AREA INTEGRAL ESTIMATES FOR THE BIHARMONIC OPERATOR IN LIPSCHITZ DOMAINS

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ABSTRACT. Let $D \subseteq \mathbf{R}^n$ be a Lipschitz domain and let u be a function biharmonic in D, i.e., $\Delta\Delta u = 0$ in D. We prove that the nontangential maximal function and the square function of the gradient of u have equivalent $L^p(d\mu)$ norms, where $d\mu \in A^\infty(d\sigma)$ and $d\sigma$ is surface measure on ∂D.

INTRODUCTION

In this paper we prove area integral and nontangential maximal function inequalities for functions biharmonic in a bounded Lipschitz domain D in $\mathbf{R}^n$. Specifically, if u satisfies $\Delta^2 u = 0$ in $D \subseteq \mathbf{R}^n$, then these functionals applied to ∇u (the gradient of u) have comparable $L^p(d\mu)$ norm $(0 < p < \infty)$, where $d\mu$ is any measure satisfying an A^∞ condition with respect to surface measure on the boundary of D. In the classical setting of functions harmonic in a halfspace or disc, such inequalities have been shown by Stein [15], Burkholder and Gundy [1], Fefferman and Stein [10], and others. For functions harmonic in a Lipschitz domain B. E. J. Dahlberg applied his results [4] comparing harmonic measure and surface measure to extend these inequalities to nonsmooth domains. And area integral estimates have been obtained in Lipschitz domains for solutions of more general second order elliptic equations in Dahlberg, Jerison and Kenig [6].

Our interest in the case of biharmonic functions has several sources. First, recent research on the Dirichlet problem for Δ^2 [2, 8, 13, and 18] has yielded much new information about the behavior of solutions, behavior which turns out to be remarkably different from that of solutions to second order equations. (Some of these results will be described in greater detail in §1.) Moreover, area integral estimates can be used to give information about Sobolev space regularity of solutions and also to give Fatou theorems (see [1]). Second, although our method of proof relies on now standard strategies such as good-λ inequalities, it has several new features. In general, solutions of the bi-Laplacian on nonsmooth domains will not satisfy a maximal principle or Harnack principle, and there

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is no positive "harmonic" measure associated with the operator. Finally, the area integral results, together with an argument of Dahlberg and Kenig [7], lead to another proof of solvability of the regularity problem for Δ^2 in $L^p(\mathbf{R}^3)$, $1 < p < 2$. In Pipher and Verchota [13], it was shown that the Dirichlet problem for Δ^2 in L^p $(p > 2)$ could be reduced to solvability of the regularity problem in the dual range of p. This regularity problem was then shown to have L^p solutions for all $1 < p < 2$ only in $\mathbf{R}^3$ and not in $\mathbf{R}^n$, $n \geq 4$ (see also [14]). Thus the inequalities presented here would give another proof of solvability of the Dirichlet problem for Δ^2 in $\mathbf{R}^3$.

As far as we know, there is no literature on theorems relating square functions and nontangential maximal functions of solutions to higher order equations. However, C. Kenig has informed us that the techniques in Dahlberg and Kenig [7] for solutions to systems of equations should also give the area integral estimates we obtain here. Their idea is to build from the case of small Lipschitz constant via an argument like that of G. David [9]. On the other hand, our method exploits the equation that the biharmonic functions satisfies in order to integrate by parts. Then we are able to use the relationship between surface measure and harmonic measure for the Laplacian much as Dahlberg does for harmonic functions in [5].

The paper is organized as follows. In §1 we state the main theorem, survey the known results on the biharmonic equation in Lipschitz domains and discuss the notation to be used throughout. In §§2 and 3 we prove the good-λ inequalities which establish the theorem.

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1. Definitions and notation

In what follows, D will always denote a bounded Lipschitz domain in $\mathbf{R}^n$, a domain whose boundary can be covered by finitely many right cylinders $Z(Q)$ centered at $Q \in \partial D$ such that there exists a rectangular coordinate system and a Lipschitz function $\varphi \colon \mathbf{R}^{n-1} \to \mathbf{R}$ with $Z \cap D = \{(x_1, \ldots, x_n) \colon x_n > \varphi(x_1, \ldots, x_{n-1})\} \cap D$. The pair (Z, φ) is called a coordinate pair. Then to each point $Q \in \partial D$ there is associated an open cone $\Gamma(Q) \subseteq D$ with vertex at Q such that for each $X \in \Gamma(Q)$ the distance of X to the boundary of D, dist$(X, \partial D)$, is comparable to $|X - Q|$. Given a function $v(X)$ defined on D the nontangential maximal function of v is defined by

$$Nv(Q) = \sup_{X \in \Gamma(Q)} |v(X)|.$$

If $v \in C^2(D)$, the area integral or square function of v at $Q \in \partial D$ is given by

$$Sv(Q) = \left\{ \int_{\Gamma(Q)} |\nabla v(X)|^2 d(X)^{2-n} \, dX \right\}^{1/2}$$

where $d(X)$ abbreviates dist$(X, \partial D)$.

A function f defined on ∂D is said to be in $L_1^p(\partial D)$ if for each coordinate pair (Z, φ) there are L^p functions $\{g_i\}$ such that

$$\int_{\mathbf{R}^{n-1}} h(x) g_i(x, \varphi(x)) \, dx = \int_{\mathbf{R}^{n-1}} \frac{\partial h}{\partial x_i}(x) f(x, \varphi(x)) \, dx$$

for all $h \in C_0^\infty(\mathbf{R}^{n-1} \cap Z)$. That is, if $f \in L_1^p(\partial D)$, a vector field, $\nabla_T f$, called the tangential gradient of f, can be defined and realized in local coordinates as

$$(g_1(x, \varphi(x)), \ldots, g_{n-1}(x, \varphi(x)), 0)$$
$$- \langle (g_1(x, \varphi(x)), \ldots, g_{n-1}(x, \varphi(x)), 0), N_{(x, \varphi(x))} \rangle$$

where $N_{(x, \varphi(x))}$ (or N_Q) denotes the unit normal vector to ∂D at $(x, \varphi(x))$ (or Q).

The study of the biharmonic equation on Lipschitz domains was begun in Dahlberg, Kenig and Verchota [8] where the following was shown.

