TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 327, Number 2, October 1991

ON SUBORDINATED HOLOMORPHIC SEMIGROUPS

ALFRED S. CARASSO AND TOSIO KATO

ABSTRACT. If $[e^{-tA}]$ is a uniformly bounded C_0 semigroup on a complex Banach space X, then $-A^{\alpha}$, $0 < \alpha < 1$, generates a holomorphic semigroup on X, and $[e^{-tA^{\alpha}}]$ is *subordinated* to $[e^{-tA}]$ through the Lévy stable density function. This was proved by Yosida in 1960, by suitably deforming the contour in an inverse Laplace transform representation. Using other methods, we exhibit a large class of probability measures such that the subordinated semigroups are always holomorphic, and obtain a necessary condition on the measure's Laplace transform for that to be the case. We then construct probability measures that do not have this property.

1. INTRODUCTION

Let X be a complex Banach space, and let $C_0(X)$ be the class of uniformly bounded C_0 semigroups $[T(t)]$, $t \geq 0$, on X. For fixed α, $0 < \alpha < 1$, let $[p_u^{\alpha}(t)]$ be the family of functions implicitly defined as follows in Laplace transform space:

$$(1) \qquad \mathscr{L}\{p_u^{\alpha}(t)\} \equiv \int_0^{\infty} p_u^{\alpha}(t) e^{-uz}\, du = e^{-tz^{\alpha}}, \qquad \operatorname{Re} z > 0.$$

The principal branch of z^{α} is understood in (1). For each fixed $t > 0$, $p_u^{\alpha}(t)$ is a Lévy 'stable' probability density function on $u \geq 0$. Given $[T(u)] \in C_0(X)$, one may use (1) to construct a new semigroup $[U(t)] \in C_0(X)$, by means of

$$(2) \qquad U(0) = I, \qquad U(t)x = \int_0^{\infty} p_u^{\alpha}(t) T(u) x\, du, \qquad t > 0,\ x \in X.$$

We express this symbolically by $U(t) = \langle p^{\alpha}(t), T \rangle$, where, for fixed t, $p^{\alpha}(t)$ is the probability distribution with density $p_u^{\alpha}(t)$. We write $T(t) = e^{-tA}$, where $-A$ is the infinitesimal generator of $[T(t)]$. Whenever multivalued functions $\psi(z)$ appear, the particular branch where $\operatorname{Re} \psi(z) > 0$ for $\operatorname{Re} z > 0$, is understood.

The above is an example of a *subordinated* semigroup: $[U(t)]$ is said to be subordinated to $[T(t)]$ through the directing process $[p^{\alpha}(t)]$. See e.g. Feller,

Received by the editors September 12, 1989.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 47D05, 60E07, 60J25.
Key words and phrases. Subordinated semigroups, holomorphic extensions.
The first author's work was supported in part by the Office of Naval Research under contract number N00014-89-F-0013; the second by NSF Grant DMS-8701724.

[5, pp. 345–349]. The concept originated with Bochner, [3, 4], who used (1) and (2) to construct A^α, $0 < \alpha < 1$. Subsequently, Phillips, [10], Nelson, [7], and Balakrishnan, [1], considered arbitrary *infinitely divisible* probability distributions on $u \geq 0$, and developed a functional calculus for semigroup generators. Alternative methods of constructing fractional powers of operators, independent of subordination, were later devised by several authors, spawning a large literature; see Pazy, [9, p. 257]. Returning to (2), Yosida, [13–15], drew attention to the fact that in that case the semigroup $[U(t)]$ is *holomorphic*, and that (1) and (2) together provide a method of constructing a large subclass of holomorphic semigroups within the class C_0. However, no other examples of families $[p(t)]$ leading to subordinated holomorphic semigroups seem generally known.

In this paper, we exhibit a rich variety of semigroups $[p(t)]$ of probability measures, such that $[U(t)] = [\langle p(t), T\rangle]$ is holomorphic whenever $[T(t)] \in C_0(X)$, and we obtain a necessary condition on $\mathscr{L}\{p(t)\}$ in order that this be the case. We also construct families $[p(t)]$ that do not have this property.

2. Semigroups of probability measures

This section summarizes known results; see Phillips, [10], Hille and Phillips, [6, pp. 660–663], and Feller, [5]. Let $B(X)$ be the Banach algebra of bounded linear operators on X. Let S be the Banach algebra of complex Borel measures μ on $\mathbf{R}^+ \equiv \{u \geq 0\}$, with convolution as multiplication, and normed by the total variation. If V is a Borel set $\subset \mathbf{R}^+$, $\mu(V)$ denotes the value of μ on V, while $\int_V g(u)\mu(du)$ is the integral with respect to μ of the Borel measurable function g. Let L be the Banach space of Borel measurable functions f on $\mathbf{R}^+$ such that

$$\|f\|_L = \int_{\mathbf{R}^+} |f(u)|\, du < \infty. \tag{3}$$

For each $\mu \in S$, define $Z_\mu \in B(L)$ by

$$Z_\mu f = (\mu * f)(\tau) \equiv \int_{\mathbf{R}^+} f(\tau - u)\mu(du), \qquad f \in L,\ \tau \geq 0. \tag{4}$$

