty} x^{1/i}$ satisfies no polynomial differential equation. This motivates us to introduce the set

$$\Omega = \left\{ \sum_{i=0}^{\infty} \alpha_i x^{\beta_i} \mid \alpha_i \in \mathbb{C}, \beta_i \in \mathbb{R}, \beta_0 > \beta_1 > \dots \text{ and } \lim \beta_i = -\infty \right\}.$$

We will show that Ω is a field that we call the field of generalized power series. In §2, we generalize the Newton polygon procedure for algebraic equations to generate solutions of differential equations. In §3, we show that after a finite number of steps the differential version of the Newton polygon process stabilizes; that is, after generating a finite number of terms of a generalized power series we will reach a point where the remaining terms are uniquely determined. In §4, we use this to enumerate, for a given differential polynomial

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$P(x, y, y', \dots, y^{(n)})$, all generalized power series solutions of $P = 0$. We must be careful when we make such a claim since the equation $y^2(xy'/y)' = 0$ has solutions $y = cx^\alpha$ for arbitrary c and α , showing that such an equation can have an uncountable number of solutions ($y' = 0$ also gives such an example). Our enumeration produces a list $\{\varphi_k\}$ of formulas in the first order language of real closed fields, with each φ_k having free real variables $\bar{\alpha}_{ik}, \bar{\alpha}_{ik}, \beta_{ik}$, $0 \leq i \leq n_k$. Given $\bar{\alpha}_{ik}, \bar{\alpha}_{ik}, \beta_{ik}$ satisfying the formula φ_k , we show that $\sum_{i=0}^{n_k} (\bar{\alpha}_{ik} + \sqrt{-1}\bar{\alpha}_{ik})x^{\beta_{ik}}$ is the initial segment of a uniquely determined generalized power series $y = \sum_{i=0}^{\infty} (\bar{\alpha}_{ik} + \sqrt{-1}\bar{\alpha}_{ik})x^{\beta_{ik}}$, provided that $\lim_{i \rightarrow \infty} \beta_{ik} = -\infty$. If the latter holds, then this generalized power series is a solution of $P = 0$. The next terms of the series are determined by a simple recursion formula (cf. Theorem 4.1 of §4). In §5, we show that the question of deciding if a system of polynomial differential equations has a generalized power series solution is undecidable and discuss some related matters.

1. THE FIELD OF GENERALIZED POWER SERIES

Before we turn to generalized power series, we will consider series of the form $\sum_{i=0}^{\infty} \alpha_i x^{\beta_i}$, where $\alpha_i \in \mathbb{C}$, $\beta_i \in \mathbb{R}$, and $\beta_0 > \beta_1 > \dots$ but with no further restriction made on the β_i . Let $\mathbb{C}((x^{-1}))$ be the field of formal Laurent series in x^{-1} . A differential polynomial equation [RI66] with coefficients in $\mathbb{C}((x^{-1}))$ is an equation of the form $0 = F(y, y', \dots, y^{(n)}) = \sum_I c_I y^{(k_0)} \dots y^{(k_m)}$, where $I = (k_0, \dots, k_m)$, $n \geq k_0 \geq \dots \geq k_m$, $c_I \in \mathbb{C}((x^{-1}))$, and distinct I may have different lengths $m+1$. We shall prove the following.

Theorem 1.1. *If $y = \sum \alpha_i x^{\beta_i}$ with $\alpha_i \in \mathbb{C}$, $\beta_i \in \mathbb{R}$, $\beta_0 > \beta_1 > \dots$ and $\lim \beta_i > -\infty$, then y satisfies no differential polynomial equation with coefficients in $\mathbb{C}((x^{-1}))$.*

We will show, in the corollary to Lemma 1.2, that substituting $y = \sum \alpha_i x^{\beta_i}$ into such an equation makes sense (because only a finite number of terms in y contribute to any term of the form εx^β in the expansion of $F(y, \dots, y^{(n)})$ in powers of x). Our proof of the above theorem relies on the following lemmas and proposition.

Lemma 1.2. *Let $\beta_i \in \mathbb{R}$, $i = 0, 1, \dots$, where $\beta_0 > \beta_1 > \dots$ and $\lim_{i \rightarrow \infty} \beta_i = 0$. Then for any integer $m \geq 0$ and $\gamma \in \mathbb{R}$, $\gamma \neq 0$, there exists an $\varepsilon_{\gamma, m} > 0$ such that if*

$$\left| \gamma - \sum_{0 \leq j \leq m} \beta_{i_j} \right| < \varepsilon_{\gamma, m}, \quad 0 \leq i_0 \leq i_1 \leq \dots \leq i_m,$$

then there exists an L , $0 \leq L \leq m$, such that $\gamma = \sum_{0 \leq j \leq L} \beta_{i_j}$.

Proof. We proceed by induction on m . If $m = 0$, we can find an $\varepsilon = \varepsilon_{\gamma, 0} > 0$ such that in the ε -neighborhood of γ there are no elements β_i except possibly

γ itself. Assume the result is true for m . Let $\varepsilon' = \frac{|\gamma|}{2(m+2)}$. There exists a j_0 such that $\beta_j < \varepsilon'$ for $j > j_0$. We now distinguish between two cases.

Case 1. $\gamma \neq \beta_j$ for $j = 0, \dots, j_0$. In this case we apply the induction hypothesis to $\gamma - \beta_0, \dots, \gamma - \beta_{j_0}$ and produce $\varepsilon_0, \dots, \varepsilon_{j_0}$, satisfying the conclusion of the lemma with respect to m . Let

$$\varepsilon_{\gamma, m+1} = \min\{\varepsilon', \varepsilon_0, \dots, \varepsilon_{j_0}\}.$$

If $|\gamma - \sum_{0 \leq j \leq m+1} \beta_{i_j}| < \varepsilon_{\gamma, m+1}$, then $\beta_{i_0} \geq \varepsilon'$. Indeed if $\beta_{i_0} < \varepsilon'$, then $\sum_{0 \leq j \leq m+1} \beta_{i_j} < |\gamma|/2$, so $|\gamma - \sum_{0 \leq j \leq m+1} \beta_{i_j}| > |\gamma|/2 \geq \varepsilon' \geq \varepsilon$. Therefore $i_0 \leq j_0$. Since $|\gamma - \beta_{i_0} - \sum_{1 \leq j \leq m+1} \beta_{i_j}| < \varepsilon \leq \varepsilon_{i_0}$, we have that there exists an L , $1 \leq L \leq m+1$, such that $\gamma - \beta_{i_0} = \sum_{1 \leq j \leq L} \beta_{i_j}$. Therefore, $\gamma = \sum_{0 \leq j \leq L} \beta_{i_j}$.

Case 2. Assume that $\gamma = \beta_k$ for some $0 \leq k \leq j_0$. For all $j \neq k$, $0 \leq j \leq j_0$, apply the induction hypothesis to $\gamma - \beta_j$ and produce ε_j satisfying the conclusion of the lemma with respect to m . Let

$$\varepsilon_{\gamma, m+1} = \min\{\varepsilon', \varepsilon_0, \dots, \varepsilon_{k+1}, \dots, \varepsilon_{j_0}\}.$$

Assume that $|\gamma - \sum_{0 \leq j \leq m+1} \beta_{i_j}| < \varepsilon_{\gamma, m+1}$. If $i_0 = k$, then we let $L = 0$ and the conclusion of the lemma is satisfied. If $i_0 < j_0$ and $i_0 \neq k$, then we have $|\gamma - \beta_{i_0} - \sum_{1 \leq j \leq m+1} \beta_{i_j}| < \varepsilon_{i_0}$ and we can argue as before. As in the first case, we show that $i_0 > j_0$ cannot happen.

In our application later of Lemma 1.2 we will assume w.l.o.g. that for any $\gamma \neq 0$, the sequence $\varepsilon_{\gamma, m}$ decreases as m increases.

Corollary. Let β_i be a decreasing sequence of real numbers. For any γ and m , there exist only a finite number of solutions $\beta_{i_0}, \dots, \beta_{i_m}$ to $\gamma = \beta_{i_0} + \dots + \beta_{i_m}$.

Proof. If $\lim \beta_i = -\infty$, the result is obvious. Assume that $\lim \beta_i = \beta > -\infty$. Replace each β_i with $\beta_i - \beta$ and γ with $\gamma - (m+1)\beta$. Note, if $\gamma - (m+1)\beta = 0$, we never have $\gamma = \beta_{i_0} + \dots + \beta_{i_m}$. We therefore can assume $\lim \beta_i = 0$ and $\gamma \neq 0$. Let N be an integer such that $\beta_i < \varepsilon_{\gamma, m}/(m+1)$ for $i > N$ and $\varepsilon_{\gamma, m}$ as in Lemma 1.2. If $\gamma = \beta_{i_0} + \dots + \beta_{i_m}$, $i_0 \leq \dots \leq i_m$, and supposing $i_m > N$, we shall rewrite this as $\gamma = \sum_{i_j \leq N} \beta_{i_j} + \sum_{i_j > N} \beta_{i_j}$. Therefore $|\gamma - \sum_{i_j \leq N} \beta_{i_j}| < \varepsilon_{\gamma, m}$, so Lemma 1.2 implies $\gamma = \sum \beta_{i_j}$ for some initial subsum of $\sum_{i_j \leq N} \beta_{i_j}$. This contradiction shows that $i_m \leq N$ and finishes the proof of the corollary.

This corollary shows that only a finite number of terms of $y = \sum a_i x^{\alpha_i}$ contribute to any term of the form cx^β in the expansion of $F(y, \dots, y^{(n)})$ in powers of x .