Theorem 1.1 [8]. *Let* $f \in L_1^2(\partial D)$ *and* $g \in L^2(\partial D)$ *where* D *is a bounded Lipschitz domain in* $\mathbf{R}^n$. *Then there exists a unique function* u *in* D *satisfying* (a)–(d):

(a) $\Delta^2 u = 0$ *in* D,
(b) $\lim_{X \to \Gamma; X \in \Gamma(Q)} u(X) = f(Q)$ *a.e.* $(d\sigma)$,
(c) $\lim_{X \to Q; X \in \Gamma(Q)} \nabla u(X) \cdot N_Q = g(Q)$ *a.e.* $d(\sigma)$,
(d) $\|N(\nabla u)\|_{L^2(\partial D, d\sigma)} < \infty$ *and moreover,*

$$\|N(\nabla u)\|_{L^2(\partial D)} + \|N(u)\|_{L^2(\partial D)} \leq C\{\|f\|_{L_1^2(\partial D)} + \|g\|_{L^2(\partial D)}\}$$

where C *is a constant that depends only on the Lipschitz character of* D.

Before describing some details of the proof of this result which we will use later, we state our main theorem. In what follows, $|\nabla\nabla u|^2$ denotes

$$\sum_{j,k} |D_j D_k u|^2.$$

Theorem 1.2. *Let* $D \subseteq \mathbf{R}^n$ *be a Lipschitz domain and let* $d\sigma$ *be surface measure on* ∂D. *Fix* $P_0 \in D$ *and let* $\{\Gamma(Q)\}$ *be a regular family of cones for* $Q \in \partial D$. *If* u *is biharmonic in* D *and* $\nabla u(P_0) = 0$ *then there exist constants* C_1, C_2 *such that*

$$\|N(\nabla u)\|_{L^p(\partial D, d\sigma)} \leq C_1 \|S(\nabla u)\|_{L^p(\partial D, d\sigma)} \leq C_2 \|N(\nabla u)\|_{L^p(\partial D, d\sigma)}$$

where C_1 *and* C_2 *depend on* P_0, *on the aperture of the cones* $\{\Gamma(Q)\}$ *and on the Lipschitz character of* D.

Theorem 1.2 is a corollary of a series of good-λ inequalities, hence the measure $d\sigma$ above may be replaced by any measure $d\mu$ which is A^∞ with respect to $d\sigma$ (see [3] and [5]). These inequalities are proved by adapting the proof of Dahlberg's theorem for harmonic functions. Modification of this approach

in the biharmonic situation is made possible in virtue of three facts: the existence of the L^2 theory for Δ^2, a representation formula for solutions of Δ^2 in D and the L^2 theory for both the Dirichlet and Neumann problems for the Laplacian.

We now briefly describe some of the aspects of L^2 theory for the biharmonic equation. The L^2 solvability of the Dirichlet problem [8] was obtained by first decomposing a biharmonic function u as $u = \tilde{u} + H$ where H is the *harmonic* extension of $u|_{\partial D}$ to D and $\tilde{u}$ is biharmonic and vanishes on the boundary of D. If $D \subseteq \mathbf{R}^n$ is assumed to be starlike with starcenter at the origin, an important representation of $\tilde{u}$ in terms of a Green's potential was found. Namely, the results of [8] give

(1.3)

    (i) $\tilde{u}(X) = \int_{Y \in D} G(X, Y)\{nv(Y) + 2Y \cdot \nabla v(Y)\}\, dY$ where $G(X, \cdot)$ is the Green's function for Δ in D, n is the dimension, and v is a *harmonic* function in D.

    (ii) There exist constants C_1, C_2 depending on the domain such that

$$\|N(\nabla \tilde{u})\|_{L^2(\partial D, d\sigma)} \leq C_1 \|v\|_{L^2(\partial D, d\sigma)} \quad \text{and} \quad \|v\|_{L^2(\partial D)} \leq C_2 \left\| \frac{\partial \tilde{u}}{\partial N} \right\|_{L^2(\partial D)}.$$

Observe that (i) says that there exists a harmonic function v such that $\Delta \tilde{u} = nv(Y) + 2Y \cdot \nabla v(Y)$ and (ii) says that the operator T defined by

$$T : v|_{\partial D} \mapsto \lim_{\substack{X \to Q \\ X \in \Gamma(Q)}} \left\langle N_Q, \nabla \int_D G(X, Y)\{nv(Y) + 2Y \cdot \nabla v(Y)\}\, dY \right\rangle$$

is *invertible* as a map from $L^2(d\sigma)$ into $L^2(d\sigma)$.

Counterexamples to the analogue of Theorem 1.1 with Dirichlet data in L^p, $p < 2$, were provided in [8] but the question of the L^2 theory for $p > 2$ remained open. In Pipher and Verchota [7] the Dirichlet problem with L^p data ($p > 2$) was solved in $\mathbf{R}^3$ and shown to be unsolvable for some Lipschitz domains in $\mathbf{R}^n$, $n \geq 4$. Because the operator T defined above is not an $L^p \to L^p$ mapping when $p > 2$, a different approach was required. This approach involved a reduction to the regularity problem for Δ^2 on D. As mentioned, a method of Dahlberg and Kenig [4] developed for systems of equations in $\mathbf{R}^3$ can be utilized to solve this problem once Theorem 1.2 is established.

In what follows, $G_\Omega(X, \cdot)$ will always denote the Green's function for the Laplacian in a domain Ω with pole at X. The harmonic measure for Ω evaluated at X is denoted dw_Ω^X. From Dahlberg's theorem [4] it is known that dw_Ω^X and $d\sigma$ are related by the following reverse Hölder inequality. For any surface ball Δ,

$$(1.4) \qquad \left\{ \int_\Delta k_X^2(Q) \frac{d\sigma(Q)}{\sigma(\Delta)} \right\}^{1/2} \lesssim \int_\Delta k_X(Q) \frac{d\sigma(Q)}{\sigma(\Delta)}$$

where $dw^X(Q) = k_X(Q)d\sigma(Q)$. For properties of harmonic measure see [4] and [12]. A ball in $\mathbf{R}^n$ centered at P with radius r is abbreviated $B(P, r)$ and $\Delta(P, r)$ denotes the surface ball $\partial D \cap B(P, r)$ for $P \in \partial D$.

