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REFLECTED BROWNIAN MOTION IN A CONE WITH RADIALLY HOMOGENEOUS REFLECTION FIELD

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ABSTRACT. This work is concerned with the existence and uniqueness of a strong Markov process that has continuous sample paths and the following additional properties.

(i) The state space is a cone in d-dimensions $(d \geq 3)$, and the process behaves in the interior of the cone like ordinary Brownian motion.

(ii) The process reflects instantaneously at the boundary of the cone, the direction of reflection being fixed on each radial line emanating from the vertex of the cone.

(iii) The amount of time that the process spends at the vertex of the cone is zero (i.e., the set of times for which the process is at the vertex has zero Lebesgue measure).

The question of existence and uniqueness is cast in precise mathematical terms as a submartingale problem in the style used by Stroock and Varadhan for diffusions on smooth domains with smooth boundary conditions. The question is resolved in terms of a real parameter α which in general depends in a rather complicated way on the geometric data of the problem, i.e., on the cone and the directions of reflection. However, a criterion is given for determining whether $\alpha > 0$. It is shown that there is a unique continuous strong Markov process satisfying (i)–(iii) above if and only if $\alpha < 2$, and that starting away from the vertex, this process does not reach the vertex if $\alpha \leq 0$ and does reach the vertex almost surely if $0 < \alpha < 2$. If $\alpha \geq 2$, there is a unique continuous strong Markov process satisfying (i) and (ii) above; it reaches the vertex of the cone almost surely and remains there. These results are illustrated in concrete terms for some special cases.

The process considered here serves as a model for comparison with a reflected Brownian motion in a cone having a *nonradially homogeneous* reflection field. This is discussed in a subsequent work by Kwon.

1. INTRODUCTION

1.1 Overview. Let Ω be a subdomain of the unit sphere S^{d-1} in $\mathbb{R}^d$, $d \geq 3$, such that $S^{d-1}\backslash\overline{\Omega}$ is nonempty and the boundary $\partial\Omega$ of Ω in S^{d-1} is C^3. Define the open cone $G = \{r\omega\colon r > 0,\ \omega \in \Omega\}$. The closure and boundary of

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G will be denoted by $\overline{G}$ and ∂G, respectively. The origin $\{0\}$ is the vertex of the cone. Let $\mathbf{v}$ be a C^2 d-dimensional vector field on $\partial G\backslash\{0\}$, such that $\mathbf{v}$ is constant on rays of the cone, i.e., for each $\omega \in \partial\Omega$, $\mathbf{v}(rw) = \mathbf{v}(\omega)$ for all $r > 0$. We assume that the component of $\mathbf{v}$ in the inward normal direction to $\partial G\backslash\{0\}$ is positive. Indeed, without loss of generality, by the scaling and continuity of $\mathbf{v}$ and compactness of $\partial\Omega$, we may and do assume that $\mathbf{v}\cdot\mathbf{n} = 1$ on $\partial G\backslash\{0\}$, where $\mathbf{n}$ is the inward unit normal to $\partial G\backslash\{0\}$. Let v_r denote the component of $\mathbf{v}$ in the direction of the radial unit vector $\mathbf{e_r}$ in $\mathbb{R}^d$, and define the vector $\mathbf{q} = \mathbf{v} - v_r\mathbf{e_r} - \mathbf{n}$. We assume $v_r \in C^3(\partial G\backslash\{0\})$ and $\mathbf{q} \in C^4(\partial G\backslash\{0\})$.

Remark. For many of the arguments in this paper, it suffices for Ω to be a $C^{2+\varepsilon}$ domain and for $\mathbf{v}$ to be a $C^{1+\varepsilon}$ vector field on $\partial G\backslash\{0\}$ for some $\varepsilon > 0$, and in some instances for $\varepsilon = 0$. For simplicity here we have rounded these up to C^3 and C^2, respectively. The strongest assumptions are needed in §2.2 for proving Lemma 2.1 and for determining whether the reflected Brownian reaches the vertex of the cone. In particular, the assumption that $v_r \in C^3(\partial G\backslash\{0\})$ is used in Lemma 2.4, and that $\mathbf{q} \in C^4(\partial G\backslash\{0\})$ is used in Lemma 2.5. It is quite likely that our smoothness hypotheses could be relaxed by working with suitable Hölder or Sobolev spaces. However, we have not focussed on obtaining the weakest possible smoothness hypotheses here, but rather have imposed mild smoothness assumptions to clearly expose the nature of the results and methods of proof.

The basic problem considered in this paper is the question of existence and uniqueness of a strong Markov process that has continuous sample paths and the following three properties.

(1.1) The state space is the (closed) cone $\overline{G}$ and the process behaves in the interior of this cone like ordinary Brownian motion.

(1.2) The process reflects instantaneously from the smooth part $\partial G\backslash\{0\}$ of the boundary of the cone, the direction of reflection being given by the vector field $\mathbf{v}$.

(1.3) The amount of time that the process spends at the vertex of the cone is zero (i.e., the set of times for which the process is at the vertex has zero Lebesgue measure).

A study of the two-dimensional analogue of this problem, where the state space is a wedge, was carried out in Varadhan-Williams [22]. A reflected Brownian motion in a cone with particular directions of reflection arises in Le Gall's [15] study of the set of times t such that, up to time t, the path of a Brownian motion W stays inside the translated cone $W(t) - \overline{G}$.

As in [22], the problem considered here does not fit within the realm of the general theory of multi-dimensional diffusions, because, at the vertex of the cone, the boundary of the state space is not smooth and the directions of reflection are discontinuous. One of the aims of the study of concrete examples is to increase the detailed knowledge about the possible boundary behaviour of multi-dimensional diffusions. Moreover, these examples can be used as "models" for comparison with diffusion having variable coefficients and radially varying directions of reflection in locally cone-like domains. As an illustration of this, in a subsequent work, Kwon [13] has used the results of this paper to study the existence and uniqueness of a reflected Brownian motion in a cone with directions of reflection that may vary along the rays of the cone.

The basic problem, described heuristically above, is formulated in precise mathematical terms as the question of existence and uniqueness of a solution of the following submartingale problem. For this, let $C_{\overline{G}}$ denote the set of continuous functions $w: [0, \infty) \to \overline{G}$, endowed with the topology of uniform convergence on compact sets in $[0, \infty)$. For each $t \geq 0$, let $\mathscr{M}_t = \sigma\{w(s): 0 \leq s \leq t\}$ denote the σ-algebra of subsets of $C_{\overline{G}}$ generated by the coordinate maps $w \to w(s) \in \mathbb{R}^d$ for $0 \leq s \leq t$, and let $\mathscr{M} = \bigvee_{t\geq 0}\mathscr{M}_t = \sigma\{w(t): 0 \leq t < \infty\}$. Equivalently, $\mathscr{M}_t$ (resp. $\mathscr{M}$) is the Borel σ-algebra associated with the topology of uniform convergence on $[0, t]$ (resp. compact subsets of $[0, \infty)$). For a probability measure P on $(C_{\overline{G}}, \mathscr{M})$, a process $\{X(t),\ t \geq 0\}$ will be called a P-(sub)-martingale on $(C_{\overline{G}}, \mathscr{M},\ \{\mathscr{M}_t\})$ if $\{X(t),\ \mathscr{M}_t,\ t \geq 0\}$ is a (sub)-martingale on $(C_{\overline{G}}, \mathscr{M}, P)$. Note in particular that this implies X is adapted to $\{\mathscr{M}_t,\ t \geq 0\}$. Let $C_b^2(\overline{G})$ denote the set of functions that are twice continuously differentiable in some domain containing $\overline{G}$ and that together with their first and second partial derivatives are bounded on $\overline{G}$.

Submartingale Problem (SP). A family $\{P_x,\ x \in \overline{G}\}$ of probability measures on $(C_{\overline{G}}, \mathscr{M})$ is a solution of the submartingale problem (SP) associated with the data $(G, \mathbf{v})$ if for each $x \in \overline{G}$, P_x has the following three properties.

$$P_x(w(0) = x) = 1. \tag{1.4}$$

(1.5) For each $f \in C_b^2(\overline{G})$ that is constant in a neighborhood of the vertex of G and satisfies

$$\mathbf{v} \cdot \nabla f \geq 0 \quad \text{on } \partial G \backslash \{0\},$$

we have

$$f(w(t)) - \frac{1}{2}\int_0^t \Delta f(w(s))\, ds$$

is a P_x-submartingale on $(C_{\overline{G}}, \mathscr{M}, \{\mathscr{M}_t\})$.

$$E^{P_x}\left[\int_0^\infty 1_{\{0\}}(w(s))\, ds\right] = 0. \tag{1.6}$$

If there is a unique solution of this submartingale problem, then it has the strong Markov property. For fixed $x \in \overline{G}$, a probability measure on $(C_{\overline{G}}, \mathscr{M})$ satisfying (1.4)–(1.6) will be referred to as a solution of the submartingale problem (SP) *starting from* x.

We now give an outline of the remainder of the paper.

§2 deals with the question of existence of a solution of the submartingale problem (SP). Our methodology follows that of Varadhan-Williams [22] in the sense that a solution is obtained as a weak limit of a family of approximating processes with a jump at the vertex. A basic building block in this is a strong Markov process that satisfies (1.1) and (1.2), but that instead of satisfying (1.3) is absorbed at the vertex. This *absorbed process* is characterized as a solution of a submartingale problem. It is shown that if a certain parameter α (depending on the geometric data of the problem) is strictly positive, then the absorbed process reaches the vertex almost surely. Conversely, if $\alpha \le 0$, then the vertex is almost surely not reached when the absorbed process starts away from it. The dependence of α on the geometric data is in general quite complicated and can only be determined explicitly in certain cases with a high degree of symmetry. This is in contrast to the two-dimensional situation treated in [22], where a simple explicit formula for α was given. However, we do derive a criterion involving an integral test for determining whether $\alpha > 0$ or $\alpha \le 0$. This criterion, which also applies to any process satisfying (1.4)–(1.5), is the main new feature in the existence proof. A harmonic function which involves the parameter α is also used to obtain estimates of some expected occupation times and hitting times. These estimates are used in a critical way to prove that if $\alpha < 2$, there is a probability measure on $C_{\overline{G}}$ satisfying (1.4)–(1.6). If $\alpha \ge 2$, there is no solution of (1.4)–(1.6).

§3 is concerned with the question of uniqueness of a solution of the submartingale problem. Uniqueness is the key to the proof of the strong Markov property. For $\alpha < 2$, there is a unique solution. To prove this, we use a parallel of the argument in §5 of Bass and Pardoux [2]. The crucial ingredient for this is a ratio limit theorem. Theorem 3.3 plays this role here, in place of Theorem 5.4 in [2]. For $\alpha \ge 2$, there is a unique solution of (1.4)–(1.5), which instead of satisfying (1.6), is concentrated on these paths that are absorbed at the vertex in finite time.

The above results are illustrated explicitly for the special case when $\overline{G}$ is a "circular cone", $\mathbf{v}$ has only normal and radial components, and the scalar radial components are all equal (in magnitude and sign). Then, if ξ denotes the azimuthal angle of the cone (measured from the axis of symmetry to the boundary), and β denotes the length of the radial component of $\mathbf{v}$ ($\beta < 0$ if the radial component is towards the vertex, $\beta \ge 0$ otherwise), then $\alpha > 0$ if and only if

$$\beta < (2-d)(\sin\xi)^{2-d}\int_0^{\xi}(\sin\theta)^{d-2}\,d\theta\,,$$

and $\alpha \ge 2$ if and only if $\xi < \cos^{-1}(1/\sqrt{d})$ and

$$\beta \le -d\sin\xi\cos\xi/(d\cos^2\xi - 1)\,.$$

1.2 Notation and terminology. The σ-algebras on the Euclidean spaces $\mathbb{R}^n$, $n \ge 1$, are taken to be the Borel σ-algebras. The Euclidean norm on $\mathbb{R}^n$ is denoted by $|\cdot|$. The unit sphere centered at the origin in $\mathbb{R}^n$ is denoted by S^{n-1}. Vectors will be denoted in bold type. In particular, if $x \in \mathbb{R}^n$, when x is to be regarded as a vector, it will be denoted by $\mathbf{x}$. The Laplacian and gradient operators in $\mathbb{R}^n$ will be denoted by Δ and ∇ respectively. For future reference, we note the expressions for the gradient and Laplace operators on $\mathbb{R}^n\backslash\{0\}$ in terms of spherical coordinates $(r, \omega) \in (0, \infty) \times S^{n-1}$. The gradient operator is given by

$$\nabla = \frac{\partial}{\partial r}\mathbf{e}_\mathbf{r} + r^{-1}\nabla_{S^{n-1}}, \tag{1.7}$$

where $\mathbf{e}_\mathbf{r}$ is the unit vector in the radial direction and $\nabla_{S^{n-1}}$ is the tangential gradient operator on the manifold S^{n-1}. The Laplacian is given by

$$\Delta = \frac{\partial^2}{\partial r^2} + (n-1)r^{-1}\frac{\partial}{\partial r} + r^{-2}\Delta_{S^{n-1}} \tag{1.8}$$

where $\Delta_{S^{n-1}}$ is the Laplace-Beltrami operator on S^{n-1}. For further expansions of these operators, see [10, p. 339].

For each Borel set $F \subset \mathbb{R}^n$, let $C(F)$ denote the set of continuous real-valued functions on F, and let $C_b(F)$ denote the functions in $C(F)$ that are bounded on F. For each $k \geq 1$, let $C^k(F)$ denote the set of real-valued functions that are defined and k-times continuously differentiable on some domain containing F. Let $C_b^k(F)$ denote those functions in $C^k(F)$ that together with their partial derivatives up to and including those of order k are bounded on F.

For a Borel set $F \subset \overline{\Omega} \subset S^{d-1}$, $C^k(F)$ and $C_b^k(F)$ can be identified with the corresponding sets of functions obtained by stereographic projection of $\overline{\Omega}$ into $\mathbb{R}^{d-1}$, where a point in $S^{d-1}\backslash\overline{\Omega}$ is chosen as the point sent to infinity by this projection. We may thus regard $C^k(F)$ and $C_b^k(F)$ as Banach spaces of functions on $F \subset \overline{\Omega}$ and apply results such as those in Gilbarg and Trudinger [8] which can be inferred from properties of their stereographic projections.

A function $\tau: C_{\overline{G}} \to [0, \infty]$ is a *stopping time* if $\{\tau \leq t\} \equiv \{w \in \overline{G}: \tau(w) \leq t\}$ is in $\mathcal{M}_t$ for all $t \geq 0$. The σ-algebra associated with τ is denoted by $\mathcal{M}_\tau = \sigma\{w(t \wedge \tau): t \geq 0\}$. Define $\tau_0: C_{\overline{G}} \to [0, \infty]$ by

$$\tau_0 = \inf\{t \geq 0: w(t) = 0\},$$

where 0 denotes the origin (vertex of the cone $\overline{G}$). We adopt the usual convention that equalities between random variables or processes are to hold almost surely.

2. Existence

2.1 Absorbed process.

Theorem 2.1. *For each $x \in \overline{G}$, there is a unique probability measure P_x^0 on $(C_{\overline{G}}, \mathcal{M})$ with the following three properties.*

$$P_x^0(w(0) = x) = 1. \tag{2.1}$$

(2.2) *For each* $f \in C_b^2(\overline{G})$ *that satisfies*

$$\mathbf{v} \cdot \nabla f \geq 0 \quad on\ \partial G \backslash \{0\},$$

we have

$$f(w(t \wedge \tau_0)) - \frac{1}{2} \int_0^{t \wedge \tau_0} \Delta f(w(s))\, ds$$

is a P_x^0*-submartingale on* $(C_{\overline{G}}, \mathscr{M}, \{\mathscr{M}_t\})$.

$$P_x^0(w(t) = 0 \quad for\ all\ t \geq \tau_0) = 1. \tag{2.3}$$

The family $\{P_x^0, x \in \overline{G}\}$ is called the solution of the submartingale problem associated with $(G, \mathbf{v})$ with absorption at the vertex.

The following lemmas are needed for the proof of the uniqueness in Theorem 2.1. The proof of the first lemma is deferred to §2.2, where the function in question is also used to determine when the vertex is reached.

Lemma 2.1. *There is a real number* α *and a function* $\psi_\alpha \in C^2(\overline{\Omega})$: $\psi_\alpha > 0$ *on* $\overline{\Omega}$ *if* $\alpha \neq 0$, *or* $\chi \in C^2(\overline{\Omega})$ *if* $\alpha = 0$, *such that* $\Phi \in C^2(\overline{G} \backslash \{0\})$ *defined for* $r > 0$, $\omega \in \overline{\Omega}$, *by*

$$\Phi(r\omega) = \begin{cases} r^\alpha \psi_\alpha(\omega), & if\ \alpha \neq 0, \\ \ln r + \chi(\omega), & if\ \alpha = 0, \end{cases} \tag{2.4}$$

satisfies

$$\Delta \Phi = 0 \quad in\ \overline{G} \backslash \{0\}, \tag{2.5}$$

$$\mathbf{v} \cdot \nabla \Phi = 0 \quad on\ \partial G \backslash \{0\}. \tag{2.6}$$

Proof. See §2.2. □

Define a function Ψ by

$$\Psi = \begin{cases} \Phi & \text{if } \alpha > 0, \\ \exp(\Phi) & \text{if } \alpha = 0, \\ \Phi^{-1} & \text{if } \alpha < 0 \end{cases} \tag{2.7}$$

on $\overline{G} \backslash \{0\}$ and $\Psi(0) = 0$. Then Ψ is continuous on $\overline{G}$, $\Psi \in C^2(\overline{G} \backslash \{0\})$, $\Psi > 0$ on $\overline{G} \backslash \{0\}$, and $\mathbf{v} \cdot \nabla \Psi = 0$ on $\partial G \backslash \{0\}$. In addition, since $\Psi(r\omega) = r^\beta h(\omega)$ where $\beta = |\alpha|$ if $\alpha \neq 0$, $\beta = 1$ if $\alpha = 0$, and $h > 0$ on $\overline{\Omega}$, we see that Ψ gives us an alternative means of measuring distance from the vertex, which will be convenient for some manipulations. We also need smooth bounded domains G_m approximating G as follows. For each positive integer m, let G_m be a C^2 domain in $\mathbb{R}^d$ such that

$$\left\{x : \frac{1}{2m} \leq \Psi(x) \leq 2m\right\} \cap G \subset G_m \subset \left\{x : \frac{1}{4m} \leq \Psi(x) \leq 4m\right\} \cap G,$$

and

$$\left\{x : \frac{1}{2m} \leq \Psi(x) \leq 2m\right\} \cap \partial G_m = \left\{x : \frac{1}{2m} \leq \Psi(x) \leq 2m\right\} \cap \partial G,$$

and such that there is a C^2 vector field $\mathbf{v_m}$ defined on ∂G_m that agrees with $\mathbf{v}$ on $\{x \in \partial G: \frac{1}{2m} \leq \Psi(x) \leq 2m\}$ and satisfies $\mathbf{v_m} \cdot \mathbf{n_m} = 1$ where $\mathbf{n_m}$ is the inward unit normal to ∂G_m. Note that we could have chosen G_m to be a C^3 domain, since $\partial\Omega$ has this property, but we shall not need the added smoothness for the proof of Theorem 2.1.

Lemma 2.2. *Fix a positive integer m and let $f_m \in C_b^2(\mathbb{R}^d)$ such that*

$$\mathbf{v_m} \cdot \nabla f_m \geq 0 \quad on\ \partial G_m. \tag{2.8}$$

Then there is $f \in C_b^2(\overline{G})$ such that f is constant in some neighborhood of the vertex of $\overline{G}$,

$$\mathbf{v} \cdot \nabla f \geq 0 \quad on\ \partial G, \tag{2.9}$$

and $f = f_m$ on $\{x \in \overline{G}: \frac{1}{m} \leq \Psi(x) \leq m\}$.