Lemma 1.3. *Let β_i be as in Lemma 1.2 and assume that $1 > \beta_0$. The following is true: $\forall j_0 \exists s_1 \forall j_1 \geq s_1 \exists s_2 \geq s_1 \forall j_2 \geq s_2 \cdots \exists s_m \geq s_{m-1} \forall j_m \geq s_m$ (if $\beta_{j_0} + \cdots + \beta_{j_m} = \beta_{j'_0} + \cdots + \beta_{j'_l} + q$, where $q \in \mathbb{Z}$, $j_0 \leq j_1 \leq \cdots \leq j_m$, $j'_0 \leq \cdots \leq j'_l$, $l \leq m$, then $l = m$, $j_i = j'_i$ for $0 \leq i \leq m$, and $q = 0$).*

Proof. Note that since $\beta_i < 1$ for all i , we must have $|q| \leq 2m + 1$. For any such q apply the previous lemma in $\beta_{j_0} - q$ and produce $\varepsilon_{j_0, m}^{(q)}$. (Note that $\beta_{j_0} - q \neq 0$.) Let s_1 be chosen so that $\beta_s < \varepsilon_{j_0, m}^{(q)}/(m+1)$ for $s \geq s_1$. We now proceed by induction on m . If $m = 0$, then $l = 0$ so $\beta_{j_0} = \beta_{j'_0} + q$. Since $\beta_i < 1$, $q = 0$. Assume $m > 0$. By the induction hypothesis, for any $j_1 \geq s_1$ the following is true: $\exists s_2 \geq s_1 \forall j_2 \geq s_2 \cdots \exists s_m \geq s_{m-1} \forall j_m \geq s_m$ (if $\beta_{j_1} + \cdots + \beta_{j_m} = \beta_{j'_1} + \cdots + \beta_{j'_l} + q$ for $q \in \mathbb{Z}$, $j_1 \leq j_2 \leq \cdots \leq j_m$, $j'_1 \leq \cdots \leq j'_l$, $l \leq m$, then $l = m$ and $j_i = j'_i$ for $1 \leq i \leq m$ and $q = 0$). Letting $\gamma = \beta_{j_0} - q$, we have $|\gamma - (\beta_{j'_0} + \cdots + \beta_{j'_l})| = |\beta_{j_1} + \cdots + \beta_{j_m}| < \varepsilon_{j_0, m}^{(q)}$. Therefore Lemma 1.2 implies that $\gamma = \beta_{j'_0} + \cdots + \beta_{j'_k}$ for $0 \leq k \leq l$. Note that we are using the remark made after Lemma 1.2 concerning monotonicity of the ε . We therefore have $\beta_{j_1} + \cdots + \beta_{j_m} = \beta_{j'_{k+1}} + \cdots + \beta_{j'_l}$; so by induction $l - k = m$ (so $k = 0$) and $j_1 = j'_1, \dots, j_m = j'_m$. Furthermore $\beta_{j_0} = \beta_{j'_0} + q$, so $q = 0$.

In the next proposition we need the following definitions. By the degree of the term $c_I y^{(k_0)} \cdots y^{(k_m)}$, where $k_0 \geq \cdots \geq k_m$ in F we mean the number $m+1$. The degree of F is the maximum of the degrees of its terms. Given two terms $t_1 = c_{I_1} y^{(k_0)} \cdots y^{(k_m)}$, $k_0 \geq \cdots \geq k_m$, and $t_2 = c_{I_2} y^{(k'_0)} \cdots y^{(k'_j)}$, $k'_0 \geq \cdots \geq k'_j$, we say t_1 is higher than t_2 if $m > j$ or $m = j$ and $(k_0, \dots, k_m) > (k'_0, \dots, k'_m)$ in the lexicographical ordering.

Proposition 1.4. *If $y = \sum b_i x^{\beta_i}$ with $\beta_i \in \mathbb{R}$, $b_i \in \mathbb{C}$, $1 > \beta_0 > \beta_1 > \cdots$ and $\lim \beta_i = 0$, then y satisfies no differential polynomial equation over $\mathbb{C}((x^{-1}))$.*

Proof. Assume not and let $F(y, \dots, y^{(n)}) = 0$. Let m be the degree of F . Lemma 1.3 implies that there are in fact functions $s_1(j_0), s_2(j_0, j_1), \dots, s_m(j_0, \dots, j_{m-1})$ such that if $j_i \geq s_i(j_0, \dots, j_{i-1})$ for $i = 1, \dots, m$ and $\beta_{j_0} + \cdots + \beta_{j_m} = \beta_{j'_0} + \cdots + \beta_{j'_l} + q$ for $q \in \mathbb{Z}$, $j_0 \leq \cdots \leq j_m$, $j'_0 \leq \cdots \leq j'_l$, $l \leq m$, then $l = m$ and $j_i = j'_i$ for $i = 0, \dots, m$. Denote by $\mathcal{S}$ the set of $(m+1)$ -tuples $(j_0, \dots, j_m)$ such that $j_0 \leq \cdots \leq j_m$ and $j_i \geq s_i(j_0, \dots, j_{i-1})$ for $i = 1, \dots, m$. Let $c_I y^{(k_0)} \cdots y^{(k_m)}$ be the highest term occurring in F where $I = (k_0, \dots, k_m)$ and let $c_I = \sum_{t \leq t_I} c_t^{(I)} x^t$ with $c_t^{(I)} \neq 0$.

Consider $(j_0, \dots, j_m) \in \mathcal{S}$ and let us determine the coefficients of powers of x in $F(y, \dots, y^{(n)})$. Any power of x in this expansion is of the form $x^{\beta_{j'_0} + \cdots + \beta_{j'_l} - k'_0 - \cdots - k'_l + t'}$ with $j'_0 \leq \cdots \leq j'_l$, $l \leq m$ and some $t' \in \mathbb{Z}$ (this comes from a term $c_{I'} y^{(k'_0)} \cdots y^{(k'_l)}$). Lemma 1.3 implies that if

$$x^{\beta_{j_0} + \cdots + \beta_{j_m} - k_0 - \cdots - k_m + t} = x^{\beta_{j'_0} + \cdots + \beta_{j'_l} - k'_0 - \cdots - k'_l + t'}$$

then $l = m$, $\beta_{j_i} = \beta_{j'_i}$, $i = 0, \dots, m$, and $k_0 + \dots + k_m - t = k'_0 + \dots + k'_m - t'$. Therefore the coefficient of $x^{\beta_{j_0} + \dots + \beta_{j_m} - k_0 - \dots - k_m + t_l}$ in the expansion of $F(y, \dots, y^{(n)})$ is

$$(b_{j_0} \cdots b_{j_m}) \sum_{k'_0, \dots, k'_m} c_{t'}^{(I')} \sum_{(h_0, \dots, h_m)} \binom{\beta_{h_0}}{k'_0} k'_0! \cdots \binom{\beta_{h_m}}{k'_m} k'_m! \\ = (b_{j_0} \cdots b_{j_m}) P(\beta_{j_0}, \dots, \beta_{j_m})$$

where the first sum is over all $(k'_0, \dots, k'_m)$ such that $k'_0 + \dots + k'_m - t' = k_0 + \dots + k_m - t_l$ and the second sum is over all $(h_0, \dots, h_m) = (j_{\sigma(0)}, \dots, j_{\sigma(m)})$ where σ is a permutation of $(0, \dots, m)$ and

$$P(Z_0, \dots, Z_m) = \sum_{(k'_0, \dots, k'_m)} c_{t'}^{(I')} \sum_{(i_0, \dots, i_m)} k'_0! \cdots k'_m! \binom{Z_{i_0}}{k'_0} \cdots \binom{Z_{i_m}}{k'_m}$$

where the first sum is as before and the second is over all permutations of $(0, \dots, m)$. Here

$$\binom{Z_i}{k_j} = \frac{Z_i(Z_i - 1) \cdots (Z_i - k_j + 1)}{k_j!}.$$

Note $P(Z_0, \dots, Z_m) \in \mathbb{C}[Z_0, \dots, Z_m]$. Order the monomials in $Z_0, \dots, Z_m$ first by degree and then, for fixed degree, lexicographically. If $(i_0, \dots, i_m)$ is a permutation of $(0, \dots, m)$, then $Z_0^{k_0} \cdots Z_m^{k_m}$ is not less than $Z_{i_0}^{k_0} \cdots Z_{i_m}^{k_m}$. Also, for any term $c_{t'}^{(I')} y^{(k'_0)} \cdots y^{(k'_m)}$ (where $k'_0 \geq \dots \geq k'_m$) occurring in F such that $(k'_0, \dots, k'_m) \neq (k_0, \dots, k_m)$, we have that $Z_0^{k_0} \cdots Z_m^{k_m}$ is larger than $Z_0^{k'_0} \cdots Z_m^{k'_m}$, since $(k_0, \dots, k_m)$ was chosen so that $(k_0, \dots, k_m) > (k'_0, \dots, k'_m)$ in the lexicographic ordering. These two facts imply that the highest term in $P(Z_0, \dots, Z_m)$ is $lc_{t'}^{(I)} Z_0^{k_0} \cdots Z_m^{k_m}$, where l is the number of permutations of $(0, \dots, m)$, leaving $(k_0, \dots, k_m)$ fixed. Therefore, P does not vanish identically. For some $\bar{j}_0$, $P(\beta_{\bar{j}_0}, Z_1, \dots, Z_m)$ does not vanish identically. Therefore for some $\bar{j}_1 > s_1(\bar{j}_0)$, $P(\beta_{\bar{j}_0}, \beta_{\bar{j}_1}, Z_2, \dots, Z_m)$ does not vanish identically. Continuing in this way, we have for some $\bar{j}_0, \bar{j}_1 > s_1(\bar{j}_0), \dots, \bar{j}_m > s_m(\bar{j}_0, \dots, \bar{j}_{m-1})$, that $P(\beta_{\bar{j}_0}, \dots, \beta_{\bar{j}_m}) \neq 0$. This means that the coefficient of $x^{\beta_{\bar{j}_0} + \dots + \beta_{\bar{j}_m} - k_0 - \dots - k_m + t_l}$ is not zero in the expansion of $F(y, \dots, y^{(n)})$. This contradiction completes the proof of the proposition.

Proof of Theorem 1.1. We assume the theorem is false and derive a contradiction. Let $\lim \alpha_j = \alpha > -\infty$ and let j_0 be an integer such that $\alpha_j < \alpha + 1$ for $j > j_0$. Let $z = (y - \sum_{j \leq j_0} a_j x^{\alpha_j}) x^{-\alpha}$. The term z satisfies a differential polynomial equation with coefficients in the differential field [RI50]

$$F = \mathbb{C}((x^{-1})) \langle x^{\alpha_0}, \dots, x^{\alpha_{j_0}}, x^\alpha \rangle \\ = \mathbb{C}((x^{-1})) \langle x^{\alpha_0}, \dots, x^{\alpha_{j_0}}, x^\alpha \rangle.$$

Therefore, $F(z, z', z'', \dots) = F\langle z \rangle$ has finite transcendence degree over F . Since F has finite transcendence degree over $\mathbb{C}((x^{-1}))$, $F\langle z \rangle$ has finite transcendence degree over $\mathbb{C}((x^{-1}))$. This implies that for some $N > 0$, $z, z', \dots, z^{(N)}$ are algebraically dependent over $\mathbb{C}((x^{-1}))$. Therefore z satisfies a polynomial differential equation over $\mathbb{C}((x^{-1}))$. If $z = \sum b_i x^{\beta_i}$, we have $1 > \beta_0 > \dots$ and $\lim \beta_i = 0$. Now apply Proposition 1.4 to reach a contradiction and prove the theorem.

