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ON THE RANGE OF THE RADON d-PLANE TRANSFORM AND ITS DUAL

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ABSTRACT. We present direct, group-theoretic proofs of the range theorem for the Radon d-plane transform $f \to \hat{f}$ on $\mathscr{S}(\mathbb{R}^n)$. (The original proof, by Richter, involves extensive use of local coordinate calculations on $G(d, n)$, the Grassmann manifold of affine d-planes in $\mathbb{R}^n$.) We show that moment conditions are not sufficient to describe this range when $d < n-1$, in contrast to the compactly supported case. Finally, we show that the dual d-plane transform maps $\mathscr{E}(G(d, n))$ surjectively onto $\mathscr{E}(\mathbb{R}^n)$.

1. INTRODUCTION

In this article we investigate the d-dimensional Radon transform $f \to \hat{f}$ on $\mathbb{R}^n$, where $d < n - 1$. This transform integrates functions on $\mathbb{R}^n$ over d-dimensional planes, and so maps functions on $\mathbb{R}^n$ to functions on $G(d, n)$, the affine Grassmann manifold of d-dimensional planes in $\mathbb{R}^n$. One of the most interesting problems concerning such transforms is how to characterize the range of certain function and distribution spaces on $\mathbb{R}^n$, such as $\mathscr{D}(\mathbb{R}^n)$, $\mathscr{S}(\mathbb{R}^n)$, and $\mathscr{E}'(\mathbb{R}^n)$.

For $d = n - 1$, the ranges of $\mathscr{D}$, $\mathscr{S}$, and $\mathscr{E}'$ are characterized by the Helgason moment conditions [9]. For $d < n - 1$, it is still possible to characterize the range of $\mathscr{D}(\mathbb{R}^n)$ by moment conditions [10], but the situation for $\mathscr{S}(\mathbb{R}^n)$ is quite different.

In fact, moment conditions do not suffice to describe the range $\mathscr{S}(\mathbb{R}^n)\,\hat{}$, as will be seen from a simple counterexample in §2. It turns out that the range $\mathscr{S}(\mathbb{R}^n)\,\hat{}$ can be described as the space of rapidly decreasing functions on $G(d, n)$ satisfying a system of second-order partial differential equations. For $d = 1$ and $n = 3$, this result was already obtained by Fritz John in 1938 [14]. For arbitrary d and n, these differential equations were first given explicitly by Gelfand, Gindikin, and Graev [2] in terms of the local coordinates on $G(d, n)$. (See also Gelfand-Graev-Shapiro [1] for the analogous results on $\mathbb{C}^n$.) However, their proof omitted many details. In 1984, Grinberg [7]

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described the range $\mathscr{S}(\mathbb{R}^n)^\frown$ in terms of both moment conditions and differential equations. Finally, in 1986, a complete proof characterizing the range $\mathscr{S}(\mathbb{R}^n)^\frown$ in terms of the differential equations of Gelfand, et al. was obtained by Richter [16]. Richter also provided a range characterization in terms of differential equations involving the infinitesimal left regular representation of the Euclidean motion group $E(n)$ on $G(d, n)$. The proof used extensive local coordinate calculations on $G(d, n)$.

In this paper, we present a more direct, group-theoretic proof of the range theorem. As a consequence of a part of the proof, we obtain a proof of the surjectivity of the dual d-plane transform on the space $\mathscr{E}(\mathbb{R}^n)$. This was proven by Hertle [13] in the case $d = n - 1$.

In §2, we define the space $\mathscr{S}(G(d, n))$ of rapidly decreasing functions on $G(d, n)$ and the partial Fourier transform on this space. We also prove certain fundamental properties of this partial Fourier transform. Our definition of rapidly decreasing functions on $G(d, n)$ is equivalent to, but quite different in formulation from, that of Richter. However, it can also be easily extended to define the Schwartz functions on a homogeneous unitary K-vector bundle, where K is a compact Lie group. In this section, we will also present the counterexample referred to above.

In §3, we investigate the infinitesimal left regular representation of the Euclidean motion group on $\mathbb{R}^n$ and on $G(d, n)$, and its behavior under the Radon transform. We also introduce the differential operators needed in the statement of the range theorem.

In §4 we prove the range theorem. Finally in §5, we prove the surjectivity of the dual d-plane transform on $\mathscr{E}(\mathbb{R}^n)$. We also present an interesting problem connected with the nullspace of the dual d-plane transform.

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2. Rapidly decreasing functions and the partial Fourier transform on $G(d, n)$

Let $1 \leq d \leq n-1$. The space $G(d, n)$ of d-planes in $\mathbb{R}^n$ is a homogeneous space of the group $E(n)$ of isometries of $\mathbb{R}^n$. It is also a vector bundle over the Grassmann manifold $G_{d,n}$ of d-dimensional subspaces of $\mathbb{R}^n$, the projection π_d of $G(d, n)$ onto $G_{d,n}$ being the mapping which associates to any $\xi \in G(d, n)$ the parallel d-plane σ through the origin. The fiber $\pi_d^{-1}(\sigma)$ of $\sigma \in G_{d,n}$ is naturally identified with $\sigma^\perp \approx \mathbb{R}^{n-d}$.

If $\sigma^\perp$ is an arbitrary fiber and $\varphi \in \mathscr{E}(G(d, n))$, then the restriction of φ to $\sigma^\perp$ will be denoted $\varphi|_{\sigma^\perp}$. Define the differential operator $\Box$ on $G(d, n)$ by

$$(\Box\varphi)|_{\sigma^\perp} = \Delta_{\sigma^\perp}(\varphi|_{\sigma^\perp}) \tag{1}$$

for all $\varphi \in \mathscr{E}(G(d, n))$, $\Delta_{\sigma^\perp}$ being the Laplacian on $\sigma^\perp$. $\Box$ is invariant under the action of $E(n)$ on $G(d, n)$ [9]. (See also §4 below.)

The d-dimensional Radon transform $f \to \hat{f}$ is defined by

$$\hat{f}(\xi) = \int_\xi f(x)\, dm(x)$$

for any function f on $\mathbb{R}^n$ integrable on d-planes, $dm(x)$ being the Euclidean measure on the d-planes ξ.

Consider the parametrization of $G(d, n)$ given by

$$\xi \leftrightarrow (\sigma, x) \tag{2}$$

where $\sigma = \pi_d(\xi)$ and $\{x\} = \xi \cap \sigma^\perp$. By [9], the range $\mathscr{D}(\mathbb{R}^n)\hat{}$ is the space $\mathscr{D}_H(G(d, n))$ consisting of all $\varphi \in \mathscr{D}(G(d, n))$ satisfying the condition that for each $m \in \mathbb{Z}^+$, there exists a homogeneous degree m polynomial P_m on $\mathbb{R}^n$ with

$$\int_{\sigma^\perp} \varphi(\sigma, x)\langle x, u\rangle^m\, d\sigma^\perp(x) = P_m(u) \tag{3}$$

for all $u \in \sigma^\perp$, $d\sigma^\perp$ being the Euclidean measure on $\sigma^\perp$. When $d = n-1$, these moment conditions also characterize the range $\mathscr{S}(\mathbb{R}^n)\hat{}$. We also refer the reader to [17], in which the range $L^2_c(\mathbb{R}^n)\hat{}$ is described in terms of the above moment and other intregrability conditions.

A natural question to ask is whether the moment conditions (3) also suffice to describe the range $\mathscr{S}(\mathbb{R}^n)\hat{}$ when $d < n-1$.

Now it is clear that the range $\mathscr{S}(\mathbb{R}^n)\hat{}$ consists of functions on $G(d, n)$ which decrease rapidly on its fibers. To make this precise, we first need to formulate a definition of the space $\mathscr{S}(G(d, n))$ of rapidly decreasing functions on the manifold $G(d, n)$.

It is difficult to define $\mathscr{S}(G(d, n))$ by means of the parametrization (2), since x and σ are not independent parameters $(x \in \sigma^\perp)$. Instead we use the fact that $G(d, n)$ is a homogeneous $O(n)$-unitary vector bundle [19].