2. THE NONTANGENTIAL MAXIMAL FUNCTION
DOMINATES THE SQUARE FUNCTION

In this section we show that

$$\|S(\nabla u)\|_{L^p(\partial D, d\sigma)} \leq C\|N(\nabla u)\|_{L^p(\partial D, d\sigma)}$$

where

$$S^2(\nabla u)(Q) = \int_{\Gamma(Q)} d(X)^{2-n}|\nabla\nabla u(X)|^2 \, dX.$$

The proof has three steps, each of which involves establishing a good-λ inequality and invoking standard arguments (see Burkholder and Gundy [1]) to derive the desired L^p inequalities. Let $\{\Gamma(Q)\}_{Q \in \partial D}$ be a regular family of cones [5] and let $\{\overline{\Gamma}(Q)\}$ be another regular family of cones with $\Gamma(Q) \subseteq \overline{\Gamma(Q)}$. In the lemmas below, the square functions will be defined in terms of integration over the cones $\{\Gamma(Q)\}$ and the nontangential maximal functions will be defined in terms of the larger ones $\{\overline{\Gamma}(Q)\}$. There are purely geometric arguments that allow one to pass from inequalities involving one family of cones to those for another family of cones which will not be discussed here. We begin by defining an intermediate square function $\widetilde{S}(u)(Q)$ by setting

$$\widetilde{S}^2(u)(Q) = \int_{\Gamma(Q)} d(X)^{2-n}|\Delta u(X)|^2 \, dX$$

for u biharmonic in D. The first lemma says that $\|N(\nabla u)\|_p$ dominates $\|\widetilde{S}(u)\|_p$.

Lemma 2.1. *If u satisfies $\Delta^2 u = 0$ in D, a Lipschitz domain contained in $\mathbf{R}^n$, then for sufficiently small γ there exists a positive constant C such that*

$$\sigma\{Q \in \partial D : \widetilde{S}(u)(Q) > 2\lambda, N(\nabla u)(Q) \leq \gamma\lambda\} \leq C\gamma^2\sigma\{Q \in \partial D : \widetilde{S}(u)(Q) > \lambda\}.$$

Proof. Let $\{\Delta_j(Q_j, r_j)\}$ be a Whitney decomposition of $\{\widetilde{S}(u) > \lambda\}$. That is, $\{Q_j\}$ is a sequence of points in $\{\widetilde{S}(u) > \lambda\}$ such that

(a) $\{\widetilde{S}(u) > \lambda\} = \bigcup \Delta(Q_j, r_j)$ where $\Delta(Q_j, r_j) = B(Q_j, r_j) \cap \partial D$.

(b) No point belongs to more than $C(n)$ of the balls $B(Q_j, r_j)$, where $C(n)$ is a constant depending only on the dimension n and the Lipschitz character of D.

(c) Each $B_j = B(Q_j, r_j)$ has the property that $\partial D \cap B_j$ is given by the graph of a Lipschitz function.

(d) There is a number $r_0 = r_0(D)$ such that if $r_j \leq r_0$, then there is a point Q_j^* in $B(Q_j, 2r_j) \cap \partial D$ such that $\widetilde{S}(u)(Q_j^*) \leq \lambda$.

As in Dahlberg, Jerison and Kenig [6], a sawtooth region Ω_j associated to $E_j = \Delta_j \cap \{\widetilde{S}(u) > 2\lambda, N(\nabla u) \leq \gamma\lambda\}$ may be constructed so that

(i) $\bigcup_{Q \in E_j}(\Gamma(Q) \cap B(Q, c_1 r_j)) \subseteq \Omega_j \subseteq \bigcup_{Q \in E_j}(\overline{\Gamma}(Q) \cap B(Q, c_2 r_j))$.

(ii) $\partial\Omega_j \cap \partial D = E_j$.

(iii) Ω_j is a starlike Lipschitz domain with Lipschitz constant a multiple of the Lipschitz constant of D.

(iv) $\operatorname{diam}(\Omega_j) \approx r_j$.

Let us fix now a particular Δ_j with $r_j \leq r_0$. Using interior estimates and property (d) above it is readily shown that given any $\tau > 0$, γ may be chosen sufficiently small so that $\tilde{S}_{\tau r_j}(u)(Q) > \lambda/2$ for $Q \in E_j$ where $\tilde{S}_{\tau r_j}(u)(Q)$ is defined by integration over the truncated cone $\Gamma_{\tau r_j}(Q) = \Gamma(Q) \cap B(Q, \tau r_j)$. That is, the main contribution in the square function takes place near the boundary. For details see [5]. Dropping the j subscripts, we wish to show that $\sigma(E) \leq C\gamma^2\sigma(\Delta)$ and the conclusion of Lemma 2.1 will follow by summing over each E_j. This inequality is shown under the assumption that $|\nabla u|^2 \leq \gamma\lambda$ in Ω and that $\tilde{S}^2_{\tau r}(u) > \lambda/2$ on E. By rescaling about the starcenter of Ω and making a change of variable we may then assume that $\sigma(\Delta) = 1$. Hence all constants in the inequalities below are absolute and depend only on the Lipschitz character of D. Let u be the harmonic function in the sawtooth region Ω given by (1.3) so that $\Delta u(Y) = nv(Y) + 2Y \cdot \nabla u(Y)$ (where we now call the starcenter of Ω the origin). Then

$$\sigma(E) \lesssim \frac{1}{\lambda^2} \int_E \int_{\Gamma_\tau(Q)} d(X)^{2-n} |\Delta u(x)|^2 \, dX \, d\sigma(Q)$$

$$\lesssim \frac{1}{\lambda^2} \left\{ \int_E \int_{\Gamma_\tau(Q)} d(X)^{2-n} |v(X)|^2 \, dX \right\}$$

$$+ \left\{ \frac{1}{\lambda^2} \int_E \int_{\Gamma_{tr}(Q)} d(X)^{2-n} |X \cdot \nabla v|^2 \, dX \right\} d\sigma(Q)$$

$$= \mathrm{I} + \mathrm{II}.$$

Let S_Ω denote the square function defined with respect to a family of cones for the new domain Ω. By the L^2 theory for harmonic functions we have

$$\mathrm{II} \leq \frac{1}{\lambda^2} \int_{\partial\Omega} S^2_\Omega(v)(Q) \, d\sigma(Q)$$

$$\lesssim \frac{1}{\lambda^2} \int_{\partial\Omega} v^2(Q) \, d\sigma(Q).$$

Using the information on v given in the estimates of (1.3)(ii), the above is bounded by