Then, the map $\mu \mapsto Z_\mu$ is an isometric isomorphism of S into $B(L)$. Let $[T(u)] \in C_0(X)$. For each $\mu \in S$, define

$$\langle \mu, T\rangle = \int_{\mathbf{R}^+} T(u)\mu(du). \tag{5}$$

Then, $\mu \mapsto \langle \mu, T\rangle$, is a continuous homomorphism of S into $B(X)$. In particular, $\langle \mu * \nu, T\rangle = \langle \mu, T\rangle\langle \nu, T\rangle$.

For each $x \geq 0$, let δ_x denote the Dirac measure at x, i.e., $\delta_x(V) = 1$ if $x \in V$, $\delta_x(V) = 0$ if $x \notin V$. Let P be the set of all algebraic semigroups $[p(t)]$, $t \geq 0$, of probability measures on $\mathbf{R}^+$. Thus, for fixed t, $p(t) \in S$, $p(t) \geq 0$, $\|p(t)\|_S = 1$, $p(t) * p(s) = p(t+s)$, $s, t \geq 0$, and $p(0) = \delta_0$. If $[p(t)] \in P$, then $[Z_p(t)] \equiv [Z_{p(t)}]$ forms an algebraic contraction semigroup on L, and for $[T(u)] \in C_0(X)$, $[U(t)] = [\langle p(t), T\rangle]$ is a uniformly bounded algebraic semigroup on X. $[U(t)]$ is subordinated to $[T(t)]$.

Definition 1. *$\mathscr{I}$ is the set of all $[p(t)] \in P$ such that, given an arbitrary complex Banach space X, $[\langle p(t), T\rangle] \in C_0(X)$ whenever $[T(t)] \in C_0(X)$.*

Theorem 1. *Let $[p(t)] \in P$. The following statements are equivalent:*

(a) $[p(t)] \in \mathscr{I}$.

(b) $[Z_p(t)] \in C_0(L)$.

(c) *For every $x > 0$, $p(t)(V_x) \to 1$ as $t \downarrow 0$, where $V_x \equiv \{0 \le u \le x\}$.*

For fixed $t \ge 0$, define the Laplace transform of $p(t) \in S$ by

$$\mathscr{L}\{p(t)\} = \int_{\mathbf{R}^+} e^{-uz} p(t)(du), \qquad \operatorname{Re} z > 0. \tag{6}$$

Theorem 2. *The following statements are equivalent:*

(a) $[p(t)] \in \mathscr{I}$.

(b) $\mathscr{L}\{p(t)\} = e^{-t\psi(z)}$, $t \ge 0$, *where $\psi(z)$ is holomorphic for* $\operatorname{Re} z > 0$ *and continuous for* $\operatorname{Re} z \ge 0$, *with* $\operatorname{Re} \psi(z) \ge 0$. *Moreover, $\psi(0) = 0$, and $\psi'(x)$ is completely monotone for $x > 0$.*

When $[p(t)] \in \mathscr{I}$, the function $\psi(z)$ is called the *exponent of* $[p(t)]$. An equivalent characterization of $\psi(z)$ is the following: *There exists a positive measure ρ on $\mathbf{R}^+$, finite or infinite, such that $\int_{u>1} u^{-1}\rho(du) < \infty$, and*

$$\psi(z) = \int_{\mathbf{R}^+} (1 - e^{-uz}) u^{-1} \rho(du), \qquad \operatorname{Re} z \ge 0. \tag{7}$$

A few objects $\in \mathscr{I}$ are known explicitly as functions of u for all $t \ge 0$. In the following examples, $p_u(t)$ denotes the density of the probability distribution $p(t)$ on $\mathbf{R}^+$.

Degenerate.

$$p(t) = \delta_t, \qquad \mathscr{L}\{p(t)\} = e^{-tz}, \qquad t > 0. \tag{8}$$

Inverse Gaussian. This is the special case $\alpha = 1/2$ in (1).

$$p_u(t) = \frac{te^{-t^2/4u}}{\sqrt{4\pi u^3}}, \qquad \mathscr{L}\{p(t)\} = e^{-t\sqrt{z}}, \qquad t > 0. \tag{9}$$

Gamma. With fixed $b > 0$,

$$p_u(t) = \frac{b^t u^{t-1} e^{-bu}}{\Gamma(t)}, \qquad \mathscr{L}\{p(t)\} = b^t (z+b)^{-t}, \qquad t > 0. \tag{10}$$