Proof. Let $g: \mathbb{R} \to [0, 1]$ be a twice continuously differentiable function such that

$$g(y) = \begin{cases} 0 & \text{for } y \leq \frac{1}{2m} \text{ and } y \geq 2m, \\ 1 & \text{for } \frac{1}{m} \leq y \leq m. \end{cases}$$

It is readily verified that $g(\Psi) \in C^2(\overline{G})$.

$$g(\Psi)(x) = \begin{cases} 0 & \text{on } \left\{x \in \overline{G}: \Psi(x) \leq \frac{1}{2m} \text{ or } \Psi(x) \geq 2m\right\}, \\ 1 & \text{on } \left\{x \in \overline{G}: \frac{1}{m} \leq \Psi(x) \leq m\right\}, \end{cases}$$

and

$$\mathbf{v} \cdot \nabla g(\Psi) = (\mathbf{v} \cdot \nabla\Psi) g'(\Psi) = 0 \quad \text{on } \partial G.$$

Since ∂G, $\mathbf{v}$, agree with ∂G_m, $\mathbf{v_m}$, respectively, on $\{x \in \overline{G}: \frac{1}{2m} \leq \Psi(x) \leq 2m\}$, it follows that $f \equiv f_m g(\Psi)$ has the desired properties. □

Proof of Theorem 2.1. We first verify the uniqueness. For each positive integer m, define

$$\tau_m = \inf\left\{t \geq 0: \Psi(w(t)) \leq \tfrac{1}{m} \text{ or } \Psi(w(t)) \geq m\right\}.$$

Note that by path continuity, $\tau_m(w) \uparrow \tau_0(w)$ as $m \to \infty$, for all $w \in C_{\overline{G}}$. Combining this with condition (2.3), we see that it suffices to show for each $x \in \overline{G}\backslash\{0\}$ and m: $m^{-1} < \Psi(x) < m$, that there is at most one probability measure P_x^m on $\mathscr{M}_{\tau_m}$ such that (2.1)–(2.2) hold with P_x^m, τ_m in place of P_x^0, τ_0, respectively. By Lemma 2.2 and the definition of τ_m, one can replace $f \in C_b^2(\overline{G})$ and $\mathbf{v}$, by $f_m \in C_b^2(\mathbb{R}^d)$ and $\mathbf{v_m}$, respectively, in the modified version of (2.2). By the theory of Stroock-Varadhan [21, p. 192] for reflected diffusions in smooth domains with smooth data, these modified conditions characterize the law of reflected Brownian motion in G_m with reflection field $\mathbf{v_m}$ on ∂G_m and starting point x, up its first exit from $\{z \in \overline{G}: \frac{1}{m} < \Psi(z) < m\}$. Note that the submartingale problem used in [21] employs functions f in (2.2) that also depend on time and $\frac{1}{2}\Delta f$ is replaced by

$$1_{G_m}(w(s))\left(\frac{\partial f}{\partial s}+\frac{1}{2}\Delta f\right)(s, w(s))$$

there. By virtue of the time homogeneity of our data, this is equivalent to the time independent formulation we have used here. Thus, uniqueness holds.

For the proof of existence, we adopt a constructive pathwise approach. One could alternatively take a projective limit of the measures P_x^m on $\mathscr{M}_{\tau_m}$ corresponding to the reflected Brownian motions with smooth data $(G_m, \mathbf{v_m})$. However, since there is no guarantee a priori that there is a projective limit on *continuous* path space, one would have to establish the limit first on say the space of right continuous paths and then verify that the limit measure actually lives on the space of continuous paths. Since the estimates for the latter are the same as for the constructive approach, we use the more concrete constructive approach here.

Let B be a d-dimensional Brownian motion starting from the origin, defined on a complete probability space with probability measure Q. Let $\{\mathscr{F}_t, t \geq 0\}$ be the right continuous filtration obtained by augmenting the raw filtration associated with B by the Q-null sets. Let $x \in \overline{G}\backslash\{0\}$. It follows from Proposition 4.1 of Lions and Sznitman [16] that for each m: $\frac{1}{m} < \Psi(x) < m$, there is a unique pair of continuous $\mathscr{F}_t$-adapted processes (X^m, L^m) such that the following two properties hold.

$$X^m(t) = x + B(t) + \int_0^t \mathbf{v_m}(X^m(s))\, dL^m(s) \in \overline{G}_m \quad \text{for all } t \geq 0, \tag{2.10}$$

and

$$\begin{gathered} L^m \text{ is a nondecreasing real-valued process such that} \\ L^m(t) = \int_0^t 1_{\partial G_m}(X^m(s))\, dL^m(s) \quad \text{for all } t \geq 0. \end{gathered} \tag{2.11}$$

For each m, define

$$\sigma_m = \inf\left\{t \geq 0\colon \Psi(X^m(t)) \leq \tfrac{1}{m} \text{ or } \Psi(X^m(t)) \geq m\right\}.$$

Then, for $k \geq m$, by Proposition 4.1 in [16] and the coincidence of the data $(G_m, \mathbf{v_m})$ with $(G_k, \mathbf{v_k})$ on $\{x \in \overline{G}\colon \frac{1}{m} \leq \Psi(x) \leq m\}$, we have Q-a.s.

$$X^k(\cdot \wedge \sigma_m) = X^m(\cdot \wedge \sigma_m) \quad \text{and} \quad L^k(\cdot \wedge \sigma_m) = L^m(\cdot \wedge \sigma_m).$$

In particular,

$$\sigma_m = \inf\left\{t \geq 0\colon \Psi(X^k(t)) \leq \tfrac{1}{m} \text{ or } \Psi(X^k(t)) \geq m\right\}, \qquad Q\text{-a.s.}$$

Thus, the following are well defined Q-a.s.:

$$\sigma = \lim_{m\to\infty} \uparrow \sigma_m, \quad L(\cdot) = \lim_{m\to\infty} \uparrow L^m(\cdot \wedge \sigma_m),$$

$$X(t) = \begin{cases} \lim_{m\to\infty} X^m(t), & \text{for all } t \in [0, \sigma), \\ 0, & \text{for all } t \in [\sigma, \infty). \end{cases}$$

On the exceptional Q-null set, we could define σ, L, X by the same expressions, but with limsup in place of lim. Note that Q-a.s., L is finite on $[0, \sigma)$, although it may take the value $+\infty$ on $[\sigma, \infty)$, and

$$X(t) = x + B(t) + \int_0^t \mathbf{v}(X(s))\, dL(s) \quad \text{for all } t \in [0, \sigma), \tag{2.12}$$

is continuous on $[0, \sigma)$. Although X is not a priori continuous at $\sigma < \infty$, we will show that this is in fact true Q-a.s., by proving that

$$Q\left(\overline{\lim_{t\uparrow\sigma}}\, \Psi(X(t)) > 0,\ \sigma < \infty\right) = 0. \tag{2.13}$$

For this, let J be a positive integer such that $J^{-1} < \Psi(x) < J$ and note that

$$\begin{aligned} &\left\{\overline{\lim_{t\uparrow\sigma}}\, \Psi(X(t)) > 0,\ \sigma < \infty\right\} \\ &\quad = \bigcup_{j\ge J} \left\{\overline{\lim_{t\uparrow\sigma}}\, \Psi(X(t)) > \frac{1}{j},\ \underline{\lim_{t\uparrow\sigma}}\, \Psi(X(t)) = 0,\ \sigma < \infty\right\} \\ &\quad\quad \cup \bigcup_{j\ge J} \left\{\overline{\lim_{t\uparrow\sigma}}\, \Psi(X(t)) = +\infty,\ \underline{\lim_{t\uparrow\sigma}}\, \Psi(X(t)) > \frac{1}{j},\ \sigma < \infty\right\}, \end{aligned} \tag{2.14}$$

since $\Psi(X(\sigma_m)) = m^{-1}$ or m on $\{\sigma_m < \infty\}$, for all m sufficiently large. Fix $j \ge J$ and define a sequence $\{\eta_k\}_{k=0}^{\infty}$ inductively such that $\eta_0 = 0$, and for $k \ge 1$,

$$\eta_{2k-1} = \inf\left\{t \ge \eta_{2k-2} \colon \Psi(X(t)) \le \frac{1}{2j}\right\},$$
$$\eta_{2k} = \inf\left\{t \ge \eta_{2k-1} \colon \Psi(X(t)) \ge \frac{1}{j}\right\},$$

with the usual convention that the infimum of the empty set is $+\infty$. Note that

$$\left\{\overline{\lim_{t\uparrow\sigma}}\, \Psi(X(t)) > \frac{1}{j},\ \underline{\lim_{t\uparrow\sigma}}\, \Psi(X(t)) = 0,\ \sigma < \infty\right\} \subset \{\eta_{2k} < \sigma < \infty \text{ for all } k\}. \tag{2.15}$$

We shall prove that the right member of (2.15) is a Q-null set. For this, consider $k \ge 1$. By (2.12), after shifting time in the dL integral, on $\{\eta_{2k} < \infty\}$ we have

$$X(t + \eta_{2k}) = X(\eta_{2k}) + B(t + \eta_{2k}) - B(\eta_{2k}) + \int_0^t \mathbf{v}(X(s + \eta_{2k}))\, dL(s + \eta_{2k})$$

for $0 \le t \le \eta_{2k+1} - \eta_{2k} = \inf\{s \ge 0 \colon \Psi(X(s + \eta_{2k})) \le \frac{1}{2j}\}$. Let $\varepsilon > 0$ be such that the distances between the compact sets $\Psi^{-1}(\frac{1}{2j})$ and $\Psi^{-1}(\frac{1}{j})$ and between $\Psi^{-1}(2j)$ and $\Psi^{-1}(\frac{1}{j})$ in $\overline{G}$ exceed ε. On $\{\eta_{2k} < \infty\}$, define

$$\tau_k = \inf\{s \ge 0 \colon |X(s + \eta_{2k}) - X(\eta_{2k})| \ge \varepsilon\}.$$

It follows from the uniqueness in Proposition 4.1 of [16] that on $\{\eta_{2k} < \infty\}$, $L((\cdot \wedge \tau_k) + \eta_{2k})$ is adapted to the filtration generated by $X(\eta_{2k})$ and

$B(\cdot+\eta_{2k})-B(\eta_{2k})$. Then by Itô's formula and the uniqueness of solutions of "stopped" submartingale problems (see Theorem 5.6 of [21]), on $\{\eta_{2k}<\infty\}$, the conditional law of $X((\cdot\wedge\tau_k)+\eta_{2k})$ given $\mathscr{F}_{\eta_{2k}}$ is the law of reflected Brownian motion in G_j with reflection field $\mathbf{v}_j$ and initial position $X(\eta_{2k})\in\Psi^{-1}(1/j)$, stopped at the first time it has gone distance ε from its starting point. From the uniform estimate contained in the Remark following Theorem 3.1 of [21], it follows that for each $\gamma\in(0,1)$ there is $t_\gamma>0$ such that for all $k\geq 1$,

$$Q(\tau_k>t_\gamma|\mathscr{F}_{\eta_{2k}})>\gamma\quad\text{on }\{\eta_{2k}<\infty\}. \tag{2.16}$$

Then,

$$\begin{aligned}\sum_{k=1}^\infty & Q(\eta_{2k}<\infty,\ \eta_{2k+1}-\eta_{2k}>t_\gamma|\mathscr{F}_{\eta_{2k}})\\ &\geq\sum_{k=1}^\infty 1_{\{\eta_{2k}<\infty\}}Q(\tau_k>t_\gamma|\mathscr{F}_{\eta_{2k}})\geq\sum_{k=1}^\infty 1_{\{\eta_{2k}<\infty\}}\gamma.\end{aligned}$$

Thus, for fixed $\gamma\in(0,1)$

$$\begin{aligned}\{\eta_{2k}<\infty\text{ for all }k\}&=\left\{\sum_{k=1}^\infty 1_{\{\eta_{2k}<\infty\}}\gamma=\infty\right\}\\ &\subset\left\{\sum_{k=1}^\infty Q(\eta_{2k}<\infty,\ \eta_{2k+1}-\eta_{2k}>t_\gamma|\mathscr{F}_{\eta_{2k}})=\infty\right\}.\end{aligned}$$

It then follows from an extension of the Borel-Cantelli lemma [9, Corollary 2.3] that the last set above is Q-a.s. equal to

$$\{\eta_{2k}<\infty,\ \eta_{2k+1}-\eta_{2k}>t_\gamma\text{ for infinitely many }k\}.$$

Thus, Q-a.s., $\{\eta_{2k}<\infty\text{ for all }k\}\subset\{\lim_{k\to\infty}\eta_{2k}=\infty\}$. Hence, the right member of (2.15) has Q-probability zero and therefore so does the left member.

To complete the proof of (2.13), we will prove a similar result for the other sets depending on j in (2.14). For this, η_k and τ_k will be defined anew. First, fix $j\geq J$ and let $\{m_k\}_{k=1}^\infty$ be a strictly increasing sequence of positive integers such that the following two conditions hold.

(i) m_1 is sufficiently large that $m_1>j$,

$$K\equiv\inf\{|x|:x\in\Psi^{-1}(m_1)\}>2,$$

and

$$\sup\left\{|x|:x\in\Psi^{-1}\left(\frac{1}{m_1}\right)\right\}<1-K^{-1}.$$

(ii) For $k\geq 2$, the distance between $\Psi^{-1}(m_{k-1})$ and $\Psi^{-1}(m_k)$ exceeds 1. By induction, define a sequence of stopping times $\{\eta_k\}_{k=0}^\infty$ such that $\eta_0=0$ and

$$\eta_k=\inf\{t\geq\eta_{k-1}:\Psi(X(t))\geq m_k\}.$$

Then,

$$\left\{\overline{\lim_{t\uparrow\sigma}}\Psi(X(t)) = +\infty,\ \underline{\lim_{t\uparrow\sigma}}\Psi(X(t)) > \frac{1}{j},\ \sigma<\infty\right\} \tag{2.17}$$
$$\subset \{\eta_k < \sigma < \infty \text{ for all } k\}.$$

On $\{\eta_k < \infty\}$, define

$$\tau_k = \inf\{s \geq 0\colon |X(s+\eta_k) - X(\eta_k)| \geq 1\}.$$

Then by (2.12), on $\{\eta_k < \infty\}$, $k \geq 1$, we have

$$X(t+\eta_k) = X(\eta_k) + B(t+\eta_k) - B(\eta_k) + \int_0^t \mathbf{v}(X(s+\eta_k))\,dL(s+\eta_k) \tag{2.18}$$

for $0 \leq t \leq \tau_k$. Here $B(\cdot+\eta_k) - B(\eta_k)$ is a Brownian motion independent of $\mathscr{F}_{\eta_k}$ and $L((\cdot\wedge\tau_k)+\eta_k)$ is adapted to the filtration generated by $X(\eta_k)$ and $B(\cdot+\eta_k) - B(\eta_k)$. Now, on $\{\eta_k < \infty\}$, define $\gamma_k = |X(\eta_k)|$,

$$\widehat{X}_k(t) = \gamma_k^{-1}X(\gamma_k^2 t + \eta_k), \quad \widehat{B}_k(t) = \gamma_k^{-1}(B(\gamma_k^2 t + \eta_k) - B(\eta_k)),$$
$$\widehat{L}_k(t) = \gamma_k^{-1}L(\gamma_k^2 t + \eta_k), \quad \hat{\tau}_k = \inf\{s \geq 0\colon |\widehat{X}_k(s) - \widehat{X}_k(0)| \geq \gamma_k^{-1}\}.$$

Then, by (2.18) and the radial homogeneity of $\mathbf{v}$, we have

$$\widehat{X}_k(t) = \widehat{X}_k(0) + \widehat{B}_k(t) + \int_0^t \mathbf{v}(\widehat{X}_k(s))\,d\widehat{L}_k(s) \tag{2.19}$$

for $0 \leq t \leq \hat{\tau}_k$. Note that by condition (i) on m_1 and the fact that $\{m_k\}$ is increasing,

$$m_1^{-1} \leq \Psi(\widehat{X}_k(\cdot\wedge\hat{\tau}_k)) \leq m_1 \quad \text{for all } k \geq 1.$$

Moreover, by Brownian scaling $\widehat{B}_k$ is a Brownian motion independent of $\mathscr{F}_{\eta_k}$, and by the radial homogeneity of $\partial G\backslash\{0\}$, $\widehat{L}_k(\cdot\wedge\hat{\tau}_k)$ can increase only when $\widehat{X}_k(\cdot\wedge\hat{\tau}_k) \in \partial G$. It then follows by applying Itô's formula to $\widehat{X}^k(\cdot\wedge\hat{\tau}_k)$ and using the uniqueness of "stopped" submartingale problems for $(G_{m_1}, \mathbf{v}_{m_1})$ that on $\{\eta_k < \infty\}$, the conditional law of $\widehat{X}_k(\cdot\wedge\hat{\tau}_k)$ given $\mathscr{F}_{\eta_k}$ is the law of reflected Brownian motion with data $(G_{m_1}, \mathbf{v}_{\mathbf{m}_1})$ and initial position $\widehat{X}_k(0) \in \{z \in \overline{G}\colon |z| = 1\}$, stopped at the first time it has gone a distance γ_k^{-1} from its starting point. Let $P_z^{m_1}$ denote the law of reflected Brownian motion in G_{m_1} with reflection field $\mathbf{v}_{\mathbf{m}_1}$ and initial position z. By [21, p. 181], we have the following uniform estimate on exit times for this diffusion:

$$P_z^{m_1}\left(\sup_{0\leq s\leq t}|w(s) - w(0)| \geq r\right) \leq Ctr^{-2}$$

where C does not depend on $z \in G_{m_1}$, t or r. Thus, combining the above we have for any $t > 0$, on $\{\eta_k < \infty\}$,

$$Q(\tau_k > t|\mathscr{F}_{\eta_k}) = Q(\hat{\tau}_k > \gamma_k^{-2} t|\mathscr{F}_{\eta_k})$$
$$= P^{m_1}_{\widehat{X}_k(0)}\left(\sup_{0\le s\le \gamma_k^{-2}t} |w(s) - w(0)| < \gamma_k^{-1}\right)$$
$$\geq 1 - (C\gamma_k^{-2}t)\gamma_k^2 = 1 - Ct.$$

Thus, for each $\gamma \in (0, 1)$, there is $t_\gamma > 0$ such that for all $k \geq 1$, $Q(\tau_k > t_\gamma|\mathscr{F}_{\eta_k}) > \gamma$ on $\{\eta_k < \infty\}$. The proof that the right and hence the left member of (2.17) has Q-probability zero then follows in a similar manner to the argument following (2.16). This completes the proof of (2.13).