Theorem 1.1 implies that if we are looking for series solutions $\sum \alpha_i x^{\beta_i}$ of polynomial differential equations, we can assume that $\lim \beta_i = -\infty$. We therefore make the following definition. A generalized power series is a formal series of the following form: $y = \sum_{i=0}^{\infty} \alpha_i x^{\beta_i}$ where $\alpha_i \in \mathbb{C}$, $\beta_i \in \mathbb{R}$, $\beta_0 > \beta_1 > \dots$, and $\lim \beta_i = -\infty$. (Note that y may consist of a finite number of terms, in which case, there is an N such that $\alpha_i = 0$ for $i > N$.) We denote by Ω the set of generalized power series. To prove that Ω is a field we need the following obvious

Lemma 1.5. *If $\beta_0 > \beta_1 > \dots$, $\beta'_0 > \beta'_1 > \dots$ and $\lim \beta_i = \lim \beta'_i = -\infty$, then for any $M \in \mathbb{R}$ there are at most a finite number of pairs (i, j) such that $\beta_i + \beta'_j > M$.*

Proof. Assume that $\beta_i + \beta'_j > M$ for some i, j . We then have $\beta_0 + \beta'_j > M$ and so $\beta'_j > M - \beta_0$. This can only be satisfied for a finite number of j . Similarly, there are only a finite number of possibilities for i . This proves the lemma.

Corollary 1. *If $\beta_0^{(1)} > \beta_1^{(1)} > \dots$, $\beta_0^{(2)} > \beta_1^{(2)} > \dots$, $\beta_0^{(m)} > \beta_1^{(m)} > \dots$, and $\lim_{i \rightarrow \infty} \beta_i^{(j)} = -\infty$ for each $1 \leq j \leq m$, then for any $M \in \mathbb{R}$ there are at most a finite number of m -tuples $i_1, \dots, i_m$ such that $\beta_{i_1}^{(1)} + \dots + \beta_{i_m}^{(m)} > M$.*

Proof. Proceed by induction of m , taking into account that it is equivalent to the following statement:

$$\lim_{i_1 \rightarrow \infty, \dots, i_m \rightarrow \infty} (\beta_{i_1}^{(1)} + \dots + \beta_{i_m}^{(m)}) = -\infty,$$

and for $m = 2$ this was proved in Lemma 1.5.

Corollary 2. Ω is a differential field.

Proof. Lemma 1.5 implies that the product of two elements of Ω belongs to Ω . Moreover,

$$\begin{aligned} y^{-1} &= \alpha_0 x^{-\beta_0} \left(1 + \sum_{i \geq 1} \frac{\alpha_i}{\alpha_0} x^{\beta_i - \beta_0} \right)^{-1} \\ &= \alpha_0^{-1} x^{-\beta_0} \sum_{j \geq 0} (-1)^j \left(\sum_{i \geq 1} \frac{\alpha_i}{\alpha_0} x^{\beta_i - \beta_0} \right)^j. \end{aligned}$$

As mentioned above, each $(\sum_{i \geq 1} x^{\beta_i - \beta_0})^j$ lies in Ω . Furthermore, for any M , $(\sum_{i \geq 1} x^{\beta_i - \beta_0})^j$ can contribute to the coefficient of any power x^{M_1} , where $M_1 \geq M$ only if $j < (M - \beta_0)/(\beta_1 - \beta_0)$. Thus the exponents in the expansion of y^{-1} have no finite limit points, proving the corollary.

We shall see that generalized power series appear naturally when one tries to apply a process similar to the classical (i.e. dealing with algebraic functions) Newton polygon process to generating a formal solution of a system of ordinary differential equations. Generalized power series have occurred in other contexts as well. The field $\mathcal{L}$ of series of the form $\sum_{i=0}^{\infty} a_i x^{\nu_i}$, where $a_i, \nu_i \in \mathbb{R}$, $\nu_0 < \nu_1 < \dots$, and $\lim \nu_i = \infty$, was studied by Levi-Civita [LC54] in the nineteenth century, by Ostrowski [OS35] and Mac Lane [MA39] in the 1930's, and more recently by Laugwitz [LA68], Robinson [RO73], and Lightstone and Robinson [LR75]. Ostrowski studied this field as part of a general investigation of valuation theory. Mac Lane showed that when we let the a_i be in $\mathbb{C}$, this field is algebraically closed. The other authors studied $\mathcal{L}$ as a nonarchimedean field (ordered with x as an infinitesimal) with the idea of doing analysis over this field. Clearly $\mathcal{L}$ (when $a_i \in \mathbb{C}$) and Ω are isomorphic. We have chosen to work with Ω because of its relation to asymptotic expansions of a function at ∞ .

In our generalization of the Newton polygon process, we will need the following technical lemma.

Lemma 1.6. *If $y = \sum_{i \geq 1} \alpha_i x^{\beta_i} \in \Omega$ and $P(x, Y, \dots, Y^{(m)})$ is a differential polynomial such that $0 \neq P(y) = \alpha x^\beta +$ terms with exponents smaller than β , $\alpha \neq 0$, then for suitable N , $P(\sum_{1 \leq i \leq N} \alpha_i x^{\beta_i}) = \alpha x^\beta +$ terms with exponents smaller than β .*

Proof. Consider any term $a_k x^{k_0} Y^{(k_1)} \dots Y^{(k_s)}$ of the polynomial P , where $a_k \in \mathbb{C}$. If some term from the expansion of $x^{k_0} y^{(k_1)} \dots y^{(k_s)}$ makes a contribution to αx^β , then $\beta = k_0 + \beta_{i_1} - k_1 + \dots + \beta_{i_s} - k_s$ for some $i_1, \dots, i_s$. Corollary 1 of Lemma 1.5 implies that there are only a finite number of such $i_1, \dots, i_s$. The lemma is proved.

2. DIFFERENTIAL VERSION OF THE NEWTON POLYGON PROCESS

Let the differential polynomial $P \in \mathbb{Q}[x, Y, Y^{(1)}, \dots, Y^{(n)}]$ be given. Write $P = P_0 + \dots + P_N$, where $P_j = \sum_K \alpha_K Y^{(k_1)} \dots Y^{(k_j)}$ is the homogeneous part of P of degree j with respect to the indeterminants $Y, Y^{(1)}, \dots, Y^{(n)}$, $K = (k_1, \dots, k_j)$ is a multi-index with $k_1 \geq k_2 \geq \dots \geq k_j$ and $\alpha_K \in \mathbb{Q}[x]$. The Newton polygon process that we will now describe produces a generalized power series y . If $y \in \Omega$, then this series will furthermore be a solution of $P = 0$. Assume that some initial segment

$$y_0 = \sum_{1 \leq i \leq s} (\bar{\alpha}_i + \sqrt{-1} \bar{\alpha}_i) x^{\beta_i}$$

of y has already been produced where $\bar{\alpha}_i, \bar{\bar{\alpha}}_i, \beta_i \in \mathbb{R}$ (we do not exclude the possibility that $y_0 = 0$). Using Taylor's formula, we define $P^{(y_0)}(Y)$ as

$$(1) \quad P^{(y_0)}(Y) = P(y_0 + Y) = \sum_K \frac{1}{|K|!} \left. \frac{\partial^K P}{\partial Y^K} \right|_{Y=y_0} Y^K,$$

where $K = (k_1, \dots, k_j)$ is a multi-index, $Y^K = Y^{(k_1)} \dots Y^{(k_j)}$, $\partial^K P / \partial Y^K = \partial^j P / \partial Y^{(k_1)} \dots \partial Y^{(k_j)}$, and $|K|! = i_1! \dots i_l!$, where i_1 is the largest t such that $k_1 = k_2 = \dots = k_t$, i_2 is the largest t such that $k_{i_1+1} = k_{i_1+2} = \dots = k_{i_1+t}$, etc.

We also define $P_K^{(y_0)}$ as

$$P_K^{(y_0)} = \left. \frac{\partial^K P}{\partial Y^K} \right|_{Y=y_0} \in \mathbb{Q}[\bar{\alpha}_1, \dots, \bar{\alpha}_s, \bar{\bar{\alpha}}_1, \dots, \bar{\bar{\alpha}}_s, \beta_1, \dots, \beta_s, \sqrt{-1}][x, x^{\beta_1}, \dots, x^{\beta_s}].$$

We define the weight w_K of the term $P_K^{(y_0)} Y^K$ as $\deg_x P_K^{(y_0)} - k_1 - \dots - k_j$. Let $w^{(j)} = \max_{K=(k_1, \dots, k_j)} w_K$. We now define the Newton polygon. For every K we mark the point (w_K, j) on (x, y) plane. The convex hull of these points and $(-\infty, 0)$ is denoted by $\mathcal{P}_{y_0}$ and is called the Newton polygon of the equation $P = 0$ corresponding to the initial segment y_0 . If $n = \deg_{Y, \dots, Y^{(m)}} P$ is the degree of P , then $\mathcal{P}_{y_0}$ is situated between the two lines $y = 0$ and $y = n$. Furthermore, note that we obtain the same convex hull if we take the points $(w^{(j)}, j)$. For any edge e of the Newton polygon, we call its slope the number $-(i_1 - i_2)/(j_1 - j_2)$ for any pair of distinct points $(i_1, j_1), (i_2, j_2) \in e$ (note that this is the negative inverse of the usual geometric slope).

We now show how, using the Newton polygon, we can choose the next term $(\bar{\alpha} + \bar{\bar{\alpha}}\sqrt{-1})x^\beta$ in the expansion of $y = y_0 + (\bar{\alpha} + \bar{\bar{\alpha}}\sqrt{-1})x^\beta + \dots$. Let v_β be a vertex of $\mathcal{P}_{y_0}$ such that the line L_β with slope β passing through v_β touches $\mathcal{P}_{y_0}$ either just at v_β or contains an edge e of $\mathcal{P}_{y_0}$ descending from v_β . We call v_β available for the exponent β . We now fix a certain available vertex v of $\mathcal{P}_{y_0}$ (and refer to it as the active vertex) such that $\beta^{(1)} < \beta_s$ where $\beta^{(1)}$ is the slope of the edge descending from v . Denote by l the ordinate of v .

For any $0 \leq j \leq n$, we define the leading polynomial

$$f_j \in \mathbb{Q}[\bar{\alpha}_1, \dots, \bar{\alpha}_s, \bar{\bar{\alpha}}_1, \dots, \bar{\bar{\alpha}}_s, \beta_1, \dots, \beta_s, \sqrt{-1}, \beta]$$

as

$$f_j = \sum_K \frac{1}{|K|!} l c \left(\left. \frac{\partial^K P}{\partial Y^K} \right|_{Y=y_0} \right) \binom{\beta}{k_1} \dots \binom{\beta}{k_j} k_1! \dots k_j!,$$

where the sum ranges over all $K = (k_1, \dots, k_j)$ such that the weight $w_K = \deg_x \frac{\partial^K P}{\partial Y^K} \big|_{Y=y_0} - k_1 - \dots - k_j = w^{(j)}$ and $\binom{\beta}{k_i} = \beta(\beta-1) \dots (\beta-k_i+1)/k_i!$. We

also define the characteristic polynomial

$$h_\beta \in \mathbb{Q}[\bar{\alpha}_1, \dots, \bar{\alpha}_s, \bar{\alpha}_1, \dots, \bar{\alpha}_s, \beta_1, \dots, \beta_s, \sqrt{-1}, \alpha, \beta]$$

as

$$h_\beta = \sum \alpha^j f_j,$$

where the sum ranges over all j such that $(w^{(j)}, j)$ belongs to L_β .

