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LINEAR TOPOLOGICAL CLASSIFICATIONS OF CERTAIN FUNCTION SPACES

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ABSTRACT. Some linear classification results for the spaces $C_p(X)$ and $C_p^*(X)$ are proved.

0. INTRODUCTION

If X is a space then $C_p(X)$ denotes the set of all continuous real-valued functions on X with the topology of pointwise convergence. We write $C_p^*(X)$ for the subspace of $C_p(X)$ consisting of all bounded functions. R stands for the usual space of real numbers, I—for the unit segment $[0, 1]$ and Q is the Hilbert cube I^ω. If $n \geq 1$ then μ^n denotes the n-dimensional universal Menger compactum. Let X be a separable metric space. A separable metric space Y is called an X-manifold if Y admits an open cover by sets homeomorphic to open subsets of X.

Results in [A1, A2 and Ps] show that the linear topological classification of the spaces $C_p(X)$ is very complicated. Below the linear topological classification results for the spaces $C_p(X)$ which I know are listed:

(1) Let X and Y be non-zero-dimensional compact polyhedra. Then $C_p(X) \sim C_p(Y)$ if and only if $\dim X = \dim Y$ [Pv]. Here the symbol "$\sim$" stands for linear homeomorphism.

(2) If X is a locally compact subset of R^n such that $\mathrm{cl}(\mathrm{Int}(X)) \cap (R^n - X) \neq \varnothing$ then $C_p(X) \sim C_p(R^n)$ [Dr1].

(3) If X is a 1-dimensional compact ANR with finite ramification points or a continuum X is a one-to-one continuous image of $[0, \infty)$ then $C_p(X) \sim C_p(I)$ [KO].

For topological classification results of the spaces $C_p(X)$ see [BGM, BGMP, GH and M].

The aim of this paper is to prove the following results:

(4) $C_p(X) \sim C_p(Q)$ if and only if X is a compact metric space containing a copy of Q.

(5) Let X be a subset of R^n. Then $C_p(X) \sim C_p(I^n)$ iff X is compact and $\dim X = n$.

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(6) $C_P(X) \sim C_P(\mu^n)$ if and only if X is an n-dimensional compact metric space containing a copy of μ^n.

(7) $C_P(X) \sim C_P(l_2)$ provided X is an l_2-manifold (by l_2 is denoted the separable Hilbert space).

(8) Let X be one of the spaces Q, I^n or μ^n, and Y be a locally compact subset of an X-manifold. Then $C_P(Y) \sim C_P(X)^\omega$ if and only if Y contains a closed copy of the topological sum $\sum X_i$ of infinitely many copies of X.

Similar results are also proved for the spaces $C_P^*(X)$.

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1. Preliminaries

All spaces under discussion are Tychonoff and all mappings between topological spaces are continuous. By $L_P(X)$ is denoted the dual linear space of $C_P(X)$ with the weak (i.e. pointwise) topology. It is known that

$$L_P(X) = \left\{ \sum_{i=1}^{k} a_i \delta_{x_i} : a_i \in R - (0) \text{ and } x_i \in X \text{ for each } i \leq k \right\}.$$

Here δ_x is the Dirac measure at the point $x \in X$. We denote

$$P_\infty(X) = \left\{ \sum_{i=1}^{k} a_i \delta_{x_i} : a_i \in (0, 1) \text{ for each } i \text{ and } \sum_{i=1}^{k} a_i = 1 \right\}$$

and $\mathrm{supp}(l) = (x_1, \dots, x_k)$, where $l = \sum_{i=1}^{k} a_i \delta_{x_i} \in L_P(X)$.

Let A be a closed subset of a space X. Consider the following conditions:

(i) There is a continuous linear extension operator $u: C_P(A) \to C_P(X)$ (recall that $u: C_P(A) \to C_P(X)$ is an extension operator if $u(f)|A = f$ for every $f \in C_P(A)$);

(ii) There is a continuous linear extension operator $u: C_P(A) \to C_P(X)$ and a positive constant c such that $\|u(f)\| \leq c \,.\, \|f\|$ for every $f \in C_P^*(A)$. Here $\|f\|$ is the supremum norm of f;

(iii) There is a regular extension operator $u: C_P(A) \to C_P(X)$ i.e. a continuous linear extension operator u with $u(1_A) = 1_X$ and $u(f) \geq 0$ provided $f \geq 0$.