Thus we have shown that X is a continuous process satisfying (2.12) and $X(t) = 0$ for all $t \geq \sigma$, where by the continuity of X, $\sigma = \inf\{s \geq 0: X(s) = 0\}$. Note that Q-a.s.,

$$\sigma_m = \inf\left\{t \geq 0: \Psi(X(t)) \leq \frac{1}{m} \text{ or } \Psi(X(t)) \geq m\right\}.$$

Now, for any $f \in C_b^2(\overline{G})$ satisfying $\mathbf{v}\cdot\nabla f \geq 0$ on $\partial G\backslash\{0\}$, we have Q-a.s. by Itô's formula

$$\begin{aligned} &f(X(t\wedge\sigma_m)) - \frac{1}{2}\int_0^{t\wedge\sigma_m} \Delta f(X(s))\,ds \\ &\quad = f(X(0)) + \int_0^{t\wedge\sigma_m} \nabla f(X(s))\,dB(s) + \int_0^{t\wedge\sigma_m} (\mathbf{v}\cdot\nabla f)(X(s))\,dL(s). \end{aligned} \tag{2.20}$$

For fixed t, as $m \to \infty$, by the continuity of X and boundedness of f and Δf, we can take the almost sure limit as $m \to \infty$ of the first three terms in (2.20), and by the boundedness of ∇f, we can take the L^2 limit of the stochastic integral, and since $L(\cdot\wedge\sigma_m)$ is nondecreasing and can increase only when X is on ∂G, we can take a monotone a.s. limit of the last term to obtain Q-a.s. for all $t \geq 0$,

$$\begin{aligned} &f(X(t\wedge\sigma)) - \frac{1}{2}\int_0^{t\wedge\sigma} \Delta f(X(s))\,ds \\ &\quad = f(X(0)) + \int_0^{t\wedge\sigma} \nabla f(X(s))\,dB(s) + \int_{[0,t\wedge\sigma)} (\mathbf{v}\cdot\nabla f)(X(s))\,dL(s). \end{aligned}$$

Since the sum of the first two terms on the right of the equals sign is a martingale and the third term is a continuous increasing process adapted to $\{\mathscr{F}_t\}$, and the sum of all three is bounded on each compact time interval, being equal to the left member, it follows that the left member is a Q-submartingale. The probability measure induced on $(C_{\overline{G}}, \mathscr{M})$ by X under Q is then readily seen to satisfy properties (2.1)–(2.3). Since $x \in \overline{G}\backslash\{0\}$ was arbitrary, and solvability for $x = 0$ is trivial, this completes the proof of existence of $\{P_x^0, x \in \overline{G}\}$. □

Remark. It follows from the uniqueness in Theorem 2.1 (cf. [21, p. 196]) that the family $\{P_x^0, x \in \overline{G}\}$ has the strong Markov property. Uniqueness, together with a tightness estimate (cf. [21, p. 181]), yields that for each bounded continuous functional f on $C_{\overline{G}}$, $x \to E^{P_x^0}[f(w(\cdot))]$ is continuous on $\overline{G}\setminus\{0\}$. The latter implies the Feller continuity on $\overline{G}\setminus\{0\}$, so that $x \to E^{P_x^0}[f(w(t))]$ is continuous on $\overline{G}\setminus\{0\}$ for each $f \in C_b(\overline{G})$ and $t \geq 0$.

Corollary 2.1. *Let F be a Borel set in $G\setminus\{0\}$ and suppose F has zero Lebesgue measure. Then,*

$$E^{P_x^0}\left[\int_0^{\tau_0} 1_F(w(s))\,ds\right] = 0 \quad \textit{for all } x \in \overline{G}. \tag{2.21}$$

In particular, this holds with $F = \partial G\setminus\{0\}$.

Proof. Let P_x^m and τ_m be as defined in the uniqueness part of the proof of Theorem 2.1. Then by the theory of Stroock-Varadhan [21], (2.21) holds with P_x^m and τ_m in place of P_x^0 and τ_0, respectively. The desired result then follows for $x \neq 0$ by letting $m \to \infty$, and it clearly holds for $x = 0$. □

The following "scaling" property plays a key role in the proof of uniqueness in §3.

Lemma 2.3. *Let $x \in \overline{G}$ and $r > 0$. Then for each $A \in \mathscr{M}$,*

$$P_x^0(A) = P_{rx}^0(r^{-1}w(r^2\cdot) \in A). \tag{2.22}$$

Proof. For each $A \in \mathscr{M}$, let $Q_x(A)$ denote the right member of (2.22). By the uniqueness in Theorem 2.1, it suffices to verify that (2.1)–(2.3) hold with Q_x in place of P_x^0.

Properties (2.1) and (2.3) for Q_x follow easily from those for P_{rx}^0 and the fact that

$$\tau_0(w) = r^2\tau_0(r^{-1}w(r^2\cdot)). \tag{2.23}$$

For (2.2), note that if $f \in C_b^2(\overline{G})$ satisfies $\mathbf{v}\cdot\nabla f \geq 0$ on $\partial G\setminus\{0\}$, then so does $f(r^{-1}\cdot)$, since $\mathbf{v}(z) = \mathbf{v}(r^{-1}z)$ for $z \in \partial G\setminus\{0\}$ and $r^{-1}z \in \partial G\setminus\{0\}$ if (and only if) $z \in \partial G\setminus\{0\}$. Then, by applying property (2.2) of P_{rx}^0 to $f(r^{-1}\cdot)$, after a change of variable $(s \to r^{-2}s)$ in the time integration, we conclude that

$$\left\{f(r^{-1}w(r^2t\wedge\tau_0(w))) - \frac{1}{2}\int_0^{t\wedge r^{-2}\tau_0(w)}(\Delta f)(r^{-1}w(r^2s))\,ds,\ \mathscr{M}_{r^2t},\ t \geq 0\right\}$$

is a P_{rx}^0-submartingale. Then, setting $\widetilde{w}(\cdot) = r^{-1}w(r^2\cdot)$, noting $Q_x(A) = P_{rx}^0(\widetilde{w}(\cdot) \in A)$ and using (2.23), we see that (2.2) holds with Q_x in place of P_x^0. □

2.2 Is the vertex reached? The harmonic function Φ, described in Lemma 2.1, plays a key role in determining whether the absorbed process reaches the vertex of the cone. This function will also be used in §2.4 to establish the existence of a continuous strong Markov process satisfying (1.1)-(1.3) when $\alpha < 2$. Now we shall prove Lemma 2.1 and in the course of this establish a criterion for determining whether α is positive or nonnegative, and hence for determining whether the vertex is reached or not (see Theorem 2.2). Our motivation for seeking a solution of the form (2.4) is the radial homogeneity of the Laplace operator and the boundary conditions.

Proof of Lemma 2.1. We first try to find a solution of the form (2.4) for $\alpha \neq 0$. By the expressions (1.7)–(1.8) for Δ and ∇ in spherical coordinates, we see that $\Phi(r\omega) = r^\alpha \psi_\alpha(\omega)$ satisfies (2.5)–(2.6) if and only if ψ_α satisfies

$$\alpha(\alpha + d - 2)\psi_\alpha + \Delta_{S^{d-1}}\psi_\alpha = 0 \quad \text{in } \Omega, \tag{2.24}$$

$$\alpha v_r \psi_\alpha + \mathbf{v_T} \cdot \nabla_{S^{d-1}} = 0 \quad \text{on } \partial\Omega, \tag{2.25}$$

where $v_r = \mathbf{v} \cdot \mathbf{e_r}$ is the (scalar) radial component of the vector $\mathbf{v}$ and $\mathbf{v_T} = \mathbf{v} - v_r\mathbf{e_r}$ is the (vector) component of $\mathbf{v}$ in the tangent bundle to S^{d-1}. Note that on $\partial\Omega$, the inward unit normal $\mathbf{n}$ to $\partial G \backslash \{0\}$ is also an inward unit normal to $\partial\Omega$ in S^{d-1}, since $\mathbf{n} \cdot \mathbf{e_r} = 0$. Hence on $\partial\Omega$ we shall continue to use the same symbol $\mathbf{n}$, whether it is regarded as a vector in $\mathbb{R}^d$ or in the tangent space to S^{d-1}. Also, by the normalization of $\mathbf{v}$, $\mathbf{v_T} \cdot \mathbf{n} = \mathbf{v} \cdot \mathbf{n} = 1$.

In general, *explicit* formulae for $\alpha \neq 0$ and a positive $\psi_\alpha \in C^2(\overline{\Omega})$ such that (2.24)–(2.25) hold cannot be found, except in very special cases of a simple geometry for G and particular directions of reflection with a high degree of symmetry. For example, suppose $\overline{G}$ is a circular cone with azimuthal angle $\xi \in (0, \pi)$ (measured from the axis of symmetry to the boundary), i.e., $\overline{G} = \{x \in \mathbb{R}^d : \mathbf{x} \cdot \mathbf{u} \geq |\mathbf{x}| \cos\xi\}$ for some unit vector $\mathbf{u}$ in $\mathbb{R}^d$, and $\mathbf{v} = \mathbf{n} + \beta\mathbf{e_r}$ for some constant $\beta \in \mathbb{R}$. Then a solution of (2.24) can be found as in Burkholder [3, p. 192]:

$$\psi_\alpha(\theta) = F\left(-\alpha,\ \alpha + d - 2;\ \tfrac{1}{2}(d-1);\ \tfrac{1}{2}(1 - \cos\theta)\right)$$

where θ denotes the azimuthal angular coordinate for points on $\overline{\Omega}$ and F is the hypergeometric function [1]. In those cases for which α is a positive integer, $F(-\alpha, \alpha + d - 2;\ \frac{1}{2}(d-1);\ \frac{1}{2}(1 - \cos\theta))$ is a Jacobi polynomial function of $\cos\theta$, and one can determine those values of ξ and β such that (2.25) holds. This is illustrated in the example at the end of §3.4 for $\alpha = 2$. Even in this simple setting, for noninteger values of α it is not easy to determine ξ and β such that $\psi_\alpha > 0$ and (2.25) holds. However, for us it will suffice to show existence of a solution and to give a criterion for determining the sign of α. For this we consider the following eigenvalue problem.

$$\Delta_{S^{d-1}}\psi_\alpha + \lambda(\alpha)\psi_\alpha = 0 \quad \text{in } \Omega, \tag{2.26}$$

$$\mathbf{v_T} \cdot \nabla_{S^{d-1}}\psi_\alpha + \alpha v_r \psi_\alpha = 0 \quad \text{on } \partial\Omega. \tag{2.27}$$

We then seek an $\alpha \neq 0$ such that there is an eigenvalue $\lambda(\alpha)$ and corresponding eigenfunction $\psi_\alpha \in C^2(\overline{\Omega})$ with $\psi_\alpha > 0$ on $\overline{\Omega}$ satisfying (2.26)–(2.27) and $\lambda(\alpha) = \alpha(\alpha + d - 2)$.

For the eigenvalue problem we have the following existence result. Here $d\Theta$ denotes integration with respect to $(d-1)$-dimensional surface measure on Ω.

Lemma 2.4. *For each $\alpha \in \mathbb{R}$ there is a unique pair $(\lambda(\alpha), \psi_\alpha) \in \mathbb{R} \times C^2(\overline{\Omega})$ such that $\psi_\alpha \in C^2(\overline{\Omega})$, $\psi_\alpha > 0$ on $\overline{\Omega}$, $\int_\Omega \psi_\alpha \, d\Theta = 1$ and* (2.26)–(2.27) *hold. The functions $\alpha \to \lambda(\alpha) \in \mathbb{R}$ and $\alpha \to \psi_\alpha \in C^2(\overline{\Omega})$ are real analytic. Moreover, $\lambda(\alpha)$ is a concave function of α and it is bounded above by the first eigenvalue λ_0 for equation* (2.26) *with Dirichlet boundary conditions in place of* (2.27).

Proof. We first convert (2.26)–(2.27) to an equivalent problem with a purely oblique derivative boundary condition (i.e., without the term $\alpha v_r \psi_\alpha$ in (2.27)). For this, resolve $\mathbf{v_T}$ into its inward normal and tangential components on $\partial\Omega$:

$$\mathbf{v_T} = \mathbf{n} + \mathbf{q}$$

where $\mathbf{n} \cdot \mathbf{q} = 0$ and $\mathbf{q}$ is in the tangent bundle for $\partial\Omega \subset S^{d-1}$. For brevity, we shall write $\frac{\partial}{\partial n}$ in place of $\mathbf{n} \cdot \nabla_{S^{d-1}}$. Since $v_r \in C^3(\partial\Omega)$ and $\partial\Omega$ is C^3, we can construct a function $g \in C^3(\overline{\Omega})$ such that $g = 0$ on $\partial\Omega$ and $\frac{\partial g}{\partial n} = v_r$ on $\partial\Omega$. Then $\mathbf{v_T} \cdot \nabla_{S^{d-1}} g = \frac{\partial g}{\partial n} = v_r$, and so $\mathbf{v_T} \cdot \nabla_{S^{d-1}}(e^{\alpha g}) = \alpha v_r e^{\alpha g}$. Thus, $\psi_\alpha \in C^2(\overline{\Omega})$ is a solution of (2.26)–(2.27) if and only if $f_\alpha = e^{\alpha g} \psi_\alpha \in C^2(\overline{\Omega})$ is a solution of

$$(L_\alpha + \lambda(\alpha)) f_\alpha = 0 \quad \text{in } \Omega, \tag{2.28}$$

$$\mathbf{v_T} \cdot \nabla_{S^{d-1}} f_\alpha = 0 \quad \text{on } \partial\Omega. \tag{2.29}$$

where

$$L_\alpha \equiv \Delta_{S^{d-1}} - 2\alpha \nabla_{S^{d-1}} g \cdot \nabla_{S^{d-1}} + \alpha^2 |\nabla_{S^{d-1}} g|^2 - \alpha \Delta_{S^{d-1}} g. \tag{2.30}$$

To prove the existence and analytic dependence on α of an eigenvalue and associated strictly positive eigenfunction for (2.28)–(2.29), we first need to establish the following functional analysis framework so that we can apply the Krein-Rutman [11] theory of positive linear operators and the perturbation theory of Crandall and Rabinowitz [5].

The operator L_α is uniformly elliptic in Ω with coefficients in $C^1(\overline{\Omega})$, and $\mathbf{v_T} \in C^3(\partial\Omega)$ with $\mathbf{v_T} \cdot \mathbf{n} = 1$. Moreover, for each $k > 0$, there is $\lambda_k \in \mathbb{R}$ such that for all $\alpha \in (-k, k)$,

$$\alpha^2 |\nabla_{S^{d-1}} g|^2 - \alpha \Delta_{S^{d-1}} g + \lambda_k < 0 \quad \text{on } \overline{\Omega}. \tag{2.31}$$

Fix $k > 0$ and $\alpha \in (-k, k)$. Let $D = \mathbf{v_T} \cdot \nabla_{S^{d-1}}$. It follows (after stereographic projection of $\overline{\Omega}$ into $\mathbb{R}^{d-1}$) from Theorem 6.31 of [8] and the remark following it, that for each $h \in C^1(\overline{\Omega})$ there is a unique $f \in C^2(\overline{\Omega})$ solving the oblique derivative problem:

$$(L_\alpha + \lambda_k) f = -h \quad \text{in } \Omega, \tag{2.32}$$

$$Df = 0 \quad \text{on } \partial\Omega, \tag{2.33}$$

and $\|f\|_2 \le C(\alpha)\|h\|_1$ for some constant $C(\alpha)$, where $\|\cdot\|_2$, $\|\cdot\|_1$ are the norms on the Banach spaces $C^2(\overline{\Omega})$, $C^1(\overline{\Omega})$, respectively, induced from the stereographic projection of $\overline{\Omega}$ into $\mathbb{R}^{d-1}$ (cf. §1.2). Thus, we can define a bounded linear operator $T_\alpha: C^1(\overline{\Omega}) \to C^2(\overline{\Omega})$ by $T_\alpha h = f$ and the operator norm satisfies $\|T_\alpha\| \le C(\alpha)$. Fix $\alpha_0 \in (-k, k)$ and let $\gamma = \alpha - \alpha_0$. Then

$$M_\gamma \equiv L_\alpha - L_{\alpha_0} = \gamma(-2\nabla_{S^{d-1}} g \cdot \nabla_{S^{d-1}} + (2\alpha_0 + \gamma)|\nabla_{S^{d-1}} g|^2 - \Delta_{S^{d-1}} g),$$

is a bounded linear operator from $C^2(\overline{\Omega})$ into $C^1(\overline{\Omega})$ with $\|M_\gamma\| \le \gamma C$ for some constant C that can be chosen independent of all γ's satisfying $\alpha_0 + \gamma \in (-k, k)$. It follows from this and the uniqueness of T_α that there is $\gamma_0 > 0$ (depending on α_0) such that for all $\gamma \in (-\gamma_0, \gamma_0)$, $\alpha_0 + \gamma \in (-k, k)$ and, in the operator topology of operators mapping $C^1(\overline{\Omega})$ into $C^2(\overline{\Omega})$, the series $\sum_{n=0}^\infty T_{\alpha_0}(M_\gamma T_{\alpha_0})^n$ is uniformly convergent to $T_{\alpha_0+\gamma}$ and $\gamma \to T_{\alpha_0+\gamma}$ is a real analytic function. Since $\alpha_0 \in (-k, k)$ was arbitrary, it follows that $\alpha \to T_\alpha$ is a real analytic function of $\alpha \in (-k, k)$.

Define

$$K = \{h \in C^1(\overline{\Omega}) : h \ge 0 \text{ on } \overline{\Omega}\}.$$

In the sense of Krein-Rutman [11], K is a cone in the Banach space $C^1(\overline{\Omega})$ with nonempty interior K^0. Since the inclusion map of $C^2(\overline{\Omega})$ into $C^1(\overline{\Omega})$ is compact (see [8, Lemma 6.36]), when T_α is viewed as a map from $C^1(\overline{\Omega})$ into $C^1(\overline{\Omega})$, it is compact. We now show, using the maximum principle and Hopf's boundary point lemma, that T_α maps K into K and is *strongly positive* (i.e., for any $0 \not\equiv h \in K$, we have $T_\alpha h \in K^0$). Clearly, $T_\alpha 0 = 0$. By the strong maximum principle [8, Theorem 3.5] and (2.31), for $0 \not\equiv h \in K$, $f = T_\alpha h$ is either strictly positive in $\overline{\Omega}$ or f attains a nonpositive minimum at a point $x_0 \in \partial\Omega$ and f exceeds this minimum value in Ω. In the latter case we would have $Df(x_0) > 0$, by the Hopf boundary point lemma (see [8, Lemma 3.4 and (3.11)]), which contradicts the boundary condition (2.33). Thus, for any $0 \not\equiv h \in K$ we have $T_\alpha h > 0$ on $\overline{\Omega}$ and so $T_\alpha h \in K^0$. We have now verified all the hypotheses of Theorem 6.3 of Krein-Rutman [11, p. 267], and so we conclude the following. Here K^* denotes the set of all continuous linear functionals ϕ on $C^1(\overline{\Omega})$ such that $\phi(h) \ge 0$ for all $h \in K$.

(a) The operator $T_\alpha: C^1(\overline{\Omega}) \to C^1(\overline{\Omega})$ has a unique eigenfunction $f_\alpha \in K^0$ such that $\int_\Omega f_\alpha \, d\Theta = 1$. The corresponding eigenvalue μ_α:

$$T_\alpha f_\alpha = \mu_\alpha f_\alpha \tag{2.34}$$

is strictly positive. It is the largest eigenvalue of T_α and it is simple.

(b) The adjoint operator T_α^* has a unique eigenfunctional ϕ_α^* in K^* such that $\phi_\alpha^*(1) = 1$. The associated eigenvalue is μ_α: $T_\alpha^* \phi_\alpha^* = \mu_\alpha \phi_\alpha^*$; it is simple and ϕ_α^* is a strictly positive functional on K, i.e., if $h \not\equiv 0$, $h \in K$, then $\phi_\alpha^*(h) > 0$.

Note that $f_\alpha \in C^2(\overline{\Omega})$ and $f_\alpha > 0$ on $\overline{\Omega}$, by (2.34) and the facts that $\mu_\alpha > 0$, $T_\alpha h \in C^2(\overline{\Omega})$ for any $h \in C^1(\overline{\Omega})$, and $T_\alpha h > 0$ for any $0 \not\equiv h \in K$.