The Newton polygon procedure then has to select the next term αx^β in such a way that $h_\beta(\alpha) = 0$. For the same reason as in the classical Newton polygon procedure for expanding an algebraic function as a Puiseux series, $h_\beta(\alpha)$ equals the coefficient of the highest term in the expansion of $P(y_0 + \alpha x^\beta)$ with indeterminate α and $\beta < \beta_s$. We now show how to select α and β .

Two different cases can occur. In the first case $f_l \equiv 0$ as a polynomial in β . Corresponding to this case are two possibilities. For the first possibility we can take any β such that $\beta_s, \beta^{(2)} > \beta > \beta^{(1)}$, where $\beta^{(2)}$ is the slope of the edge of the polygon $\mathcal{P}_{y_0}$ whose lower endpoint is v , and an arbitrary $\alpha \neq 0$ for this choice of β . The second possibility for this case is to take $\beta = \beta^{(1)}$ and $\alpha \neq 0$ such that $h_\beta(\alpha) = 0$, provided that h_β has nonzero roots and $\beta^{(1)} \neq -\infty$. In the second case, $f_l \not\equiv 0$ and either we take β to be a root of $f_l(\beta) = 0$ such that $\beta_s, \beta^{(2)} > \beta > \beta^{(1)}$, provided that such a β exists, and an arbitrary $\alpha \neq 0$, or we take $\beta = \beta^{(1)}$ and $\alpha \neq 0$ being a root of $h_\beta(\alpha) = 0$, provided such an α exists. If the Newton process generates a sequence of β_i 's such that $\lim \beta_i = -\infty$ the formula 1 shows that the corresponding series is a solution of $P(y) = 0$. Note that different choices (at any stage) of an active vertex v for β or α will lead to all possible generalized series solutions. We illustrate this Newton polygon process with the following

Example. Let $P = xY''Y + Y'Y - xY'' - Y' - x(Y')^2 = 0$. Figure 1 shows the Newton polygon for this equation. We start by letting $y_0 = 0$. This puts no restrictions on β and there are two available vertices: $(-1, 2)$ and $(-1, 1)$. If we let $(-1, 2)$ be the active vertex, then $f_2 \equiv 0$, $f_1 = -\beta^2$, and $f_0 = 0$. Since $f_2 \equiv 0$, we can take β such that $\infty > \beta > 0$ and α arbitrary or we can let $\beta = \beta^{(1)} = 0$ and any α satisfying $h_\beta(\alpha) = -\alpha\beta^2 = -\alpha(0) = 0$, i.e. α arbitrary. If we let $(-1, 1)$ be the active vertex, then $f_1 = -\beta^2 \neq 0$ and $f_0 = 0$. Since β must satisfy $0 > \beta$ and α must satisfy $h_\beta(\alpha) = -\beta^2\alpha = 0$, we see that letting $(-1, 1)$ be an active vertex yields no possible β . Therefore, we have determined that $y_0 = \alpha_0 x^{\beta_0}$, with $\beta_0 > 0$ and α_0 arbitrary or $y_0 = \alpha_0$ with α_0 arbitrary (note that $y = 0$ is a solution of $P(y) = 0$). We shall now attempt to extend each of these initial segments by adding a term αx^β .

Let $y_0 = \alpha_0$ and replace y by $\alpha_0 + Y$ in $P(Y)$. We obtain $P(\alpha_0 + Y) = (\alpha_0 - 1)xY'' + (\alpha_0 - 1)Y' + xY''Y + Y'Y - x(Y')^2$. The Newton polygon for this is shown in Figure 1 (if $\alpha_0 \neq 1$) and in Figure 2 (if $\alpha_0 = 1$). Let us assume

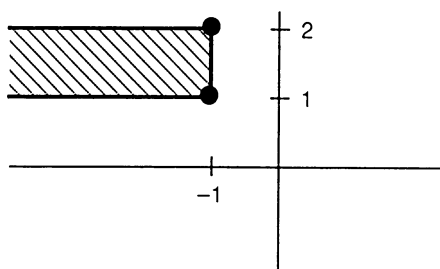


FIGURE 1

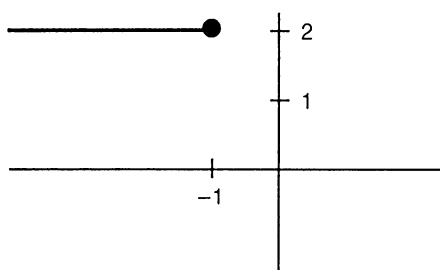


FIGURE 2

that $\alpha_0 \neq 1$. Since $\beta_0 = 0$, we must have $\beta < 0$. This implies that the only available vertex is $(-1, 1)$. We have $f_1 = (\alpha_0 - 1)\beta^2$ and $f_0 = 0$. Therefore $h_\beta(\alpha) = (\alpha_0 - 1)\beta^2\alpha$. Since $\alpha_0 \neq 0$ and $\beta < 0$, this has no nonzero root α . Now assume $\alpha_0 = 1$. Referring to Figure 2, we see that $(-1, 2)$ is an available vertex. We have $f_2 \equiv 0$, so we can select any $\beta < 0$ and any α . Therefore, for $y_0 = \alpha_0$ we can extend y_0 to $y_1 = \alpha_0 + \alpha x^\beta$ with $\beta < 0$ only if $\alpha_0 = 1$ in which case $y_1 = 1 + \alpha x^\beta$ where α and β are arbitrary with $\beta < 0$. Note that $y_0 = \alpha_0$ is a solution of $P(y) = 0$.

We now let $y_0 = \alpha_0 x^{\beta_0}$, $\beta_0 > 0$, and try to extend this to $y_1 = \alpha_0 x^{\beta_0} + \alpha x^\beta$, $\beta_0 > \beta$. We have

$$\begin{aligned} P(y_0 + Y) &= P(\alpha_0 x^{\beta_0} + Y) = -\alpha_0 \beta_0^2 x^{\beta_0-1} + (\alpha_0 x^{\beta_0+1} - x)Y'' \\ &\quad + (\alpha_0 x^{\beta_0} - 1 - 2\alpha_0 \beta_0 x^{\beta_0})Y' + (\alpha_0 \beta_0^2 x^{\beta_0-1})Y \\ &\quad + Y''Y + YY' - x(Y')^2. \end{aligned}$$

Figure 3 shows the Newton polygon for this differential polynomial. Note that the coefficient of Y'' is nonzero since $\beta_0 + 1 > 1$, the coefficient of Y' is nonzero since $\beta_0 > 1$, and the coefficient of Y is nonzero since $\beta_0 > 0$ and $\alpha_0 \beta_0^2 \neq 0$. The only available vertices are $(\beta_0 - 1, 1)$ and $(\beta_0 - 1, 0)$. Note that if we let $(\beta_0 - 1, 0)$ be the active vertex, then $f_0 = -\alpha_0 \beta_0^2 \neq 0$ and $h_\beta(\alpha) = -\alpha_0 \beta_0^2$, which has no roots (in general, letting a vertex of the form $(w_K, 0)$ be an active vertex leads nowhere). Therefore we let $(\beta_0 - 1, 1)$ be the

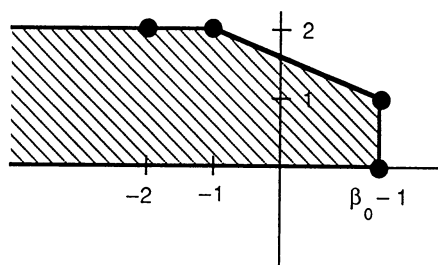


FIGURE 3

active vertex. We have $f_1 = \alpha_0[\beta_0^2 - 2\beta_0\beta + \beta^2] = \alpha_0(\beta - \beta_0)^2$ and $f_0 = -\alpha_0\beta_0^2$. We have $f_1 \neq 0$ and we cannot take β as a root of $f_1 = 0$. Therefore, we must let $\beta = 0$. We then have $h_\beta(\alpha) = \alpha_0\beta_0^2\alpha - \alpha_0\beta_0^2$, so $\alpha = 1$. Therefore, if we try to extend $y_0 = \alpha_0x^{\beta_0}$, $\beta_0 > 0$, to $y_1 = \alpha_0x^{\beta_0} + \alpha x^\beta$, we must have $\beta = 0$ and $\alpha = 1$.

In summary, we see that after applying the Newton procedure twice, we have generated the following possible initial segments: $1 + \alpha x^\beta$, with $\beta < 0$ and α arbitrary; α , with α arbitrary; and $\alpha x^\beta + 1$, with $\beta > 0$ and α arbitrary. The reader should show that each of these is actually a solution of $P(Y) = 0$ and that none of these can be further extended.

The Newton polygon procedure described above allows us to obtain an extension $y_1 = y_0 + \alpha x^\beta$ of the given initial segment y_0 . We now construct the new Newton polygon $\mathcal{P}_{y_1}$ corresponding to y_1 . By Taylor's formula, we have

$$\begin{aligned} P^{(y_1)} &= P(y_1 + Y) = \sum_K \frac{1}{|K|!} \left. \frac{\partial^K P}{\partial Y^K} \right|_{Y=y_1} Y^K \\ (2) \quad &= \sum_K \frac{1}{|K|!} \sum_{K^{(1)}} \frac{1}{|K^{(1)}|!} \left. \frac{\partial^{K+K^{(1)}} P}{\partial Y^{K+K^{(1)}}} \right|_{Y=y_0} (\alpha x^\beta)^{K^{(1)}} Y^K. \end{aligned}$$

Hence, the degree in x of the coefficient of Y^K in $P^{(y_1)}$ (where $K = (k_1, \dots, k_j)$) is at most

$$\max_{K^{(1)}=(k_1^{(1)}, \dots, k_{j_1}^{(1)})} \left\{ \deg_x \left. \frac{\partial^{K+K^{(1)}} P}{\partial Y^{K+K^{(1)}}} \right|_{Y=y_0} + j_1\beta - k_1^{(1)} - \dots - k_{j_1}^{(1)} \right\}.$$

The following lemma compares $\mathcal{P}_{y_1}$ to $\mathcal{P}_{y_0}$.