Negative binomial. This is a discrete family consisting of a weighted sum of Dirac measures. With fixed $0 < b < 1$ and $a = 1 - b$,

$$p(t) = b^t \sum_{j=0}^{\infty} \binom{-t}{j} (-a)^j \delta_j, \qquad \mathscr{L}\{p(t)\} = b^t (1 - ae^{-z})^{-t}, \qquad t > 0. \tag{11}$$

Poisson. This is also a discrete family. With fixed $c > 0$

$$(12) \qquad p(t) = e^{-ct}\sum_{j=0}^{\infty}\frac{(ct)^j}{j!}\delta_j, \qquad \mathscr{L}\{p(t)\} = e^{ct(e^{-z}-1)}, \qquad t > 0.$$

Compound Poisson. Let q be an arbitrary probability measure on $\mathbf{R}^+$, and let $Q(z) = \mathscr{L}\{q\}$. With $\{q\}^{*0} \equiv \delta_0$ and fixed $c > 0$,

$$(13) \qquad p(t) = e^{-ct}\sum_{j=0}^{\infty}\frac{(ct)^j}{j!}\{q\}^{*j}, \qquad \mathscr{L}\{p(t)\} = e^{ct(Q(z)-1)}, \qquad t > 0.$$

This construction includes many explicitly known semigroups $\in \mathscr{I}$ as special cases. Thus, (12) corresponds to the choice $q = \delta_1$. Similarly, (11) is a special case of (13) with $c = -\log b > 0$, and

$$(14) \qquad cQ(z) = -\log(1 - ae^{-z}) = \sum_{j=1}^{\infty}\frac{a^j e^{-jz}}{j}, \qquad \operatorname{Re} z \geq 0,$$

so that $cq = \sum_{j=1}^{\infty} a^j\delta_j/j$. As another example, let q have the density $q_u = be^{-bu}$, $b > 0$. Then

$$(15) \qquad \{q_u\}^{*n} = \frac{b^n u^{n-1} e^{-bu}}{\Gamma(n)}, \qquad n \geq 1,$$

and $p(t)$ can be expressed in terms of the modified Bessel function I_1. With $c = 1$, $p(t) = e^{-t}\delta_0 + r(t)$, where $r(t)$ has the density

$$(16) \qquad r_u(t) = e^{-t}(bt/u)^{1/2}e^{-bu}I_1(2\sqrt{btu}), \qquad t > 0,$$

and

$$(17) \qquad \mathscr{L}\{p(t)\} = e^{-t}e^{bt(z+b)^{-1}}, \qquad t > 0.$$

3. Holomorphic semigroups

We consider bounded holomorphic semigroups $[S(t)]$ on X, for which t can assume complex values in a sector

$$(18) \qquad \Sigma_\omega = \{t \in \mathbf{C}\colon \operatorname{Re} t > 0, |\operatorname{Arg}(t)| < \omega\}, \qquad 0 < \omega \leq \pi/2,$$

with ω fixed. The family $[S(t)]$ is assumed to satisfy the following:

(a) $S(t)$ is a holomorphic function of $t \in \Sigma_\omega$.

(b) $S(t_1)S(t_2) = S(t_1 + t_2)$, $t_1, t_2 \in \Sigma_\omega$.

(c) If $0 < \varepsilon < \omega$, then $\|S(t)\|_X \leq M_\varepsilon < \infty$, for $t \in \Sigma_{\omega-\varepsilon}$.

(d) $S(0) = I$, and, within any sector $\Sigma_{\omega-\varepsilon}$ with $0 < \varepsilon < \omega$, $S(t)$ is strongly continuous at $t = 0$.

The following result, due to Yosida, [12], tells us when a given semigroup $[U(t)] \in C_0(X)$, defined on $t \geq 0$, can be extended to a bounded holomorphic semigroup $[S(t)]$ in some sector Σ_ω. Note that (20) below together with $\|U(t)\|_X \leq M < \infty$, imply

$$(19) \qquad \sup_{t>0}\{t\|(e^{-\beta t}U(t))'\|_X\} \leq C_\beta < \infty,$$

for any $\beta > 0$.

Theorem 3. *Let* $[U(t)]$, $t \geq 0$, $\in C_0(X)$ *with infinitesimal generator* $-A$. *Let* $U(t)X \subset D(A)$ *for all* $t > 0$, *and let*

$$\limsup_{t\downarrow 0}\{t\|AU(t)\|_X\} < \infty. \tag{20}$$

Then, for any $\beta > 0$, $[e^{-\beta t}U(t)]$ *can be extended to a bounded holomorphic semigroup* $[S(t)]$ *in some sector* Σ_ω.

4. Subordination and the class $\mathscr{H}$

Definition 2. *For any complex Banach space* X, $H(X) \subset C_0(X)$ *is the class of semigroups on* X *satisfying the hypotheses of Theorem* 3; $G(X) \subset H(X)$ *is the class of semigroups with bounded generators;* $\mathscr{H}$ *[resp.* $\mathscr{G}$*] is the set of all* $[p(t)] \in \mathscr{I}$ *such that for every* X, $[\langle p(t), T\rangle] \in H(X)$ *[resp.* $G(X)$*] whenever* $[T(t)] \in C_0(X)$.