Let $e_1, e_2, \dots, e_n$ be the usual basis of $\mathbb{R}^n$, and let σ_0 denote the subspace $\mathbb{R}e_1 + \mathbb{R}e_2 + \cdots + \mathbb{R}e_d$. We identify $\sigma_0^\perp$ with $\mathbb{R}^{n-d} = \mathbb{R}e_{d+1} + \cdots + \mathbb{R}e_n$. Let $\pi\colon k \mapsto k \cdot \sigma_0$ be the canonical projection of $O(n)$ onto $G_{d,n}$. A compact subset $M \subset G_{d,n}$ is called *full* if M is the closure of its interior. If M is a full compact subset of $G_{d,n}$, we say that it *admits a local cross section into* $O(n)$ if M is contained in an open set $V \subset G_{d,n}$ such that there exists a local C^∞ cross section η of V into a submanifold of $O(n)$ (i.e., $\pi \circ \eta$ is the identity map on V). Note that $G_{d,n}$ is a finite union of full compact sets which admit local cross sections.

By definition, the function φ belongs to $\mathscr{S}(G(d, n))$ if and only if $\varphi \in \mathscr{E}(G(d, n))$ and satisfies the estimate

$$\sup_{\substack{\sigma \in M \\ x \in R^{n-d}}} (1 + \|x\|)^r |E_\sigma D_x \varphi(\sigma, \eta(\sigma) \cdot x)| < \infty \tag{4}$$

for all $r \in \mathbb{Z}^+$, for all differential operators E on $G_{d,n}$, for all constant coefficient differential operators D on $\mathbb{R}^{n-d}$, and for all full compact subsets M of $G_{d,n}$ admitting local cross sections η into $O(n)$. Note that $\eta(\sigma)\cdot(\sigma_0, x) = (\sigma, \eta(\sigma)\cdot x)$. In the estimate (4), we could also have replaced D by a power of the operator $\square$. Note also that a local trivialization of the vector bundle $G(d, n)$ is given by

$$\Gamma\colon V \times \mathbb{R}^{n-d} \to \pi_d^{-1}(V), \qquad (\sigma, x) \mapsto (\sigma, \eta(\sigma)\cdot x) \tag{5}$$

for all $x \in \mathbb{R}^{n-d}$, $\sigma \in V$, where $V \subset G_{d,n}$ is an open set admitting a local cross section into $O(n)$.

The advantage in using local cross sections to parametrize $G(d, n)$ locally, as in (5), is that the distance from the origin to the d-plane $(\sigma, \eta(\sigma)\cdot x)$ is $\|x\|$. Moreover, the transition matrices between adjacent local trivial bundles are all orthogonal. The definition (4) may also be used to define the rapidly decreasing functions on a homogeneous unitary K-vector bundle, where K is a compact Lie group [19].

Now let $\varphi \in \mathcal{S}(G(d, n))$. The *partial Fourier transform* of φ is the function $\tilde{\varphi}$ on $G(d, n)$ given by

$$\tilde{\varphi}(\sigma, u) = \int_{\sigma^\perp} \varphi(\sigma, x) e^{-i\langle x, u\rangle}\, d\sigma^\perp(x), \qquad u \in \sigma^\perp.$$

Clearly, $\tilde{\varphi} \in \mathcal{E}(G(d, n))$. The inverse partial Fourier transform is defined likewise.

Proposition 2.1. *If $\varphi \in \mathcal{S}(G(d, n))$, then $\tilde{\varphi} \in \mathcal{S}(G(d, n))$.*

The proof proceeds as in the Euclidean case and will be omitted.

Next, for each function $f \in \mathcal{S}(\mathbb{R}^n)$, we define the function φ_f on $G(d, n)$ by

$$\varphi_f(\sigma, x) = f(x), \qquad \sigma \in G_{d,n},\ x \in \sigma^\perp.$$

Clearly, $\varphi_f \in \mathcal{E}(G(d, n))$.

Lemma 2.2. $\varphi_f \in \mathcal{S}(G(d, n))$.

The proof is tedious but straightforward. For details, see [3].

Proposition 2.3. *If $f \in \mathcal{S}(\mathbb{R}^n)$, then $\hat{f} \in \mathcal{S}(G(d, n))$.*

Proof. The Fourier transform $\tilde{f}$ belongs to $\mathcal{S}(\mathbb{R}^n)$, so by Lemma 2.2, $\varphi_{\tilde{f}} \in \mathcal{S}(G(d, n))$. According to the Projection-slice Theorem [10], $\varphi_{\tilde{\varphi}}$ is the partial Fourier transform of $\hat{f}$. Taking inverse partial Fourier transform, Proposition 2.1 implies that $\hat{f} \in \mathcal{S}(G(d, n))$.

With the definition of $\mathcal{D}_H(G(d, n))$ in mind, we define $\mathcal{S}_H(G(d, n))$ to be the space of all $\varphi \in \mathcal{S}(G(d, n))$ satisfying the moment conditions (3). Clearly, the Radon transform $f \mapsto \hat{f}$ is a 1-1 map of $\mathcal{S}(\mathbb{R}^n)$ into $\mathcal{S}_H(G(d, n))$ and for $d = n-1$, it is a bijection [10]. On the other hand, it is not onto for $d < n-1$. In fact, we produce below a function $\varphi \in \mathscr{S}_H(G(d, n))$ which does not belong to the range $\mathscr{S}(\mathbb{R}^n)\widehat{\ }$.

If $\sigma \in G_{d,n}$, any $u \in \sigma^\perp$ may be considered as a vector field acting on $\sigma^\perp$. This vector field will still be denoted by u. For $\varphi \in \mathscr{S}_H(G(d, n))$, the moment conditions translate to the following differentiability conditions for the partial Fourier transform at the origin of each $\sigma^\perp$:

$$(u^m \tilde{\varphi}|_{\sigma^\perp})(0) = P_m(u), \qquad u \in \sigma^\perp.$$

This is a condition that holds only at the origin in each $\sigma^\perp$. Consequently, one can find a function $\psi \in \mathscr{D}(G(d, n))$ satisfying the following two conditions:

(i) For some $x_0 \in \mathbb{R}^n$, there exist $\sigma_0, \sigma_1 \in G_{d,n}$ with $x_0 \in \sigma_0^\perp$, $x_0 \in \sigma_1^\perp$ such that $\psi(\sigma_0, x_0) \neq \psi(\sigma_1, x_0)$.

(ii) $\psi(\sigma, x) = 0$ for all $\sigma \in G_{d,n}$ and all $x \in \sigma^\perp$ such that $\|x\| \leq 1$.

Now by Proposition 2.1, $\psi = \tilde{\varphi}$ for some $\varphi \in \mathscr{S}(G(d, n))$. By (ii), $\varphi \in \mathscr{S}_H(G(d, n))$ with each polynomial P_m identically zero. On the other hand, (i) implies that $\varphi \notin \mathscr{S}(\mathbb{R}^n)\widehat{\ }$, because if $\varphi = \hat{f}$, then $\psi(\sigma_0, x_0) = \tilde{f}(x_0) = \psi(\sigma_1, x_0)$.

We remark that for $d = n-1$, condition (i) above cannot be fulfilled. We also remark that if $\varphi \in \mathscr{D}_H(G(d, n))$, its partial Fourier transform $\tilde{\varphi}$ is analytic on each fiber $\sigma^\perp$, and so is determined by its derivatives at the origin. Hence the global behavior of $\tilde{\varphi}$ is determined by the moment conditions on φ. This explains the difference between the results for $\mathscr{D}(\mathbb{R}^n)\widehat{\ }$ and $\mathscr{S}(\mathbb{R}^n)\widehat{\ }$.

3. The infinitesimal action of the Euclidean motion group

In what follows we assume $d < n-1$. In order to facilitate later calculations we also adopt the following conventions. Suppose M is a manifold of dimension m. Let f be a function, D a differential operator, and T a distribution on M. If $\tau: M \to M$ is a diffeomorphism, f^τ will denote the function $f \circ \tau^{-1}$, D^τ the differential operator $f \to (D(f \circ \tau)) \circ \tau^{-1}$, and T^τ the distribution $f \to T(f \circ \tau)$. Then $(Df)^\tau = D^\tau f^\tau$, and if D' is another differential operator, $(DD')^\tau = D^\tau D'^\tau$. If τ preserves a nonvanishing m-form ω on M (so the adjoint D^* of each differential operator D is defined), then $(D^*)^\tau = (D^\tau)^*$ and $(DT)^\tau = D^\tau T^\tau$.