Lemma 2.1. *Let $(w^{(l)}, l)$ be the active vertex for β in $\mathcal{P}_{y_0}$ and let $\mathcal{P}_{y_1}$ be the new Newton polygon corresponding to $y_1 = y_0 + \alpha x^\beta$.*

(i) *Any point (p, j) such that $p + j\beta > w^{(l)} + l\beta$ does not belong to $\mathcal{P}_{y_1}$.*

- (ii) If $(p, j) \in \mathcal{P}_{y_0}$ lies on the boundary of $\mathcal{P}_{y_0}$ and $j \geq l$, then $(p, j) \in \mathcal{P}_{y_1}$ and the leading polynomial f_j is the same for $\mathcal{P}_{y_1}$ as for $\mathcal{P}_{y_0}$.
- (iii) $(w^{(l)} + l\beta, 0)$ does not belong to $\mathcal{P}_{y_1}$. Therefore $\mathcal{P}_{y_1}$ contains at least one edge with a slope less than β and so, at the next step, we can pick out an active vertex with ordinate less than or equal to l .

Proof. (i) First note that

$$\deg_x \left. \frac{\partial^{K+K^{(1)}} P}{\partial Y^{K+K^{(1)}}} \right|_{Y=y_0} + j_1 \beta - k_1^{(1)} - \dots - k_{j_1}^{(1)} \leq w^{(j+j_1)} + (k_1 + \dots + k_{j_1}) + j_1 \beta \leq w^{(l)} + l\beta + (k_1 + \dots + k_{j_1}) - j\beta.$$

This last inequality is because $w^{(j+j_1)} + (j+j_1)\beta \leq w^{(l)} + l\beta$ by the choice of L_β . If $(p, j) \in \mathcal{P}_{y_1}$ then $p = \max_{K=(k_1, \dots, k_j)} w_K$, where w_K is the weight of the term $P_K Y^K$ in $P(y_1 + Y)$. Therefore

$$p \leq \deg_x \left. \frac{\partial^{K+K^{(1)}} P}{\partial Y^{K+K^{(1)}}} \right|_{Y=y_0} + j_1 \beta - k_1^{(1)} - \dots - k_{j_1}^{(1)}$$

for an appropriate $K^{(1)}$ and so, by the above, $p \leq w^{(l)} + l\beta - j\beta$. This contradicts the assumption that $p + j\beta > w^{(l)} + l\beta$ and proves (i).

(ii) Fix some $K = (k_1, \dots, k_j)$. If $w_K = w^{(j)}$, we obtain, from formula (2) (taking $K^{(1)} = 0$), that the term

$$\frac{1}{|K|!} \left. \frac{\partial^K P}{\partial Y^K} \right|_{Y=y_0} \binom{\beta}{k_1} \dots \binom{\beta}{k_j} k_1! \dots k_j!$$

gives a contribution to f_j . If $K^{(1)} = (k_1^{(1)}, \dots, k_{j_1}^{(1)}) \neq 0$, then the term

$$\left. \frac{\partial^{K+K^{(1)}} P}{\partial Y^{K+K^{(1)}}} \right|_{Y=y_0} (\alpha x^\beta)^{K^{(1)}}$$

does not contribute to f_j . To see this note first that

$$\deg_x \left. \frac{\partial^{K+K^{(1)}} P}{\partial Y^{K+K^{(1)}}} \right|_{Y=y_0} + \beta_{j_1} - k_1^{(1)} - \dots - k_{j_1}^{(1)} \leq w^{(j+j_1)} + \beta_{j_1} + k_1 + \dots + k_{j_1}.$$

Since any edge of $\mathcal{P}_{y_0}$ above the line $Y = l$ has slope $> \beta$, then we have $w^{(j+j_1)} + \beta(j+j_1) < w^{(j)} + \beta_j$. We therefore have

$$\deg_x \left. \frac{\partial^{K+K^{(1)}} P}{\partial Y^{K+K^{(1)}}} \right|_{Y=y_0} + \beta_{j_1} - k_1^{(1)} - \dots - k_{j_1}^{(1)} < w^{(j)} + k_1 + \dots + k_j;$$

and so if $K^{(1)} \neq 0$, the term

$$\left. \frac{\partial^{K+K^{(1)}} P}{\partial Y^{K+K^{(1)}}} \right|_{y=y_0} (\alpha x^\beta)^{K^{(1)}}$$

does not contribute to f_j . Thus f_j , and by the same token, (p, j) , do not change in $\mathcal{P}_{y_1}$.

(iii) Consider the point $(w^{(l)} + l\beta, 0)$. To this point corresponds the coefficient (put $K = 0$ in (2))

$$(3) \quad \sum_{K^{(1)}=(k_1^{(1)}, \dots, k_{j_1}^{(1)})} \frac{1}{|K^{(1)}|!} l c \left(\left. \frac{\partial^{K^{(1)}} P}{\partial Y^{K^{(1)}}} \right|_{y_0=Y} \right) \alpha^{j_1} \binom{\beta}{k_1^{(1)}} \cdots \binom{\beta}{k_{j_1}^{(1)}} k_1^{(1)}! \cdots k_{j_1}^{(1)}!,$$

where the summation ranges over all $K^{(1)}$ such that

$$\deg \left(\left. \frac{\partial^{K^{(1)}} P}{\partial Y^{K^{(1)}}} \right|_{y_0=Y} \right) + \beta_{j_1} - k_1^{(1)} - \cdots - k_{j_1}^{(1)} = w^{(l)} + l\beta.$$

In particular, for k_1 we have $w^{(j_1)} + \beta_{j_1} = w^{(l)} + l\beta$, and this proves that the coefficient (3) equals $h_\beta(\alpha)$, which is 0. This shows that the point $(w^{(l)} + l\beta, 0)$ does not belong to $\mathcal{P}_{y_1}$. The second part of (iii) follows from this.

The case when at the next step we can take the same v as an active vertex can only happen if the edge descending from the endpoint v in the polygon $\mathcal{P}_{y_1}$ has a slope less than β . The following lemma tells us when this can continue to happen.

Lemma 2.2. *Let $y = \sum_{i=0}^{\infty} \alpha_i x^{\beta_i}$ be generated by the Newton polygon process as above and assume $y \in \Omega$. If, in this process, a vertex v with ordinate l becomes available and stays available at each step thereafter, then $\partial^K P / \partial Y^K|_{Y=y} = 0$ for all $K = (k_1, \dots, k_j)$ for which $j < l$.*

Proof. Assume that after the t th step the vertex $v = (w^{(l)}, l)$ becomes active and stays active. For any s and vertex (p, j) of the $s + t$ th Newton polygon with $j \leq l$ we have $p + \beta_{t+s}j \leq w^{(l)} + \beta_{t+s}l$, so this point lies to the left of $X + \beta_{t+s}Y = w^{(l)} + \beta_{t+s}l$ and between $Y = 0$ and $Y = l$. If $\partial^K P / \partial Y^K|_{Y=y} = \alpha x^\beta +$ terms with smaller exponent, $\alpha \neq 0$, then by Lemma 1.6, $\partial^K P / \partial Y^K|_{Y=y_{t+s}} = \alpha x^\beta +$ terms with smaller exponents for sufficiently large s . Since we are assuming that $\lim_{s \rightarrow \infty} \beta_{t+s} = -\infty$, formula (1) shows that this would produce a point in the Newton polygon to the right of $X + \beta_{t+s}Y = w^{(l)} + \beta_{t+s}l$. Therefore, $\partial^K P / \partial Y^K|_{Y=y} = 0$.

We end this section by noting that a Newton polygon type process has been previously used by several authors (e.g. [RA78 and DE82]) to find solutions of

linear differential equations and to find solutions of some nonlinear first order differential equations (e.g. [HI76]).

3. STABILIZATION OF THE NEWTON POLYGON PROCESS

We begin with the following definition. A solution $y = \sum (\bar{\alpha}_i + \sqrt{-1\bar{\alpha}_i})x^{\beta_i} \in \Omega$ of $P(Y) = 0$ is called *stabilizable* for P if after some steps of the Newton polygon process we come to a Newton polygon $\mathcal{P}_{y_0}$ corresponding to an initial segment $y_0 = \sum_{1 \leq i \leq s} (\bar{\alpha}_i + \sqrt{-1\bar{\alpha}_i})x^{\beta_i}$ with active vertex v having ordinate 1 and only the edge descending from the vertex v has a slope less than β_s . Furthermore, we require that β_s is not greater than any real root of the polynomial $f_1(\beta)$. We also describe this situation as y is *stable* for P after y_0 . We make the following two remarks. The first is that $f_1 \not\equiv 0$ as a polynomial in β . Indeed,

$$f_1(\beta) = \sum_q \frac{1}{q!} \gamma_q \beta(\beta-1) \cdots (\beta-q+1),$$

where $\gamma_q = lc(\partial P / \partial Y^{(q)})|_{y_0=Y}$ and the summation ranges over all q such that the weight

$$\deg_x \left(\frac{\partial P}{\partial Y^{(q)}} \Big|_{y_0=Y} \right) - q = w^{(1)},$$

which is the abscissa of the point v . Obviously, $f_1 \not\equiv 0$.

The second remark is that after stabilizing there is a unique possibility for choosing the next term αx^β in the expansion $y = y_0 + \alpha x^\beta + \cdots$; namely β is the slope of the edge e of the Newton polygon descending from v and α satisfies the equation $h_\beta(\alpha) = f_1(\beta)\alpha + lc(P(y_0)) = 0$. Therefore the Newton polygon process, after stabilizing, can be continued uniquely and correctly and leads to the unique solution $y \in \Omega$ of $P(Y) = 0$ with initial segment y_0 , provided that the slopes tend to $-\infty$. If the latter does not hold, we get no solution with initial segment y_0 (by Theorem 1.1). The next lemma states that although a solution y (with an infinite number of terms) of $P = 0$ may not be stabilizable for P , y is stabilizable for some partial derivative of P .

Lemma 3.1. *For any solution $y = \sum (\bar{\alpha}_i + \sqrt{-1\bar{\alpha}_i})x^{\beta_i} \in \Omega$ of $P(Y) = 0$ (provided that the series has an infinite number of terms so $\lim \beta_i = -\infty$ by Theorem 1.1) consider an index I for which*

$$\frac{\partial^I P}{\partial Y^I} \Big|_{Y=y} = 0, \quad \frac{\partial^I P}{\partial Y^I} \not\equiv 0, \quad \text{and} \quad \deg_{Y, Y^{(1)}, \dots, Y^{(m)}} \left(\frac{\partial^I P}{\partial Y^I} \right)$$

is the least possible. Then y is a stabilizable solution for $\frac{\partial^I P}{\partial Y^I}$.

Proof. First we show that in the Newton polygon process for expanding y as a solution of $\partial^I P / \partial Y^I = 0$, we eventually come to an active point with ordinate 1. Assume the contrary, then starting from some step, a certain vertex v with ordinate $l > 1$ would become and stay an active point. By virtue of Lemma 2.2,

$\partial^{I+K}P/\partial Y^{I+K}|_{y=Y} = 0$ for all $K = (k_1, \dots, k_j)$ such that $j < l$. For some $K = (k_1, \dots, k_j)$ we have $\partial^{I+K}P/\partial Y^{I+K} \neq 0$, hence the multi-index $I + K$ satisfies the conditions of the lemma, contradicting the choice of I . Thus, we have proved that a certain vertex v with ordinate 1 eventually would become and stay an active point.