A is said to be l-embedded (resp., l^*-embedded) in X if the condition (i) (resp., the condition (ii)) holds. If (iii) is satisfied then A is called strongly l-embedded in X. Dugundji [D] proved that every closed subset of a metric space X is strongly l-embedded in X (he did not state this explicitly in this form). It is known (see [AČ, Dr1]) that A is l-embedded (resp., strongly l-embedded) in X if and only if there is a mapping $r: X \to L_P(A)$ (resp., $r: X \to P_\infty(A)$) such that $r(x) = \delta_x$ for every $x \in A$. Such a mapping will be called an L_P-valued (resp., a P_∞-valued) retraction. Every L_P-valued retraction $r: X \to L_P(A)$ defines a continuous linear extension operator $u_r: C_P(A) \to C_P(X)$ by setting

$u_r(f)(x) = r(x)(f)$. If the operator u_r satisfies the condition (ii), r is said to be a bounded L_P-valued retraction.

Let $u: C_P(A) \to C_P(X)$ be a continuous linear extension operator. Then the mapping $v(f, g) = u(f) + g$ is a linear homeomorphism from $C_P(A) \times C_P(X; A)$ onto $C_P(X)$, where

$$C_P(X; A) = \{g \in C_P(X): g|A = 0\}.$$

Analogously, if A is l^*-embedded in X then $C_P^*(A) \times C_P^*(X; A)$ is linearly homeomorphic to $C_P^*(X)$.

Let $\mathscr{K}$ be a family of bounded subsets of a space X (i.e. $f|K$ is bounded for every $K \in \mathscr{K}$ and $f \in C_P(X)$) and E be a linear topological subset of $C_P(X)$. Then we set

$$\left(\prod E\right)_{\mathscr{K}} = \left\{(f_1, \dots, f_n, \dots) \in E^\omega: \lim_n \|f_n\|_K = 0 \text{ for every } K \in \mathscr{K}\right\}$$

and

$$\left(\prod E\right)^*_{\mathscr{K}} = \left\{(f_1, \dots, f_n, \dots) \in \left(\prod E\right)_{\mathscr{K}}: \sup_n \|f_n\| < \infty\right\}.$$

$(\prod E)_{\mathscr{K}}$ and $(\prod E)^*_{\mathscr{K}}$ are considered as topological linear subspaces of $C_P(X)^\omega$. We write $(\prod E)_b$ and $(\prod E)_b$ (resp. $(\prod E)_c$ and $(\prod E)^*_c$) if $\mathscr{K}$ is the family of all bounded (resp., of all compact) subsets of X. In the above notations $\|f\|_K$ stands for the set $\sup\{|f(X)|: x \in K\}$. Let us note that if X is pseudocompact and E is a linear subset of $C_P(X)$, the space

$$\left(\prod E\right)_0 = \left\{(f_1, \dots, f_n, \dots) \in E^\omega: \lim_n \|f_n\| = 0\right\}$$

is considered in [GH].

We need also the following notion: a space X is said to be a k_R-space [N] if every function $f: X \to R$ is continuous provided that $f|K$ is continuous for each compact subset K of X.

2. Linear topological classifications of $C_P(X)$

2.1 **Lemma.** *Let A be a strongly l-embedded (resp., l-embedded or l^*-embedded) subset of a space X. Then $A \times Y$ is strongly l-embedded (resp., l-embedded or l^*-embedded) in $X \times Y$ for every space Y.*

Proof. Suppose A is strongly l-embedded in X. So, there exists a P_∞-valued retraction $r_1: X \to P_\infty(A)$. Define a mapping $r: X \times Y \to P_\infty(A \times Y)$ by setting

$$r(x, y) = \sum_{i=1}^k a_i \delta_{(x_i, y)}, \quad \text{where } r_1(x) = \sum_{i=1}^k a_i \delta_{x_i}.$$

It is easily shown that r is a P_∞-valued retraction. Thus, $A \times Y$ is strongly l-embedded in $X \times Y$. One can also prove that r is a (bounded) L_P-valued retraction provided r_1 is a (bounded) L_P-valued retraction. Hence, if A is l (resp., l^*)-embedded in X then $A \times Y$ is l (resp., l^*)-embedded in $X \times Y$.