$$\frac{1}{\lambda^2} \int_{\partial\Omega} \left| \frac{\partial\tilde{u}}{\partial N} \right|^2 d\sigma$$

where $u = \tilde{u} + H$ in Ω; $H|_{\partial\Omega} = u|_{\partial\Omega}$. Now

$$\int_{\partial\Omega} \left| \frac{\partial\tilde{u}}{\partial N} \right|^2 \lesssim \int_{\partial\Omega} \left| \frac{\partial u}{\partial N} \right|^2 + \left| \frac{\partial H}{\partial N} \right|^2 d\sigma$$

and $|\nabla u|^2 \leq \gamma^2 \lambda^2$ in Ω. From the L^2 theory for the Neumann problem (see Jerison and Kenig [11] and also Verchota [17]) we see that

$$\int_{\partial\Omega} \left|\frac{\partial H}{\partial n}\right|^2 d\sigma \lesssim \int_{\partial\Omega} |\nabla_T H|^2 d\sigma = \int_{\partial\Omega} |\nabla_T u|^2 d\sigma$$

and hence

$$\mathrm{II} \lesssim \gamma^2 \sigma(\partial\Omega) \approx \gamma^2.$$

It remains to estimate the first lower term.

$$\mathrm{I} = \frac{1}{\lambda^2} \int_E \int_{\Gamma_{\tau r}(Q)} d(X)^{2-n} |v(X)|^2 \, dX \, d\sigma \leq \int_\Omega d(X) |v(X)|^2 \, dX.$$

Fix some small $\delta > 0$ and let $\Omega_\delta = \{X \in \Omega : \mathrm{dist}(X, \partial\Omega) > \delta\}$. We integrate separately over Ω_δ and $\Omega \setminus \Omega_\delta$.

$$\int_{\Omega\setminus\Omega_\delta} d(X) |v(X)|^2 \, dX \leq \delta^2 \int_{\partial\Omega} N^2 v(Q) \, d\sigma(Q) \lesssim C\gamma^2\lambda^2 \sigma(\Delta)$$

with the last inequality following just as in the previous estimate. For the integration over Ω_δ, recall that the representation of Δu in Ω gives

$$|v(X)| \leq C(|\nabla v(X)| + |\Delta u(X)|).$$

For $X \in \Omega_\delta$, interior estimates give $|\Delta u(X)| \lesssim \frac{\gamma\lambda}{\delta}$. Therefore

$$\int_{\Omega_\delta} d(X) |v(X)|^2 \lesssim \frac{\gamma^2\lambda^2}{\delta^2} |\Omega| + \int_{\Omega_\delta} d(X) |\nabla v(X)|^2 \, dX$$

$$\lesssim C_D \left\{ \gamma^2\lambda^2 + \int_{\partial\Omega_\delta} S_\Omega^2 v(Q) \, d\sigma(Q) \right\}$$

$$\lesssim \gamma^2\lambda^2 + \int_{\partial\Omega} N^2(v) \, d\sigma(Q)$$

$$\lesssim \gamma^2\lambda^2.$$

This proves the lemma.

Corollary 2.2. *If u is biharmonic in $D \subseteq \mathbf{R}^n$, then for any measure $d\mu \in A^\infty(d\sigma)$ and any $p > 0$, there exists a constant $C > 0$ depending only on p, the domain D and the A^∞ constant for $d\mu$ such that*

$$\|\widetilde{S}(u)\|_{L^p(\partial D, d\mu)} \leq C \|N(\nabla u)\|_{L^p(\partial D, d\mu)}.$$

Proof. Standard.

Lemma 2.3. *Suppose u is biharmonic in D and vanishes on ∂D. Then for sufficiently small $\gamma > 0$ there exist constants $C, \eta > 0$ depending only on D and γ such that*

$$\sigma\{S(\nabla u) > 2\lambda, N(\nabla u) \leq \gamma\lambda\} \leq C\gamma^\eta \sigma\{S(\nabla u) > \lambda\}.$$

Before giving the proof of Lemma 2.3 we formulate and prove a corollary which gives one of the two main inequalities we seek relating the area integral and nontangential maximal function.

Corollary 2.4. *If u is biharmonic in D and $\gamma > 0$ is sufficiently small then there are constants C, $\eta > 0$ depending only on D, γ such that*

$$\sigma\{S(\nabla u) > 2\lambda, N(\nabla u) \le \gamma\lambda\} \le C\gamma^\eta \sigma\{S(\nabla u) > \lambda\}.$$

Proof of corollary. As before let Δ be one of the surface balls in the Whitney decomposition of $\{S(\nabla u) > \lambda\}$ and let Ω be the sawtooth region associated to $\Delta \cap \{S(\nabla u) > 2\lambda, N(\nabla u) \le \gamma\lambda\}$. Within Ω, we decompose u as $u = \tilde{u} + H$, where $\Delta^2 \tilde{u} = 0$ with $\tilde{u}|_{\partial\Omega} = 0$ and where H is harmonic with $H|_{\partial\Omega} = u|_{\partial\Omega}$. It suffices to prove that

$$(2.5) \qquad \frac{1}{\lambda^2} \int_{\partial\Omega} S_\Omega^2(\nabla u)\, d\sigma(Q) \le \gamma^2 \sigma(\Delta).$$

But, invoking Lemma 2.3 for $\tilde{u}$, we have

$$\int_{\partial\Omega} S_\Omega^2(\nabla u)\, d\sigma \lesssim \int_{\partial\Omega} S_\Omega^2(\nabla \tilde{u}) + S_\Omega^2(\nabla H)\, d\sigma$$

$$\lesssim \int_{\partial\Omega} N^2(\nabla \tilde{u}) + N^2(\nabla H)\, d\sigma$$

$$\lesssim \int_{\partial\Omega} N^2(\nabla u) + N^2(\nabla H)\, d\sigma$$

$$\lesssim \gamma^2 \lambda^2 \sigma(\Delta) + \int_{\partial\Omega} |\nabla_T H|^2\, d\sigma$$

which is bounded by $\gamma^2 \lambda^2 \sigma(\Delta)$ since $|\nabla_T H| = |\nabla_T u|$ on $\partial\Omega$. □

Proof of Lemma 2.3. Fix Δ, one of the Whitney balls of $\{S(\nabla u) > \lambda\}$ and let $E = \Delta \cap \{S(\nabla u) > 2\lambda, N(\nabla u) \le \gamma\lambda\}$. As in the proof of Lemma 2.1, given $\tau > 0$ we may choose $\delta > 0$ so that $S_{\tau r}(\nabla u)(Q) > \lambda/2$ for $Q \in E$ if $r = \operatorname{rad}(\Delta)$. We will obtain the estimate on $\sigma(E)$ by showing, as Dahlberg [5] does in the harmonic case, that $\omega_\Omega(E) \le \gamma^2$. The harmonic measure $d\omega_\Omega$ is evaluated at some fixed point $P^* \in \Omega$ and from $\omega_\Omega(E) \le \gamma^2$ it follows by the A^∞ relation that

$$\frac{\sigma(E)}{\sigma(\Delta)} \le C\left[\frac{\omega_\Omega(E)}{\omega_\Omega(\Delta)}\right]^\theta \quad \text{for some } \theta$$

and the lemma will be proven.