Since μ_α is simple and $\alpha \to T_\alpha$ is real analytic, it follows from Lemma 1.3 of Crandall and Rabinowitz [5] that $\alpha \to \mu_\alpha$ and $\alpha \to f_\alpha \in C^2(\overline{\Omega})$ are real analytic functions of $\alpha \in (-k, k)$.

Now f_α is a solution of (2.28)–(2.29) with $\lambda(\alpha) = \lambda_k + \mu_\alpha^{-1}$. Although T_α and μ_α depend on k, by uniqueness of the eigenfunction f_α, it follows that $\lambda(\alpha)$ and f_α are uniquely determined, independent of k. Thus, since k was arbitrary, for each $\alpha \in \mathbb{R}$ we obtain a unique pair $(\lambda(\alpha), f_\alpha)$ such that $\lambda(\alpha) \in \mathbb{R}$, $f_\alpha \in C^2(\overline{\Omega})$, $\int_\Omega f_\alpha \, d\Theta = 1$, $f_\alpha > 0$ on $\overline{\Omega}$ and (2.28)–(2.29) hold. Moreover, as functions of α, $\lambda(\alpha)$ and f_α are real analytic, as is $\alpha \to e^{-\alpha g} \in C^2(\overline{\Omega})$. Setting $\psi_\alpha = e^{-\alpha g} f_\alpha / \int_\Omega e^{-\alpha g} f_\alpha \, d\Theta$, we obtain the unique pair $(\lambda(\alpha), \psi_\alpha)$ as described in Lemma 2.4, except that it remains to prove the concavity and boundedness of $\lambda(\alpha)$. For this, note that it follows from Protter and Weinberger [19] and the existence of ψ_α, that for each $\alpha \in \mathbb{R}$,

$$\lambda_0 \geq \sup_{\psi \in \mathscr{A}_\alpha} \inf_{x \in \overline{\Omega}} \left(-\frac{L\psi}{\psi}(x) \right) = \lambda(\alpha) \tag{2.35}$$

where $L = \Delta_{S^{d-1}}$,

$$M_\alpha \psi = \mathbf{v}_\mathbf{T} \cdot \nabla_{S^{d-1}} \psi + \alpha v_r \psi \quad \text{for all } \psi \in C^2(\overline{\Omega}),$$

$$\mathscr{A}_\alpha = \{\psi \in C^2(\overline{\Omega}) : M_\alpha \psi = 0 \text{ on } \partial\Omega \text{ and } \psi > 0 \text{ on } \overline{\Omega}\},$$

and λ_0 is the first eigenvalue for (2.26) with Dirichlet boundary conditions on $\partial\Omega$ in place of (2.27). To verify the concavity of $\lambda(\alpha)$, fix $\alpha_1 < \alpha_2$ and let $\beta \in (0, 1)$. A straightforward calculation shows that $\psi_0 = \psi_{\alpha_1}^\beta \psi_{\alpha_2}^{1-\beta}$ satisfies $M_{\alpha_0} \psi_0 = 0$ on $\partial\Omega$ with $\alpha_0 = \beta\alpha_1 + (1-\beta)\alpha_2$. Moreover, writing ψ_i for ψ_{α_i}, $i = 1, 2$ and ∇ for $\nabla_{S^{d-1}}$ we have

$$\begin{aligned} \frac{L\psi_0}{\psi_0} &= \gamma \frac{L\psi_1}{\psi_1} + (1-\gamma) \frac{L\psi_2}{\psi_2} \\ &\quad + \gamma(\gamma - 1) \left(\left(\frac{|\nabla \psi_1|}{\psi_1} \right)^2 + 2 \frac{\nabla \psi_1 \cdot \nabla \psi_2}{\psi_1 \psi_2} + \left(\frac{|\nabla \psi_2|}{\psi_2} \right)^2 \right) \\ &\leq \gamma \frac{L\psi_1}{\psi_1} + (1-\gamma) \frac{L\psi_2}{\psi_2} = -\gamma\lambda(\alpha_1) - (1-\gamma)\lambda(\alpha_2), \end{aligned} \tag{2.36}$$

where we used the fact that the term multiplying the negative factor $\gamma(\gamma - 1)$ is a complete square. Thus, by (2.35),

$$\lambda(\alpha_0) \geq \inf_{x \in \overline{\Omega}} \left(\frac{-L\psi_0}{\psi_0} \right) \geq \gamma\lambda(\alpha_1) + (1-\gamma)\lambda(\alpha_2),$$

and hence $\lambda(\alpha)$ is concave. □

Remark 1. The results and proof of Lemma 2.4 hold for more general elliptic equations and boundary conditions than those treated here. We are grateful to

Roger Nussbaum for helpful discussions on his investigations on this subject and their consequences in our context.

Remark 2. For $\alpha = 0$, by uniqueness, $\lambda(\alpha) = 0$ and $\psi_\alpha = 1/\int_\Omega d\Theta$.

Lemma 2.5. *There is a unique* $\psi_0^* \in C^2(\overline{\Omega})$ *such that* $\psi_0^* > 0$ *on* $\overline{\Omega}$, $\int_\Omega \psi_0^* \, d\Theta = 1$ *and*

$$\Delta_{S^{d-1}} \psi_0^* = 0 \quad \text{in } \Omega, \tag{2.37}$$

$$\mathbf{v}_\mathbf{T}^* \cdot \nabla_{S^{d-1}} \psi_0^* - (\operatorname{div}_{\partial\Omega} \mathbf{q}) \psi_0^* = 0 \quad \text{on } \partial\Omega, \tag{2.38}$$

where $\mathbf{v}_\mathbf{T}^* = \mathbf{n} - \mathbf{q}$ *and* $\operatorname{div}_{\partial\Omega} \mathbf{q}$ *is the divergence of the vector field* $\mathbf{q}$ *on the manifold* $\partial\Omega$ (*recall that* $\mathbf{q} \in C^4(\partial\Omega)$ *is the vector component of* $\mathbf{v}_\mathbf{T}$ *in the tangent bundle to* $\partial\Omega \subset S^{d-1}$).

Proof. By very similar reasoning to that in the proof of Lemma 2.4, with $\mathbf{v}_\mathbf{T}^*$ in place of $\mathbf{v}_\mathbf{T}$ and $-\operatorname{div}_{\partial\Omega} \mathbf{q} \in C^3(\partial\Omega)$ in place of αv_r, we conclude that there is a unique pair $(\lambda^*(0), \psi_0^*) \in \mathbb{R} \times C^2(\overline{\Omega})$ such that $\psi_0^* > 0$, $\int_\Omega \psi_0^* \, d\Theta = 1$ and (2.38) holds together with

$$\Delta_{S^{d-1}} \psi_0^* + \lambda^*(0) \psi_0^* = 0 \quad \text{in } \Omega. \tag{2.39}$$

Now, by the normalization of ψ_0^*, (2.39), and the divergence theorem on Ω [14, IX.4], we have

$$\lambda^*(0) = \int_\Omega \lambda^*(0) \psi_0^* \, d\Theta = -\int_\Omega \Delta_{S^{d-1}} \psi_0^* \, d\Theta = \int_{\partial\Omega} \frac{\partial \psi_0^*}{\partial n} \, d\sigma,$$

where $d\sigma$ denotes integration with respect to $(d-2)$-dimensional surface measure on $\partial\Omega$. Since $\mathbf{v}_\mathbf{T}^* = \mathbf{n} - \mathbf{q}$ and $\operatorname{div}_{\partial\Omega}(\psi_0^* \mathbf{q}) = \mathbf{q} \cdot \nabla_{\partial\Omega} \psi_0^* + (\operatorname{div}_{\partial\Omega} \mathbf{q}) \psi_0^*$ [14; p. 205, 126], where $\nabla_{\partial\Omega}$ denotes the tangential gradient on the manifold $\partial\Omega$, it follows from (2.38) that

$$\frac{\partial \psi_0^*}{\partial n} - \operatorname{div}_{\partial\Omega}(\psi_0^* \mathbf{q}) = 0 \quad \text{on } \partial\Omega, \tag{2.40}$$

and hence

$$\lambda^*(0) = \int_{\partial\Omega} \operatorname{div}_{\partial\Omega}(\psi_0^* \mathbf{q}) \, d\sigma = 0,$$

by the divergence theorem on the compact manifold $\partial\Omega$ which has no boundary. Thus, ψ_0^* satisfies (2.37) and uniqueness follows from the uniqueness of $(\lambda^*(0), \psi^*(0))$. □

Lemma 2.6. *For each* $\alpha \in \mathbb{R}$,

$$\lambda(\alpha) = \frac{-\alpha \int_{\partial\Omega} v_r \psi_\alpha \psi_0^* \, d\sigma}{\int_\Omega \psi_\alpha \psi_0^* \, d\Theta}, \tag{2.41}$$

and

$$\lambda'(0) = -\int_{\partial\Omega} v_r \psi_0^* \, d\sigma. \tag{2.42}$$

Proof. By (2.26) and the divergence theorem applied twice on Ω we have

$$\begin{aligned}\lambda(\alpha)\int_\Omega \psi_\alpha\psi_0^*\,d\Theta &= -\int_\Omega(\Delta_{S^{d-1}}\psi_\alpha)\psi_0^*\,d\Theta\\ &= -\int_\Omega \psi_\alpha\Delta_{S^{d-1}}\psi_0^*\,d\Theta - \int_{\partial\Omega}\left(\frac{\partial\psi_0^*}{\partial n}\psi_\alpha - \frac{\partial\psi_\alpha}{\partial n}\psi_0^*\right)d\sigma.\end{aligned}$$

By (2.27), (2.37)–(2.38), (2.40), and the divergence theorem on $\partial\Omega$, the last line above simplifies to

$$-\int_{\partial\Omega}(\mathrm{div}_{\partial\Omega}(\mathbf{q}\psi_0^*\psi_\alpha)+\alpha v_r\psi_\alpha\psi_0^*)\,d\sigma = -\int_{\partial\Omega}\alpha v_r\psi_\alpha\psi_0^*\,d\sigma.$$

Thus, (2.41) holds, the denominator being nonzero by the positivity of ψ_α and ψ_0^*. By the differentiability properties of ψ_α proved in Lemma 2.4, we can readily justify differentiating with respect to α under the integral signs in (2.41) to obtain $\lambda'(\alpha)$. Then noting that $\psi_0 = \text{constant}$ and $\int_\Omega \psi_0^*\,d\Theta = 1$, on setting $\alpha = 0$ we obtain (2.42). □

Remark. There is an intuitive probabilistic interpretation of the relationship between $\lambda(\alpha)$ and α as follows. Consider spherical Brownian on $\overline{\Omega}$ with oblique reflection at the boundary $\partial\Omega$, where the direction of reflection is given by the vector field $\mathbf{v_T}$. If mass is "created" at the rate αv_r on $\partial\Omega$, then $-\lambda(\alpha)$ is the rate of "killing" in Ω that just balances the creation on the boundary to produce a process that has a nontrivial steady state distribution. At points on $\partial\Omega$ where $\alpha v_r < 0$, this negative rate of "creation" is to be interpreted as killing at the rate $-\alpha v_r$, and similarly if $\lambda(\alpha) > 0$, this corresponds to creation at the rate $\lambda(\alpha)$ in Ω. The "average" of v_r over $\partial\Omega$, in the weighted sense of (2.41), determines whether killing $(\lambda(\alpha) \le 0)$ or creation $(\lambda(\alpha) > 0)$ is needed to balance the effect of the creation and/or killing on $\partial\Omega$.

We now return to the question that motivated our analysis of $\lambda(\alpha)$. By Lemma 2.4, $\gamma(\alpha) \equiv \lambda(\alpha) - \alpha(\alpha + d - 2)$ is a strictly concave function of α, bounded above by $\lambda_0 - \alpha(\alpha + d - 2)$, and so $\gamma(\alpha)$ tends to $-\infty$ as $\alpha \to \pm\infty$. Moreover, $\gamma(0) = 0$. It follows that there is (a unique) $\alpha^* > 0$ (resp. $\alpha^* < 0$) such that $\lambda(\alpha^*) = \alpha^*(\alpha^* + d - 2)$ if and only if $\lambda'(0) > d - 2$ (resp. $\lambda'(0) < d - 2$). Indeed, we have the following.

Lemma 2.7. *If* $\lambda'(0) > d - 2$ (*resp.* $\lambda'(0) < d - 2$), *there is a unique* $\alpha^* > 0$ (*resp.* $\alpha^* < 0$) *such that*

$$\lambda(\alpha^*) = \alpha^*(\alpha^* + d - 2).$$

Moreover, Φ_{α^*} *defined by*

$$\Phi_{\alpha^*}(r\omega) = r^{\alpha^*}\psi_{\alpha^*}(\omega) \quad \textit{for all } r > 0,\ \omega \in \overline{\Omega}, \tag{2.43}$$

is a solution in $C^2(\overline{G}\backslash\{0\})$ *of* (2.5)–(2.6), *and* $\psi_{\alpha^*} > 0$ *on* $\overline{\Omega}$. *If* $\lambda'(0) = d - 2$, *there is* $\chi \in C^2(\overline{\Omega})$ *such that*

$$\Phi_0(r\omega) \equiv \ln r + \chi(\omega) \quad \textit{for all } r > 0,\ \omega \in \overline{\Omega}, \tag{2.44}$$

is a solution in $C^2(\overline{G}\backslash\{0\})$ *of* (2.5)–(2.6).

Notation. If $\lambda'(0) = d - 2$, we define $\alpha^* = 0$.

Proof. Only the case $\lambda'(0) = d - 2$ needs proof. By similar analysis to that which led to (2.24)–(2.25), $\Phi_0 \in C^2(\overline{G}\backslash\{0\})$ of the form (2.44) satisfies (2.5)–(2.6) if and only if $\chi \in C^2(\overline{\Omega})$ satisfies the following

$$\left\{\begin{array}{ll} \Delta_{S^{d-1}}\chi = -(d-2) & \text{in } \Omega, \\ \mathbf{v_T} \cdot \nabla_{S^{d-1}}\chi = -v_r & \text{on } \partial\Omega. \end{array}\right. \tag{2.45}$$

For $g \in C^3(\overline{\Omega})$ as in the proof of Lemma 2.4, the above are equivalent to the following equations for $\tilde{\chi} = \chi + g \in C^2(\overline{\Omega})$:

$$\left\{\begin{array}{ll} \Delta_{S^{d-1}}\tilde{\chi} = 2 - d + \Delta_{S^{d-1}} g & \text{in } \Omega, \\ \mathbf{v_T} \cdot \nabla\tilde{\chi} = 0 & \text{on } \partial\Omega. \end{array}\right. \tag{2.46}$$

Regarding the function ψ_0^* as a functional on $C^1(\overline{\Omega})$, defined by

$$\psi_0^*(f) \equiv \langle f, \psi_0^* \rangle \equiv \int_\Omega f\psi_0^*\, d\Theta \quad \text{for all } f \in C^1(\overline{\Omega}),$$

we have $\psi_0^*(\cdot) \in K^*$, where K^* was defined in Lemma 2.4. Using the notation of the proof of Lemma 2.4, fix $\lambda_k < 0$. Then for T_0 as defined there, by (2.32) we have

$$\lambda_k\langle T_0 h, \psi_0^* \rangle = -\langle \Delta_{S^{d-1}}(T_0 h), \psi_0^* \rangle - \langle h, \psi_0^* \rangle \quad \text{for any } h \in C^1(\overline{\Omega}).$$

Now, using the divergence theorem on Ω and $\partial\Omega$ and (2.33), we obtain in a similar manner to that in Lemma 2.6 that $\langle \Delta_{S^{d-1}}(T_0 h), \psi_0^* \rangle = 0$. Thus,

$$\langle T_0 h, \psi_0^* \rangle = -\lambda_k^{-1} \langle h, \psi_0^* \rangle$$

and so $T_0^* \psi_0^* = -\lambda_k^{-1} \psi_0^*$. Thus ψ_0^* is the unique (up to a positive scalar multiple) eigenfunction for T_0^* in K^* and since the associated eigenvalue $\mu_0 = -\lambda_k^{-1}$ is simple (by (b) following (2.34)), it follows from the Fredholm alternative [8, §5.4] that there is a $\tilde{\chi} \in C^2(\overline{\Omega})$ satisfying (2.46) if and only if

$$\langle 2 - d + \Delta_{S^{d-1}} g, \psi_0^* \rangle = 0. \tag{2.47}$$

But by the divergence theorem and the properties of ψ_0^* and g we have

$$\langle \Delta_{S^{d-1}} g, \psi_0^* \rangle = -\int_{\partial\Omega} \frac{\partial g}{\partial n} \psi_0^*\, d\sigma = -\int_{\partial\Omega} v_r \psi_0^*\, d\sigma.$$

Thus, by the formula for $\lambda'(0)$, (2.47) is equivalent to $\lambda'(0) = d - 2$. □

Lemma 2.1 now follows by setting $\alpha = \alpha^*$. □

Notation. Henceforth we shall denote the distinguished α^* of Lemma 2.7 simply by α.

We now give a definitive criterion for determining whether the vertex of the cone is hit by the absorbed process.

Theorem 2.2. *Let* $x \in \overline{G}\backslash\{0\}$. *Then*

$$P_x^0(\tau_0 < \infty) = \begin{cases} 1 & \text{if } \alpha > 0, \\ 0 & \text{if } \alpha \le 0, \end{cases} \tag{2.48}$$

where $\alpha > 0$ *or* ≤ 0 *as* $\lambda'(0)$ *given by* (2.42) *is* $> (d-2)$ *or* $\le (d-2)$.

Proof. Let Ψ be defined by (2.7) and for each $r \ge 0$ let

$$\tau_r = \inf\{t \ge 0 \colon \Psi(w(t)) = r\}.$$

For $0 \le \varepsilon < \Psi(x) < R$, define $\tau_{\varepsilon R} = \tau_\varepsilon \wedge \tau_R$. We need the following result which is a consequence of the proof of Lemma 2.8 below. Since that proof does not depend on Theorem 2.2, there is no danger of circularity in using the result here.

$$P_x^0(\tau_{0R} < \infty) = 1. \tag{2.49}$$

For $\varepsilon > 0$, multiplying Φ by a function of the form $g(\Psi)$ where $g \colon \mathbb{R} \to [0, 1]$ is twice continuously differentiable and satisfies $g = 1$ on $\{y \colon \varepsilon \le y \le R\}$ and $g = 0$ on $\{y \colon y \le \frac{\varepsilon}{2}$ or $y \ge 2R\}$ (cf. Lemma 2.2), we obtain a $C_b^2(\overline{G})$ function that agrees with Ψ on $\overline{G}_{\varepsilon R} \equiv \{x \in \overline{G} \colon \varepsilon \le \Psi(x) \le R\}$ and for which the submartingale property of (2.2) holds. It follows from this, together with Doob's stopping theorem and (2.5), that $\Phi(w(\cdot \wedge \tau_{\varepsilon R}))$ is a P_x^0-martingale. Then, since Φ is bounded on $\overline{G}_{\varepsilon R}$ and (2.49) holds, after taking expectations and letting $t \to \infty$, we conclude that

$$E^{P_x^0}[\Phi(w(\tau_{\varepsilon R}))] = \Phi(x). \tag{2.50}$$

If $\alpha > 0$, then $\Phi = \Psi$ and the above yields

$$P_x^0(\tau_\varepsilon < \tau_R) = \frac{R - \Phi(x)}{R - \varepsilon}.$$

Letting $\varepsilon \downarrow 0$ and using (2.49), it follows that

$$P_x^0(\tau_0 < \tau_R) = 1 - R^{-1}\Phi(x) \quad \text{if } \alpha > 0.$$

Letting $R \to \infty$ yields (2.48) for the $\alpha > 0$ case. If $\alpha \le 0$, then $\Psi = \Phi^{-1}$ if $\alpha < 0$ or $\Psi = \exp(\Phi)$ if $\alpha = 0$ and (2.50) yields

$$P_x^0(\tau_\varepsilon < \tau_R) = \begin{cases} \dfrac{R^{-1} - \Phi(x)}{R^{-1} - \varepsilon^{-1}} & \text{if } \alpha < 0, \\[2ex] \dfrac{\ln R - \Phi(x)}{\ln R - \ln \varepsilon} & \text{if } \alpha = 0. \end{cases}$$

Letting $\varepsilon \downarrow 0$ gives $P_x^0(\tau_0 < \tau_R) = 0$ and (2.48) follows for $\alpha \le 0$.