The Euclidean motion group $E(n) = O(n) \times \mathbb{R}^n$ is the set of all ordered pairs (k, v) where $k \in O(n)$ and $v \in \mathbb{R}^n$, with group law $(k, v) \cdot (k', v') = (kk', k \cdot v' + v)$. We have $\mathbb{R}^n = E(n)/O(n)$ and $G(d, n) = E(n)/E(d) \times O(n-d)$. The left action of $E(n)$ on $\mathbb{R}^n$ and $G(d, n)$ will both be denoted by τ. If $k \in O(n)$, we will, as we have been previously doing, replace $\tau(k)x$ by the simpler expression $k \cdot x$.

$E(n)$ may also be identified with the subgroup of $\mathrm{GL}(n+1, \mathbb{R})$ consisting of the matrices $\left(\begin{smallmatrix} k & v \\ 0 & 1 \end{smallmatrix}\right)$, where $k \in O(n)$ and $v \in \mathbb{R}^n$. The left regular representation of $E(n)$ on $\mathscr{E}(\mathbb{R}^n)$ and $\mathscr{E}(G(d, n))$ will be denoted by λ and

ν, respectively: if $g \in E(n)$, then $\lambda(g)f = f^{\tau(g)}$ and $\nu(g)\varphi = \varphi^{\tau(g)}$ for $f \in \mathscr{E}(\mathbb{R}^n)$ and $\varphi \in \mathscr{E}(G(d, n))$.

Let $\mathfrak{e}$ be the Lie algebra of $E(n)$. Then the infinitesimal left regular representations $d\lambda$ and $d\nu$ of $\mathfrak{e}$ will be extended to the universal enveloping algebra $\mathfrak{U}(\mathfrak{e})$. Explicitly, if $X_1, \dots, X_r \in \mathfrak{e}$, then

$$(6) \quad (d\lambda(X_1 \cdots X_r)f)(x) = \left\{ \frac{\partial^r}{\partial t_1 \cdots \partial t_r} f(\tau(\exp(-t_r X_r) \cdots \tau(-t_1 X_1)) \cdot x) \right\}_{t_i=0}$$

for all $f \in \mathscr{E}(\mathbb{R}^n)$. A similar expression can be written for $d\nu$.

If $f \in \mathscr{S}(\mathbb{R}^n)$ and $g \in E(n)$, it is immediate that $(\lambda(g)f)\widehat{\ } = \nu(g)\hat{f}$. By differentiating inside the integral sign we have

$$(7) \qquad (d\lambda(U)f)\widehat{\ } = d\nu(U)\hat{f}$$

for all $U \in \mathfrak{U}(\mathfrak{e})$. (By (8) below, $d\lambda(U)f \in \mathscr{S}$ for all $U \in \mathfrak{U}(\mathfrak{e})$, so differentiation inside the integral sign is indeed permissible.)

We may identify the Lie algebra $\mathfrak{e}$ with the Lie subalgebra of $gl(n+1, \mathbb{R})$ consisting of the matrices $\left(\begin{smallmatrix} T & v \\ 0 & 0 \end{smallmatrix}\right)$, where $T \in \mathrm{so}(n)$, $v \in \mathbb{R}^n$. Let E_{ij} denote the $(n+1) \times (n+1)$ matrix $(\delta_{ir}\delta_{sj})_{r,s=1}^n$. Write

$$X_{ij} = E_{ij} - E_{ji}, \qquad 1 \le i, j \le n,$$
$$E_k = E_{k, n+1}, \qquad 1 \le k \le n+1.$$

Note that $X_{ii} = 0$. $\mathfrak{e}$ has basis X_{ij} $(i < j)$ and E_k $(1 \le k \le n)$.

If $U \in \mathfrak{U}(\mathfrak{e})$, $d\lambda(U)$ and $d\nu(U)$ are differential operators on $\mathbb{R}^n$ and $G(d, n)$, respectively. In particular,

$$(8) \qquad d\lambda(X_{ij}) = x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i}, \quad d\lambda(E_k) = -\frac{\partial}{\partial x_k}.$$

For $1 \le i, j, l \le n$, let $V_{ijl} = E_i X_{jl} + E_j X_{li} + E_l X_{ij} \in \mathfrak{U}(\mathfrak{e})$. (If any of the indices i, j, l coincide, $V_{ijl} = 0$.) By (7),

$$(9) \qquad d\nu(V_{ijl})\hat{f} = 0$$

for all $f \in \mathscr{S}(\mathbb{R}^n)$.

Following Richter [16], we define $\mathscr{S}_D(G(d, n))$ to be the space of all $\varphi \in \mathscr{S}(G(d, n))$ satisfying the differential equations $d\nu(V_{ijl})\varphi = 0$. By (9), $\mathscr{S}(\mathbb{R}^n)\widehat{\ } \subset \mathscr{S}_D(G(d, n))$.

4. The range of the d-plane transform

In this section we show that $\mathscr{S}(\mathbb{R}^n)\widehat{\ } = \mathscr{S}_D(G(d, n))$. First we require a lemma about the adjoint action of $E(n)$ on the elements $V_{ijl} \in \mathfrak{U}(\mathfrak{e})$.

Lemma 4.1. *Let* $k = (k_{rs})_{r,s=1}^{n} \in O(n)$ *and* $v \in \mathbb{R}^n$. *Suppose* $1 \le i, j, l \le n$. *Then*

(i) $$\mathrm{Ad}(k)V_{ijl} = \sum_{u<s<r} \det \begin{bmatrix} k_{ui} & k_{uj} & k_{ul} \\ k_{si} & k_{sj} & k_{sl} \\ k_{ri} & k_{rj} & k_{rl} \end{bmatrix} \cdot V_{usr},$$

(ii) $$\mathrm{Ad}(v)V_{ijl} = V_{ijl}.$$

Proof. If we consider $E(n)$ as a matrix group, then the adjoint representation is just given by conjugation: $\mathrm{Ad}(g)X = gXg^{-1}$ $(g \in E(n),\ X \in \mathfrak{e})$. Thus, by a routine computation,

$$\mathrm{Ad}(k)E_i = \sum_{j=1}^{n} k_{ji}E_j, \quad \mathrm{Ad}(k)X_{jl} = \sum_{i,r=1}^{n} k_{ij}k_{rl}X_{ir}.$$

Hence

$$\begin{aligned}\mathrm{Ad}(k)V_{ijl} &= \sum_{u,s,r} k_{ui}k_{sj}k_{rl}(E_uX_{sr} + E_sX_{ru} + E_rX_{us}) \\ &= \sum_{u,s,r} k_{ui}k_{sj}k_{rl}V_{usr}.\end{aligned}$$

For each fixed $u < s < r$ in the above sum, we have $V_{urs} = -V_{usr}$, etc., proving (i). For (ii), write $v = \sum_{r=1}^{n} v_rE_r$. Then $\mathrm{Ad}(v)X_{jl} = X_{jl} + v_jE_l - v_lE_j$ and $\mathrm{Ad}(v)E_i = E_i$ for all i, j, l. Hence

$$\begin{aligned}\mathrm{Ad}(v)V_{ijl} = E_i(X_{jl} + v_jE_l - v_lE_j) + E_j(X_{li} + v_lE_i - v_iE_l) \\ + E_l(X_{ij} + v_iE_j - v_jE_i) = V_{ijl}.\end{aligned}$$

The next lemma describes how $\mathfrak{U}(\mathfrak{e})$ behaves under the partial Fourier transform.

Lemma 4.2. *Let* $T \in so(n)$ *and* $v \in \mathbb{R}^n$. *If* $\varphi \in \mathscr{S}(G(d, n))$, *then*

(i) $(d\nu(T)\varphi)^{\sim} = d\nu(T)\tilde{\varphi}$, *and*
(ii) $(d\nu(v)\varphi)^{\sim}(\sigma, x) = -i\langle v, x\rangle\tilde{\varphi}(\sigma, x)$.