   Now

$$(2.6) \quad \omega_\Omega(E) \lesssim \frac{1}{\lambda^2} \int_E \int_{\Gamma_{\tau r}(Q)} d(X)^{2-n} |\nabla\nabla u(X)|^2\, dX\, d\omega_\Omega(Q)$$

$$\lesssim \frac{1}{\lambda^2}\left\{\int_\Omega G(P^*, X)|\nabla\nabla u(X)|^2\, dX + \sup_{X \in K \subset\subset \Omega} |\nabla\nabla u(X)|^2\right\}$$

since $G(P^*, X)$ (or just $G(X)$) is comparable to $\omega_\Omega(\Delta(X^*, d(X)))/d(X)^{n-2}$ if X is near $\partial\Omega$ and X^* is the radial projection of X onto $\partial\Omega$. Because $|\nabla\nabla u(X)|^2 \le C_\Omega \gamma^2 \lambda^2$ when $X \in K \subset\subset \Omega$ we have only to bound the integral.

The following identity will allow us to use Green's theorem and is satisfied by any biharmonic function.

$$|\nabla\nabla u|^2 = \Delta|\nabla u|^2 + \tfrac{1}{2}|\Delta u|^2 - \tfrac{1}{4}\Delta\Delta u^2 .$$

After substituting this into (2.6), there are three integrals to estimate. In the first we use the estimate on $G(X)$:

(i) $\dfrac{1}{\lambda^2}\displaystyle\int_\Omega G(X)|\Delta u(X)|^2\, dX$

$$\lesssim \frac{1}{\lambda^2}\int_{P\in\partial\Omega}\int_{\Gamma_\Omega(P)} d(X)^{2-n}|\Delta u(X)|^2\, dX\, d\omega_\Omega(P) + \frac{1}{\lambda^2}|\Delta u(P^*)|^2$$

$$\lesssim \frac{1}{\lambda^2}\int_{\partial\Omega} \widetilde{S}_\Omega^2(u)\, d\omega_\Omega(P) + \gamma^2 .$$

Let $K_\Omega(P)\, d\omega_\Omega(P) = d\sigma_\Omega(P)$ and use Corollary 2.2 with respect to the measure $d\omega_\Omega$ to obtain

$$\frac{1}{\lambda^2}\int_{\partial\Omega} \widetilde{S}_\Omega^2(u)\, d\omega_\Omega \lesssim \frac{1}{\lambda^2}\int_{\partial\Omega} N^2(\nabla u)\, d\omega_\Omega \lesssim \gamma^2 ,$$

(ii) $\left|\displaystyle\int_\Omega G(X)\Delta|\nabla u|^2\, dX\right| \lesssim \left|\displaystyle\int_{\partial\Omega} |\nabla u(P)|^2 - |\nabla u(P^*)|^2\, d\omega_\Omega(P)\right| \leq \gamma^2\lambda^2 ,$

(iii) $\left|\displaystyle\int_\Omega G(X)\Delta\Delta u^2(X)\, dX\right| \lesssim \left|\displaystyle\int_{\partial\Omega} \Delta u^2(P) - \Delta u^2(P^*)\, d\omega_\Omega\right|$

and

$$\left|\int_{\partial\Omega} \Delta u^2(P)\, d\omega_\Omega(P)\right| \leq \left|\int_{\partial\Omega} 2u(P)\Delta u(P)\, d\omega_\Omega(P)\right| + \int_{\partial\Omega} |\nabla u(P)|^2\, d\omega_\Omega(P) .$$

For this term we use the fact that $u \equiv 0$ on $\partial\Omega \cap \partial D$. By expressing $u(P)$ in terms of an integral of its derivatives and using interior estimates which give $d(P)|\Delta u(P)| \leq \sup_{X\in B(P,d(P))} |\nabla u(X)|$, the expression $|u(P)\Delta u(P)|$ is bounded by $\gamma^2\lambda^2$. This gives the desired bound for $\omega_\Omega(E)$.

3. THE SQUARE FUNCTION DOMINATES
THE NONTANGENTIAL MAXIMAL FUNCTION

In this last section we show that $\|N(\nabla u)\|_{L^p(\partial D, d\sigma)}$ is dominated by $\|S(\nabla u)\|_{L^p(\partial D, d\sigma)}$ finishing the proof of Theorem 1.2. We now assume that the square function is defined with respect to $\overline{\Gamma}(Q)$ and the nontangential maximal function is defined with respect to the smaller cone $\gamma(Q)$. We will use the following abbreviation: if $E \subseteq \partial D$ is any set, then

$$E_\delta^* = \left\{ Q \in \partial D : \sup_{\Delta\ni Q} \frac{\sigma(\Delta\cap E)}{\sigma(\Delta)} \leq \delta \right\} .$$

Fix a point $P_0 \in D$.

Lemma 3.1. *If* $\Delta^2 u = 0$ *in* D *and* $\nabla u(P_0) = 0$, *then, for sufficiently small* γ *and* δ, *there is a constant* $C > 0$ *such that*

$$\sigma(\{N(\nabla u) > 4\lambda,\, S(\nabla u) \le \gamma\lambda\} \cap \{S(\nabla u) > \gamma\lambda\}_\delta^*) \le C\gamma^2 \sigma\{N(\nabla u) > \lambda\}.$$

Remark. The assumption that $\nabla u(P_0) = 0$ implies that if $\|S(\nabla u)\|_{L^p(d\sigma)} = L < \infty$, then for any compact subset K of D,

$$(3.2) \qquad \sup_{X \in K}\{|\nabla u(x)| + |\nabla\nabla u(X)|\} \le C(K, L).$$

The argument is standard by interior estimates and the Agmon-Miranda maximum principle for functions biharmonic on a sufficiently smooth domain. The estimate (3.2) is invoked to handle the "large" balls in the Whitney decomposition of $\{N(\nabla u) > \lambda\}$ below. See Dahlberg [5] for details.