2.2 **Lemma.** *Let A be an l^*-embedded subset of a space X. Then $(\prod C_P(X))_b$ is linearly homeomorphic to $(\prod C_P(A))_b \times (\prod C_P(X; A))_b$.*

Proof. Let $u: C_P(A) \to C_P(X)$ be a continuous linear extension operator such that $\|u(f)\| \le c . \ \|f\|$ for every $f \in C_P^*(A)$, where $c > 0$. Since $\|f\| = \infty$ provided $f \in C_P(A) - C_P^*(A)$, the inequality $\|u(f)\| \le c . \ \|f\|$ holds for every $f \in C_P(A)$. Then the mapping $r: X \to L_P(A)$, defined by $r(x)(f) = u(f)(x)$, is an L_P-valued retraction. Consider the linear homeomorphism v from $C_P(A) \times C_P(X; A)$ onto $C_P(X)$, $v(f, g) = u(f) + g$. Suppose $(f_1, \dots, f_n, \dots) \in C_P(A)^\omega$ and $(g_1, \dots, g_n, \dots) \in C_P(X; A)^\omega$. Put

$$H(K) = \mathrm{cl}_A \left(\bigcup \{\mathrm{supp}(r(x)) : x \in K\} \right),$$

where K is a subset of X. Obviously, $\|u(f_n)\|_K \le c . \|f_n\|_{H(K)}$ for every $n \in N$. By a result of Arhangel′skii [A2], $H(K)$ is a bounded subset of A provided K is a bounded subset of X. Hence, $(f_1, \dots, f_n, \dots) \in (\prod C_P(A))_b$ if and only if $(u(f_1), \dots, u(f_n), \dots,)$ belongs to $(\prod C_P(X))_b$. Consequently, $(v(f_1, g_1), \dots, v(f_n, g_n), \dots)$ belongs to $(\prod C_P(X))_b$ if $(g_1, \dots, g_n, \dots) \in (\prod C_P(X; A)))_b$ and $(f_1, \dots, f_n, \dots) \in (\prod C_P(A))_b$. Suppose

$$(v(f_1, g_1), \dots, v(f_n, g_n), \dots) \in \left(\prod C_P(X)\right)_b.$$

Then $(f_1, \dots, f_n, \dots) \in (\prod C_P(A))_b$ because $v(f_n, g_n)|A = f_n$ for every n. Therefore $(u(f_1), \dots, u(f_n), \dots) \in (\prod C_P(X))_b$. So we have $(g_1, \dots, g_n, \dots) \in (\prod C_P(X; A))_b$. Thus, $(v(f_1, g_1), \dots, v(f_n, g_n), \dots)$ belongs to $(\prod C_P(X))_b$ iff $(g_1, \dots, g_n, \dots) \in (\prod C_p(X; A))_b$ and $(f_1, \dots, f_n, \dots) \in (\prod C_P(A))_b$. Hence, the formula $v_0((f_1, \dots, f_n, \dots), (g_1, \dots, g_n, \dots)) = (v(f_1, g_1), \dots, v(f_n, g_n) \dots)$ defines a linear mapping from $(\prod C_P(A))_b \times (\prod C_P(X; A))_b$ onto $(\prod C_P(X))_b$ which is a homeomorphism.