Continuing the Newton polygon process we would eventually have that β_0 would be less than any real root of the polynomial $f_1(\beta)$ by Theorem 1.1. This means that y is a stabilizable solution for $\frac{\partial^I P}{\partial Y^I}$ and completes the proof of the lemma.

Corollary. *For any solution*

$$y = \sum (\bar{\alpha}_i + \sqrt{-1}\bar{\alpha}_i)x^{\beta_i}$$

of $P(Y)$, the exponents $\beta_0, \beta_1, \dots$ generate a finite $\mathbb{Z}$ module.

Proof. Any solution y of $P(y) = 0$ is stabilizable for $\partial^I P/\partial Y^I$ for some I . If y is stable for $\partial^I P/\partial Y^I$ after $y_0 = \sum_{0 \leq i \leq s} (\bar{\alpha}_i + \sqrt{-1}\bar{\alpha}_i)x^{\beta_i}$, then all the exponents β_i lie in the $\mathbb{Z}$ -module generated by $1, \beta_0, \dots, \beta_s$.

Note that the set $\Omega^{\text{fin}} = \{\sum (\bar{\alpha}_i + \sqrt{-1}\bar{\alpha}_i)x^{\beta_i} \in \Omega \mid \{\beta_i\} \text{ generate a finite } \mathbb{Z}\text{-module}\}$ forms a differential field.

4. ENUMERATING THE GENERALIZED SERIES SOLUTIONS OF AN ORDINARY DIFFERENTIAL EQUATION

The Newton polygon process described above can be used to enumerate (in a sense made precise below) the generalized series solutions of a differential equation $P = 0$, where P is the same as the beginning of §2. If we consider an initial segment $y_0 = \sum_{1 \leq i \leq s} (\bar{\alpha}_i + \sqrt{-1}\bar{\alpha}_i)x^{\beta_i}$ of $P = 0$ where we are thinking of the $\bar{\alpha}_i$, $\bar{\alpha}_i$, and β_i as real parameters, then the next iteration of the Newton polygon procedure used to extend y_0 to $y_0 + (\bar{\alpha} + \sqrt{-1}\bar{\alpha})x^\beta$ will place restrictions on these parameters. These restrictions can be expressed as a certain quantifier-free formula $\varphi(\bar{\alpha}_1, \dots, \bar{\alpha}_s, \bar{\alpha}_1, \dots, \bar{\alpha}_s, \beta_1, \dots, \beta_s, \bar{\alpha}, \bar{\alpha}, \beta)$ in the first-order language of ordered fields. We consider the validity of such formulas in the first-order theory of real closed fields (sometimes called the Tarski algebra) and we shall refer to these as Tarski algebra formulas. It is well known that the Tarski algebra is decidable [TA51, CO75, GR88]. The rest of this section is devoted to the proof of the following:

Theorem 4.1. *Let an ordinary differential equation $P = 0$ be given. There is a procedure that enumerates all the generalized series solutions of $P = 0$ in the following sense. It enumerates a succession of consistent Tarski algebra formulas*

$$\Psi(\bar{\alpha}_1, \dots, \bar{\alpha}_{s_j}, \bar{\alpha}_1, \dots, \bar{\alpha}_{s_j}, \beta_1, \dots, \beta_{s_j})$$

with parameters and ordinary differential polynomials $P_j \in Q[x]\{Y\}$ such that for each j and any $\bar{\alpha}_1^{(0)}, \dots, \bar{\alpha}_{s_j}^{(0)}, \beta_1^{(0)}, \dots, \beta_{s_j}^{(0)} \in \mathbb{R}$ satisfying this formula,

$y_0 = \sum_{1 \leq i \leq s_j} (\bar{\alpha}_i^{(0)} + \sqrt{-1} \bar{\alpha}_i^{(0)}) x^{\beta_i^{(0)}}$ is an initial segment of an expansion of solution $y = y_0 + \dots \in \Omega$ of the equation $P_j = 0$, provided that the slopes in the expansion tend to $-\infty$. If the latter holds this solution y is stable for P_j after producing the initial segment y_0 via the Newton polygon procedure (so y is uniquely defined from y_0 on) and $P(y) = 0$. Any generalized series solution of $P(y) = 0$ appears in this way.

In the procedure that we now describe, the Tarski algebra formulas are generated when we attempt to extend an initial segment $y_0 = \sum_{1 \leq i \leq s} (\bar{\alpha}_i + \sqrt{-1} \bar{\alpha}_i) x^{\beta_i}$. We will proceed by induction on s , the number of terms in the initial segment y_0 . So, we are given a set of parameters $\bar{\alpha}_1, \dots, \bar{\alpha}_s, \bar{\alpha}_1, \dots, \bar{\alpha}_s, \beta_1, \dots, \beta_s$ and a certain Tarski algebra formula $\varphi(\bar{\alpha}_1, \dots, \bar{\alpha}_s, \bar{\alpha}_1, \dots, \bar{\alpha}_s, \beta_1, \dots, \beta_s)$. At the base of the induction $s = 0$ and the formula φ is $(0 = 0)$. In addition to φ we will also be given an equation $Q = 0$, an inequality $Q_0 \neq 0$, a Newton polygon $\mathcal{P}_{y_0}^{(Q)}$ for the equation $Q = 0$ corresponding to the initial segment y_0 , and lastly an active vertex at the previous step of the Newton polygon process. Here $Q(Y), Q_0(Y)$ are differential polynomials such that $Q_0 < Q$ where the latter relations mean that either the order of Q_0 is less than the order of Q or that they both have the same order m and the degree of Q_0 in $Y^{(m)}$ is less than the degree of Q in $Y^{(m)}$. When $s = 0$, $Q = P$ and $Q_0 = 1$. The polygon $\mathcal{P}_{y_0}^{(Q)}$ is represented by specifying the integers $0 \leq t_1 < \dots < t_u \leq n$ and a subset of them $0 \leq t'_1 < \dots < t'_u \leq n$ such that the ordinates of the vertices belonging to the boundary of $\mathcal{P}_{y_0}^{(Q)}$ are exactly $t_1, \dots, t_u$ and the endpoints of the edges of $\mathcal{P}_{y_0}^{(Q)}$ have exactly the ordinates $t'_1, \dots, t'_u$. For each t_j we are given the abscissa p_j of the corresponding vertex, where $p_j = p_j^{(1)} \beta_1 + \dots + p_j^{(s)} \beta_s + p_j^{(0)}$, where $p_j^{(1)}, \dots, p_j^{(s)}, p_j^{(0)} \in \mathbb{Z}$, and $p_j^{(1)}, \dots, p_j^{(s)} \geq 0$. The algorithm also produces the leading polynomial $f_{t_j} \in \mathbb{Q}[\bar{\alpha}_1, \dots, \bar{\alpha}_s, \bar{\alpha}_1, \dots, \bar{\alpha}_s, \beta_1, \dots, \beta_s, \sqrt{-1}]$ corresponding to the vertex (p_j, t_j) of $\mathcal{P}_{y_0}^{(Q)}$. The active vertex at the previous step is specified by indicating some t'_j .

We now describe the next step of the enumeration algorithm, during which the next term αx^β of $y = y_0 + \alpha x^\beta + \dots$ will be produced. Firstly, we select an active vertex v with ordinate $l = t'_j$, less than or equal to t'_j . We then choose the exponent β according to the following possibilities (cf. §2). Let $f_l = \sum_{0 \leq j \leq j_0} f_l^{(j)} \beta^j$ be a leading polynomial, where $f_l^{(j)} \in \mathbb{Q}[\bar{\alpha}_1, \dots, \bar{\alpha}_s, \bar{\alpha}_1, \dots, \bar{\alpha}_s, \beta_1, \dots, \beta_s, \sqrt{-1}]$. The first case is when the line with the next slope β passing through the node v lies strictly outside $P_{y_0}^{(Q)}$ and strictly between two edges of the Newton polygon, with slopes $\delta_1 > \delta_2$ respectively, adjacent to v . We then add to the formula φ the following conditions: $\delta_1 > \beta > \delta_2$, $\beta_s > \beta$, $f_l(\beta) = 0$ (and in this case $\bar{\alpha}$ and $\bar{\alpha}$ are arbitrary). In the second case, when

$\beta = \delta_2$, the algorithm adds to φ this latter equality as well as the condition $\beta_s > \beta$ and does the following. The algorithm produces the characteristic polynomial $h_{\delta_2} \in Q[\overline{\alpha}_1, \dots, \overline{\alpha}_m, \overline{\alpha}_1, \dots, \overline{\alpha}_m, \beta_1, \dots, \beta_m, \sqrt{-1}, \alpha]$, (see above) corresponding to the edge with slope δ_2 and with endpoint v . Note that h_{δ_2} involves t_j , p_j and f_j . The expression $\overline{\alpha} + \sqrt{-1}\overline{\alpha}$ is then substituted for α in h_{δ_2} , and we write

$$h_{\delta_2}(\overline{\alpha} + \sqrt{-1}\overline{\alpha}) = \overline{h_{\delta_2}}(\overline{\alpha}, \overline{\alpha}) + \sqrt{-1}\overline{h_{\delta_2}}(\overline{\alpha}, \overline{\alpha}).$$

Finally, we add to the formula φ the equations $\overline{h_{\delta_2}}(\overline{\alpha}, \overline{\alpha}) = \overline{\overline{h_{\delta_2}}}(\overline{\alpha}, \overline{\alpha}) = 0$. The resulting Tarski formula is denoted by $\hat{\varphi}$. We can check whether such β , $\delta_1 > \beta > \delta_2$ exists. If not, we do not consider v as an available node. We wish to continue this process and our initial segment stabilizes. Since an initial segment may not stabilize for $Q = 0$ but only for some derivative $\partial^K Q / \partial Y^K$ we are forced to consider various branchings in our procedure. Consider all the differential polynomials $\partial^K Q / \partial Y^K$ for all $K = (k_1, \dots, k_j)$ with $j < l$. Observe that $\partial^K Q / \partial Y^K \prec Q$. Now the process branches in two ways.