Proof of Lemma 3.1. Fix Δ, one of the surface balls in the Whitney decomposition of $\{N(\nabla u) > \lambda\}$, with radius $r \le r_0$. Let

$$E = \Delta \cap \{N(\nabla u) > 4\lambda,\, S(\nabla u) \le \gamma\lambda\} \cap \{S(\nabla u) > \gamma\lambda\}_\delta^*.$$

By choosing γ sufficiently small we can ensure that $N_{\tau r}(\nabla u)(Q) > 2\lambda$ when $Q \in E$, where

$$N_{\tau r}(\nabla u)(Q) = \sup_{X \in \Gamma(Q) \cap B(Q, \tau r)} |\nabla u(X)|.$$

As in the proof of Lemma 2.1, we may rescale and change variables to assume that $\sigma(\Delta) = 1$. Also, if δ is small enough, then there is a $C > 0$ such that

$$(3.3) \qquad \int_\Omega d(X)|\nabla\nabla u(X)|^2\, dX \le C\gamma^2\lambda^2$$

and

$$\int_{\partial\Omega} S^2(\nabla H)\, d\sigma \lesssim \int_{\partial\Omega} S^2(\nabla u)\, d\sigma + \int_{\partial\Omega} S^2(\nabla\tilde{u})\, d\sigma.$$

Now Corollary 2.4 says that $\|N(\nabla\tilde{u})\|_{L(\partial\Omega)}$ dominates $\|S(\nabla\tilde{u})\|_{L^2(\partial\Omega)}$. Moreover, interchanging the order of integration in the first term on the right above gives

$$\int_{\partial\Omega} S^2(\nabla u)\, d\sigma \lesssim \int_\Omega \operatorname{dist}(X, \partial\Omega)|\nabla\nabla u(X)|^2\, dX.$$

Since $\operatorname{dist}(X, \partial\Omega)$ is bounded above by $d(X)$, the distance of X to ∂D, the estimate in (3.3) shows that $\int_{\partial\Omega} S^2(\nabla u)\, d\sigma \lesssim \gamma^2\lambda^2$. Hence altogether

$$\int_{\partial\Omega} N_\Omega^2(\nabla u)\, d\sigma \lesssim |\nabla H(P^*)|^2 \sigma(\Delta) + \gamma^2\lambda^2 + \int_{\partial\Omega} N^2(\nabla\tilde{u})\, d\sigma.$$

Therefore, to complete the proof we need the following two estimates:

$$(3.4) \qquad |\nabla H(P^*)|^2 \le C\gamma^2\lambda^2$$

and

$$(3.5) \qquad \int_{\partial\Omega} N^2(\nabla\tilde{u})\, d\sigma \le C\gamma^2\lambda^2.$$

We begin with (3.4). Clearly $|\nabla H(P^*)| \leq |\nabla u(P^*)| + |\nabla \tilde{u}(P^*)|$ and $|\nabla \tilde{u}(P^*)|$ can be estimated using the representation formula for $\tilde{u}(X)$ in Ω. Because

$$\tilde{u}(X) = \int_{Y \in \Omega} G(X, Y) \Delta u(Y) \, dY,$$

we see that

$$|\nabla \tilde{u}(X)| \lesssim \int_{Y \in \Omega} |\nabla_X G(X, Y)| \, |\Delta u(Y)| \, dY.$$

Choose ε small and let $\Omega_\varepsilon = \{Y \in \Omega : \mathrm{dist}(Y, \partial\Omega) > \varepsilon\}$. Recall that $G(X, Y)$ is the Green's function for Δ in Ω and that $\mathrm{dist}(P^*, \partial\Omega) \approx r = \mathrm{rad}(\Delta)$. Then, (dropping from now on the "$\sim$" on $\tilde{u}$) we have

$$|\nabla \tilde{u}(P^*)| \leq \int_{Y \in \Omega \setminus \Omega_\varepsilon} |\nabla_X G(\cdot, Y)(P^*)| \, |\Delta u(Y)| \, dY$$

$$+ \int_{\Omega_\varepsilon} |\nabla_X G(\cdot, Y)(P^*)| \, |\Delta u(Y)| \, dY.$$

The bound in (3.3) follows from the fact that

$$\gamma^2 \lambda^2 \sigma(E) \geq \int_E S^2(\nabla u)(Q) \, d\sigma(Q)$$

$$\geq \int_\Omega d(X)^{2-n} |\nabla \nabla u(X)|^2 \sigma\{Q \in E : X \in \Gamma(Q)\} \, dX$$

and from the assumption that, for $Q \in E$,

$$\sup_{\Delta' \ni Q} \frac{\sigma(\Delta' \cap \{S(\nabla u) > \gamma\lambda\})}{\sigma(\Delta')} \lesssim \delta,$$

that is, for any Δ' containing Q,

$$\sigma(\Delta' \cap \{S(\nabla u) \leq \gamma\lambda\}) > (1 - \delta)\sigma(\Delta').$$

Again we refer to [5] for details. We wish to show the estimate: $\sigma(E) \leq C\gamma^2$.

We first claim that ∇u may be assumed to vanish at P^*, the starcenter of Ω. For consider $u^*(X) \equiv u(X) - X \cdot \nabla u(P^*)$, biharmonic in Ω with $\Delta u^* = \Delta u$. Then $\nabla u^*(x) = \nabla u(X) - \nabla u(P^*)$ and $S(\nabla u^*)(Q) = S(\nabla u)(Q)$ for all Q. Let P be a point in Ω such that $|\nabla u(P)| \leq \lambda$. The existence of P is guaranteed by the maximality of the surface ball Δ in the set $\{N(\nabla u) > \lambda\}$ and the construction of Ω. Then if L is a line segment joining P and P^*,

$$|\nabla u(P^*)| \leq |\nabla u(P)| + |P - P^*| \sup_{X \in L} |\nabla \nabla u(X)|$$

follows by integration. And so $|\nabla u(P^*)| \leq \lambda + C\gamma\lambda$, which is bounded by 2λ for γ sufficiently small. Therefore $N(\nabla u^*)(Q) > 2\lambda$ when $Q \in E$ since $N(\nabla u)(Q) > 4\lambda$ and it would suffice to do all estimates with u^* replacing u.