Proof. An easy computation shows that

$$(\nu(k)\varphi)^{\sim} = \nu(k)\tilde{\varphi} \tag{10}$$

for all $k \in O(n)$. Differentiating, we obtain (i). For (ii), we note that $\exp(tv) = tv$ so that

$$\begin{aligned}(\nu(tv)\varphi)^{\sim}(\sigma, x) &= \int_{\sigma^\perp} \varphi(\sigma, u - P_{\sigma^\perp}(tv))e^{-i\langle u, x\rangle}\, d\sigma^\perp(u) \\ &= e^{-i\langle P_{\sigma^\perp}(tv), x\rangle}\tilde{\varphi}(\sigma, x) = e^{-it\langle v, x\rangle}\tilde{\varphi}(\sigma, x).\end{aligned} \tag{11}$$

Here $P_{\sigma^\perp}(tv)$ is the orthogonal projection of tv on $\sigma^\perp$. Differentiating both sides of (11) proves (ii).

By Proposition 2.1 and Lemma 4.2, $\psi \in \mathscr{S}_D(G(d, n))^{\sim}$ if and only if

$$(x_i\, d\nu(X_{jl}) + x_j\, d\nu(X_{li}) + x_l\, d\nu(X_{ij}))\psi(\sigma, x) = 0 \tag{12}$$

for all i, j, l. Let Λ_{ijl} be the differential operator in (12). Since $d\nu(\mathrm{Ad}(g)U) = d\nu(U)^{\tau(g)}$ for all $g \in E(n)$, $U \in \mathfrak{U}(\mathfrak{e})$, we have by Lemma 4.1

$$\Lambda_{ijl}^{\tau(k)} = \sum_{u<s<r} \det \begin{bmatrix} k_{ui} & k_{uj} & k_{ul} \\ k_{si} & k_{sj} & k_{sl} \\ k_{ri} & k_{rj} & k_{rl} \end{bmatrix} \Lambda_{usr}. \tag{13}$$

Equation (13) may also be obtained by direct computation.

The next three lemmas, due to Richter [16], show that the elements of $\mathscr{S}_D(G(d, n))^{\sim}$ are of the form φ_F, where $F \in \mathscr{S}(\mathbb{R}^n)$. Our proofs are based on(13) and are essentially coordinate-free.

Lemma 4.3. *Let $\psi \in \mathscr{S}_D(G(d, n))^{\sim}$. Then there exists $F \in \mathscr{E}(\mathbb{R}^n \backslash \{0\}) \cap C(\mathbb{R}^n)$ such that $\psi(\sigma, x) = F(x)$ for all $\sigma \in G_{d,n}$, $x \in \sigma^{\perp}$.*

Proof. Fix $x \in \mathbb{R}^n$. Define $O(x^{\perp})$ to be the subgroup of $O(n)$ consisting of all k such that $k \cdot x = x$. (Note $O(0^{\perp}) = O(n)$.) Let $so(x^{\perp})$ denote its Lie algebra. $so(x^{\perp})$ consists of all infinitesimal rotations fixing x. Assume first $x \neq 0$. We will prove below that for any $Z \in so(x^{\perp})$,

$$(d\nu(Z)\psi)(\sigma, x) = 0 \tag{14}$$

for all $\sigma \in G_{d,n}$ such that $x \in \sigma^{\perp}$. It will then follow from (14) that

$$\psi(\sigma, x) = \psi(\sigma', x) \tag{15}$$

for all $\sigma, \sigma' \in G_{d,n}$ such that $x \in \sigma^{\perp} \cap \sigma'^{\perp}$. Thus ψ does not depend on the argument σ and (15) can be used to define the function F away from the origin. It is easily seen that $F \in \mathscr{E}(\mathbb{R}^n \backslash \{0\})$. Also since $so((tx)^{\perp}) = so(x^{\perp})$ for all $t \neq 0$, (14) implies that

$$(d\nu(Z)\psi)(\sigma, tx) = 0$$

for all $Z \in so(x^{\perp})$, $t \neq 0$. Letting $t \to 0$, we obtain

$$(d\nu(Z)\psi)(\sigma, 0) = 0 \tag{16}$$

for all $Z \in so(x^{\perp})$. Since x is an arbitrary point in $\mathbb{R}^n \backslash \{0\}$, (16) holds for all $Z \in \sum_{x \in R^n - 0} so(x^{\perp}) = so(n)$. Hence (14) holds even when $x = 0$, so it can be used to define $F(0)$. It is easy to see that F is continuous at 0.

We now prove (14). Let $r = \|x\|$, and let σ_0 be the d-dimensional subspace spanned by $e_1, \dots, e_d$. Since $O(n)$ is transitive on the set of d-planes at a given distance from 0, there exists $k \in O(n)$ such that $k \cdot re_n = x$ and $k \cdot \sigma_0 = \sigma$. Note that $O(x^{\perp}) = k\left(\begin{smallmatrix} O(n-1) & 0 \\ 0 & 1 \end{smallmatrix}\right)k^{-1}$. Thus, since $Z \in so(x^{\perp})$,

$$\mathrm{Ad}(k^{-1})Z = \sum_{1 \le i < j < n} c_{ij}(k) X_{ij} \in \begin{pmatrix} so(n-1) & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence

$$\begin{aligned}(d\nu(Z)\psi)(\sigma, x) &= (d\nu(Z)\psi)(k\cdot(\sigma_0, re_n))\\ &= (d\nu(Z)\psi)^{\tau(k^{-1})}(\sigma_0, re_n)\\ &= (d\nu(\mathrm{Ad}(k^{-1})Z)\psi^{\tau(k^{-1})})(\sigma_0, re_n)\\ &= \sum_{1\le i<j<n} c_{ij}(k)(d\nu(X_{ij})\psi^{\tau(k^{-1})})(\sigma_0, re_n).\end{aligned}$$

To prove (14) it will suffice to prove that each of the above summands is zero. But

$$\begin{aligned}(r d\nu(X_{ij})\psi^{\tau(k^{-1})})(\sigma_0, re_n) &= (\Lambda_{ijn}\psi^{\tau(k^{-1})})(\sigma_0, re_n)\\ &= (\Lambda_{ijn}^{\tau(k)}\psi)^{\tau(k^{-1})}(\sigma_0, re_n)\\ &= (\Lambda_{ijn}^{\tau(k)}\psi)(\sigma, x).\end{aligned}$$

By (13), $\Lambda_{ijn}^{\tau(k)}$ is a linear combination of operators of the form Λ_{usr}. Since $\psi \in \widetilde{\mathscr{S}}_D$, the last expression above equals zero. This proves (14) and the lemma.

The next lemma shows that F is also differentiable at the origin.

Lemma 4.4. *Let $\psi \in \mathscr{E}(G(d, n))$ satisfy the condition $\psi(\sigma, x) = \psi(\sigma', x)$ for all $x \in \mathbb{R}^n$, and $\sigma, \sigma' \in G_{d,n}$ with $x \in \sigma^\perp \cap \sigma'^\perp$. Define the function F on $\mathbb{R}^n$ by $F(x) = \psi(\sigma, x)$ for all $x \in \sigma^\perp$. Then $F \in \mathscr{E}(\mathbb{R}^n)$.*

Proof. As in the preceding lemma, $F \in \mathscr{E}(\mathbb{R}^n\backslash\{0\}) \cap C(\mathbb{R}^n)$. In particular, F is continuous at the origin. We intend to show that for each $i = 1, \dots, n$, there exists a function $\Psi_i \in \mathscr{E}(G(d, n))$ such that

$$\frac{\partial F}{\partial x_i}(x) = \Psi_i(\sigma, x) \tag{17}$$

for all $x \neq 0$ in $\sigma^\perp$ and all $\sigma \in G_{d,n}$. Since Ψ_i satisfies the hypothesis of the present lemma, $\partial F/x_i$ can be extended continuously to the origin. An elementary induction then proves that all partial derivatives of F can be continuously extended to the origin.

The proof of (17) consists of several steps.