The correspondence between the sign of $\lambda'(0) - d + 2$ and α was established in Lemma 2.7. □

Example. Consider the case when $\mathbf{q} = 0$, i.e., $\mathbf{v} = \mathbf{n} + v_r\mathbf{e_r}$ for $v_r \in C^3(\partial\Omega)$. Then $\psi_0^* = 1/\int_\Omega d\Theta$ (cf. Lemma 2.5), and so

$$\lambda'(0) = -\int_{\partial\Omega} v_r \, d\sigma \Big/ \int_\Omega d\Theta.$$

In particular, if $v_r = \beta \equiv$ constant, and $\overline{G}$ is a circular cone of azimuthal angle ξ, then using the spherical coordinate expressions for $d\Theta$ and $d\sigma$ [10, p. 339], we obtain

$$\lambda'(0) = -\beta(\sin\xi)^{d-2} \Big/ \int_0^{\xi} (\sin\theta)^{d-2}\, d\theta .$$

Thus, by Theorem 2.2, for $x \in \overline{G}\backslash\{0\}$, $P_x^0(\tau_0 < \infty) = 1$ (resp. 0) as β is less than (resp. greater than or equal to)

$$(2-d)(\sin\xi)^{2-d} \int_0^{\xi} (\sin\theta)^{d-2}\, d\theta .$$

2.3 Estimates of expected occupation and hitting times. The following lemma is needed in the later proof of the existence when $\alpha < 2$ of a process that spends zero time at the vertex. A consequence of the proof is (2.49). This lemma and Corollary 2.2 following it are the counterparts of Theorem 2.3 and Corollary 2.3 in Varadhan-Williams [22], where the state space is a two-dimensional wedge.

Let Ψ and τ_r be defined as in the proof of Theorem 2.2. For $0 \le r \le R$, define $\tau_{rR} = \tau_r \wedge \tau_R$.

Lemma 2.8. *Let $x \in \overline{G}\backslash\{0\}$, $R \ge \Psi(x)$ and $0 \le \varepsilon \le R$. Then,*

$$E^{P_x^0}\left[\int_0^{\tau_{0R}} 1_{(0,\varepsilon)}(\Psi(w(s)))h(w(s))\, ds\right] = g(\Phi(x)) \tag{2.51}$$

where for $z = r\omega \in \overline{G}\backslash\{0\}$, $r > 0$, $\omega \in \overline{\Omega}$,

$$h(z) = \begin{cases} \psi_\alpha^{(2/\alpha)-2}(\alpha^2\psi_\alpha^2 + |\nabla_{S^{d-1}}\psi_\alpha|^2)(\omega) & \text{if } \alpha \ne 0, \\ e^{2\chi(\omega)}(1 + |\nabla_{S^{d-1}}\chi|^2)(\omega) & \text{if } \alpha = 0 \end{cases}$$

and if $\alpha \in (0, \infty)\backslash\{2\}$,

$$g(y) = \begin{cases} 0 & \text{for } y = 0, \\ \alpha^2 y(\varepsilon^{(2/\alpha)-1}(2\alpha^{-1}(1-\varepsilon/R) + \varepsilon/R) - y^{(2/\alpha)-1})/(2-\alpha) & \text{for } 0 < y \le \varepsilon, \\ \alpha\varepsilon^{2/\alpha}(1 - y/R) & \text{for } y \ge \varepsilon \end{cases} \tag{2.52}$$

if $\alpha = 2$,

$$g(y) = \begin{cases} 0 & \text{for } y = 0, \\ 2y((\ln\varepsilon + 1 - \varepsilon/R) - \ln y) & \text{for } 0 < y \le \varepsilon, \\ 2\varepsilon(1 - y/R) & \text{for } y \ge \varepsilon \end{cases} \tag{2.53}$$

if $\alpha = 0$,

$$g(y) = \begin{cases} \frac{1}{2}(\varepsilon^2 - e^{2y}) + \varepsilon^2(\ln R - \ln\varepsilon) & \text{for } -\infty < y \le \ln\varepsilon, \\ \varepsilon^2(\ln R - y) & \text{for } y \ge \ln\varepsilon \end{cases} \tag{2.54}$$

if $\alpha < 0$

$$g(y) = \begin{cases} -2\alpha\varepsilon^{1-(2/\alpha)}(y - R^{-1})/(2-\alpha) & \text{for } 0 \le y \le \varepsilon^{-1}, \\ (2-\alpha)^{-1}(\alpha^2(\varepsilon^{-2/\alpha} - y^{2/\alpha}) - 2\alpha\varepsilon^{1-(2/\alpha)}(\varepsilon^{-1} - R^{-1})) & \text{for } y \ge \varepsilon^{-1} . \end{cases} \tag{2.55}$$

Remark. We note that h is positive and bounded away from zero on $\overline{G}\backslash\{0\}$, since $\psi_\alpha > 0$ on $\overline{\Omega}$.

Proof. Consider $f = g(\Phi)$ defined on $\overline{G}\backslash\{0\}$. Then $f \in C^1(\overline{G}\backslash\{0\})$, $f \in C^2(\overline{G}\backslash(\{0\}\cup\Psi^{-1}(\varepsilon)))$ and by (2.5)–(2.6), f satisfies $\mathbf{v}\cdot\nabla f = (\mathbf{v}\cdot\nabla\Phi)g'(\Phi) = 0$ on $\partial G\backslash\{0\}$, and on $\overline{G}\backslash(\{0\}\cup\Psi^{-1}(\varepsilon))$,

$$\Delta f = g''(\Phi)|\nabla\Phi|^2. \tag{2.56}$$

Now, for $z = r\omega \in \overline{G}\backslash\{0\}$, $r > 0$, $\omega \in \overline{\Omega}$,

$$|\nabla\Phi(z)|^2 = \begin{cases} r^{2\alpha-2}(\alpha^2\psi_\alpha^2 + |\nabla_{S^{d-1}}\psi_\alpha|^2)(\omega) = (\Phi^{2-(2/\alpha)}h)(z) & \text{if } \alpha \neq 0, \\ r^{-2}(1 + |\nabla_{S^{d-1}}\chi(\omega)|^2) = (e^{-2\Phi}h)(z) & \text{if } \alpha = 0. \end{cases} \tag{2.57}$$

As in [22, Theorem 2.3], g was chosen so that (2.56)–(2.57) reduce to

$$\frac{1}{2}\Delta f(z) = \begin{cases} -h(z) & \text{for } 0 < \Psi(z) < \varepsilon, \\ 0 & \text{for } \Psi(z) > \varepsilon. \end{cases}$$

It is readily verified that f is nonnegative and bounded on $\{z \in \overline{G}: 0 < \Psi(z) < R\}$, $f(z) = 0$ on $\{z \in \overline{G}: \Psi(z) = R\}$, and $\lim_{z\to 0} f(z) = 0$ if $\alpha > 0$. Consistent with the latter, we define $f(0) = 0$ if $\alpha > 0$.

For $0 < r < \Psi(x)$, let $\overline{G}_{rR} = \{z \in \overline{G}: r \leq \Psi(z) \leq R\}$. By convoluting f with an approximate identity, we can obtain a sequence of functions $\{f_n\}$ such that $f_n \in C_b^2(\overline{G}_{rR})$, f_n and ∇f_n converge uniformly on $\overline{G}_{rR}$ to f and ∇f, respectively, and $\{\Delta f_n\}$ is bounded on $\overline{G}_{rR}$ and converges pointwise to Δf on $\overline{G}_{rR}\backslash\Psi^{-1}(\varepsilon)$. By the proof of Theorem 2.1, the process X defined by (2.12) and $X(t) = 0$ for all $t \geq \sigma$, has the law of P_x^0. When we apply Itô's formula to f_n and $X(\cdot\wedge\sigma_{rR})$ where $\sigma_{rR} = \inf\{t \geq 0: \Psi(X(t)) \leq r \text{ or } \geq R\}$, we obtain a.s. for all $t \geq 0$:

$$\begin{aligned} &f_n(X(t\wedge\sigma_{rR})) - f_n(x) \\ &\quad = \int_0^{t\wedge\sigma_{rR}} \nabla f_n(X(s))\,dB(s) + \int_0^{t\wedge\sigma_{rR}} \mathbf{v}\cdot\nabla f_n(X(s))\,dL(s) \\ &\qquad + \frac{1}{2}\int_0^{t\wedge\sigma_{rR}} \Delta f_n(X(s))\,ds. \end{aligned}$$

By Corollary 2.1, the amount of time that X spends in $\Psi^{-1}(\varepsilon)$ has Lebesgue measure zero, a.s. Combining this with the nature of the convergence of the f_n's, we can let $n \to \infty$ in the above to obtain the same equation with f, ∇f, in place of f_n, ∇f_n, respectively, and Δf_n replaced by $1_{\overline{G}_{rR}\backslash\Psi^{-1}(\varepsilon)}\Delta f$. Since $\mathbf{v}\cdot\nabla f = 0$ on $\partial G\cap\overline{G}_{rR}$, and ∇f is bounded on $\overline{G}_{rR}$, after taking expectations in this equation and representing the result in terms of the law P_x^0 of X, we obtain for all $t \geq 0$,

$$E^{P_x^0}[f(w(t\wedge\tau_{rR}))] + E^{P_x^0}\left[\int_0^{t\wedge\tau_{rR}} 1_{(0,\varepsilon)}(\Psi(w(s)))h(w(s))\,ds\right] = f(x). \tag{2.58}$$

On letting $t \to \infty$, then $r \to 0$, we conclude by Fatou's lemma and the non-negativity of f, that

$$E^{P_x^0}\left[\int_0^{\tau_{0R}} 1_{(0,\varepsilon)}(\Psi(w(s)))h(w(s))\,ds\right] \le f(x). \tag{2.59}$$

Setting $\varepsilon = R$ and recalling that h is bounded below by a strictly positive constant on $\overline{G}\backslash\{0\}$, we conclude that

$$E^{P_x^0}[\tau_{0R}] < \infty. \tag{2.60}$$

Hence (2.49) holds and Theorem 2.2 is validated. In particular, for $\alpha \le 0$, $\tau_{0R} = \tau_R$ P_x^0-a.s. It follows that for all α, $f(w(\tau_{0R})) = 0$ P_x^0-a.s. Then, letting $t \to \infty$ followed by $r \to 0$ in (2.58), by bounded and monotone convergence we obtain (2.51). □

Corollary 2.2. *For $\alpha \le 2$ and $x \in \overline{G}\backslash\{0\}$, $E^{P_x^0}[\tau_0] = +\infty$. For $\alpha > 2$, there are constants b_1 and b_2 depending on the data $(G, \mathbf{v})$ such that $0 < b_1 \le b_2 < \infty$ and*

$$b_1|x|^2 \le E^{P_x^0}[\tau_0] \le b_2|x|^2.$$

Proof. This follows from Lemma 2.8 on letting $\varepsilon = R \to \infty$ in the same manner as Corollary 2.3 of [22] follows from Theorem 2.3 there. □

2.4 Existence. In this section it is shown that if $\alpha < 2$, there is a solution of the submartingale problem with "reflection" rather than absorption at the vertex of the cone. This is obtained as a weak limit from a family of approximating processes which have a jump at the vertex.

Let $D_{\overline{G}}$ denote the space of all functions $w\colon [0, \infty) \to \overline{G}$ which are right continuous on $[0, \infty)$ and have finite left limits on $(0, \infty)$. Endow $D_{\overline{G}}$ with the Skorohod topology (cf. Kurtz [12, p. 7]). The Borel σ-algebra $\mathcal{M}^D$ associated with this metric topology on $D_{\overline{G}}$ is the same as that generated by the coordinate maps: $\mathcal{M}^D = \sigma\{w(s)\colon 0 \le s < \infty\}$. The restriction of $\mathcal{M}^D$ to $C_{\overline{G}}$ is $\mathcal{M}$ and we may think of P_x^0 as a probability measure on $(D_{\overline{G}}, \mathcal{M}^D)$ concentrated on $(C_{\overline{G}}, \mathcal{M})$. Let $\mathcal{M}_t^D = \sigma\{w(s)\colon 0 \le s \le t,\, w \in D_{\overline{G}}\}$ for each $t \ge 0$.

For each $\delta \in \overline{G}\backslash\{0\}$ and $x \in \overline{G}$, we shall define a probability measure P_x^δ on $D_{\overline{G}}$ such that P_x^δ is the law of an approximating process that, starts from x, behaves like the process with absorption at the vertex prior to hitting the vertex but rather than being absorbed there, jumps instantaneously to the point δ in the cone and continues from there as if it had started there. Let $\sigma_0 = 0$ and define $\{\sigma_n,\, n \ge 1\}$ inductively by

$$\sigma_n^m = \inf\{t \ge \sigma_{n-1} \colon \Psi(w(t)) \le 1/m\} \quad \text{for all } m \in \mathbb{N},$$
$$\sigma_n = \sup_m \sigma_n^m.$$

If $P_x^0(\tau_0 < \infty) = 1$, we define P_x^δ as follows. Let $Q_x^0 = P_x^0$ if $x \ne 0$ or $Q_x^0 = P_\delta^0$ if $x = 0$, and having defined Q_x^{n-1} for some $n \ge 1$, define Q_x^n

to equal Q_x^{n-1} on $\mathscr{M}^D_{\sigma_n-}$ and the regular conditional probability distribution (r.c.p.d.) of Q_x^n given $\mathscr{M}^D_{\sigma_n-}$ to be given by

$$Q_x^n(w(\cdot+\sigma_n)\in A|\mathscr{M}^D_{\sigma_n-}) = P_\delta^0(w(\cdot)\in A) \quad \text{for all } A\in\mathscr{M}^D \text{ on } \{\sigma_n- <\infty\}.$$

By the consistency of the Q_x^n, we can define P_x^δ to be equal to Q_x^n on $\mathscr{M}^D_{\sigma_n-}$ for all n. From the fact that $P_\delta^0(\sup_{0\le s\le t}|w(s)-w(0)|\ge \frac{1}{2}|\delta|) < 1$ for t sufficiently small, it follows that there is a probability measure P_x^δ on $(D_{\overline{G}}, \mathscr{M}^D)$ such that $P_x^\delta = Q_x^n$ on $\mathscr{M}^D_{\sigma_n-}$ for all n and $P_x^\delta(\lim_{n\to\infty}\sigma_n = +\infty) = 1$. If $P_x^0(\tau_0<\infty) = 0$, we define $P_x^\delta = P_x^0$ for $x\neq 0$, and P_0^δ is defined to equal P_δ^0. By construction, for each $\delta\in\overline{G}\backslash\{0\}$, the family $\{P_x^\delta, x\in\overline{G}\}$ on $(D_{\overline{G}}, \mathscr{M}^D)$ has the strong Markov property.

Lemma 2.9. *For each fixed $x\in\overline{G}$, the family $\{P_x^\delta\}$ on $(D_{\overline{G}}, \mathscr{M}^D)$ is tight as $|\delta|\to 0$, i.e., for any sequence $\{\delta_n\}_{n=1}^\infty\subset\overline{G}\backslash\{0\}$ such that $|\delta_n|\to 0$, there is a subsequence $\{\delta_{n_k}\}_{k=1}^\infty$ such that $\{P_x^{\delta_{n_k}}\}_{k=1}^\infty$ converges weakly to a probability measure on $(D_{\overline{G}}, \mathscr{M}^D)$.*

Proof. Since $\overline{G}$ is locally compact and the trajectories have jumps of size at most $|\delta|$, it is sufficient (cf. Prokhorov [18, p. 182]) to prove

$$\lim_{t\to 0}\overline{\lim_{|\delta|\to 0}}\sup_{x\in\overline{G}} P_x^\delta(\eta_x(\gamma)\le t) = 0 \tag{2.61}$$

for each $\gamma>0$, where

$$\eta_x(\gamma) = \inf\{s\ge 0 : |w(s)-x|\ge\gamma\}.$$

Assume that $|\delta|$ is so small that $|\delta|\le\gamma/3$. Then, by the strong Markov property of the family $\{P_x^\delta, x\in\overline{G}\}$ we have

$$\begin{aligned}\sup_{x\in\overline{G}} P_x^\delta(\eta_x(\gamma)\le t) &\le \sup_{|x|\ge\gamma/3} P_x^\delta(\eta_x(\gamma/6)\le t)\\ &= \sup_{|x|\ge\gamma/3} P_x^0(\eta_x(\gamma/6)\le t).\end{aligned} \tag{2.62}$$

By Lemma 2.3, for $|x|\ge\gamma/3$ and $\hat{x} = x/|x|$,

$$P_x^0\left(\inf_{0\le s\le t}|w(s)-w(0)|\ge\gamma/6\right) = P_{\hat{x}}^0\left(\inf_{0\le s\le t/|x|^2}|w(s)-w(0)|\ge\gamma/6|x|\right).$$

Then, as in the proof of Theorem 2.1, for m sufficiently large, $P_{\hat{x}}^0$ in the right member above can be replaced by the law $P_{\hat{x}}^m$ of a reflected Brownian motion with data $(G_m, \mathbf{v}_m)$, where for $P_{\hat{x}}^m$ we have the uniform exit time estimate (cf. [21, p. 181]):

$$P_{\hat{x}}^m\left(\inf_{0\le s\le t/|x|^2}|w(s)-w(0)|\ge\gamma/6|x|\right)\le Ct|x|^{-2}(\gamma/6|x|)^{-2} = 36Ct/\gamma^2$$

for all $\hat{x}\in\overline{\Omega}$. It follows that the right member of (2.62) is $o(1)$ as $t\to 0$. □

The next theorem states that any weak limit point P_x of P_x^δ as $|\delta| \to 0$ is a solution of the submartingale problem (SP) defined by (1.4)–(1.6). The method of proof is the same as that in Varadhan-Williams [22, Theorem 2.5], and so only a sketch of the proof is given here.

Theorem 2.3. *Let $x \in \overline{G}$ and suppose P_x is a weak limit point of $\{P_x^\delta\}$ as $|\delta| \to 0$. Then P_x is a probability measure on $(C_{\overline{G}}, \mathscr{M})$ satisfying* (1.4) *and* (1.5). *If, in addition, $\alpha < 2$, then* (1.6) *holds.*

Sketch of proof. Since the P_x^δ only allow jumps of size $|\delta|$ from the origin and $|\delta| \to 0$, it follows that P_x is supported on the space of continuous paths and (1.4) holds. For any f as described in (1.5), $f(0) = f(\delta)$ for all sufficiently small $|\delta|$, and it follows that for such δ, the submartingale property in (1.5) holds for P_x^δ on $(D_{\overline{G}}, \mathscr{M}^D, \{\mathscr{M}_t^D\})$. This submartingale property is preserved in the limit as $|\delta| \to 0$. The final result that (1.6) holds when $\alpha < 2$ follows from the next two lemmas, which can be proved in the same way as in [22, Lemmas 2.2, 2.3]. The estimates obtained in Lemma 2.8 play a key role in the proof of Lemma 2.11. □

For $r \geq 0$, define $\tau_r(w) = \inf\{t \geq 0 : \Psi(w(t)) = r\}$ for $w \in D_{\overline{G}}$.