We proceed to estimate $\sigma(E)$ under the assumption that $\nabla u(P^*) = 0$.

$$\sigma(E) \lesssim \frac{1}{\lambda^2} \int_E N_\tau^2(\nabla u)(Q) \, d\sigma(Q) \leq \frac{1}{\lambda^2} \int_{\partial\Omega} N_\Omega^2(\nabla u)(Q) \, d\sigma(Q)$$

where N_Ω denotes the nontangential maximal function for the Lipschitz domain Ω. We now decompose u into $u = \tilde{u} + H$ with $\Delta^2 \tilde{u} = 0$ in Ω, $\tilde{u}|_{\partial\Omega} = 0$, $\Delta H = 0$ and $H|_{\partial\Omega} = u|_{\partial\Omega}$. Then using the L^2 theory for ∇H, we have

$$\int_{\partial\Omega} N_\Omega^2(\nabla u)\, d\sigma \lesssim \int_{\partial\Omega} N^2(\nabla \tilde{u})\, d\sigma + \int_{\partial\Omega} N^2(\nabla H)\, d\sigma$$

$$\lesssim \int_{\partial\Omega} N^2(\nabla \tilde{u})\, d\sigma + |\nabla H(P^*)|^2 \sigma(\partial\Omega) + \int_{\partial\Omega} S^2(\nabla H)\, d\sigma.$$

In Ω_ε, $|\nabla_X G(\cdot, Y)(P^*)| \leq C|Y - P^*|^{1-n}$. Then, by interior Schauder estimates,

$$\sup_{Y \in \Omega_\varepsilon} \|\Delta u\|_\infty \leq \sup_{Y \in \Omega_\varepsilon} \left(\int_{Z \in B(Y, d(Y)/2)} |\nabla\nabla u(Z)|^2\, dZ \right)^{1/2}$$

and these L^2 averages are in turn dominated by $S(\nabla u)(Q)$ for some $Q \in E$. Hence

$$\int_{\Omega_\varepsilon} |\nabla_X G(\cdot, Y)(P^*)| \, |\Delta u(Y)|\, dY$$

$$\leq C\gamma\lambda \int_{\Omega_\varepsilon} |Y - P^*|^{1-n}\, dY$$

$$\leq C\gamma\lambda.$$

In $\Omega \setminus \Omega_\varepsilon$, we use the estimate $|\nabla_X G(\cdot, Y)(P^*)| \lesssim CG(P^*, Y)/R$ and hence
(3.6)
$$\int_{\Omega \setminus \Omega_\varepsilon} |G(P^*, Y)| \, |\Delta u(Y)|\, dY \lesssim \int_{\Omega \setminus \Omega_\varepsilon} [G(P^*, Y)/d_\Omega(Y)] d_\Omega(Y) |\Delta u(Y)|\, dY.$$

If Y is near the boundary of Ω and Y^* is the projection of Y onto $\partial\Omega$, the Green's function has the estimate [11]

$$\frac{G(P^*, Y)}{d_\Omega(Y)} \approx \frac{\omega_\Omega(\Delta(Y^*, d_\Omega(Y)))}{\sigma(\Delta(Y^*, d_\Omega(Y)))}.$$

Here $\Delta(Y^*, d_\Omega(Y))$ is, as usual, the surface ball with center Y^* and radius $d_\Omega(Y)$. Hence the ratio $G(P^*, Y)/d_\Omega(Y)$ is dominated by $M_\sigma(K_\Omega)(Y^*)$, the Hardy-Littlewood maximal function of the density $K_\Omega = d\omega_\Omega/d\sigma$. By the reverse Hölder inequality (1.4) satisfied by K_Ω and Hölder's inequality in the integral (3.6) we obtain

$$\int_{\Omega \setminus \Omega_\varepsilon} |G(P^*, Y)| \, |\Delta u(Y)|\, dY \leq C\gamma\lambda.$$

This proves (3.4).

We turn now to the proof of (3.5). Invoking inequality (3.3) and the estimate $d_\Omega(X)|\nabla\nabla u(X)| \leq \gamma\lambda$ it suffices to show the following inequality. (The estimate $d_\Omega(X)|\nabla\nabla u(X)| \leq \gamma\lambda$ follows by dominating $|\nabla\nabla u(X)|$ by $S(\nabla u)/d(X)$ using interior estimates to bound $\nabla\nabla u$ pointwise by L^2 averages.)

(3.7) $$\int_{\partial\Omega} N^2(\nabla u)\, d\sigma \lesssim d_\Omega^2(P^*)|\Delta u(P^*)|^2 \sigma(\Delta) + \int_{\partial\Omega} S^2(\nabla u)\, d\sigma.$$

Since the estimate is scale invariant, we will assume for convenience that $\text{diam}(\Omega) = 1$ and that the starcenter of Ω is the origin. Let v be the harmonic function in Ω given by [8] satisfying $\Delta u(Y) = nv(Y) + 2Y \cdot \nabla v(Y)$. Then by Theorem 1.1 and (1.3)(ii), there is a constant $C = C(\Omega)$ such that

$$\int_{\partial\Omega} N^2(\nabla u)(Q)\, d\sigma(Q) \lesssim C \int_{\partial\Omega} v^2(Q)\, d\sigma(Q).$$

Because $\text{diam}(\Omega) = 1$, the constant C above is absolute. The L^2 norm of v will be estimated using $\|Sv\|_{L^2}$.

However, instead of defining the square function Sv using the full gradient, $|\nabla v|$, a lemma of Stein [16, p. 213] is easily modified to show that the radial derivative $|X \cdot \nabla v(Y)|$ may be used instead. Hence

(3.8)
$$\int_{\partial\Omega} v^2\, d\sigma(Q) \lesssim |v(P^*)|^2 \sigma(\Delta) + \int_{\partial\Omega} S^2 v(Q)\, d\sigma(Q)$$

$$\lesssim |\Delta u(P^*)|^2 + \int_{\Omega} d_{\Omega}(X)|X \cdot \nabla v(X)|^2\, dX.$$

Then, because $d(X)|\nabla\nabla u(X)| \lesssim C\gamma\lambda$ in Ω, the first term above is bounded by $C\gamma^2\lambda^2$ and it remains to estimate the solid integral. Again by (1.3), with $d_{\Omega}(X) = \text{dist}(X, \partial\Omega)$,

$$\int_{\Omega} d_{\Omega}(X)|X \cdot \nabla v(X)|^2\, dX \lesssim \int_{\Omega} d_{\Omega}(X)[|\Delta u(X)|^2 + |v(X)|^2]\, dX.$$

By (3.3), $\int_{\Omega} d_{\Omega}(X)|\Delta u(X)|^2\, dX \leq C\gamma^2\lambda^2$.