(A) We first remark that ψ satisfies the relation $\Lambda_{ijl}\psi = 0$. In fact, using the notation in the proof of Lemma 4.3, we have

$$\begin{aligned}(\Lambda_{ijl}\psi)(\sigma, x) &= (\Lambda_{ijl}\psi)^{\tau(k^{-1})}(\sigma_0, re_n)\\ &= (\Lambda_{ijl}^{\tau(k^{-1})}\psi^{(\tau(k^{-1})})(\sigma_0, re_n)\\ &= \sum_{u<s<n} c_{usn}(k)(\Lambda_{usn}\psi^{\tau(k^{-1})})(\sigma_0, re_n)\\ &= \sum_{u<s<n} c_{usn}(k) r(d\nu(X_{us})\psi^{\tau(k^{-1})})(\sigma_0, re_n).\end{aligned}$$

Here $c_{usn}(k)$ is an appropriate 3×3 determinant. Now $\psi^{\tau(k^{-1})}$ satisfies the hypothesis of the lemma, and $X_{us} \in \left(\begin{smallmatrix} so(n-1) & 0 \\ 0 & 0 \end{smallmatrix}\right)$, so $(d\nu(X_{us})\psi^{\tau(k^{-1})})(\sigma_0, re_n) = 0$.

(B) Next we examine the relation between $\partial F/\partial x_i$ and the corresponding derivatives of ψ. Let $\sigma \in G_{d,n}$ and let $u_1, \dots, u_d$ be an orthonormal basis of σ. We may assume that each u_j is a column vector with components u_{ij} $(1 \le i \le n)$. Recall that we identify each vector $v \in \sigma^\perp$ with the corresponding vector field in $\sigma^\perp$. We have

$$\begin{aligned}
(d\nu(e_i)\psi)(\sigma, x) &= -P_{\sigma^\perp}(e_i)\psi(\sigma, x) = -\langle P_{\sigma^\perp}(e_i), \operatorname{grad} F(x)\rangle \\
&= -\langle e_i, P_{\sigma^\perp}(\operatorname{grad} F(x))\rangle \\
&= -\left\langle e_i, \operatorname{grad} F(x) - \sum_{j=1}^{d} \langle \operatorname{grad} F(x), u_j\rangle u_j \right\rangle \\
&= -\frac{\partial F}{x_i} + \sum_{j=1}^{d}\sum_{r=1}^{n} \frac{\partial F}{\partial x_r} u_{rj} u_{ij} \\
&= -\frac{\partial F}{x_i}\left(1 - \sum_{j=1}^{d} u_{ij}^2\right) + \sum_{j=1}^{d} u_{ij} \sum_{\substack{r=1 \\ r \neq i}} \frac{\partial F}{x_r} u_{rj}
\end{aligned} \tag{18}$$

Now if $X \in so(n)$, $F(\exp(tX)\cdot x) = \psi(\exp(tX)\cdot\sigma,\ \exp(tX)\cdot x)$. Thus

$$(d\lambda(X)F)(x) = (d\nu(X)\psi)(\sigma, x) \tag{19}$$

for all $X \in so(n)$. Hence by part (A) and (12),

$$\begin{aligned}
&x_i(d\nu(X_{rj}))\psi(\sigma, x) + x_r(d\nu(X_{ji})\psi)(\sigma, x) \\
&\quad = -x_j(d\lambda(X_{ir})F)(x) = -x_j\left(x_i\frac{\partial F}{\partial x_r} - x_r\frac{\partial F}{\partial x_i}\right)(x).
\end{aligned}$$

Therefore,

$$\frac{\partial F}{\partial x_r} = \frac{1}{x_i x_j}\left(x_j x_r \frac{\partial F}{\partial x_i} - x_i\, d\nu(X_{rj})\psi - x_r\, d\nu(X_{ji})\psi\right) \tag{20}$$

In (20), the derivatives of F are understood as being evaluated at $x \in \sigma^\perp$, and the derivatives of ψ at (σ, x). Substituting (20) into (18), and then using

(12), we have

$$\begin{aligned}(d\nu(e_i)\psi)(\sigma, x) &= -\frac{\partial F}{\partial x_i}\left(1-\sum_{j=1}^{d} u_{ij}^2 - \sum_{\substack{r=1\\ r\neq i}}\sum_{j=1}^{d}\frac{u_{ij}u_{rj}x_r}{x_i}\right)\\
&\quad -\sum_{\substack{r=1\\ r\neq i}}\sum_{j=1}^{d}\left(\frac{u_{ij}u_{rj}}{x_j}\,d\nu(X_{rj})\psi + \frac{u_{ij}u_{rj}x_r}{x_i x_j}\,d\nu(X_{ji})\psi\right)\\
&= -\frac{\partial F}{\partial x_i}\left(1-\sum_{j=1}^{d} u_{ij}^2 - \sum_{j=1}^{d}\frac{u_{ij}}{x_i}\sum_{\substack{r=1\\ r\neq i}}^{n} u_{rj}x_r\right)\\
&\quad -\sum_{\substack{r=1\\ r\neq i}}^{n}\sum_{j=1}^{d}\frac{u_{ij}u_{rj}}{x_i x_j}(x_i\,d\nu(X_{rj}) + x_r\,d\nu(X_{ji}))\psi\\
&= -\frac{\partial F}{\partial x_i}\left(1-\sum_{j=1}^{d} u_{ij}^2 - \sum_{j=1}^{d}\frac{u_{ij}}{x_i}(-u_{ij}x_i)\right)\\
&\quad +\sum_{r=1}^{n}\sum_{j=1}^{d}\frac{u_{ij}u_{rj}}{x_i}\,d\nu(X_{ir})\psi\\
&= -\frac{\partial F}{\partial x_i} + \frac{1}{x_i}\sum_{r=1}^{n} A_{ir}(\sigma)\,d\nu(X_{ir})\psi\end{aligned} \tag{21}$$

Here $A_{ir}(\sigma) = \sum_{j=1}^{d} u_{ij}u_{rj}$ depends only on $\sigma \in G_{d,n}$. By (21) we have

$$\frac{\partial F}{\partial x_i} = -\nu(e_i)\psi + \frac{1}{x_i}\sum_{r=1}^{n} A_{ir}(\sigma)\,d\nu(X_{ir})\psi\,. \tag{22}$$

(C) Let Ψ_i be the right-hand side of (22). To show that $\Psi_i \in \mathscr{E}(G(d, n))$, it suffices to show that the second expression on the right-hand side of (22) equals a C^∞ function Φ_i on $G(d, n)$. This will prove the lemma. In order to prove this claim, we first assert that

$$x_m\sum_{r=1}^{n} A_{ir}(\sigma)\,d\nu(X_{ir})\psi = x_i\sum_{r=1}^{n} A_{ir}(\sigma)\,d\nu(X_{mr})\psi \tag{23}$$

for all $1 \le i,\ m \le n$. For this it suffices to prove that for each $1 \le j \le d$

$$x_m\sum_{r=1}^{n} u_{rj}\,d\nu(X_{ir})\psi = x_i\sum_{r=1}^{n} u_{rj}\,d\nu(X_{mr})\psi\,. \tag{24}$$

For then, each side of (24) can be multiplied by u_{ij} and summed over j. Now to prove (24), it may be assumed that $i = 1$ and $m = 2$. The proof for the other cases is analogous. By (12) we have the following system of n equations.

$$\begin{array}{lllll} x_2u_{2j} & + \; x_3u_{3j} & + & \cdots \; + x_nu_{nj} & = -x_1u_{1j} \\ x_2\,d\nu(X_{31})\psi & + \; x_3\,d\nu(X_{12})\psi & & & = -x_1\,d\nu(X_{23})\psi \\ x_2\,d\nu(X_{41})\psi & + & + \; x_4\,d\nu(X_{12})\psi & & = -x_1\,d\nu(X_{24})\psi \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_2\,d\nu(X_{n1})\psi & + & & +x_n\,d\nu(X_{12})\psi & = -x_1\,d\nu(X_{2n})\psi \end{array}$$

Eliminating the variables $x_3, \ldots, x_n$, we obtain

$$\begin{aligned} &x_2(u_{2j}\,d\nu(X_{12})\psi - u_{3j}\,d\nu(X_{31})\psi - u_{4j}\,d\nu(X_{41})\psi - \cdots - u_{nj}\,d\nu(X_{n1})\psi \\ &\quad = x_1(-u_{1j}\,d\nu(X_{12})\psi + u_{3j}\,d\nu(X_{23})\psi + u_{4j}\,d\nu(X_{24})\psi + \cdots + u_{nj}\,d\nu(X_{2n})\psi \end{aligned}$$

This equation simplifies to (24) for $i = 1$ and $m = 2$. This proves (24) and (23).