Lemma 2.10. *Suppose $0 < \varepsilon \leq R$ and $x \in \overline{G}$ such that $\Psi(x) < R$. Let $\{\delta_n\}$ be a sequence of points in $\overline{G}\backslash\{0\}$ such that $|\delta_n| \to 0$ and $\{P_x^{\delta_n}\}$ converges weakly to P_x. Then*

$$E^{P_x}\left[\int_0^{\tau_R} 1_{[0,\varepsilon)}(\Psi(w(s)))\, ds\right] \leq \varliminf_{n\to\infty} E^{P_x^{\delta_n}}\left[\int_0^{\tau_R} 1_{[0,\varepsilon)}(\Psi(w(s)))\, ds\right]. \tag{2.63}$$

Lemma 2.11. *Suppose $\delta \in \overline{G}\backslash\{0\}$, $x \in \overline{G}$, $R > \Psi(x) \vee \Psi(\delta)$, and $0 < \varepsilon \leq R$. Let g and h be defined as in Lemma* 2.8. *Then if $x \neq 0$,*

$$\begin{aligned} &E^{P_x^\delta}\left[\int_0^{\tau_R} 1_{(0,\varepsilon)}(\Psi(w(s)))h(w(s))\, ds\right] \\ &\qquad = g(\Phi(x)) + \frac{P_x^0(\tau_0 < \tau_R)}{P_\delta^0(\tau_R < \tau_0)} g(\Phi(\delta)), \end{aligned}$$

or if $x = 0$,

$$E^{P_0^\delta}\left[\int_0^{\tau_R} 1_{(0,\varepsilon)}(\Psi(w(s)))h(w(s))\, ds\right] = \frac{g(\Phi(\delta))}{P_\delta^0(\tau_R < \tau_0)}.$$

Here, for $0 < \Psi(z) \leq R$ we have by the proof of Theorem 2.2,

$$P_z^0(\tau_R < \tau_0) = \begin{cases} 1 & \text{if } \alpha \leq 0, \\ \Phi(z)/R & \text{if } \alpha > 0. \end{cases} \tag{2.64}$$

3. Uniqueness

3.1 Support Theorem.

Theorem 3.1. *Let $\varepsilon > 0$, $t > 0$ and $x_0 \in \overline{G}$ such that $|x_0| > \varepsilon$. Suppose $\zeta: [0, t] \to \{x \in \overline{G}: |x| > \varepsilon\}$ is continuous with $\zeta(0) = x_0$ and $\zeta(s) \in G$ for all $s \neq 0$. Then*

$$P^0_{x_0}\left(\sup_{0\le s\le t} |w(s) - \zeta(s)| < \varepsilon\right) > 0. \tag{3.1}$$

Proof. If $x_0 \in G$, then there is ε': $0 < \varepsilon' < \varepsilon$ and $d(\zeta(s), \partial G) > \varepsilon'$ for $0 \le s \le t$. Then

$$P^0_{x_0}\left(\sup_{0\le s\le t} |w(s) - \zeta(s)| < \varepsilon\right) \ge P\left(\sup_{0\le s\le t} |B(s) - \zeta(s)| < \varepsilon'\right), \tag{3.2}$$

where B is a Brownian motion starting from x_0 under P. By [20, Exercise 6.7.5], the right member of (3.2) is strictly positive.

Now consider $x_0 \in \partial G$. Choose $t_1 \in (0, t)$ such that

$$P^0_{x_0}\left(\sup_{0\le s\le t_1} |w(s) - x_0| \le \frac{\varepsilon}{4}\right) > 0. \tag{3.3}$$

Let $t_2 \in (0, t_1]$ such that

$$|\zeta(s) - x_0| \le \frac{\varepsilon}{4} \quad \text{for } 0 \le s \le t_2. \tag{3.4}$$

It follows from Corollary 2.1 with $F = \partial G\backslash\{0\}$ and (3.3), that there is $\kappa \in (0, \varepsilon/4)$ and $t_3 \in (0, t_2]$ such that

$$P^0_{x_0}\left(\sup_{0\le s\le t_3} |w(s) - x_0| \le \frac{\varepsilon}{4},\ d(w(t_3), \partial G) \ge \kappa\right) > 0, \tag{3.5}$$

where $d(x, \partial G)$ denotes the distance of x from ∂G. Since ζ is continuous and $\zeta(s) \in G$ for all $s \in (0, t]$, there is $\varepsilon' \in (0, \varepsilon)$ such that

$$d(\zeta(s), \partial G) \ge \varepsilon' \quad \text{for all } s \in [t_3, t].$$

Let $\tilde{t} = t - t_3$ and $\tilde{\zeta}(s) = \zeta(s + t_3)$ for $s \in [0, \tilde{t}]$. Then by the Markov property of $P^0_{x_0}$ at the time t_3 and (3.4), we have

$$\begin{aligned}
&P^0_{x_0}\left(\sup_{0\le s\le t} |w(s) - \zeta(s)| < \varepsilon\right) \\
&\quad \ge E^{P^0_{x_0}}\left[P^0_{w(t_3)}\left(\sup_{0\le s\le \tilde{t}} |w(s) - \tilde{\zeta}(s)| < \varepsilon\right);\right. \\
&\qquad\qquad \left.\sup_{0\le s\le t_3} |w(s) - x_0| \le \frac{\varepsilon}{4},\ d(w(t_3), \partial G) \ge \kappa\right].
\end{aligned} \tag{3.6}$$

For each $x \in G$ satisfying $|x - x_0| \le \varepsilon/4$ and $d(x, \partial G) \ge \kappa$, let

$$\rho(x) = P_x^0 \left(\sup_{0 \le s \le \tilde{t}} |w(s) - \tilde{\zeta}(s)| < \varepsilon \right).$$

Then,

$$\rho(x) \ge P_x^0 \left(\sup_{0 \le s \le \tilde{t}} |w(s) - \tilde{\zeta}(s)| < \varepsilon, \inf_{0 \le s \le \tilde{t}} d(w(s), \partial G) > \frac{1}{2}(\kappa \wedge \varepsilon') \right) \tag{3.7}$$

where

$$|x - \tilde{\zeta}(0)| \le |x - x_0| + |x_0 - \zeta(t_3)| \le \varepsilon/2,$$

$d(x, \partial G) \ge \kappa$ and $d(\tilde{\zeta}(\cdot), \partial G) \ge \varepsilon'$. Since w under P_x^0 behaves like Brownian motion in G until it reaches ∂G, it follows that the above probability is the same as when w has the law of Brownian motion starting from x. Then, by the support theorem for Brownian motion [20, Exercise 6.7.5], the right member of (3.7) is bounded below by a strictly positive constant which is independent of x satisfying $|x - x_0| \le \varepsilon/4$ and $d(x, \partial G) \ge \kappa$. Combining this with (3.5) and (3.6) yields the desired result. □

3.2 Preliminary lemmas. The following two lemmas are preparatory to the proof of the ergodic result in §3.3, but they are also of independent interest.

In the following, $C_{\mathbb{R}^d}$ denotes the set of continuous functions $w: [0, \infty) \to \mathbb{R}^d$, endowed with the topology of uniform convergence on compact sets in $[0, \infty)$. Also, $\mathscr{M}$ and $\{\mathscr{M}_t\}$ denote the natural σ-field and filtration on $C_{\mathbb{R}^d}$ generated by the coordinate maps. For any domain $U \subset \mathbb{R}^d$ with a C^3 boundary and a C^2 vector field $\mathbf{u}$ defined on ∂U, a solution of the submartingale problem for $(U, \mathbf{u})$ starting from $x \in \overline{U}$, is a probability measure Q_x on $C_{\mathbb{R}^d}$ such that the following three conditions hold.

$$Q_x(w(0) = x) = 1. \tag{3.8}$$

$$Q_x(w(t) \in \overline{U} \text{ for all } t \ge 0) = 1. \tag{3.9}$$

(3.10) For each $f \in C_b^2(\mathbb{R}^d)$ satisfying $\mathbf{u} \cdot \nabla f \ge 0$ on ∂U,

$$f(w(t)) - \frac{1}{2} \int_0^t \Delta f(w(s))\, ds$$

is a Q_x-submartingale on $(C_{\mathbb{R}^d}, \mathscr{M}, \{\mathscr{M}_t\})$. Uniqueness of such a Q_x is guaranteed by the theory of Stroock-Varadhan [21], since in the above $f \in C_b^2(\mathbb{R}^d)$ can be replaced by $f \in C_b^2(\overline{U})$ (cf. [8, Lemma 6.37]). Existence of such a Q_x is guaranteed if U is bounded, and also in the cases of $U = H$ and $U = G_r$ of Lemmas 3.1 and 3.2 below, since there local existence follows from [21] and global existence (nonexplosion) follows from the homogeneity of the boundary and of the boundary conditions outside a compact set (cf. the proof of Theorem 2.1).

Lemma 3.1. *Let H be an open half-space in $\mathbb{R}^d$ and fix $x_0 \in \partial H$. Let $\mathbf{u}$ be a fixed vector in $\mathbb{R}^d$ such that $\mathbf{u} \cdot \mathbf{n} = 1$, where $\mathbf{n}$ is the inward unit normal to ∂H. For each $r > 0$, define $\eta_r = \inf\{t \geq 0 \colon |w(t) - x_0| \geq r\}$, where w is the canonical process on $C_{\mathbb{R}^d}$. Fix $r_0 > 0$ and let $x \in \overline{H}$ such that $|x - x_0| < r_0$. Let Q_x denote the probability measure on $C_{\mathbb{R}^d}$ associated with the solution of the submartingale problem for $(H, \mathbf{u})$ with starting point x. Then the following hold.*

(i) *Q_x-a.s., η_{r_0} is a continuous functional on $C_{\mathbb{R}^d}$.*

(ii) *For any sequence of probability measures $\{Q^n\}$ that converges weakly to Q_x on $(C_{\mathbb{R}^d}, \mathcal{M})$, we have*

$$\lim_{n\to\infty} E^{Q^n}[f(w(\eta_{r_0}))] = E^{Q_x}[f(w(\eta_{r_0}))]$$

for all $f \in C(\partial B(x_0, r_0))$, where $B(x_0, r_0) = \{y \in \mathbb{R}^d \colon |y - x_0| < r_0\}$.

Proof. (i) By analogy with [12, pp. 13–14], it suffices to show that

$$Q_x\left(\lim_{r\to r_0} \eta_r = \eta_{r_0}\right) = 1.$$

Now, Q_x-a.s., $\eta_r \uparrow \eta_{r_0}$ as $r \uparrow r_0$, since $w(\cdot)$ has continuous paths starting from x. On the other hand, for $r \downarrow r_0$, under Q_x conditioned on $w(\eta_{r_0}) \notin \partial H$, $w(\cdot + \eta_{r_0})$ almost surely hits $\{y \colon |y - x_0| > r_0\}$ immediately and so $\eta_r \downarrow \eta_{r_0}$. It remains to consider what happens when we condition on $w(\eta_{r_0}) \in \partial H$. For this, suppose $z \in \partial H$ and $|z - x_0| = r_0$. Now, w has the following scaling property under Q_z (this can be verified from scaling of the submartingale problem as for Lemma 2.3): for each $\gamma > 0$, $w(\gamma\cdot) - z$ has the same distribution as $\sqrt{\gamma}(w(\cdot) - z)$. By choosing coordinates appropriately, we may suppose $x_0 = 0$, $\mathbf{n} = (0, \ldots, 0, 1)$ and $z = (r_0, 0, \ldots, 0)$. Then for $\eta_{r_0}^+ \equiv \inf\{t \geq 0 \colon |w(t)| > r_0\}$,

$$\begin{aligned} Q_z\left(\lim_{r\downarrow r_0} \eta_r = 0\right) &= Q_z(\eta_{r_0}^+ = 0) = \lim_{n\to\infty} Q_z\left(\eta_{r_0}^+ \leq \frac{1}{n}\right) \\ &\geq \overline{\lim_{n\to\infty}}\, Q_z\left(w_1\left(\frac{1}{n}\right) > r_0\right) \end{aligned}$$

where w_1 denotes the first component of w. By the scaling property of Q_z, the last expression above is equal to

$$\overline{\lim_{n\to\infty}}\, Q_z\left(\frac{1}{\sqrt{n}}(w_1(1) - r_0) > 0\right) = Q_z(w_1(1) > r_0) > 0.$$

By the zero-one law, it follows that $Q_z(\lim_{r\downarrow r_0} \eta_r = 0) = 1$. Combining this with the previous result for $w(\eta_{r_0}) \notin \partial H$, and the strong Markov property, we obtain

$$Q_x\left(\lim_{r\downarrow r_0} \eta_r = \eta_{r_0}\right) = 1.$$

(ii) For this part of the proof, we denote r_0 simply by r. Fix $f \in C(\partial B(x_0, r))$. Let $\hbar(x) = f(x_0 + r(x - x_0)/|x - x_0|)$ for $|x - x_0| \geq \frac{1}{2}r$ and extend h to be continuous on all of $\mathbb{R}^d$. Since $\mathbf{n}\cdot\mathbf{w}(\cdot)$ under Q_x is a one-dimensional reflected Brownian motion, we have

$$E^{Q_x}(\eta_s) < \infty \quad \text{for all } s > 0. \tag{3.11}$$

Fix $\varepsilon > 0$ and for $N \in \mathbb{N}$, let $A_N = \{\eta_r > N\}$. Then for all N sufficiently large,

$$Q_x(A_N) \le N^{-1} E^{Q_x}(\eta_{r+1}) < \varepsilon, \tag{3.12}$$

and

$$\begin{aligned}\varlimsup_{n\to\infty} Q^n(A_N) &\le \varlimsup_{n\to\infty} Q^n\left(\sup_{0\le s\le N}|w(s)-x_0| \le r\right)\\ &\le Q_x\left(\sup_{0\le s\le N}|w(s)-x_0|\le r\right)\\ &\le Q_x(\eta_{r+1} > N) \le N^{-1}E^{Q_x}(\eta_{r+1}) < \varepsilon.\end{aligned}\tag{3.13}$$

By (i) above and the continuous mapping theorem of weak convergence, $w(\cdot\wedge\eta_r)$ under Q^n converges weakly to $w(\cdot\wedge\eta_r)$ under Q_x and so

$$\lim_{n\to\infty} E^{Q^n}[h(w(\eta_r\wedge N))] = E^{Q_x}[h(w(\eta_r\wedge N))]. \tag{3.14}$$

On the other hand,

$$f(w(\eta_r)) = f(w(\eta_r))1_{A_N} - h(w(\eta_r\wedge N))1_{A_N} + h(w(\eta_r\wedge N)).$$

Thus,

$$\begin{aligned}&|E^{Q^n}[f(w(\eta_r))] - E^{Q_x}[f(w(\eta_r))]|\\ &\quad\le (\|f\|_\infty + \|h\|_\infty)(Q^n(A_N)+Q_x(A_N))\\ &\qquad + |E^{Q^n}[h(w(\eta_r\wedge N))] - E^{Q_x}[h(w(\eta_r\wedge N))]|.\end{aligned}\tag{3.15}$$

Fix N such that (3.12)–(3.13) hold. Then from these inequalities and (3.14), we see that the $\varlimsup_{n\to\infty}$ of the left member of (3.15) is bounded by $2(\|f\|_\infty + \|h\|_\infty)\varepsilon$. Since $\varepsilon > 0$ was arbitrary, (ii) follows. □

Lemma 3.2. *Let $x_0 \in \partial\Omega$. For $r > 0$, define*

$$\begin{aligned}B(x_0, r) &= \{x\in\mathbb{R}^d : |x-x_0| < r\},\\ I_r &= \left\{x \in G\cap\partial B(x_0, r) : d(x, \partial G) > \tfrac{r}{4}\right\},\\ \eta_r &= \inf\{t\ge 0 : |w(t)-x_0| \ge r\}.\end{aligned}$$

Then there are κ, $\gamma > 0$ such that for all $0 < r \le \gamma$ and $x \in \overline{G}$ satisfying $|x-x_0| \le \frac{1}{2}r$, we have

$$P_x^0(w(\eta_r)\in I_r) > \kappa > 0,$$

where κ and γ are independent of r.

Proof. The idea of the proof is to approximate w under P_x^0 locally near x_0 by a reflected Brownian motion in a half-space with boundary that is the tangent plane to ∂G at x_0 and with constant reflection field given by $\mathbf{v}(x_0)$. Since the above estimate holds for such a reflected Brownian motion, it will then follow for w under P_x^0 for all r sufficiently small.

Let $\mathbf{n_0}$ be the inward unit normal to ∂G at x_0 and let $\mathbf{v_0} = \mathbf{v}(x_0)$. Define

$$H = \{x \in \mathbb{R}^d : \mathbf{x} \cdot \mathbf{n_0} > 0\}.$$

Then ∂H is the tangent plane to ∂G at x_0. Since the boundary of G is C^2 (in fact it is C^3) near x_0, in a sufficiently small neighborhood of x_0 this boundary can be represented as the graph of a C_b^2 function defined on a neighborhood of x_0 in ∂H. More precisely, choose Cartesian coordinates $y = (y_1, \ldots, y_d)$ centered at x_0 such that the positive y_d-axis is parallel to $\mathbf{n_0}$. Then $\tilde{y} \equiv (y_1, \ldots, y_{d-1}) \in \partial H$, and there is an $r_0 > 0$ and a C_b^2 function ζ_0 defined on the ball $\{\tilde{y} : |\tilde{y}| \le r_0\}$ in ∂H such that for $\phi_0(y) = y_d - \zeta_0(\tilde{y})$,

$$G \cap B(x_0, r_0) = \{y \in B(x_0, r_0) : \phi_0(y) > 0\}$$

and

$$\partial G \cap B(x_0, r_0) = \{y \in B(x_0, r_0) : \phi_0(y) = 0\}.$$

We say ϕ_0 is a defining function for G in $B(x_0, r_0)$. Since the inward normal to ∂G at x_0 is parallel to the y_d-axis and in the direction of $\nabla\phi_0(0)$, it follows that $\nabla_{\mathbb{R}^{d-1}}\zeta_0(0) = 0$. Then, since ζ_0 is a C_b^2 function, there is a constant $C > 0$ such that for $|\tilde{y}| \le r_0$,

$$|\zeta_0(\tilde{y})| \le C|\tilde{y}|^2 \quad \text{and} \quad |\nabla\zeta_0(\tilde{y})| \le C|\tilde{y}|.$$

For $0 < r < \frac{1}{2}r_0$, let $h_r(y) = y/r$ and for $|\tilde{y}| < 2$, let $\zeta_r(\tilde{y}) = \frac{1}{r}\zeta_0(r\tilde{y})$ and $\phi_r(y) = y_d - \zeta_r(\tilde{y})$. Then h_r dilates the set $G \cap B(x_0, 2r)$ to $\{y \in B(x_0, 2) : \phi_r(y) > 0\}$, and the vector field $\mathbf{v}$ on $\partial G \cap B(x_0, 2r)$ determines a vector field $\mathbf{v_r}$ on $\{y \in B(x_0, 2) : \phi_r(y) = 0\}$ by $\mathbf{v_r}(y) = \mathbf{v}(ry)$. Because of the C_b^2 nature of ζ_0 and $\mathbf{v}$, and the above bounds on ζ_0 and $\nabla\zeta_0$, for all r sufficiently small, ζ_r can be extended to a C_b^2 function on ∂H and $\mathbf{v_r}$ can be extended to a C_b^2 vector field on $\{y \in \mathbb{R}^d : \phi_r(y) \equiv y_d - \zeta_r(\tilde{y}) = 0\}$, in such a way that

$$\zeta_r(\tilde{y}) = 0 \text{ and } \mathbf{v_r}(\tilde{y}, \zeta_r(\tilde{y})) = \mathbf{v_0} \quad \text{for } |\tilde{y}| \ge 4,$$

$$|\zeta_r| \le C'r,\ |\nabla\zeta_r| \le C'r \text{ and } \left|\frac{\partial^2\zeta_r}{\partial\tilde{y}_i\partial\tilde{y}_j}\right| \le C'r, \qquad i, j \in \{1, \ldots, d\},\ \text{on } \partial H,$$

$$|\mathbf{v_r} - \mathbf{v_0}| \le C'r \text{ and } \mathbf{v_r} \cdot \nabla\phi_r \ge c > 0 \quad \text{on } \{y \in \mathbb{R}^d : \phi_r(y) = 0\},$$

for some constants C' and c that are independent of r. Let $g(y_d)$ be a nondecreasing C_b^2 function on $\mathbb{R}$ that agrees with y_d on $|y_d| \le 1$ and equals $2\,\mathrm{sgn}(y_d)$ for $|y_d| \ge 2$. By replacing y_d by $g(y_d)$ in the definition of ϕ_r, we obtain a C_b^2 function which for all r sufficiently small defines the same domain $G_r = \{y \in \mathbb{R}^d : \phi_r(y) > 0\}$ and has the same properties mentioned above as the original ϕ_r. We shall henceforth consider only such sufficiently small r and use $\phi_r(y) = g(y_d) - \zeta_r(\tilde{y})$. Then the data $(G_r, \mathbf{v_r})$ satisfy the hypotheses of Stroock-Varadhan [21], and so for each $x \in \overline{G}_r$, there is a unique solution P_x^r of the submartingale problem on $C_{\mathbb{R}^d}$ associated with $(G_r, \mathbf{v_r})$ starting from x. Note also that since $G_r \cap B(x_0, 2) = h_r(G \cap B(x_0, 2r))$, the image of I_r under h_r is $I_1^r \equiv \{x \in G_r \cap \partial B(x_0, 1) \colon d(x, \partial G_r) > \frac{1}{4}\}$.