(A) First, assume $l > 1$ and continue the Newton polygon process applying it to Q . At this point we have an initial segment

$$\overline{y}_0 = \sum_{1 \leq j \leq s} (\overline{\alpha}_j + \sqrt{-1}\overline{\alpha}_j) x^{\beta_j} + (\overline{\alpha} + \overline{\alpha}\sqrt{-1}) x^{\beta},$$

the same equation $Q = 0$, and the same inequality $Q_0 \neq 0$. And we yield a new Newton polygon $\mathcal{P}_{\overline{y}_0}^{(Q)}$ in the following way. Recall (see (1)) that

$$Q = \sum_K \frac{1}{|K|!} \frac{\partial^K Q}{\partial Y^K} \bigg|_{\overline{y}_0=Y} Y^K = \sum_{\overline{K}} \overline{\rho}_{\overline{K}} x^{k_0} Y^K,$$

where $\overline{K} = (k_0, K) = (k_0, k_1, \dots, k_j)$, $\overline{\rho}_{\overline{K}} \in Q[\overline{\alpha}_1, \dots, \overline{\alpha}_s, \overline{\alpha}_1, \dots, \overline{\alpha}_s, \beta_1, \dots, \beta_s, \sqrt{-1}, \overline{\alpha}, \overline{\alpha}, \beta]$, $k_0 = \sum \kappa_j \beta_j + \kappa_0$, where $0 \leq \kappa_1, \dots, \kappa_s \in \mathbb{Z}$, $\kappa_0 \in \mathbb{Z}$, $0 \leq k_1, \dots, k_j \in \mathbb{Z}$. Now we consider all the finite number of possible shapes of $\mathcal{P}_{\overline{y}_0}^{(Q)}$. A shape means two lists of integers $0 \leq \overline{t}_1 \leq \dots \leq \overline{t}_u < l$, $0 \leq \overline{t}'_1 < \dots < \overline{t}'_{u'} < l$ such that $\overline{t}'_1, \dots, \overline{t}'_{u'}$ are the ordinates of the extremal vertices, with ordinates less than l , of $\mathcal{P}_{\overline{y}_0}^{(Q)}$ (recall that the vertices of the Newton polygon with ordinates not less than l do not change), and $\overline{t}_1, \dots, \overline{t}_u$ are all the vertices, with ordinates less than l , of $\mathcal{P}_{\overline{y}_0}^{(Q)}$. Observe that the shapes can be expressed in terms of a quantifier-free Tarski algebra formula in the parameters $\overline{\alpha}_1, \dots, \overline{\alpha}_s, \overline{\alpha}_1, \dots, \overline{\alpha}_s, \beta_1, \dots, \beta_s, \overline{\alpha}, \overline{\alpha}, \beta$ (actually, it is a formula of linear programming, i.e. all the inequalities are linear).

(B) Second, provided again that $l > 1$, we consider a system of differential equations $\mathcal{S} = \{Q = 0\} \cup \{\partial^K Q / \partial Y^K = 0\}$ for all $K = (k_1, \dots, k_j)$, $j < l$.

We apply the reduction process from [SE56] to the system $\mathcal{S} \cup \{Q_0 \neq 0\}$ and obtain as a result that the latter system is equivalent (over a differential closure) to a disjunction of the following form: $\bigvee_i (Q_i = 0, Q_{i_0} \neq 0)$, where $Q_i \prec \partial^K Q / \partial Y^K \preceq Q$, for $K = (k_1, \dots, k_j)$, $j < l$, and $Q_{i_0} \prec Q_i$. For any i we consider the system $Q_i = 0, Q_{i_0} \neq 0$ as a new starting point for the Newton polygon process. We consider $(\bar{\alpha}_1 + \bar{\alpha}_1 \sqrt{-1})x^{\beta_1}$ as the first term, then substituting it in Q_i gives the next Newton polygon and Tarski algebra formula; if it is consistent, we substitute the next term $(\bar{\alpha}_2 + \sqrt{-1}\bar{\alpha}_2)x^{\beta_2}$, etc., up to the term $(\bar{\alpha}_s + \sqrt{-1}\bar{\alpha}_s)x^{\beta_s}$. After this we start to apply the recursive process, under description, to the system $Q_i = 0, Q_{i_0} \neq 0$, the constructed Newton polygon and the Tarski algebra formula.

Now assume that the ordinate of the active vertex v is 1. Fix some natural number N and consider the following condition. Recall that $f_1(Z) \in Q[\bar{\alpha}_1, \dots, \bar{\alpha}_s, \bar{\alpha}_1, \dots, \bar{\alpha}_s, \beta_1, \dots, \beta_s, \sqrt{-1}][Z]$ is the leading polynomial corresponding to the active node v and suppose that $\deg_Z f_1 = l$ for some l and $f_1(Z) = a_l Z^l + \dots + a_0$, $a_l \neq 0$. The condition we are considering is that the Newton polygon process applied to Q correctly gives, after N steps, the successive terms $(\bar{\alpha}_{s+1} + \sqrt{-1}\bar{\alpha}_{s+1})x^{\beta_{s+1}}, \dots, (\bar{\alpha}_{s+N} + \sqrt{-1}\bar{\alpha}_{s+N})x^{\beta_{s+N}}$ and, in addition, $|\beta_{s+N}| > |\alpha_0/\alpha_l| + \dots + |\alpha_l/\alpha_l|$. This latter inequality just implies that β_{s+N} is less than any real root of the polynomial f_1 . The described condition can be represented by a quantifier-free Tarski algebra formula ψ_N in $\bar{\alpha}_j, \bar{\alpha}_j, \beta_j$, $1 \leq j \leq s+N$ (cf. above). Note that for any solution y of $Q(Y) = 0$ and some initial segment y_0 for which the next active vertex v has ordinate 1, we know by Theorem 1.1 that an N exists such that the next N terms of this solution satisfy such a Tarski algebra formula.

At this point we have generated, for a fixed N , a Tarski formula that states that a segment of length $s+N$ is an initial segment of a possible solution y of $P = 0$ and is stable for some equation $Q = 0$ provided that the slopes after stabilization tend to $-\infty$. In the process we have generated an auxiliary equation $Q_0 \neq 0$ which must be satisfied to guarantee that y satisfies $P = 0$. To see if $Q_0(y) \neq 0$, we carry out the following subprocedure that looks for an exponent M_0 such that $Q_0(y) \neq 0 \pmod{x^{M_0}}$. For the N fixed above, we fix some N_0 and some term of Q_0 of the form $x^{l_0} y^{(l_1)} \dots y^{(l_s)}$, namely, some term of its expansion of the form $x^{l_0} x^{\beta_{j_1} - l_1} \dots x^{\beta_{j_s} - l_s}$ where $0 \leq j_1, \dots, j_s \leq N_0$. The subprocedure looks over all $N_0, j_1, \dots, j_s$ (and also $k_1, \dots, k_n$ introduced below). Denote $M = l_0 + \beta_{j_1} - l_1 + \dots + \beta_{j_s} - l_s$. We are at this point considering M as a candidate for M_0 . If some term contributes x^M in the expansion of $Q_0(y)$, then $M = l'_0 + \beta'_{j'_1} - l'_1 + \dots + \beta'_{j'_s} - l'_s$, for a certain term $x^{l'_0} Y^{(l'_1)} \dots Y^{(l'_s)}$ of Q_0 . We therefore must consider representations of the form $M + l = \beta_{j_1^{(0)}} + \dots + \beta_{j_s^{(0)}}$ for a finite number (less than $2(\deg(Q_0) + 1)(\text{ord}(Q_0))$)

of integers l such that $|l| < (\deg(Q_0) + 1)(\text{ord}(Q_0))$ and $0 \leq s^{(0)} \leq \text{ord}(Q_0) \leq n = \text{ord}(Q)$. Fix some integer l within this range (the subprocedure looks over all such l). For simplicity of notation assume that $\beta_{j_1^{(0)}}, \dots, \beta_{j_{s^{(0)}}^{(0)}} < 0$. Then

$$\beta_{j_1^{(0)}} \geq \frac{M - (\deg(Q) + 1)(\text{ord}(Q))}{n + 1} = M_1.$$

The subprocedure now considers all k_1 and for a fixed k_1 , adds the condition to the Tarski algebra formula ψ_N that k_1 is the largest index satisfying the inequality $\beta_{k_1} \geq M_1$. It then looks over all $j_1^{(0)} \leq k_1$. Fix some such $j_1^{(0)}$ and consider the possible representations of $M + l - \beta_{j_1^{(0)}}$ in the form $\beta_{j_2^{(0)}} + \dots + \beta_{j_{s^{(0)}}^{(0)}} = M + l - \beta_{j_1^{(0)}}$ for $0 \leq s^{(0)} \leq n$, in the same way as above. After $n + 1$ steps we will obtain some Tarski algebra formula (for fixed $k_0, k_1, \dots, k_n$). The formula yields all representations of $M + l$ in the form $M + l = \beta_{j_1^{(0)}} + \dots + \beta_{j_{s^{(0)}}^{(0)}}$ for $0 \leq s^{(0)} \leq n$. Now write down the coefficient of x^M in the expansion of $Q_0(y)$ and the condition that this coefficient is not zero. This yields some additional Tarski algebra formula which we combine with the existing Tarski algebra formula. Test its satisfiability [TA51, CO75, GR88] and if it is satisfiable, we have $Q_0(y) \not\equiv 0 \pmod{x^M}$ so $Q_0(y) = 0$. Our procedure will then output this formula as one of the Ψ in Theorem 4.1. If it is not satisfiable, continue the subprocedure described above for the next $(n + 1)$ -tuple $(k_0, \dots, k_n)$. If $Q_0(y) \not\equiv 0$, then for an appropriate choice of $k_0, \dots, k_n$ we will discover that $Q_0(y) \not\equiv 0 \pmod{x^M}$.

To prove Theorem 4.1 we must show that the above procedure enumerates all the generalized series solutions y of the input equation $P = 0$. In expanding y by the Newton polygon process either the active vertex eventually would have ordinate 1 and then y is a solution of the equation $P = 0$ (by Lemma 2.2), or the active vertex eventually would have ordinate $l > 1$ and $\partial^K P / \partial Y^K = 0$ for all $K = (k_1, \dots, k_j)$ with $j < l$ (again by Lemma 2.2). In the second case (see B in the present section) we obtain using [SE56] a new system $Q_i = 0$, $Q_{i0} \neq 0$ which has y as a solution and for which $Q_i \prec P$. After some steps of this reduction the algorithm comes to a certain system $Q = 0$, $Q_0 \neq 0$ for which y is a stabilizable solution. This completes the proof of Theorem 4.1.

Corollary. *Let $P = 0$ be an ordinary differential equation as above. If $P = 0$ has a generalized series solution, then it has a solution of the form*

$$y = \sum_{i=0}^{\infty} (\bar{\alpha}_i + \sqrt{-1}\bar{\bar{\alpha}}_i) x^{\beta_i}$$

where $\beta_0 > \beta_1 > \dots$, $\lim_{i \rightarrow \infty} \beta_i = -\infty$, and $\bar{\alpha}_i, \bar{\bar{\alpha}}_i, \beta_i \in \tilde{\mathbb{Q}}$, where $\tilde{\mathbb{Q}}$ is the real closure of $\mathbb{Q}$.