From (24) we see that $x_m = 0$ implies that $\sum_{r=1}^n u_{rj}\,d\nu(X_{mr})\psi = 0$, for all $j = 1, \ldots, d$ so that

$$x_m = 0 \Rightarrow \sum_{r=1}^n A_{ir}(\sigma)\,d\nu(X_{mr})\psi = 0. \tag{25}$$

Now by [2], $G(d, n)$ is a finite union of trivial bundles W_α with the following local coordinate representation:

$$\omega_\alpha\colon (\sigma, x) \to (y(\sigma), x_\alpha).$$

Here α is a choice of $n-d$ indices $i_1, \ldots, i_{n-d}$ in $\{1, \ldots, n\}$, $x_\alpha = (x_{i_1}, \ldots, x_{i_{n-d}})$, and $y(\sigma)$ is a local coordinate system for $\sigma \in \pi_d(W_\alpha) \subset G_{d,n}$.

To prove that $\Phi_i \in \mathscr{E}(G(d, n))$, it suffices to prove that $\Phi_i|_{W_\alpha} \in \mathscr{E}(W_\alpha)$ for each α. Suppose first i is one of the indices $i_1, \ldots, i_{n-d}$. Since x_i is a local coordinate in W_α, it follows by (25) for $m = i$ that

$$\Phi_i(\sigma, x) = \frac{\sum_{r=1}^n A_{ir}(\sigma)\,d\nu(X_{ir})\psi}{x_i}$$

is a C^∞ function on W_α. Suppose next that i is not one of the indices $i_1, \ldots, i_{n-d}$. Choose any m in $\{i_1, \ldots, i_{n-d}\}$. Since x_m is a local coordinate in W_α, it follows from (25) that the function

$$\frac{\sum_{r=1}^n A_{ir}(\sigma)\,d\nu(X_{mr})\psi}{x_m} \tag{26}$$

is a C^∞ function on W_α. But by (23), this function equals Φ_i. (Strictly speaking, it equals Φ_i at the points (σ, x) where $x_i \neq 0$, a dense open set in W_α. But then (26) extends Φ_i to be a C^∞ function on all of W_α.) This proves that $\Phi_i \in \mathscr{E}(G(d, n))$ and thus the lemma.

Lemma 4.5. *Let the function F be as in Lemma 4.3. Then $F \in \mathscr{S}(\mathbb{R}^n)$.*

Proof. By Lemma 4.4, $F \in \mathscr{E}(\mathbb{R}^n)$. We must now prove the estimate

$$\sup_{x \in R^n} \|x\|^k |\Delta^m F(x)| < \infty$$

where Δ denotes the Laplacian on $\mathbb{R}^n$. For any $x \neq 0$ in $\mathbb{R}^n$, write $r = \|x\|$ and $x' = x/r$. Then $x' \in S^{n-1}$ and we have

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}L_{x'}$$

where $L_{x'}$ represents the Laplace operator on S^{n-1}. Then

$$\|x\|^k \cdot \Delta^m F(x) = \sum_{\substack{u,v,w \\ \text{finite}}} h_{uvw} r^u \frac{\partial^v}{\partial r^v} L_{x'}^w F(rx'). \tag{27}$$

Since the Casimir operator on $\mathrm{SO}(n)$ projects to the Laplace operator on S^{n-1}, we have $L_{x'} = c\sum_{i<j} d\lambda(X_{ij}^2)$. Thus each summand in (27) is itself a linear combination of terms of the form

$$r^u \frac{\partial^v}{\partial r^v} d\lambda(X_1^2 \cdots X_s^2) F(rx') \tag{28}$$

where each X_i is a basis element of $so(n)$. But since $F(x) = \psi(\sigma, x)$ for $x \in \sigma^\perp$ and since, by (19), $d\lambda(U)F(x) = d\nu(U)\psi(\sigma, x)$ for all $U \in so(n)$, (28) equals

$$r^u \partial^v / \partial r^v \, d\nu(X_1^2 \cdots X_s^2)\psi(\sigma, rx') \tag{29}$$

where $x' \in \sigma^\perp \cap S^{n-1}$. We will now estimate (29). Let $\Gamma: V \times \mathbb{R}^{n-d} \to \pi_d^{-1}(V)$ be the parametrization of a local trivial subbundle of $G(d, n)$ as in (5), and let $M \subset G_{d,n}$ be a full compact subset of V admitting a local cross section η into $O(n)$. We will show that

$$\sup_{\substack{\sigma\in M \\ x\in R^{n-d}}} \left| r^u \frac{\partial^v}{\partial r^v} d\nu(X_1^2 \cdots X_s^2)\psi(\Gamma(\sigma, x)) \right| < \infty. \tag{30}$$

Now

$$\frac{\partial}{\partial r} = \sum_{d+1}^{n} \frac{x_i}{\|x\|}\frac{\partial}{\partial x_i}.$$

Also,

$$\begin{aligned}
&(d\nu(X_1^2 \cdots X_s^2)\psi)(\Gamma(\sigma, x)) \\
&\quad = \frac{\partial^{2s}}{\partial t_1^2 \cdots \partial t_s^2} \psi(\exp(-t_s X_s) \cdots \exp(-t_1 X_1) \cdot (\sigma, \eta(\sigma)\cdot x))\Big|_{t_i=0} \\
&\quad = \frac{\partial^{2s}}{\partial t_1^2 \cdots \partial t_s^2} \psi(\exp(-t_s X_s) \cdots \exp(-t_1 X_1) \cdot \sigma, \exp(-t_s X_s) \cdots \\
&\qquad\qquad \exp(-t_1 X_1) \cdot \eta(\sigma) \cdot x))\Big|_{t_i=0}.
\end{aligned}$$

But $\exp(-t_sX_s)\cdots\exp(-t_1X_1)\cdot\eta(\sigma)\cdot x = \eta(\exp(-t_sX_s)\cdots\exp(-t_1X_1)\cdot\sigma)\cdot x'$ for some $x'\in\mathbb{R}^{n-d}$ so

$$\begin{aligned} x' &= \eta(\exp(-t_sX_s)\cdots\exp(-t_1X_1)\cdot\sigma)^{-1}\cdot\exp(-t_sX_s)\cdots\exp(-t_1X_1)\cdot\eta(\sigma)\cdot x \\ &= k(\sigma, t_1, \dots, t_s)\cdot x, \end{aligned}$$

where $k(\sigma, t_1\dots, t_s)\in O(n-d)$ is a C^∞ function of σ and $t_1\dots, t_s$ such that $k(\sigma, 0, \dots, 0) = I_{n-d}$, the $(n-d)\times(n-d)$ identity matrix. Thus,

$$\begin{aligned} &d\nu(X_1^2\cdots X_s^2)\psi(\Gamma(\sigma, x)) \\ &\quad = \frac{\partial^{2s}}{\partial t_1^2\cdots\partial t_s^2}(\psi\circ\Gamma)(\exp(-t_sX_s)\cdots\exp(-t_1X_1)\cdot\sigma, k(\sigma, t)\cdot x)\bigg|_{t_i=0} \\ &\quad = \sum_{\beta, E} C_{\beta, E}x^\beta E_\sigma(\psi\circ\Gamma)(\sigma, x) \end{aligned}$$

where the sum runs through a finite set of multi-indices $\beta = (\beta_{d+1}, \dots, \beta_n)$ and differential operators E on $G_{d,n}$. Applying

$$r^u\frac{\partial^v}{\partial r^v} = \|x\|^u\left(\sum_{d+1}^n \frac{x_i}{\|x\|}\frac{\partial}{\partial x_i}\right)^v$$

to the above sum yields a finite linear combination of terms of the form $x^{\beta'}\|x\|^\kappa D_xE_\sigma(\psi\circ\Gamma)(\sigma, x)$. Here κ may be a negative exponent but this does not matter. Since $\psi\in\mathscr{S}(G(d, n))$, each of the above terms is bounded for all $(\sigma, x)\in M\times\mathbb{R}^{n-d}$, with $\|x\|\geq 1$, say. This shows that (30), and hence (28) and (27) are bounded.

Theorem 4.6. $\mathscr{S}(\mathbb{R}^n)^\wedge = \mathscr{S}_D(G(d, n))$.