2.3 **Lemma.** *Let A be an l^*-embedded subset of a space X. If every closed and bounded subset of A is compact then $(\prod C_P(X \times Y))_c \sim (\prod C_P(A \times Y))_c \times (\prod C_P(X \times Y; A \times Y))_c$ for any space Y.*

Proof. Let $u_1: C_P(A) \to C_P(X)$ be a continuous linear extension operator such that $\|u_1(f)\| \le c . \ \|f\|$ for every $f \in C_P^*(A)$, where $c > 0$, and $r_1: X \to L_P(A)$ be defined by $r_1(x)(f) = u_1(f)(x)$. Obviously, r_1 is an L_P-valued retraction. For a given space Y the equality $r(x, y) = \sum_{i=1}^k a_i \delta_{(x_i, y)}$, where $r_1(x) = \sum_{i=1}^k a_i \delta_{x_i}$, defines an L_P-valued retraction from $X \times Y$ into $L_P(A \times Y)$. Next, set $u(f)(x, y) = r(x, y)(f)$ for every $(x, y) \in X \times Y$ and $f \in C_P(A \times Y)$. It is easily shown that $u: C_P(A \times Y) \to C_P(X \times Y)$ is a continuous linear extension operator.

Claim 1. $\|u(f)\| \le c \,.\, \|f\|$ for every $f \in C_p^*(A \times Y)$.

Fix a point $(x, y) \in X \times Y$ and an $f \in C_p^*(A \times Y)$. It follows from the definition of u that

$$u(f)(x, y) = \sum_{i=1}^{k} a_i f(x_i, y), \quad \text{where } r_1(x) = \sum_{i=1}^{k} a_i \delta_{x_i}.$$

So, $|u(f)(x, y)| \le \sum_{i=1}^{k} |a_i| \,.\, \|f\|$. Take a function $g \in C_p^*(A)$ with $\|g\| = 1$ and $g(x_i) = \operatorname{sgn}(a_i)$ for each $i = 1, \dots, k$. Then $u_1(g)(x) = r_1(x)(g) = \sum_{i=1}^{k} |a_i|$. Since $\|u_1(g)\| \le c.\|g\|$, we have $\sum_{i=1}^{k} |a_i| \le c$. Hence, $|u(f)(x, y)| \le c.\|f\|$. Claim 1 is proved.

Claim 2. For every compact subset K of $X \times Y$ the set

$$H(K) = \mathrm{cl}_{A \times Y}\left(\bigcup\{\operatorname{supp}(r(x, y)) \colon (x, y) \in K\}\right),$$

is also compact.

Let $n_X \colon X \times Y \to X$ and $n_Y \colon X \times Y \to Y$ be the natural projections. Then $n_X(K)$ and $n_Y(K)$ are compact subsets of X and Y respectively. By a result of Arhangel′skii [A2],

$$H_1(K) = \mathrm{cl}_A\left(\bigcup\{\operatorname{supp}(r_1(x)) \colon x \in n_X(K)\}\right)$$

is a bounded subset of A. Thus, $H_1(K)$ is compact. So $H_1(K) \times n_Y(K)$ is a compact subset of $A \times Y$. Since $r(x, y) = (\operatorname{supp}(r_1(x))) \times \{y\}$ for every point $(x, y) \in X \times Y$, we have $H(K) \subset H_1(K) \times n_Y(K)$. Hence, $H(K)$ is compact as a closed subset of $H_1(K) \times n_Y(K)$. Claim 2 is proved.

Now, the proof of Lemma 2.3 follows form the above two claims and the arguments used in the proof of Lemma 2.2.

2.4 **Corollary.** *Let X be a product of metric spaces and A be an l^*-embedded subset of X. Then $(\prod C_p(X))_c \sim (\prod C_p(A))_c \times (\prod C_p(X; A))_c$.*

Proof. Since A is closed in X, every closed bounded subset of A is compact. Thus, the proof follows from Lemma 2.3, where Y is the one-point space.