We now choose some $\varepsilon > 0$ and set

$$\Omega_{\varepsilon} = \{X \in \Omega : \text{dist}(X, \partial\Omega) > \varepsilon\}.$$

Then

(3.9)
$$\int_{\Omega} d_{\Omega}(X)|v(X)|^2\, dX = \int_{\Omega\backslash\Omega_{\varepsilon}} d_{\Omega}(X)|v(X)|^2\, dX + \int_{\Omega_{\varepsilon}} d_{\Omega}(x)|v(X)|^2\, dX$$

$$\lesssim \varepsilon^2 \int_{\partial\Omega} N^2 v(Q)\, d\sigma(Q) + \int_{\Omega_{\varepsilon}} d_{\Omega}(X)|v(X)|^2\, dX.$$

The term $\varepsilon^2 \int_{\partial\Omega} N^2 v(Q)\, d\sigma(Q)$ is less than $C\varepsilon^2 \int_{\partial\Omega} v^2(Q)\, d\sigma(Q)$ and if ε is small enough this will be less than one half of the quantity on the left-hand side of (3.8). And

$$\int_{\Omega_{\varepsilon}} d_{\Omega}(X)|v(X)|^2\, dX \lesssim \int_{\Omega_{\varepsilon}} d_{\Omega}(X)\{|\Delta u(X)|^2 + |\nabla v(X)|^2\}\, dX$$

$$\lesssim \gamma^2\lambda^2\sigma(\Delta) + \frac{1}{\varepsilon} \int_{\Omega_{\varepsilon}} d_{\Omega_{\varepsilon/2}}(X)|\nabla v(X)|^2\, dX$$

$$\lesssim \gamma^2\lambda^2 + \frac{1}{\varepsilon} \int_{\partial\Omega_{\varepsilon/2}} S^2_{\Omega_{\varepsilon/2}} v(P)\, d\sigma(P)$$

$$\lesssim \gamma^2\lambda^2 + \frac{1}{\varepsilon} \int_{\partial\Omega_{\varepsilon/2}} v^2(P)\, d\sigma(P).$$

Because $\Omega_{\varepsilon/2}$ is starlike there exists a constant $C_0 > 0$ such that $\langle P, N_P \rangle \geq C_0$ for all $P \in \partial\Omega_{\varepsilon/2}$. Then

$$\int_{\partial\Omega_{\varepsilon/2}} v^2(P)\, d\sigma(P) \leq \frac{1}{C_0} \int_{\partial\Omega_{\varepsilon/2}} \langle P, N_P \rangle v^2(P)\, d\sigma(P)$$

$$\lesssim C(C_0) \int_{\Omega_{\varepsilon/2}} \operatorname{div}(Y v^2(Y))\, dY$$

$$\lesssim \int_{\Omega_{\varepsilon/2}} v(Y)\Delta u(Y)\, dY.$$

If $X \in \Omega_{\varepsilon/2}$, there is a constant B_ε such that $d_\Omega(X) > B_\varepsilon$ and so by Cauchy-Schwarz,

$$\int_{\Omega_{\varepsilon/2}} |v(Y)|\,|\Delta u(Y)|\, dY$$

$$\leq \frac{1}{B_\varepsilon} \left(\int_{\Omega_{\varepsilon/2}} |v(Y)|^2 d_\Omega(Y)\, dY \right)^{1/2} \cdot \left(\int_{\Omega_{\varepsilon/2}} d_\Omega(Y)|\Delta u(Y)|^2\, dY \right)^{1/2}$$

$$\leq \frac{1}{2} \int_\Omega d_\Omega(Y)|v(Y)|^2\, dY + C_\varepsilon \int_\Omega d_\Omega(Y)|\Delta u(Y)|^2\, dY.$$

The first term above can be subtracted from the left-hand side of inequality (3.9). The second term can be estimated in terms of the square function:

$$\int_\Omega d_\Omega(Y)|\Delta u(Y)|^2\, dY \lesssim |\Delta u(0)|^2 + \int_{\partial\Omega} S^2(\nabla u)\, d\sigma.$$

This proves (3.7) and establishes Lemma 3.1.   □

Concluding Remarks. We conclude with some brief remarks about Sobolev space regularity of biharmonic functions. Recall that, for $0 < \alpha < 1$ and $1 < p,\ q < \infty$, the Besov space $\Lambda_\alpha^{p,q}$ consists of those functions $F \in L^p(D)$ such that the norm

$$\|F\|_{L^p(D)} + \left\{ \int_D \left\{ \int_D |F(X) - F(Y)|^p\, dY \right\}^{q/p} \frac{dX}{|X-Y|^{n+\alpha q}} \right\}^{1/q}$$

is finite. It is known (see for example [11]) that square function estimates lead to regularity (in terms of Besov norms) of solutions. So, for example, in Verchota [18] it was shown that a biharmonic solution u to the L^p Dirichlet problem $(1 < p < \infty)$ on a C^1 domain D has its gradient, ∇u, in the space $\Lambda_{1/p}^{p,2}(D)$ when $1 < p \leq 2$ and in the space $\Lambda_{1/p}^{p,p}(D)$ when $2 \leq p < \infty$. The equivalence in L^p of the square function and nontangential maximal function established above gives another proof of the result in [18] above. And given the L^2 solvability of the Dirichlet problem for Δ^2 in $D \subseteq \mathbf{R}^n$ we can use Theorem 1.2 to prove the following regularity result.

Corollary (of Theorems 1.1 and 1.2). *Let D be a bounded Lipschitz domain in $\mathbf{R}^n$ and let u be the unique function satisfying (a), (b), (c) and (d) of Theorem 1.1. Then $|\nabla u|$ belongs to $\Lambda_{1/2}^{2,2}$ on D.*

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