Proof. We have already seen that $\mathscr{D}(\mathbb{R}^n)^\wedge\subset\mathscr{S}_D(G(d, n))$. So let $\varphi\in\mathscr{S}_D(G(d, n))$, let $\psi = \tilde\varphi$, and let F be as in Lemma 4.3. By lemma 4.5, $F\in\mathscr{S}(\mathbb{R}^n)$. By projection-slice, the partial Fourier transform $(\hat f)^\sim$ satisfies

$$(\hat f)^\sim(\sigma, x) = \tilde f(x) = F(x) = \tilde\varphi(\sigma, x).$$

Taking inverse partial Fourier transform, we obtain $\hat f = \varphi$, as desired.

5. The range of the dual d-plane transform

Consider the dual d-plane transform $\varphi\to\check\varphi$ from functions on $G(d, n)$ to functions on $\mathbb{R}^n$, given by

$$\check\varphi(x) = \int_{G_{d,n}}\varphi(\tau(x)\sigma)\,d\sigma, \qquad \varphi\in\mathscr{E}(G(d, n)),$$

where $d\sigma$ is the normalized $O(n)$-invariant measure on $G_{d,n} = O(n)/O(d)\times O(n-d)$. In this section we will prove the following theorem.

Theorem 5.1. $\mathscr{E}(G(d, n))^\vee = \mathscr{E}(\mathbb{R}^n)$.

This was proven by Hertle [13] in the case $d = n - 1$.

Before proceeding with the proof, let us first gather some preliminary facts. Since $G(d, n) = E(n)/E(d) \times O(n-d)$ is a quotient of unimodular groups, there exists a unique (up to constant multiple) C^∞ measure μ on $G(d, n)$ which is invariant under $\tau(E(n))$. We fix the measure μ to satisfy

$$\int_{G(d,n)} \varphi(\xi)\, d\mu(\xi) = \int_{G_{d,n}} \int_{\sigma^\perp} \varphi(\sigma, x)\, d\sigma^\perp(x)\, d\sigma, \qquad \varphi \in \mathscr{D}(G(d, n)).$$

Then by general principles [11],

$$\int_{G(d,n)} \hat{f}(\xi)\varphi(\xi)\, d\mu(\xi) = \int_{\mathbb{R}^n} f(x)\check{\varphi}(x)\, dx \tag{31}$$

for all $f \in \mathscr{D}(\mathbb{R}^n)$, $\varphi \in \mathscr{E}(G(d, n))$. If D is a differential operator on $G(d, n))$, the adjoint operator with respect to μ will be denoted D^*. Since μ is preserved under $\tau(E(n))$, we have $d\nu(X)^* = -d\nu(X)$ for all $X \in \mathfrak{e}$. From this one can easily see that $d\nu(V_{ijl})^* = d\nu(V_{ijl})$ for all i, j, l.

By [12], the map $f \to \hat{f}$ is continuous from $\mathscr{D}(\mathbb{R}^n)$ into $\mathscr{D}(G(d, n))$ and the map $\varphi \to \check{\varphi}$ is continuous from $\mathscr{E}(G(d, n))$ to $\mathscr{E}(\mathbb{R}^n)$. Using the relation (31), these maps have natural extensions to continuous maps $S \to \widehat{S}$ from $\mathscr{E}'(\mathbb{R}^n)$ into $\mathscr{E}'(G(d, n))$ and $T \to \check{T}$ from $\mathscr{D}'(G(d, n))$ to $\mathscr{D}'(\mathbb{R}^n)$ [10].

If $\gamma \in \mathbb{C}_n = \mathbb{C} - \{n, n+2, n+4, \ldots\}$, I^γ will denote the *Riesz potential*

$$I^\gamma f(x) = \frac{1}{H_n(\gamma)} \int_{\mathbb{R}^n} f(y)\|x - y\|^{\gamma - n}\, dy, \tag{32}$$

for $f \in \mathscr{S}(\mathbb{R}^n)$, where $H_n(\gamma) = 2^\gamma \pi^{n/2}\Gamma(\frac{1}{2}\gamma)\Gamma(\frac{1}{2}(n-\gamma))^{-1}$. When $\operatorname{Re}(\gamma) \le 0$, this is interpreted as analytic continuation. For any real number p such that $-2p \in \mathbb{C}_n$, the fractional power Δ^p of the Laplacian Δ on $\mathbb{R}^n$ is then defined by $(-\Delta)^p f = I^{-2p} f$ $(f \in \mathscr{S}(\mathbb{R}^n))$. The Fourier transform $\tilde{f}$ of f satisfies $((-\Delta)^p f)^\sim(u) = \|u\|^{2p}\tilde{f}(u)$.

According to Ortner [15, Satz 9], the following formula holds: if $\gamma, \mu \in \mathbb{C}_n$ such that $\operatorname{Re}(\gamma + \mu) < n$, then

$$I^\gamma I^\mu f = I^{\gamma+\mu} f \tag{33}$$

for all $f \in \mathscr{S}(\mathbb{R}^n)$. Suppose now $\varphi \in \mathscr{S}(G(d, n))$. In accordance with (32), the fractional power $\Box^k \varphi$ is defined by

$$\begin{aligned}(-\Box)^k \varphi(\sigma, x) &= I^{-2k}\varphi(\sigma, x)\\ &= H_{n-d}(-2k)^{-1} \int_{\sigma^\perp} \varphi(\sigma, y)\|x - y\|^{-2k-(n-d)}\, d\sigma^\perp(y)\end{aligned}$$

interpreted, as above, for $k \ge 0$ by analytic continuation. Using the parametrization (5) of $G(d, n)$, we have

$$(-\Box)^k \varphi(\Gamma(\sigma, x)) = H_{n-d}(-2k)^{-1} \int_{\mathbb{R}^{n-d}} \varphi \circ \Gamma(\sigma, u)\|x - u\|^{-2k-(n-d)}\, du.$$

Thus, under the parametrization Γ, the operator $(-\Box)^k$ is given by convolution with a tempered distribution in $\mathbb{R}^{n-d}$. Hence $(-\Box)^k\varphi \in \mathscr{E}(G(d, n))$ and the map $\varphi \to \check{\varphi}$ is continuous from $\mathscr{S}(G(d, n))$ into $\mathscr{E}(G(d, n))$.

Lemma 5.2. *Let* $\varphi \in \mathscr{S}(G(d, n))$ *and* $k \geq 0$. *Then there exists a constant* C *such that* $|(-\Box)^k\varphi(\sigma, u)| \leq C(1 + \|u\|)^{-2k-(n-d)}$ *for all* $\sigma \in G_{d,n}$, $u \in \sigma^\perp$.

Proof. This follows directly from Lemma 1 of [15], applied to each local parametrization Γ of $G(d, n)$, since the set of functions $\{x \to \varphi(\Gamma(\sigma, x)) | \sigma \in V\}$ is a bounded subset (in the topology) of $\mathscr{S}(\mathbb{R}^{n-d})$.

By Lemma 5.2, the partial Fourier transform $((-\Box)^k\varphi)^\sim$ is given by an absolutely convergent integral for $\varphi \in \mathscr{S}(G(d, n))$ and satisfies

$$((-\Box)^k\varphi)^\sim(\sigma, u) = \|u\|^{2k}\tilde{\varphi}(\sigma, u) \tag{34}$$

Lemma 5.3. *Let* $f \in \mathscr{S}(\mathbb{R}^n)$. *Then for any* $k \geq 0$,

$$((-\Delta)^k f)\,\widehat{} = (-\Box)^k \hat{f} \tag{35}$$

Proof. If $k \in \mathbb{Z}^+$, (35) is obvious, so we assume $k \notin \mathbb{Z}^+$. By Lemma 1 of [15], $(-\Delta)^k f(x) = O(\|x\|^{-n-2k})$ as $\|x\| \to \infty$ so $((-\Delta)^k f)\,\widehat{}$ is well-defined. (35) follows, since the partial Fourier transform of both sides is $\|u\|^{2k}\tilde{f}(u)$.

By (33) and (35) we have the following inversion formula for the transform $f \to \hat{f}$ (see [10]):

$$c_n f = (((-\Delta)^{d/2} f)\,\widehat{}\,)\check{} = ((-\Box)^{d/2}\hat{f})\check{} \tag{36}$$

where $c_n = 4\pi^{d/2}\Gamma(\frac{1}{2}n)\Gamma(\frac{1}{2}(n-d))^{-1}$.