Let $w^r(\cdot) = x_0 + (w(r^2\cdot) - x_0)/r$, a deterministic time-change of the image of w under h_r, and let $\eta_1^r = \inf\{t \geq 0 \colon w^r(t) \notin B(x_0, 1)\}$. Then, for each $x \in \overline{G}$ satisfying $|x - x_0| \leq \frac{1}{2}r$, by using a semimartingale representation for $w(\cdot \wedge \eta_r)$ under P_x^0, together with the correspondence between $\mathbf{v}$ and $\mathbf{v_r}$, and the local uniqueness of the solution $P_{x_r}^r$ of the submartingale problem for $(G_r, \mathbf{v_r})$ with starting point $x_r = x_0 + (x - x_0)/r$ [21, Theorem 5.6], we conclude that $w^r(\cdot \wedge \eta_1^r)$ under P_x^0 is equivalent in law to $w(\cdot \wedge \eta_1)$ under $P_{x_r}^r$, and consequently

$$P_x^0(w(\eta_r) \in I_r) = P_x^0(w^r(\eta_1^r) \in I_1^r) = P_{x_r}^r(w(\eta_1) \in I_1^r). \tag{3.16}$$

By construction, for all r sufficiently small, $\phi_r \in C_b^2$, and bounds on ϕ_r and its first and second derivatives can be chosen independent of r. Moreover, $\mathbf{v_r} \cdot \nabla\phi_r \geq c > 0$ on ∂G_r. It then follows by similar reasoning to that in Lemma 3.3 of Stroock-Varadhan [21] that if $r_n \to 0$ and $x_{r_n} \in \overline{G}_{r_n}$ such that $|x_{r_n} - x_0| \leq \frac{1}{2}$ and $x_{r_n} \to x \in \overline{H}$, then the sequence $\{P^n \equiv P_{x_{r_n}}^{r_n}\}$ is tight. Let P be a limit point of $\{P^n\}$. Then $P(w(0) = x) = 1$, and since $\overline{H}$ is closed and $\overline{G}_{r_n} \to \overline{H}$, we have $P\{w(t) \in \overline{H}$ for all $t \geq 0\} = 1$. Let $f \in C_b^2(\mathbb{R}^d)$ such that $\mathbf{v_0} \cdot \nabla f \geq 0$ on ∂H. Since ∂H and $\mathbf{v_0}$ coincide with ∂G_r and $\mathbf{v_r}$, respectively, on $\{y \in \mathbb{R}^d \colon |\tilde{y}| \geq 4\}$, and $\sup_{\tilde{y}} |\zeta_r(\tilde{y})| \to 0$, $\sup_{y \in \partial G_r} |\mathbf{v_r}(y) - \mathbf{v_0}| \to 0$ as $r \to 0$, it follows that for all r sufficiently small there is $\mu > 0$ (not depending on r) such that $\mathbf{v_0} \cdot \mathbf{v_r} \geq \mu > 0$ on ∂G_r, and for each $k > 0$ there is γ_k such that for all $r \leq \gamma_k$,

$$\mathbf{v_r} \cdot \nabla f \geq -\frac{1}{k} \quad \text{on } \partial G_r.$$

Let $f_k(z) = f(z) + \frac{1}{k\mu}\mathbf{v_0} \cdot \mathbf{z}$ for $z \in \mathbb{R}^d$. Then on ∂G_r, for all r sufficiently small,

$$\mathbf{v_r} \cdot \nabla f_k = \mathbf{v_r} \cdot \nabla f + \frac{1}{k\mu}\mathbf{v_0} \cdot \mathbf{v_r} \geq -\frac{1}{k} + \frac{1}{k\mu}\mu = 0.$$

Then, for such r, by modifying f_k off large compact sets to be in $C_b^2(\mathbb{R}^d)$ and still satisfy the above inequality (cf. Lemma 2.2), we may apply the submartingale property of P_z^r to conclude that for each $z \in \overline{G}_r$,

$$f_k(w(t)) - \frac{1}{2}\int_0^t \Delta f_k(w(s))\, ds \tag{3.17}$$

is a local submartingale under P_z^r on $(C_{\mathbb{R}^d}, \mathscr{M}, \{\mathscr{M}_t\})$. For each $R > 0$, let η_R be defined on $C_{\mathbb{R}^d}$ by $\eta_R = \inf\{t \geq 0 \colon |w(t) - x_0| \geq R\}$. Now, $\{P^{n_l}\}$ converges weakly to P for some subsequence $\{n_l\}$ of $\{n\}$. Then, for all but countably many R, $(w(\cdot\wedge\eta_R), \eta_R)$ under P^{n_l} converges weakly to $(w(\cdot\wedge\eta_R, \eta_R))$ under P, (cf. [12, p. 13], [7, p. 355]). Thus, by stopping the local submartingale in (3.17) at η_R for a nonexceptional R, we obtain a submartingale under P^{n_l} which retains this property for the weak limit P. Letting $R \to \infty$ yields that (3.17) is a *local* submartingale *under* P. Since $f_k \to f$ uniformly on compact sets in $\mathbb{R}^d$ and $\Delta f_k = \Delta f$, it follows on letting $k \to \infty$ that

$$f(w(t)) - \frac{1}{2}\int_0^t \Delta f(w(s))\,ds$$

is a local submartingale under P on $(C_{\mathbb{R}^d}, \mathscr{M}, \{\mathscr{M}_t\})$. In fact it is a submartingale since f and Δf are bounded. Then P is a solution of the submartingale problem for $(H, \mathbf{v_0})$ starting from x. By uniqueness of this solution, henceforth denoted by Q_x, it follows that $\{P^n\}$ converges weakly to Q_x.

Let $J_1 = \{z \in H \cap B(x_0, 1) \colon d(x, \partial H) > \frac{1}{2}\}$. Since $\zeta_r \to 0$ uniformly as $r \to 0$, $J_1 \subset I_1^r$ for all sufficiently small r. Moreover, using a semimartingale representation for w under Q_x, it can be shown that there is $\kappa > 0$:

$$Q_x(w(\eta_1) \in J_1) > \kappa \quad \text{for all } x \in \overline{H} \quad \text{such that } |x - x_0| \le \tfrac{1}{2}. \tag{3.18}$$

We now prove Lemma 3.2 by contradiction. For this, suppose there are sequences $\{r_n\} \subset [0, \infty)$ and $\{x_n\}$ such that $r_n \to 0$ and for each n, $x_n \in \overline{G}$, $|x_n - x_0| \le \frac{1}{2} r_n$ and

$$P^0_{x_n}(w(\eta_{r_n}) \in I_{r_n}) \le \kappa. \tag{3.19}$$

After transformation by h_{r_n}, we obtain $x_{r_n} \equiv x_0 + (x_n - x_0)/r_n \in \overline{G}_{r_n}$ satisfying $|x_{r_n} - x_0| \le \frac{1}{2}$ and for $P^n \equiv P^{r_n}_{x_{r_n}}$, by (3.16), we have $P^n(w(\eta_1) \in I_1^{r_n}) \le \kappa$. By choosing a subsequence if necessary, we may suppose $x_{r_n} \to x \in \overline{H}$. Then by the preceding analysis, $\{P^n\}$ converges weakly to Q_x, the solution of the submartingale problem for $(H, \mathbf{v_0})$ with starting point x. By Lemma 3.1(i), $w(\eta_1)$ under P^n converges weakly to $w(\eta_1)$ under Q_x, and so since J_1 is open [7, Theorem 3.1, p. 108],

$$\begin{aligned} Q_x(w(\eta_1) \in J_1) &\le \varliminf_{n\to\infty} P^n(w(\eta_1) \in J_1) \\ &\le \varliminf_{n\to\infty} P^n(w(\eta_1) \in I_1^{r_n}) \\ &\le \kappa, \quad \text{by (3.19).} \end{aligned}$$

But this contradicts (3.18). Hence the lemma is proved. □

Remark. Although we shall not need it here, since $\inf_{x_0\in\partial\Omega}(\mathbf{v_0}\cdot\mathbf{n_0}) > 0$, the constant $\kappa > 0$ in the above lemma (see (3.18)) can be chosen independent of $x_0 \in \partial\Omega$.

Lemma 3.3. *For each* $x_0 \in \overline{\Omega}$, *there are* κ, $\gamma > 0$ *such that for all* $0 < r \leq \gamma$, $\eta_r \equiv \inf\{t \geq 0 \colon |w(t) - x_0| \geq r\}$, *and* $x \in \overline{G}$ *satisfying* $|x - x_0| \leq \frac{1}{4}r$, *we have*

$$P_x^0(w(\eta_r) \in A_r) > \kappa, \tag{3.20}$$

whenever $A_r \subset \overline{G} \cap \partial B(x_0, r)$ *such that* $|A_r| \geq \frac{1}{2}|\overline{G} \cap \partial B(x_0, r)|$, *where* $|\cdot|$ *denotes surface measure on* $\partial B(x_0, r)$. *The constants* κ, γ *can be chosen independent of* r.

Proof. For $x_0 \in \Omega$, there is $\gamma = \gamma(x_0) > 0$ such that $B(x_0, \gamma) \subset G$. Then for each $r \colon 0 < r \leq \gamma$ and all $x \colon |x - x_0| \leq r/4$, $w(\cdot)$ under P_x^0 behaves like a Brownian motion up to the time η_r. It follows from the equivalence of harmonic measure to surface measure on $\partial B(x_0, r)$ and Harnack's inequality, that there is $\kappa_1 > 0$ (not depending on A_r, r, x or x_0), such that

$$P_x^0(w(\eta_r) \in A_r) \geq \kappa_1 > 0 \quad \text{for all } x \colon |x - x_0| \leq r/4,$$

whenever $|A_r| \geq \frac{1}{2}|\partial B(x_0, r)| = \frac{1}{2}|\overline{G} \cap \partial B(x_0, r)|$.

For $x_0 \in \partial\Omega$, by Lemma 3.2, there is $\kappa_2 > 0$ not depending on A_r, r or x such that for all sufficiently small r,

$$P_x^0(w(\eta_{r/2}) \in I_{r/2}) > \kappa_2$$

whenever $|x - x_0| \leq \frac{1}{4}r$. Let $\mathbf{n_0}$ denote the inward unit normal to ∂G at x_0 and let ∂H denote the tangent plane to ∂G at x_0. As in the proof of Lemma 3.2, in a sufficiently small neighborhood of x_0, the boundary ∂G can be represented as the graph of a C_b^2 function defined on ∂H whose gradient is zero at x_0. It follows that for all r sufficiently small, $(\mathbf{x} - \mathbf{x_0}) \cdot \mathbf{n_0} > r/16$ for all $x \in I_{r/2}$ and $|(\mathbf{x} - \mathbf{x_0}) \cdot \mathbf{n_0}| < r/32$ for all $x \in \partial G \cap B(x_0, r)$. Then for $U_r \equiv \{x \in B(x_0, r) \colon (\mathbf{x} - \mathbf{x_0}) \cdot \mathbf{n_0} > r/32\}$, $I_{r/2} \subset U_r$, $I_{r/2}$ is at least distance $r/32$ from ∂U_r, and $U_r \cap \partial G = \phi$. Moreover, if $A_r \subset \overline{G} \cap \partial B(x_0, r)$ such that $|A_r| \geq \frac{1}{2}|\overline{G} \cap \partial B(x_0, r)|$, then $|\partial U_r \cap A_r| \geq \frac{1}{8}|\partial U_r|$ where $|\cdot|$ denotes surface measure on ∂U_r. Now, for a fixed point in U_1, harmonic measure on ∂U_1 is bounded below by a constant times a power of the surface measure on ∂U_1 (cf. Dahlberg [6, Corollary 3]). It follows from this, together with Harnack's inequality [4, §4.7, Theorem 1] and Brownian scaling, that there is a constant $\kappa_3 > 0$ (not depending on A_r, r, x or x_0) such that for all r sufficiently small and $\tau_r = \inf\{t \geq 0 \colon w(t) \in \partial U_r\}$,

$$\inf_{x \in I_{r/2}} P_x^0(w(\eta_r) \in A_r) \geq \inf_{x \in I_{r/2}} P_x^0(w(\tau_r) \in A_r) > \kappa_3.$$

Thus, by the above and the strong Markov property of w under P_x^0, there is $\gamma = \gamma(x_0) > 0$ such that for $0 < r \leq \gamma$ and $|x - x_0| \leq \frac{1}{4}r$,

$$\begin{aligned} P_x^0(w(\eta_r) \in A_r) &\geq P_x^0(P^0_{w(\eta_{r/2})}(w(\eta_r) \in A_r); w(\eta_{r/2}) \in I_{r/2}) \\ &\geq \kappa_3\kappa_2 > 0, \end{aligned}$$

whenever $A_r \subset \overline{G} \cap \partial B(x_0, r)$ satisfies $|A_r| \geq \frac{1}{2}|\overline{G} \cap \partial B(x_0, r)|$. □

Lemma 3.4. *Suppose* $\alpha < 2$. *Let* $\{x_n\}$ *be a sequence in* $\overline{G}$ *such that* $x_n \to x \in \overline{G}$ *and let* $\{P^n\}$ *be a sequence of probability measures on* $(C_{\overline{G}}, \mathscr{M})$ *such that for each* n, P^n *is a solution of the submartingale problem* (SP) *starting from* x_n. *Then* $\{P^n\}$ *has a weak limit point and any such limit point is a solution of the submartingale problem* (SP) *starting from* x.

Proof. For $0 < \alpha < 2$, this is proved in the same manner as Theorem 3.7 of [22]. For $\alpha \leq 0$, we use a modification of the argument in [22, Theorem 3.7], as follows. Tightness of the family is proved by similar arguments to those in Lemma 2.9, and by analogous reasoning to that in Theorem 2.3, any weak limit point P^* of $\{P^n\}$ is a probability measure on $(C_{\overline{G}}, \mathscr{M})$ satisfying conditions (1.4)–(1.5). Thus it remains to verify that P^* satisfies (1.6).

By similar reasoning to that in the proof of Theorem 2.1, $P^* = P_x^0$ on $\mathscr{M}_{\tau_0}$. In particular, by Theorem 2.2, we have $P^*(\tau_0 < \infty) = 0$ if $x \neq 0$, and then (1.6) holds. On the other hand, suppose $x = 0$ and by restricting to a subsequence if necessary, suppose $\{P^n\}$ converges weakly to P^*. Note that x_n may equal 0 for infinitely many n. For each $r > 0$, let $\tau_r = \inf\{t \geq 0 \colon \Psi(w(t)) \geq r\}$. Then (cf. Lemma 2.10), for $0 < \varepsilon < R$, $R > \sup_n |x_n|$, we have

$$(3.21) \qquad E^{P^*}\left[\int_0^{\tau_R} 1_{[0,\varepsilon)}(\Psi(w(s))\, ds\right] \leq \varliminf_{n\to\infty} E^{P^n}\left[\int_0^{\tau_R} 1_{[0,\varepsilon)}(\Psi(w(s))\, ds\right].$$

It follows from condition (1.6) for P^n, by considering the cases $x_n \neq 0$ and $x_n = 0$ separately, that

$$(3.22) \qquad E^{P^n}\left[\int_0^{\tau_R} 1_{[0,\varepsilon)}(\Psi(w(s))\, ds\right] = \lim_{r\downarrow 0} E^{P^n}\left[\int_{\tau_r}^{\tau_R} 1_{(0,\varepsilon)}(\Psi(w(s))\, ds\right].$$

For $0 < r < R$, let $P_w^{n,r}$ be a regular conditional probability distribution (r.c.p.d) of $P^n|_{\mathscr{M}_{\tau_r}}$ and define $\widehat{P}_w^{n,r}$ on $\{w \colon \tau_r(w) < \infty\}$ by

$$\widehat{P}_w^{n,r}(A) = P_w^{n,r}(\widetilde{w}(\cdot + \tau_r) \in A) \quad \text{for all } A \in \mathscr{M},$$

where $\widetilde{w}$ denotes a generic element of $C_{\overline{G}}$. Then on $\{w \colon \tau_r(w) < \infty\}$, $\widehat{P}_w^{n,r}$ is a solution of the submartingale problem starting from $w(\tau_r)$, and as for $x \neq 0$, $\widehat{P}_w^{n,r} = P^0_{w(\tau_r)}$. Thus, by conditioning on $\mathscr{M}_{\tau_r}$ and shifting time by τ_r, we obtain

$$
\begin{aligned}
&E^{P^n}\left[\int_{\tau_r}^{\tau_R} 1_{(0,\varepsilon)}(\Psi(w(s)))\,ds\right] \\
&= E^{P^n}\left[E^{\widehat{P}^{n,r}_w}\left[\int_0^{\tau_R} 1_{(0,\varepsilon)}(\Psi(\widetilde{w}(s)))\,ds\right] 1_{\{\tau_r(w)<\infty\}}\right] \\
&= E^{P^n}\left[E^{P^0_{w(\tau_r)}}\left[\int_0^{\tau_R} 1_{(0,\varepsilon)}(\Psi(\widetilde{w}(s)))\,ds\right] 1_{\{\tau_r(w)<\infty\}}\right] \\
&\leq \begin{cases} cg(\ln r) & \text{if } \alpha = 0, \\ cg(r^{-1}) & \text{if } \alpha < 0, \end{cases}
\end{aligned} \tag{3.23}
$$

where the last line follows by Lemma 2.8 and the fact that g is decreasing for $\alpha = 0$ and increasing for $\alpha < 0$. Here $c = (\inf_{\omega\in\overline{\Omega}} h(\omega))^{-1} \in (0, \infty)$, and h and g are as in Lemma 2.8. Letting $r \to 0$, it follows from (3.21)–(3.23) that

$$
\begin{aligned}
&E^{P^*}\left[\int_0^{\tau_R} 1_{[0,\varepsilon)}(\Psi(w(s)))\,ds\right] \\
&\quad\leq \begin{cases} c\left(\frac{1}{2}\varepsilon^2 + \varepsilon^2(\ln R - \ln\varepsilon)\right) & \text{if } \alpha = 0, \\ c(2-\alpha)^{-1}(\alpha^2\varepsilon^{-2/\alpha} - 2\alpha\varepsilon^{1-2/\alpha}(\varepsilon^{-1} - R^{-1})) & \text{if } \alpha < 0. \end{cases}
\end{aligned}
$$

By Fatou's lemma, on letting $\varepsilon \downarrow 0$ and then $R \uparrow \infty$, we deduce that (1.6) holds for P^*. □

Lemma 3.5. *Suppose $\alpha < 2$, $R > 0$ and let $\tau_R = \inf\{t \geq 0 : \Psi(w(t)) \geq R\}$. Then for any solution P_x of the submartingale problem* (SP) *starting from $x \in \{y \in \overline{G} : \Psi(y) < R\}$, we have*

$$E^{P_x}[\tau_R] < CR^\beta,$$

where $\beta = 2/|\alpha|$ if $\alpha \neq 0$, $\beta = 2$ if $\alpha = 0$ and C is a constant that does not depend on x or R.