We have $\mathscr{G} \subset \mathscr{H} \subset \mathscr{I}$. The degenerate family (8) is evidently $\notin \mathscr{H}$, while the inverse Gaussian (9), and all other one-sided Lévy families (1), belong to $\mathscr{H}$ as shown by Yosida, [13].

Theorem 4. *Let* $[p(t)] \in \mathscr{I}$. *The following conditions are equivalent:*

(a) $[p(t)] \in \mathscr{H}$.

(b) $[Z_p(t)] \in H(L)$.

(c) $p(t)$ *is continuously differentiable* $\in S$ *for* $t > 0$, *with* $\|p'(t)\|_S = O(t^{-1})$ *as* $t \downarrow 0$.

Moreover, $[p(t)] \in \mathscr{H}$ *only if* $\psi(z)$ *maps* $\operatorname{Re} z > 0$ *into a truncated sector of opening* $< \pi$, *and there exist constants* $K > 0$, *and* γ, $0 < \gamma < 1$, *such that*

$$|\psi(z)| \leq K|z|^\gamma, \qquad |z| \geq 1, \qquad \operatorname{Re} z \geq 0. \tag{21}$$

Proof. (a) $\Rightarrow$ (b). Let $[p(t)] \in \mathscr{H}$. Using (4) and (5), $Z_p(t) = \langle p(t), T\rangle$, where $[T(u)]$ is the semigroup of right translations on L. Hence, $[Z_p(t)] \in H(L)$.

(b) $\Rightarrow$ (c). Since $[Z_p(t)]$ satisfies the hypotheses of Theorem 3 on L, the $B(L)$ limit as $h \to 0$ of $h^{-1}\{Z_p(t+h) - Z_p(t)\}$ exists for each fixed $t > 0$. By the isometric isomorphism $p(t) \mapsto Z_p(t)$, $h^{-1}\{p(t+h) - p(t)\}$ has a corresponding S limit $p'(t)$, and $\|p'(t)\|_S = \|Z_p'(t)\|_L$, $t > 0$. Therefore, from (20)

$$\limsup_{t\downarrow 0}\{t\|p'(t)\|_S\} < \infty. \tag{22}$$

If $p'(t) \in S$ for each $t > 0$, the same is true of $p'(t/2) * p'(t/2)$. By considering $\mathscr{L}\{p(t)\} = e^{-t\psi(z)}$, it follows that $p'(t/2) * p'(t/2) = p''(t)$, so that $\|p''(t)\|_S \leq \|p'(t/2)\|_S^2$. In particular, $\|p'(t)\|_S$ is a bounded continuous function of t on any interval $0 < t_0 \leq t \leq t_1 < \infty$.

(c) $\Rightarrow$ (a) Given any $[T(u)] \in C_0(X)$, differentiation with respect to t under the integral sign is justified in $\langle p(t), T\rangle$. Hence, $U'(t) = \langle p'(t), T\rangle$, $t > 0$, and

$$\|U'(t)\|_X \leq \text{const.}\, \|p'(t)\|_S, \qquad t > 0. \tag{23}$$

From (23) and (22), it follows that $\{t\|U'(t)\|_X\}$ remains bounded as $t \downarrow 0$, so that $[p(t)] \in \mathscr{H}$. This proves the first part of Theorem 4.

The second part is proved in two steps. First, a function-theoretic argument is used to obtain (21) for z on the positive real axis. Next, the representation (7) is used to extend the estimate to the right half-plane. Fix any $\beta > 0$. Since $[Z_p(t)] \in H(L)$, $[e^{-\beta t}Z_p(t)]$ can be continued analytically in t, in a sector $\Sigma_t \equiv \{\operatorname{Re} t > 0, |\operatorname{Arg}(t)| \leq \omega/2 < \pi/2\}$, with $e^{-\beta t}\|Z_p(t)\|_L$ bounded in Σ_t. In fact, with C_β the constant in (19)

$$\begin{aligned} \|(e^{-\beta t}Z_p(t))^{(n)}\|_L &\leq \|(e^{-\beta(t/n)}Z_p(t/n))'\|_L^n \leq (nt^{-1}C_\beta)^n \\ &\leq n!(et^{-1}C_\beta)^n, \qquad t > 0, n \geq 1. \end{aligned} \tag{24}$$