Now it is easy to see that if $k \geq 0$,

$$\int_{G(d,n)} (-\Box)^k\varphi(\xi) \cdot \psi(\xi)\, d\mu(\xi) = \int_{G(d,n)} \varphi(\xi) \cdot (-\Box)^k\psi(\xi)\, d\mu(\xi) \tag{37}$$

for all $\varphi, \psi \in \mathscr{D}(G(d, n))$. Thus for any $T \in \mathscr{E}'(G(d, n))$, we can define the distribution $(-\Box)^k T \in \mathscr{D}'(G(d, n))$ by the formula $((-\Box)^k T)(\varphi) = T((-\Box)^k\varphi)$, for any $\varphi \in \mathscr{D}(G(d, n))$. Since the map $\varphi \to (-\Box)^k\varphi$ is continuous from $\mathscr{D}(G(d, n))$ into $\mathscr{E}(G(d, n))$, the adjoint map $T \to (-\Box)^k T$ is continuous from $\mathscr{E}'(G(d, n))$ to $\mathscr{D}'(G(d, n))$.

Lemma 5.4. *We have*

(i) $((-\Box)^k\varphi)^{\tau(g)} = (-\Box)^k\varphi^{\tau(g)}$,

(ii) $((-\Box)^k T)^{\tau(g)} = (-\Box)^k T^{\tau(g)}$,

for all $g \in E(n)$, $\varphi \in \mathscr{D}(G(d, n))$, $T \in \mathscr{E}'(G(d, n))$.

Proof. Upon taking the partial Fourier transform of both sides, (i) follows from (34) and (10) and (11) in Lemma 4.2. (ii) follows by applying both sides to a given $\varphi \in \mathscr{D}$ and using (i).

Now Theorem 5.1 is a consequence of the following two lemmas.

Lemma 5.5 (Helgason [12]). *Let G/K and G/H be homogeneous spaces in duality, K compact. Let N denote the kernel of $S \to \widehat{S}$ on $\mathscr{E}'(G/K)$. Assume the subspace $\mathscr{E}'(G/K)^{\wedge} \subset \mathscr{E}'(G/H)$ is closed, $\mathscr{E}'(G/H)$ carrying the strong topology. Then*

$$\mathscr{E}(G/H)^{\vee} = N^{\perp}, \tag{38}$$

where $N^{\perp} = \{f \in \mathscr{E}(G/K) | S(f) = 0 \text{ for all } S \in N\}$.

A special case of Lemma 5.5 was first proven by Hertle [13]. In the present situation $G/K = E(n)/O(n) = \mathbb{R}^n$ and $G/H = E(n)/E(d) \times O(n-d) = G(d, n)$.

Lemma 5.6. $\mathscr{E}'(\mathbb{R}^n)^{\wedge} = \{T \in \mathscr{E}'(G(d, n)) | d\nu(V_{ijl})T = 0$ *for all* $i, j, l\}$.

Assuming the above two lemmas, Theorem 5.1 is proven as follows. By Lemma 5.6, $\mathscr{E}'(\mathbb{R}^n)^{\wedge}$ is weakly closed, hence strongly closed in $\mathscr{E}'(G(d, n))$. By the injectivity of the map $S \to \widehat{S}$ [10], we have $N = 0$, so Theorem 5.1 follows from (38).

Proof of Lemma 5.6. Lemma 5.6 is obtained from Theorem 4.6 by an approximation argument. We proceed as follows. If $\phi \in \mathscr{D}(E(n))$ and Ψ is a distribution on $\mathbb{R}^n$ or $G(d, n)$, we write $\phi * \Psi = \int_{E(n)} \phi(g) \cdot \Psi^{\tau(g)}\, dg$, where dg is the Haar measure on $E(n)$. Then $\phi * \Psi$ is a C^{∞} function [11], compactly supported if Ψ is.

Now let $\{\phi_m\}_{m=1}^{\infty} \subset \mathscr{D}(E(n))$ be any sequence converging in $\mathscr{E}'(E(n))$ to δ_e, the δ-function at the identity of $E(n)$. If $S \in \mathscr{E}'(\mathbb{R}^n)$, then

$$\lim_{m \to \infty} \phi_m * S = S$$

in the (strong) topology of $\mathscr{E}'(\mathbb{R}^n)$, so that

$$\lim_{m \to \infty} (\phi_m * S)^{\wedge} = \widehat{S}$$

in $\mathscr{E}'(G(d, n))$. Thus,

$$d\nu(V_{ijl})\widehat{S} = \lim_{m \to \infty} d\nu(V_{ijl})(\phi_m * S)^{\wedge} = 0$$

by Theorem 4.6. On the other hand, let $T \in \mathscr{E}'(G(d, n))$ satisfy $d\nu(V_{ijl})T = 0$ for all i, j, l. We have $\lim_{m \to \infty} \phi_m * T = T$ in $\mathscr{E}'(G(d, n))$ and

$$\begin{aligned} d\nu(V_{ijl})(\phi_m * T) &= \int_{E(n)} \phi_m(g)\, d\nu(V_{ijl})(T^{\tau(g)})\, dg \\ &= \int_{E(n)} \phi_m(g)((d\nu\, \mathrm{Ad}(g^{-1})V_{ijl})T)^{\tau(g)}\, dg \end{aligned}$$

By Lemma 4.1, $\mathrm{Ad}(g^{-1})V_{ijl} = \sum c_{usr}V_{usr}$, so by the hypothesis on T, the right-hand side equals zero. Thus, by Theorem 4.6 and the Support Theorem

[9], $\phi_m * T = \hat{f}_m$, for some $f_m \in \mathscr{D}(\mathbb{R}^n)$. From the inversion formula (36), we have

$$c_n f_m = ((-\Box)^{d/2}(\phi_m * T))^{\vee}. \tag{39}$$

Now by Lemma 5.3, $(-\Box)^{d/2}(\phi_m * T) = \phi_m * (-\Box)^{d/2}T$, which converges to $(-\Box)^{d/2}T$ in $\mathscr{D}'(G(d, n))$. Thus, $f_m = c_n^{-1}(\phi_m * (-\Box)^{d/2}T)^{\vee}$ converges to the distribution $S = ((-\Box)^{d/2}T))^{\vee}$ in $\mathscr{D}'(\mathbb{R}^n)$. But the functions $\phi_m * T$ are all supported on a common compact subset of $G(d, n)$, so by the Support Theorem, the functions f_m are all supported in a common subset of $\mathbb{R}^n$. Thus $S \in \mathscr{E}'(\mathbb{R}^n)$ and $\{f_m\}$ converges to S in the space $\mathscr{E}'(\mathbb{R}^n)$. It follows that

$$T = \lim_{m\to\infty} \phi_m * T = \lim_{m\to\infty} \hat{f}_m = \widehat{S},$$

the convergence being in $\mathscr{E}'(G(d, n))$. This proves Lemma 5.6.

The proof of Theorem 5.1 is now complete.

Remark. For $d < n-1$, it is difficult to formulate a range theorem for $\mathscr{E}'(\mathbb{R}^n)^{\wedge}$ in terms of moment conditions. The reason is that σ and $x \in \sigma^{\perp}$ are not independent parameters.

Let H be the nullspace of the transform $\varphi \to \check{\varphi}$ on $\mathscr{E}(G(d, n))$, i.e., the annihilator of $\mathscr{E}'(\mathbb{R}^n)^{\wedge}$. Since the operators $d\nu(V_{ijl})$ are all selfadjoint, Theorem 5.1 implies that H is the double annihilator, that is to say the weak closure, of the subspace $\sum_{i<j<1} d\nu(V_{ijl})\mathscr{E}(G(d, n))$. In particular, H contains functions of compact support. By contrast, the dual transform is injective on $\mathscr{D}(G(n-1, n))$ [4, 18]. For $d = 1$ and $n = 3$, H is thus the weak closure of $d\nu(V_{123})\mathscr{E}(G(1, 3)) \subset \mathscr{E}(G(1, 3))$. It is an interesting problem to determine whether $H = d\nu(V_{123}\mathscr{E}(G(1, 3))$, i.e., whether or not there are functions in H that do not belong to the range of the operator $d\nu(V_{123})$.

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