2.5 **Lemma.** *Suppose X is a space such that both $X \times I$ and $X \times T$ are k_R-spaces, where $T = \{0, 1/n \colon n \in N\}$. Then $C_p(X \times I)$ is linearly homeomorphic to $(\prod C_p(X \times I))_c$.*

Proof. Since, by Lemma 2.1, $X \times T$ is strongly l-embedded in $X \times I$ we have

$$(1) \qquad C_p(X \times I) \sim C_p(X \times T) \times C_p(X \times I; X \times T).$$

Let $I_n = [1/n+1, 1/n]$ and $E_n = C_p(X \times I_n; X \times \{1/n+1, 1/n\})$ for every $n \in N$. Consider the set

$$\left(\prod E_n\right)_c = \Big\{(f_1, \dots, f_n, \dots) \in \prod E_n \colon \lim_n \|f_n\|_{K \times I_n} = 0 \quad \text{for every compact subset } K \text{ of } X\Big\}$$

as a topological linear subset of $\prod\{E_n \colon n \in N\}$. Since $X \times I$ is a k_R-space we have $C_p(X \times I; X \times T) \sim (\prod E_n)_c$. Identifying each E_n with the space $E = C_p(X \times I; X \times \{0, 1\})$ we get

$$C_p(X \times I; X \times T) \sim \left(\prod E\right)_c. \tag{2}$$

Analogously, $C_p(X \times T) \sim C_p(X \times \{0\}) \times C_p(X \times T; X \times \{0\})$ and

$$C_p(X \times T; X \times \{0\}) \sim \left(\prod C_p(X)\right)_c.$$

Thus,

$$C_p(X \times T) \sim C_p(X \times \{0\}) \times \left(\prod C_p(X)\right)_c \sim \left(\prod C_p(X)\right)_c. \tag{3}$$

By Lemma 2.3, the following holds

$$\left(\prod C_p(X \times I)\right)_c \sim \left(\prod C_p(X \times \{0, 1\})\right)_c \times \left(\prod E\right)_c. \tag{4}$$

Obviously,

$$\left(\prod C_p(X \times \{0, 1\})\right)_c \sim \left(\prod C_p(X)\right)_c \times \left(\prod C_p(X)\right)_c \sim \left(\prod C_p(X)\right)_c. \tag{5}$$

So we have

$$\begin{aligned} C_p(X \times I) &\sim C_p(X \times T) \times C_p(X \times I; X \times T) \quad \text{by (1)} \\ &\sim \left(\prod C_p(X)\right)_c \times \left(\prod E\right)_c \quad \text{by (2) and (3)} \\ &\sim \left(\prod C_p(X \times I)\right)_c \quad \text{by (4) and (5).} \end{aligned}$$

2.6 **Corollary.** *Let X be as in Lemma 2.5. Then $C_p(X \times I)$ is homeomorphic to $C_p(X \times I)^\omega$.*

Proof. S. Gul′ko and T. Hmyleva [GH] proved that $(\prod C_p(X))_0$ is homeomorphic to $C_p(X)^\omega \times (\prod C_p(X))_0$ for every pseudocompact space X. Using the same arguments one can see that $(\prod C_p(X))_c$ is homeomorphic to $C_p(X)^\omega \times (\prod C_p(X))_c$ for each X. Now, the proof of Corollary 2.6 follows from Lemma 2.5.

2.7 **Lemma.** *Suppose a space X contains an l-embedded copy F_1 of a space Y and Y contains an l^*-embedded copy F_2 of X. Then $C_p(X) \sim C_p(Y)$ provided one of the following conditions is fulfilled:*

(i) $C_p(Y) \sim (\prod C_p(Y))_b$:
(ii) $C_p(Y) \sim (\prod C_p(Y))_c \sim (\prod C_p(F_2))_c \times (\prod C_p(Y; F_2))_c$.