Fix ω with $0 < \omega < 2\tan^{-1}\{1/(eC_\beta)\}$. Then, for $\operatorname{Re} t > 0$, $|\operatorname{Arg}(t)| \leq (\omega/2)$, the Taylor series

$$e^{-\beta t}Z_p(t) = e^{-\beta \operatorname{Re} t}Z_p(\operatorname{Re} t) + \sum_{n=1}^{\infty}(n!)^{-1}(t - \operatorname{Re} t)^n(e^{-\beta \operatorname{Re} t}Z_p(\operatorname{Re} t))^{(n)}, \tag{25}$$

converges uniformly in $B(L)$. Using the isometric isomorphism of S into $B(L)$, it follows that $p(t)$ is holomorphic $\in S$ for $t \in \Sigma_t$, and

$$\|e^{-\beta t}p(t)\|_S \leq (1 - eC_\beta(\operatorname{Re} t)^{-1}|t - \operatorname{Re} t|)^{-1} \leq K_\beta < \infty, \qquad t \in \Sigma_t. \tag{26}$$

Taking the Laplace transform of $p(t)$, we get

$$|e^{-t(\beta+\psi(z))}| \leq \|e^{-\beta t}p(t)\|_S \leq K_\beta, \qquad \operatorname{Re} z \geq 0, t \in \Sigma_t. \tag{27}$$

Let $\xi(z) = \beta + \psi(z)$. It follows from (27) that ξ maps the half-plane $\Pi \equiv \operatorname{Re} z > 0$, into the sector $\{\Sigma_z \equiv |\operatorname{Arg}(z)| \leq (\pi - \omega)/2\}$. From Theorem 2, $\psi(z)$ is holomorphic in Π with $\psi(1) \geq 0$. Hence

$$f(z) \equiv z^{\omega/\pi}\xi(z)/\xi(1), \tag{28}$$

maps Π conformally into itself with $f(1) = 1$. Put

$$z = \frac{1+w}{1-w}, \qquad h(w) = f\left(\frac{1+w}{1-w}\right), \qquad g(w) = \frac{h(w)-1}{h(w)+1}. \tag{29}$$

Then, $h(w)$ maps the unit disc into Π with $h(0) = 1$, and $g(w)$ maps the unit disc into itself with $g(0) = 0$. From the Schwarz Lemma applied to $g(w)$, we get

$$|f(z)| = |h(w)| \leq \frac{1+|w|}{1-|w|} = \frac{|z+1| + |z-1|}{|z+1| - |z-1|}. \tag{30}$$

Hence, for real $x \geq 1$, $0 < f(x) \leq x$. Therefore, from (28)

$$0 \leq \psi(x) \leq Ax^\gamma, \qquad x \geq 1; \qquad A = \psi(1) + \beta, \qquad \gamma = (\pi - \omega)/\pi. \tag{31}$$

We now use the representation (7) to obtain a similar estimate valid in the half-plane $\operatorname{Re} z \geq 0$. The following elementary estimates will be needed:

$$|1 - e^{-z}| \leq \operatorname{Min}\{|z|, 2\}, \qquad \operatorname{Re} z \geq 0; \tag{32}$$

(33) $$1-e^{-x} \geq \sigma, \qquad x \geq 1, \qquad 1-e^{-x} \geq \sigma x, \qquad 0 \leq x \leq 1,$$

where $\sigma = 1 - e^{-1}$. From (7) and (31), we have for $x \geq 1$,

(34) $$\begin{aligned} Ax^{\gamma} &\geq \left(\int_0^{1/x} + \int_{1/x}^{\infty}\right)(1-e^{-ux})u^{-1}\rho(du) \\ &\geq \sigma x \int_0^{1/x} \rho(du) + \sigma \int_{1/x}^{\infty} u^{-1}\rho(du). \end{aligned}$$

Therefore, with $\varepsilon = 1/x \leq 1$,

(35) $$\int_0^{\varepsilon} \rho(du) \leq \sigma^{-1}A\varepsilon^{1-\gamma}, \qquad \int_{\varepsilon}^{\infty} u^{-1}\rho(du) \leq \sigma^{-1}A\varepsilon^{-\gamma}.$$

If $\operatorname{Re} z \geq 0$, we obtain using (32)

(36) $$\begin{aligned} |\psi(z)| &\leq |z| \int_0^{\varepsilon} \rho(du) + 2\int_{\varepsilon}^{\infty} u^{-1}\rho(du) \\ &\leq \sigma^{-1}A(|z|\varepsilon^{1-\gamma} + 2\varepsilon^{-\gamma}), \end{aligned}$$

if $\varepsilon \leq 1$, on using (35). Setting $\varepsilon = 1/|z|$, $|z| \geq 1$, we get

(37) $$|\psi(z)| \leq 3\sigma^{-1}A|z|^{\gamma}, \qquad |z| \geq 1, \qquad \operatorname{Re} z \geq 0.$$

This concludes the proof of Theorem 4.

Remark. The restriction $|z| \geq 1$ in (21) is natural: if $\psi(z) = z^{1/3}$ for example, the estimate $|\psi(z)| \leq K|z|^{1/2}$ is not valid as $z \to 0$.