Proof. First suppose that $0 < \alpha < 2$. Then the result follows for $x = 0$ in the same manner as Theorem 3.6 of [22]. For $x \neq 0$, it follows from Lemma 2.8 with $\varepsilon = R$ and the result for $x = 0$, since an r.c.p.d of P_x given $\mathscr{M}_{\tau_0}$ and shifted by τ_0 is a solution of the submartingale problem starting from the origin.

Now suppose $\alpha \leq 0$. For $x \neq 0$, $P_x = P_x^0$, in particular $P_x(\tau_0 < \infty) = 0$, and the result follows from Lemma 2.8 with $\varepsilon = R$. For $x = 0$, in a similar manner to the last lemma, we have

$$
\begin{aligned}
E^{P_0}[\tau_R] &= \lim_{r\downarrow 0} E^{P_0}\left[\int_{\tau_r}^{\tau_R} 1_{[0,R)}(\Psi(w(s)))\,ds\right] \\
&= \lim_{r\downarrow 0} E^{P_0}\left[E^{P^0_{w(\tau_r)}}\left[\int_0^{\tau_R} 1_{[0,R)}(\Psi(\widetilde{w}(s)))\,ds\right] 1_{\{\tau_r(w)<\infty\}}\right] \\
&\leq \begin{cases} \frac{1}{2}cR^2 & \text{if } \alpha = 0, \\ c\alpha^2(2-\alpha)^{-1}R^{-2/\alpha} & \text{if } \alpha < 0. \end{cases} \quad \square
\end{aligned}
$$

3.3 An ergodic theorem. In this section and the next, for each $r \geq 0$ we let

$$\sigma_r(w) \equiv \inf\{t \geq 0: |w(t)| = r\}. \tag{3.24}$$

For each $x \in \overline{\Omega} = \{y \in \overline{G}: |y| = 1\}$, define the subprobability measure $Q(x, \cdot)$ on the Borel σ-field $\mathscr{B}(\overline{\Omega})$ on $\overline{\Omega}$ by:

$$Q(x, A) = P_x^0(w(\sigma_2)/2 \in A, \sigma_2 < \sigma_0) \quad \text{for all } A \in \mathscr{B}(\overline{\Omega}). \tag{3.25}$$

The following scaling lemma is an immediate consequence of Lemma 2.3 and the fact that for $r > 0$ and $\hat{w}(\cdot) = r^{-1}w(r^2\cdot)$,

$$\sigma_{2r}(w) = r^2\sigma_2(\hat{w}).$$

Lemma 3.6. *For $x \in \overline{G}\backslash\{0\}$ and $r = |x|$,*

$$P_x^0(w(\sigma_{2r})/2r \in A, \sigma_{2r} < \sigma_0) = Q\left(\frac{x}{|x|}, A\right) \quad \textit{for all } A \in \mathscr{B}(\overline{\Omega}).$$

For $f \in C(\overline{\Omega})$, define

$$Qf(x) \equiv \int_{\overline{\Omega}} Q(x, dy)f(y) \quad \textit{for all } x \in \overline{\Omega}. \tag{3.26}$$

Theorem 3.2. *Q is a compact operator on $C(\overline{\Omega})$.*

Proof. We must show that $\{Qf: f \in C(\overline{\Omega}), \|f\| \leq 1\}$ is relatively compact in $C(\overline{\Omega})$, where $\|f\| = \sup_{x\in\overline{\Omega}}|f(x)|$. Suppose $f \in C(\overline{\Omega})$ and $\|f\| \leq 1$. For $x \in \overline{G}$, define

$$h(x) = E^{P_x^0}[f(w(\sigma_2)/2); \sigma_2 < \sigma_0].$$

Fix $x_0 \in \overline{\Omega}$. Then for $0 < r < \frac{1}{2}$, $x \in \overline{G}: |x - x_0| \leq \frac{1}{4}r$, and

$$\eta_r \equiv \inf\{t \geq 0: |w(t) - x_0| \geq r\},$$

by the strong Markov property of $\{P_z^0: z \in \overline{G}\}$, we have

$$\begin{aligned} h(x) &= E^{P_x^0}[h(w(\eta_r))] \\ &= E^{P_x^0}[h(w(\eta_r)); w(\eta_r) \in A_r] + E^{P_x^0}[h(w(\eta_r)); w(\eta_r) \in B_r], \end{aligned} \tag{3.27}$$

where $A_r \equiv \{y \in \overline{G}\cap\partial B(x_0, r): h(y) \geq 0\}$ and $B_r \equiv \{y \in \overline{G}\cap\partial B(x_0, r): h(y) < 0\}$. Note that either $|A_r| \geq \frac{1}{2}|\overline{G} \cap \partial B(x_0, r)|$ or $|B_r| \geq \frac{1}{2}|\overline{G} \cap \partial B(x_0, r)|$. It follows by Lemma 3.3 that there is $\kappa > 0$ (not depending on r) and $\gamma \in (0, \frac{1}{2})$ such that for each $0 < r \leq \gamma$, either

$$P_x^0(w(\eta_r) \in A_r) \geq \kappa \quad \text{or} \quad P_x^0(w(\eta_r) \in B_r) \geq \kappa$$

for all $x \in \overline{G}: |x - x_0| \leq \frac{1}{4}r$. For $0 < r \leq \gamma$, define

$$M_r = \sup\{h(x): x \in \overline{G}\cap\partial B(x_0, r)\} \quad \text{and} \quad m_r = \inf\{h(x): x \in \overline{G}\cap\partial B(x_0, r)\}.$$

Note that $-1 \le m_r \le M_r \le 1$ since $\|f\| \le 1$. If we define $h_r = h - \frac{1}{2}(M_r + m_r)$, then (3.27) still holds with h_r in place of h and

$$\sup_{x \in \overline{G} \cap \partial B(x_0, r)} h_r(x) = \frac{M_r - m_r}{2} = - \inf_{x \in \overline{G} \cap \partial B(x_0, r)} h_r(x).$$

Applying the above with h_r in place of h, and setting $C_r = (M_r - m_r)/2$, we see that if $P_x^0(w(\eta_r) \in A_r) \ge \kappa$ for all $x \in \overline{G}$: $|x - x_0| \le \frac{1}{4}r$, then for these x,

$$h_r(x) \ge E^{P_x^0}[h_r(w(\eta_r)); w(\eta_r) \in B_r] \ge -C_r P_x^0(w(\eta_r) \in B_r) \ge -C_r(1-\kappa),$$

or if $P_x^0(w(\eta_r) \in B_r) \ge \kappa$, then

$$h_r(x) \le E^{P_x^0}[h_r(w(\eta_r)); w(\eta_r) \in A_r] \le C_r(1-\kappa).$$

Since one of these alternatives must hold for all $x \in \overline{G}$: $|x - x_0| \le \frac{1}{4}r$, it follows that

$$\sup_{x, y \in \overline{G} \cap \overline{B(x_0, r/4)}} |h_r(x) - h_r(y)| \le C_r(2-\kappa).$$

Thus, unless $h \equiv$ constant on $\overline{B(x_0, r)}$, we have

$$\sup_{x, y \in \overline{G} \cap \overline{B(x_0, r/4)}} |h_r(x) - h_r(y)| \le \frac{C_r(2-\kappa)}{2C_r} \sup_{x, y \in \overline{G} \cap \overline{B(x_0, r)}} |h_r(x) - h_r(y)|$$

where $\sup_{x, y \in \overline{G} \cap \overline{B(x_0, r)}} |h_r(x) - h_r(y)| = 2C_r$, by the harmonic averaging property (3.27) of h and the definitions of M_r and m_r. Since differences for h are the same as for h_r, it follows that if we denote the oscillation of h on $\overline{G} \cap \overline{B(x_0, r)}$ by

$$\operatorname{osc}(r) \equiv \sup_{x, y \in \overline{G} \cap \overline{B(x_0, r)}} |h(x) - h(y)|,$$

then

$$\operatorname{osc}\left(\frac{r}{4}\right) \le \left(1 - \frac{\kappa}{2}\right) \operatorname{osc}(r) \quad \text{for } 0 < r \le \gamma. \tag{3.28}$$

(Note this holds even when $h \equiv$ constant on $\overline{B(x_0, r)}$.) Iteration of (3.28) yields

$$\operatorname{osc}(4^{-n}\gamma) \le \left(1 - \frac{\kappa}{2}\right)^n \operatorname{osc}(\gamma), \qquad n = 1, 2, \ldots.$$

Since $\operatorname{osc}(\gamma) \le 2\|f\| \le 2$ and h agrees with Qf on $\overline{\Omega}$, it follows that if $\operatorname{osc}_Q(r)$ denotes the oscillation of Qf on $\overline{\Omega} \cap \overline{B(x_0, r)}$, then

$$\operatorname{osc}_Q(4^{-n}\gamma) \le 2\left(1 - \frac{\kappa}{2}\right)^n, \qquad n = 1, 2, \ldots.$$

Hence, in a similar manner to that in Moser [17], there are positive constants C and β, not depending on f, but possibly depending on x_0, such that

$$|Qf(x) - Qf(x_0)| \le C|x - x_0|^\beta$$

for all $x \in \overline{\Omega}$: $|x - x_0| \leq \gamma$. Since $\overline{\Omega}$ is compact, it follows that $\{Qf: f \in C(\overline{\Omega}), \|f\| \leq 1\}$ is equicontinuous and so is relatively compact, by the Ascoli-Arzela theorem. □

Theorem 3.3. *Suppose g and h are bounded continuous functions on $\overline{\Omega}$ such that $h \geq 0$, but $h \not\equiv 0$. Suppose $\{\nu_n\}$ is a sequence of probability measures on $(\overline{\Omega}, \mathscr{B}(\overline{\Omega}))$. Then*

$$\frac{\int_{\overline{\Omega}} Q^n g(x)\nu_n(dx)}{\int_{\overline{\Omega}} Q^n h(x)\nu_n(dx)} \to C(g, h) \quad \text{as } n \to \infty$$

where $C(g, h)$ is a finite constant depending only on Q, g and h, and not on the sequence $\{\nu_n\}$.

Proof. The crux of the proof is to verify that Q satisfies the hypotheses of the Krein-Rutman theorem. The compactness of $Q: C(\overline{\Omega}) \to C(\overline{\Omega})$ was proved in Theorem 3.2. Let $K = \{f \in C(\overline{\Omega}): f \geq 0 \text{ on } \overline{\Omega}\}$. To prove that Q is strongly positive on K, suppose $f \in K$, $f \not\equiv 0$. Then there is $y_0 \in \Omega$, $\varepsilon \in (0, \frac{1}{2})$ and $c_1 > 0$ such that $f(x) > c_1$ whenever $|x - y_0| < \varepsilon$. Fix $x_0 \in \overline{\Omega}$ and define $\zeta: [0, 3] \to \mathbb{R}^d$ by

$$\zeta(s) = \begin{cases} \rho(s) & 0 \leq s \leq 1, \\ s y_0 & 1 \leq s \leq 3, \end{cases}$$

where $\rho: [0, 1] \to \overline{G}$, $\rho(0) = x_0$, $\rho(1) = y_0$, $\rho(s) \in G$ for $s \neq 0$ and $|\rho(s)| < 2 - \varepsilon$ for $0 \leq s \leq 1$. By the support theorem 3.1,

$$c_2 \equiv P^0_{x_0}\left(\sup_{0 \leq s \leq 3} |w(s) - \zeta(s)| < \varepsilon\right) > 0.$$

Then,

$$Qf(x_0) \geq E^{P^0_{x_0}}[f(w(\sigma_2)/2); |\tfrac{1}{2}w(\sigma_2) - y_0| < \varepsilon] \geq c_1 c_2 > 0.$$

The Krein-Rutman theorem can then be applied precisely as in Theorem 5.4 of [2] to conclude the desired result. □

3.4 Uniqueness.

Lemma 3.7. *The following two statements are equivalent.*

(i) *For each $x \in \overline{G}$, there is at most one solution of the submartingale problem* (SP) *starting from x.*

(ii) *There is at most one family of probability measures $\{P_x, x \in \overline{G}\}$ satisfying the submartingale problem* (SP) *that has the strong Markov property.*

Proof. This follows by the same kind of argument as in Theorems 12.2.4 and 12.2.3 in [20]. The only hypotheses of substance that need to be checked occur in the supporting Lemmas 12.2.1 and 12.2.2 in [20]. This amounts to showing the weak compactness and measurability in (s, x) of the set $\mathscr{C}(s, x)$ of solutions of the submartingale problem starting from x at time s. But these can be verified using Lemma 3.4 and the time homogeneity of our problem. □

Theorem 3.4. *Let* $\alpha < 2$. *For each* $x \in \overline{G}$ *there is at most one solution to the submartingale problem* (SP) *starting from* x.

Proof. By Lemma 3.7, it suffices to prove that there is at most one strong Markov family $\{P_x, x \in \overline{G}\}$ solving (SP). It follows from Theorem 2.1 and the proof of Lemma 2.2 that $P_x = P_x^0$ on $\mathscr{M}_{\tau_0}$. Since by (1.6), $\{t \geq 0: w(t) = 0\}$ has zero Lebesgue measure P_0-a.s., we can mimic the proof of uniqueness in Theorem 5.5 of [2, p. 567 ff.], with the following modifications: (i) replace B, the ball of radius $\frac{1}{2}\delta$ about x_0, by its intersection with $\overline{G}$; (ii) use Lemma 3.5 and the fact that given $M > 0$, $\inf\{|x|: x \in \overline{G}, \Psi(x) = R\} > M$ for all sufficiently large R, to conclude that $\sup_{\|x\| \leq M} E^{P_x}[\sigma_M] < \infty$, where

$$\sigma_M = \inf\{t \geq 0: |w(t)| \geq M\};$$

(iii) use the support theorem 3.1 and ergodic theorem 3.3, in place of Theorems 5.2 and 5.4 in [2]; and (iv) note that the required continuity in lines 1–2, 16–17 of [2, p. 569] can be proved in a similar manner to the equicontinuity in the proof of Theorem 3.2 (this is needed since the Krylov-Safanov result does not apply to reflected diffusions). □

Theorem 3.5. *If* $\alpha < 2$, *then for each* $x \in \overline{G}$, *there is a unique solution* P_x *of the submartingale problem* (SP) *starting from* x. *Hence, there is a unique family* $\{P_x, x \in \overline{G}\}$ *solving* (SP). *If* $\alpha \leq 0$ *and* $x \in \overline{G}\backslash\{0\}$, *then* $P_x = P_x^0$. *If* $\alpha \geq 2$, *then for each* $x \in \overline{G}$, $P_x = P_x^0$ *is the unique probability measure on* $(C_{\overline{G}}, \mathscr{M})$ *satisfying properties* (1.4) *and* (1.5). *For any value of* α, *the approximating family* $\{P_x^\delta\}$ *defined in* §2.4 *converges weakly to* P_x *as* $|\delta| \to 0$.

Proof. Combine Theorems 2.3 and 3.4 for the $\alpha < 2$ case. For $\alpha \geq 2$, the result follows in the same manner as Theorem 3.11 of [22]. □

Remark. Using the uniqueness, it can be shown that $\{P_x, x \in \overline{G}\}$ has the strong Markov property and has the Feller continuity property: $x \to E^{P_x}[f(w(t))]$ is continuous on $\overline{G}$ for each $f \in C_b(\overline{G})$ and $t \geq 0$ (cf. [21, p. 196]).

Example. Suppose $\overline{G}$ is a circular cone with azimuthal angle $\xi \in (0, \pi)$, and the vector field $\mathbf{v}$ has only normal and radial components: $\mathbf{v} = \mathbf{n} + v_r \mathbf{e_r}$, and $v_r = \beta \equiv$ constant. Then we can determine the value of β such that $\alpha = 2$, as follows. Let θ denote the azimuthal angular coordinate for points in $\overline{\Omega}$, measured from the axis of G. We have already remarked in §2.2 that when α is a positive integer, there is a solution of (2.24) in terms of a Jacobi polynomial function of $\cos\theta$. When $\alpha = 2$, after simplification [1, pp. 779, 793], this is given by $p(\theta) = d\cos^2\theta - 1$. This solution will be positive on $\overline{\Omega}$ if and only if $\xi < \cos^{-1}(1/\sqrt{d})$. Indeed, by [3, (3.10)] and the fact that $\cos^{-1}(1/\sqrt{d})$ is the smallest zero of $p(\theta)$ on $(0, \pi)$, it follows that there is a function $\psi_2 \in C^2(\overline{\Omega})$ satisfying (2.24) with $\alpha = 2$ and $\psi_2 > 0$ on $\overline{\Omega}$ if and only if the cone angle $\xi < \cos^{-1}(1/\sqrt{d})$. Assuming this condition, the function $p(\theta)$ satisfies (2.25) if and only if

$$\beta = \beta^* \equiv \frac{-d\sin\xi\cos\xi}{d\cos^2\xi - 1}.$$

If $\beta \le \beta^*$, it can be shown in a similar manner to that in Theorem 3.11 of [22], that for any solution P_0 of the submartingale problem (SP) starting from 0, the function $q(r, \theta) = r^2 p(\theta)$ applied to $w(\cdot)$ yields a local supermartingale. But, the minimum value of this supermartingale is taken at 0, and so it would follow that $P_0(w(t)) = 0$ for all $t \ge 0) = 1$, contradicting (1.6). Consequently, for $\beta \le \beta^*$ we must have $\alpha \ge 2$. If $\beta > \beta^*$, then $q(w(\cdot \wedge \tau_0))$ will be a local submartingale under P_x^0 for all $x \in \overline{G}$, and this can be used in a similar manner to the local martingale $\Phi(w(\cdot \wedge \tau_0))$, to obtain estimates such as in §2.3 that imply the existence of a solution of the submartingale problem (SP), from which it follows that $\alpha < 2$. In summary, for this example we have $\alpha \ge 2$ if and only if $\xi < \cos^{-1}(1/\sqrt{d})$ and $\beta \le \beta^*$.

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