Proof. Our assumptions and Theorem 4.1 imply that some Tarski formula Ψ (as in Theorem 4.1) is satisfiable in $\mathbb{R}$. It is well known [TA51] that this implies that Ψ is satisfiable with $\bar{\alpha}_i, \bar{\bar{\alpha}}_i, \beta_i$ in $\tilde{\mathbb{Q}}$. Theorem 4.1 now implies that the initial segment $\sum_{i=0}^s (\bar{\alpha}_i + \sqrt{-1}\bar{\bar{\alpha}}_i)x^{\beta_i}$ can be extended to a solution y of $P(y) = 0$.

Note that the above argument also shows that if $P = 0$ has a solution $y = \sum (\bar{\alpha}_i + \sqrt{-1}\bar{\bar{\alpha}}_i)x^{\beta_i}$ where $\bar{\alpha}_i, \bar{\bar{\alpha}}_i \in \mathbb{R}$, $\beta_0 > \beta_1 > \dots$, and $\beta_i \in \mathbb{Z}$, then it has a solution of the same form with $\bar{\alpha}_i, \bar{\bar{\alpha}}_i \in \tilde{\mathbb{Q}}$. This yields a result (for a single equation) similar to Theorem 2.7 of [DL84].

5. DECISION PROBLEMS

In [DL84] Denef and Lipschitz show that there is an algorithm to decide if a system of polynomial differential equations with coefficients in $\mathbb{Q}[x]$ has a solution in $\mathbb{C}[[x]]$, the ring of formal power series. They also show that, given a system of differential polynomial equations $P_i(y_1, y_1', y_1'', \dots, y_m, y_m', \dots) = 0$, $i = 1, 2, 3, \dots$, with coefficients in $\mathbb{C}[[x]]$ such that for every $n \in \mathbb{N}$ there exists $\bar{y}_1, \dots, \bar{y}_n$ in $\mathbb{C}[[x]]$ such that $P_i(\bar{y}_1, \bar{y}_1', \dots, \bar{y}_m, \dots) = 0 \pmod{x^n}$, $i = 1, 2, 3, \dots$, then there exist $\bar{\bar{y}}_1, \dots, \bar{\bar{y}}_m \in \mathbb{C}[[x]]$ such that $P_i(\bar{\bar{y}}_1, \bar{\bar{y}}_1', \dots, \bar{\bar{y}}_m, \dots) = 0$. In this section we show that this fact is not true in Ω and also that there is no algorithm to decide if such a system has a solution in Ω . It will be convenient to work with the field

$$\Omega_0 = \left\{ \sum_{i=0}^{\infty} \alpha_i x^{\beta_i} \mid \alpha_i \in \mathbb{C}, \beta_i \in \mathbb{R}, \beta_0 < \beta_1 < \dots, \text{ and } \lim_{n \rightarrow \infty} \beta_i = \infty \right\}.$$

This field is obviously a differential field with derivation $\frac{d}{dx}$. Using the map that sends x to x^{-1} , we see that Ω_0 with the derivation $\frac{d}{dx}$ is isomorphic to Ω with the derivation $D = -x^2 \frac{d}{dx}$. This observation allows us to transfer results concerning solution of differential equations in Ω_0 to similar results about Ω (and vice versa).

We start by considering the system of differential equations

$$(4) \quad y'x = \beta y, \quad \beta' = 0, \quad z'yx + z'' = y + x.$$

We first show that if $\beta = \frac{1}{n}$ for some $n \in \mathbb{N}$, then (4) has a solution in Ω_0 . If $n = 1$, we may let $y = -x$ and $z = 0$, so assume $n > 1$. We will show that there exist $a_1, \dots, a_n$ and c in $\mathbb{C}$ such that $y = cx^{1/n}$ and $z = a_1x^{1/n} + \dots + a_{n-1}x^{(n-1)/n}$ are solutions of (4). If we substitute these expressions in the last equation of (4) and compare exponents of x , we see

that

$$\begin{aligned}
 a_{n-1}c \left(\frac{n-1}{n} \right) &= 1, \\
 a_{n-2}c \left(\frac{n-2}{2} \right) + a_{n-1} \left(\frac{n-1}{n} \right) \left(\frac{n-1}{n} - 1 \right) &= 0, \\
 &\vdots \\
 a_1c + a_2 \frac{2}{n} \left(\frac{2}{n} - 1 \right) &= 0, \\
 a_1 \frac{1}{n} \left(\frac{1}{n} - 1 \right) &= c.
 \end{aligned}$$

These equations determine c^n uniquely. Once we select c , then $a_1, \dots, a_{n-1}$ are uniquely determined.

Conversely, assume that (4) has a solution in Ω_0 . We will show that $\beta = \frac{1}{n}$ for some $n \in \mathbb{N}$. The first two equations of (4) imply that $y = cx^\beta$ for some $c \in \mathbb{C}$. If $c = 0$, then the third equation reduces to $z''x^2 = x$ and this latter equation has no solution in Ω_0 . Therefore $c \neq 0$ and $\beta \in \mathbb{R}$. Let $\beta \neq 0$, $y = cx^\beta$, and $z = \sum_{i=0}^{\infty} \alpha_i x^{\beta_i} \in \Omega_0$ be solutions of (4). We shall assume that $\beta \neq \frac{1}{n}$ and derive a contradiction. First we shall show $\beta > 0$. If $\beta < 0$, comparing lowest powers of x in $z'yx + z''x = y + x$ yields $c\alpha_0\beta_0x^{\beta_0+\beta} = cx^\beta$, which is impossible. Next we shall show that $x^{1-\beta}, x^{1-2\beta}, x^{1-3\beta}, \dots$ must all occur in z with nonzero coefficients. Comparing coefficients of x^1 in $z'yx + z''x = y + x$, we have $c\alpha_{i_1}(1-\beta) = 1$, where α_{i_1} is the coefficient of $x^{1-\beta}$ in z . Therefore $\alpha_{i_1} \neq 0$. Comparing coefficients of $x^{1-n\beta}$, we have $c\alpha_{i_n}(1-n\beta) + \alpha_{i_{n-1}}(1-(n-1)\beta)(-(n-1)\beta) = 0$ where α_{i_n} is the coefficient of $x^{1-n\beta}$ in z . Induction shows that $\alpha_{i_n} \neq 0$ for all n . Since $\lim_{n \rightarrow \infty} 1-n\beta = -\infty$, we see that z cannot lie in Ω_0 , a contradiction.

We can now show that the following problem is undecidable: given a system $P_i(y_1, y_1', y_1'', \dots, y_m, y_m', y_m'', \dots) = 0$, $i = 1, 2, \dots$, of differential polynomial equations with coefficients in $\mathbb{Q}[x]$, do there exist solutions $y_1, \dots, y_m \in \Omega_0$? Given any diophantine equation $f(x_1, \dots, x_n) = 0$ with coefficients in $\mathbb{N}$, consider the system

$$\begin{aligned}
 f(\gamma_1, \dots, \gamma_n) &= 0, \\
 \gamma_1\beta_1 - 1 &= 0, \quad \gamma_2\beta_2 - 1 = 0, \dots, \quad \gamma_n\beta_n - 1 = 0, \\
 \beta'_1 &= 0, \quad y'_1x = \beta_1y_1, \quad z'_1y_1x + z''_1x = y_1 + x, \\
 &\vdots \\
 \beta'_n &= 0, \quad y'_nx = \beta_ny_n, \quad z'_ny_nx + z''_nx = y_n + x.
 \end{aligned}$$

For our discussion above, this system has a solution in Ω_0 if and only if $f(x_1, \dots, x_n) = 0$ has a solution in $\mathbb{N} - \{0\}$. Since this latter problem is

known to be undecidable [MA70], our problem is also undecidable. Note that equations (4) have a solution in Ω_0 if and only if they have a solution in Ω_0^+ where $\Omega_0^+ = \{\sum_{i=0}^{\infty} \alpha_i x^{\beta_i} \in \Omega_0 \mid \beta_0 \geq 0\}$. Therefore the above problem is also undecidable for Ω_0^+ , that is, even if we restrict solutions to have only nonnegative powers of x .

The system (4) can also be used to show that a Strong Approximation Theorem (cf. [DL84, p. 224]) does not hold in Ω_0 . If $P(y_1, y_1', \dots, y_m, y_m', \dots)$ is a differential polynomial with coefficients in Ω_0 , $\bar{y}_1, \dots, \bar{y}_m \in \Omega_0$, and n is an integer, we say $P(\bar{y}_1, \dots, \bar{y}_m, \dots) = 0 \pmod{x^n}$ if the lowest power of x occurring in $P(\bar{y}_1, \dots, \bar{y}_m, \dots)$ has exponent $\geq n$. We say in this case that $\bar{y}_1, \dots, \bar{y}_m$ is a solution of $P = 0 \pmod{x^n}$. The Strong Approximation Theorem of Denef and Lipschitz states that if, for each positive integer n , a system of differential polynomial equations with coefficients in $\mathbb{C}[[x]]$ has a solution $\pmod{x^n}$ in $\mathbb{C}[[x]]$, then the system has a solution in $\mathbb{C}[[x]]$. This is not true for Ω_0 . The equations $xy' - (\alpha + x)y - 1 = 0$, $\alpha' = 0$ have solutions

$$y_\alpha = x^\alpha e^x \int x^{-\alpha-1} e^{-x} dx + cx^\alpha e^x,$$

where c is an arbitrary constant (cf. [DL84, p. 224]). If α is not in $\mathbb{N}$, then $y \in \Omega_0$. If α is in $\mathbb{N}$, then this differential equation does not have a solution in Ω_0 (look at the x^{-1} term in $x^{-\alpha-1} e^{-x}$; it yields $\log(x)$ when integrated). On the other hand, given n we can find a large integer α such that the first α terms of y_α is a solution $\pmod{x^n}$. Therefore the system

$$\begin{aligned} \alpha' &= 0, & u'x &= \alpha u, & \alpha\beta &= 1, \\ z'ux + z''x^2 &= u + x, & xy' - (\beta + x)y - 1 &= 0 \end{aligned}$$

has no solution in Ω_0 but does have a solution $\pmod{x^n}$ for all positive n .

Added in proof. In Lemma 4 of *Rational approximation to solutions of algebraic differential equations*, Proc. Amer. Math. Soc. **10** (1959), pp. 238–244, the author, E. R. Kolchin, considers simple zeroes of differential polynomials in valued differential fields (a zero u of P is simple if $\partial P / \partial Y^{(i)}(u) \neq 0$ for some i). He shows that there is an element γ_0 of the value group such that $|v - u| \geq \gamma_0$ for every v in the field which is a zero of P different from u . In particular, this implies that generalized series that are simple zeroes of polynomial differential equations are finitely determined. This latter paper uses valuation theoretic techniques but no Newton polygon process.

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