Theorem 5. *Let* $[p(t)] \in \mathscr{I}$. *The following statements are equivalent*:

(a) $[p(t)] \in \mathscr{G}$.
(b) $[Z_p(t)] \in G(L)$.
(c) $p(t)$ *is continuously differentiable* $\in S$ *for* $t > 0$, *with* $\|p'(t)\|_S = O(1)$ *as* $t \downarrow 0$.
(d) $\psi(z)$ *is bounded on* $\operatorname{Re} z \geq 0$.
(e) $\psi(x)$ *is bounded on* $x \geq 0$.
(f) $[p(t)]$ *is a Compound Poisson family.*

Proof. (a) $\Rightarrow$ (b) $\Rightarrow$ (c). The argument is the same as that in the first part of Theorem 4, using $\|Z_p'(t)\|_L = O(1)$ as $t \downarrow 0$.

(c) $\Rightarrow$ (d) $\Rightarrow$ (e). For sufficiently small $t > 0$, we have, on differentiating under the integral sign in $e^{-t\psi(z)} = \mathscr{L}\{p(t)\}$,

(38) $$|\psi(z)e^{-t\psi(z)}| \leq \|p'(t)\|_S < K < \infty, \qquad \operatorname{Re} z \geq 0.$$

Hence, $|\psi(z)| < K < \infty$, $\operatorname{Re} z \geq 0$.

(e) $\Rightarrow$ (f). Let $0 \leq \psi(x) < K$ on $x \geq 0$. Since $[p(t)] \in \mathscr{I}$, we know from Theorem 2 that $\psi(0) = 0$ and $\psi'(x)$ is completely monotone for $x > 0$. Define

(39) $$Q(z) = 1 - \psi(z)/K, \qquad \operatorname{Re} z \geq 0.$$

Then, $Q(0) = 1$, and $Q(x)$ is completely monotone for $x > 0$. It follows from Bernstein's theorem, (Feller, [5, p. 439]), that $Q(z)$ is the Laplace transform of some probability measure q on $\mathbf{R}^+$. Since $\psi(z) = K(1 - Q(z))$, $K > 0$, $p(t)$ has the form (13).

(f) $\Rightarrow$ (a). Let $p(t)$ have the form (13) and let $U(t) = \langle p(t), T\rangle$ for given $[T(t)] \in C_0(X)$. Using the continuous homomorphism $q \mapsto \langle q, T\rangle$ of S into $B(X)$, we get

$$U(t) = e^{-ct} \sum_{n=0}^{\infty} \frac{(ct)^n}{n!} \langle q, T\rangle^n, \qquad t > 0. \tag{40}$$

Thus, $[U(t)]$ has the bounded operator $c\{\langle q, T\rangle - I\}$ as its infinitesimal generator, and $[p(t)] \in \mathscr{G}$. This concludes the proof of Theorem 5.

Theorem 6. *If* $[p(t)], [q(t)] \in \mathscr{I}$ *[resp.* $\mathscr{H}$, $\mathscr{G}$*], then* $[p(t) * q(t)] \in \mathscr{I}$ *[resp.* $\mathscr{H}$, $\mathscr{G}$*].*

Proof. That $\mathscr{I}$ is closed under convolution is immediate from Theorem 2. Using $(p(t) * q(t))' = p'(t) * q(t) + p(t) * q'(t)$, together with statement (c) in Theorem 4 [resp. Theorem 5], it follows that $\mathscr{H}$ [resp. $\mathscr{G}$] is closed under convolution.

5. Applications

Example 1. *Gamma families* $\in \mathscr{H}$.

With fixed $b > 0$, let

$$p_u(t) = \frac{b^t u^{t-1} e^{-bu}}{\Gamma(t)}, \qquad t > 0. \tag{41}$$

Then

$$(\partial/\partial t) p_u(t) = \left\{ \log b + \log u - \frac{\Gamma'(t)}{\Gamma(t)} \right\} p_u(t), \qquad t > 0. \tag{42}$$

For $0 < u < 1$, write

$$(\log u) p_u(t) = \{2 b^{t/2} \Gamma(1 + t/2) u^{t/2} (\log u) p_u(t/2)\} \{\Gamma(1+t)\}^{-1}, \tag{43}$$

and for $u \geq 1$, write

$$(\log u) p_u(t) = \{2^t (e^{-bu/2} \log u)(b/2)^t u^{t-1} e^{-bu/2}\} \{\Gamma(t)\}^{-1}. \tag{44}$$

Let

$$K_1 = \max_{0 \le v \le 1} \{v |\log v|\}, \qquad K_2 = \sup_{u \ge 1} \{e^{-bu/2} \log u\}. \tag{45}$$

From (42)–(45), we have

$$\|p'(t)\|_S \le 2^t K_2 + |\log b| + \frac{|\Gamma'(t)|}{\Gamma(t)} + \frac{4 K_1 b^{t/2} \Gamma(1 + t/2)}{t \Gamma(1+t)}, \qquad t > 0. \tag{46}$$

The only singularity in $\Gamma'(t)/\Gamma(t)$, $t \geq 0$, is a simple pole at $t = 0$; see Olver, [8, p. 39]. Also, $p''(t) = p'(t/2) * p'(t/2)$. Thus, $p(t)$ is continuously differentiable $\in S$ for $t > 0$, and $\{t\|p'(t)\|_S\}$ remains bounded as $t \downarrow 0$. By Theorem 4, $[p(t)] \in \mathscr{H}$.

Example 2 (Corollary). *If* $[T(t)] = [e^{-tA}] \in C_0(X)$, *then* $-\mathrm{Log}(A+I)$, *where*

$$\{\mathrm{Log}(A+I)\}x = \int_1^\infty s^{-1}(A+sI)^{-1}Ax\,ds, \qquad x \in D(A), \tag{47}$$

is the infinitesimal generator of $[S(t)] = [(A+I)^{-t}] \in H(X)$.

Let $[p(t)] \in \mathscr{I}$ have the exponent $\psi(z)$, and let ρ be the measure on $\mathbf{R}^+$ in (7). Let $U(t) = \langle p(t), T\rangle$. A formula for the generator of $[U(t)]$ is known, which generalizes Theorem 2 and the representation (7); see Phillips, [10], Nelson, [7], and Feller, [5, p. 458]. We have

$$\begin{aligned} &[U(t)] = [e^{-t\psi(A)}], \\ &\psi(A)x = \int_{\mathbf{R}^+} u^{-1}(I - e^{-uA})x\rho(du), \qquad x \in D(A). \end{aligned} \tag{48}$$

In fact, $\psi(A)$ is the closure of its restriction to $D(A)$. Choosing $[p(t)]$ to be the Gamma family (41) with $b = 1$, we have $\psi(z) = \mathrm{Log}(1+z)$, $\rho(du) = e^{-u}du$, and

$$\begin{aligned} \{\mathrm{Log}(A+I)\}x &= \int_{\mathbf{R}^+} (I - e^{-uA})x \left\{\int_1^\infty e^{-us}\,ds\right\} du \\ &= \int_1^\infty s^{-1}(A+sI)^{-1}Ax\,ds, \qquad x \in D(A). \end{aligned} \tag{49}$$

The result follows on viewing $[S(t)]$ as being subordinated to $[T(t)]$ through the Gamma family.

The Hausdorff-Young theorem on Fourier transforms may be combined with Theorem 4 to obtain another proof of the fact that the one-sided Lévy stable families $\in \mathscr{H}$:

Example 3. *Fix* α *with* $0 < \alpha < 1$, *and let* $[p^\alpha(t)]$ *have the exponent* $\psi(z) = z^\alpha$. *Then,* $[p^\alpha(t)] \in \mathscr{H}$.

From Theorem 2, $[p^\alpha(t)] \in \mathscr{I}$ for $0 < \alpha < 1$. We verify statement (c) of Theorem 4 for $(d/dt)p^\alpha(t)$. Let $b = \cos(\alpha\pi/2) > 0$, and let C be a generic positive constant. Let $q_{t,\alpha}(u) \equiv (\partial/\partial t)p_u^\alpha(t)$, $u \geq 0$. For real y, $Q_{t,\alpha}(y) \equiv -(iy)^\alpha e^{-t(iy)^\alpha}$ and $(\partial/\partial y)Q_{t,\alpha}(y)$ are, respectively, the Fourier transforms of the densities $q_{t,\alpha}(u)$ and $g_{t,\alpha}(u) \equiv iuq_{t,\alpha}(u)$. Moreover

$$|Q_{t,\alpha}(y)| = |y|^\alpha e^{-bt|y|^\alpha}, \tag{50}$$

$$|(\partial/\partial y)Q_{t,\alpha}(y)| \leq \alpha|y|^{\alpha-1}(1+t|y|^\alpha)e^{-bt|y|^\alpha}. \tag{51}$$

Let $1 < r < \min\{2, (1-\alpha)^{-1}\}$, let $s = r/(r-1)$, and let $\| \|_r$ denote the $L^r(-\infty, \infty)$ norm. The change of variables $v = ty^\alpha$ shows that

$$t\|Q_{t,\alpha}\|_r \le Ct^{-1/\alpha r}, \qquad t\|(\partial/\partial y)Q_{t,\alpha}\|_r \le Ct^{1/\alpha s}. \tag{52}$$

Using the Hausdorff-Young inequality, (Rudin, [11, p. 247]), we obtain

$$t\|q_{t,\alpha}\|_s \le Ct^{-1/\alpha r}, \qquad t\|g_{t,\alpha}\|_s \le Ct^{1/\alpha s}. \tag{53}$$

Next, for any $v > 0$, Hölder's inequality gives

$$\int_0^v |q_{t,\alpha}(u)|\,du \le \|q_{t,\alpha}\|_s v^{1/r}, \tag{54}$$

$$\int_v^\infty u|q_{t,\alpha}(u)|(1/u)\,du \le \|g_{t,\alpha}\|_s v^{-1/s}. \tag{55}$$

Therefore, on choosing $v = t^{1/\alpha}$, it follows from (53), (54), and (55), that $q_{t,\alpha}(u) \in L^1(\mathbf{R}^+)$ for each $t > 0$, and $\|q_{t,\alpha}\|_1 \equiv \|(d/dt)p^\alpha(t)\|_S = O(t^{-1})$ as $t \downarrow 0$. Hence, $[p^\alpha(t)] \in \mathscr{H}$.

Evidently, a large number of objects $\in \mathscr{H}$ can be created by convolutions. Further objects $\in \mathscr{H}$ may be generated by means of the following construction:

Example 4. *Subordination of convolution semigroups.*

Let $[p_1(t)] \in \mathscr{H}$, $[p_2(t)] \in \mathscr{I}$, and consider the convolution semigroup $[Z_{p_2}(t)] \in C_0(L)$. Let $U(t) = \langle p_1(t), Z_{p_2}\rangle$. Then, $[U(t)] \in H(L)$ is the convolution semigroup $[Z_{p_3}(t)]$, where

$$p_3(t) = \int_{\mathbf{R}^+} p_2(u)p_1(t)(du), \qquad t > 0. \tag{56}$$

Using Laplace transforms, it is easily seen that if $\psi_j(z)$ is the exponent $[p_j(t)]$, $j = 1, 2$, then $[p_3(t)]$ has the exponent $\psi_3(z) = \psi_1(\psi_2(z))$. Since $\psi_3(x)$ vanishes at 0 and has a completely monotone derivative on $x > 0$, it follows from Theorem 2 that $[p_3(t)] \in \mathscr{I}$. Statement (b) of Theorem 4 shows that $[p_3(t)] \in \mathscr{H}$.

We now construct two distinct classes of objects $\in \mathscr{I}\backslash\mathscr{H}$.

Example 5. *If* $[p(t)] \in \mathscr{I}$, *then* $[q(t)] = [p(t) * \delta_t] \in \mathscr{I}\backslash\mathscr{H}$.

If $[p(t)]$ has the exponent $\psi(z)$, $[q(t)]$ has the exponent $z + \psi(z)$. From (21) in Theorem 4, $[q(t)] \notin \mathscr{H}$.

Example 6. *Let* $\{\beta_n\}_{n=0}^\infty$ *and* $\{a_n\}_{n=0}^\infty$ *be any two sequences satisfying*

$$0 < \beta_n < 1; \qquad 0 < a_n; \qquad \lim_{n\to\infty} \beta_n = 1; \qquad \sum_{n=0}^\infty a_n < \infty; \tag{57}$$

and define

$$\psi(z) = \sum_{n=0}^\infty a_n z^{\beta_n}, \qquad \operatorname{Re} z > 0. \tag{58}$$

Then, $\psi(z)$ is the exponent of some $[p(t)] \in \mathcal{I}\backslash\mathcal{H}$. The same is true for $\psi(\varphi(z))$, whenever $\varphi(z)$ is a function of the form (7) *that does not satisfy* (21).

The infinite series of holomorphic functions (58) converges uniformly on compact subsets of the half-plane $\operatorname{Re} z > 0$, to a holomorphic $\psi(z)$ with $\psi(0) = 0$. In particular, termwise differentiation is permissible. It follows that $\psi(x)$ has a completely monotone derivative on $x > 0$. By Theorem 2, $[p(t)] \in \mathcal{I}$. Since each $a_n > 0$, $\psi(z)$ cannot satisfy (21) on $x > 0$, and $[p(t)] \notin \mathcal{H}$. Similarly, $\psi(\varphi(z))$ is the exponent of some $[r(t)] \in \mathcal{I}$, since it vanishes at zero, and has a completely monotone derivative for $x > 0$. That $[r(t)] \notin \mathcal{H}$ may be seen by examining the series:

$$\psi(\varphi(x)) = \sum_{n=0}^{\infty} a_n \{\varphi(x)\}^{\beta_n}, \qquad x > 0, \tag{59}$$
$$> a_m \{\varphi(x)\}^{\beta_m}, \qquad m > 0.$$

Fix any γ with $0 < \gamma < 1$, fix $m > 0$ such that $\gamma < \beta_m < 1$, and let $\alpha_m = (\gamma/\beta_m) < 1$. Then, $x^{-\gamma}\psi(\varphi(x)) > a_m\{x^{-\alpha_m}\varphi(x)\}^{\beta_m}$, and cannot remain bounded as $x \to \infty$.

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CENTER FOR COMPUTING AND APPLIED MATHEMATICS, NATIONAL INSTITUTE OF STANDARDS AND TECHNOLOGY, GAITHERSBURG, MARYLAND 20899

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA AT BERKELEY, BERKELEY, CALIFORNIA 94720