Proof. We have $C_p(X) \sim C_p(F_1) \times E_1$ and $C_p(Y) \sim C_p(F_2) \times E_2$, where $E_1 = C_p(X; F_1)$ and $E_2 = C_p(Y; F_2)$. Thus, $C_p(X) \sim C_p(Y) \times E_1$. Suppose $C_p(Y) \sim (\prod C_p(Y))_b$. By Lemma 2.2,

$$\left(\prod C_p(Y)\right)_b \sim \left(\prod C_p(F_2)\right)_b \times \left(\prod E_2\right)_b,$$

so

$$\left(\prod C_p(Y)\right)_b \sim \left(\prod C_p(X)\right)_b \times \left(\prod E_2\right)_b.$$

Therefore,

$$C_P(Y) \sim \left(\prod C_P(Y)\right)_b \sim C_P(Y) \times \left(\prod C_P(Y)\right)_b$$
$$\sim C_P(Y) \times \left(\prod C_P(X)\right)_b \times \left(\prod E_2\right)_b .$$

Hence, $C_P(X) \sim E_1 \times C_P(Y) \sim E_1 \times C_P(Y) \times (\prod C_P(X))_b \times (\prod E_2)_b \sim C_P(X) \times (\prod C_P(X))_b \times (\prod E_2)_b \sim (\prod C_P(X))_b \times (\prod E_2)_b \sim C_P(Y)$.

If condition (ii) is fulfilled we use the same arguments.

2.8 **Theorem.** (i) *Let X be a subspace of R^n. Then $C_P(X) \sim C_P(I^n)$ if and only if X is compact and* $\dim X = n$;

(ii) $C_P(X) \sim C_P(Q)$ *if and only if X is a compact metric space containing a copy of Q.*

Proof. We prove only the first part of Theorem 2.8. The proof of (ii) is analogous to that of (i).

Suppose $C_P(X) \sim C_P(I^n)$. Then by [A2 and A3] X is a compact metric space. Next, it follows from a result of Pavlovskii [Pv] that there is a nonempty open subset of I^n which can be embedded in X. Thus, $\dim X = n$.

Now, let X be a compact n-dimensional subset of R^n. Then X contains a copy of I^n. On the other hand X can be considered as a subset of I^n. Hence, by Corollary 2.4, $(\prod C_P(I^n))_c \sim (\prod C_P(X))_c \times (\prod C_P(I^n; X))_c$. Since $C_P(I^n) \sim (\prod C_P(I^n))_c$ (see Lemma 2.5), we derive from Lemma 2.7(ii) that $C_P(X) \sim C_P(I^n)$.

2.9 **Theorem.** *Let μ^n be the n-dimensional universal Menger compactum. Then $C_P(X) \sim C_P(\mu^n)$ if and only if X is an n-dimensional compact metric space containing a copy of μ^n.*

Proof. Let $C_P(X) \sim C_P(\mu^n)$. Then, by results of Arhangel′skii [A2, A3] and Pestov [Ps], X is an n-dimensional compact metric space. It follows from [Pv] that there exists an open subset of μ^n which can be embedded in X. But each open subset of μ^n contains a copy of μ^n [Bt]. Thus, X contains a copy of μ^n.

Suppose X is an n-dimensional compact metric space containing a copy of μ^n. Since X can be embedded in μ^n, by Lemma 2.7(ii) and Corollary 2.4 it is enough to show that $C_P(\mu^n) \sim (\prod C_P(\mu^n))_c$. For proving this fact we need the following result of Dranishnikov [Dr2]: There is a mapping f_n from μ^n onto Q such that $f_n^{-1}(P)$ is homeomorphic to μ^n for every $LC^{n-1}\&C^{n-1}$-compact subspace P of Q. Now, consider Q as a product $Q_1 \times I$, where Q_1 is a copy of Q. Let $T = \{0, 1/k; k \in N\}$ and $T^* = f_n^{-1}(Q_1 \times T)$. Then

$$C_P(\mu^n) \sim C_P(T^*) \times C_P(\mu^n; T^*) \tag{6}$$

and

$$C_P(T^*) \sim C_P(f_n^{-1}(Q_1 \times \{0\})) \times C_P(T^*; f_n^{-1}(Q_1 \times \{0\})).$$

Since each of the sets $f_n^{-1}(Q_1 \times \{1/k\})$, $k \in N$, and $f_n^{-1}(Q_1 \times \{0\})$ is homeomorphic to μ^n, we have

$$C_P(T^*; f_n^{-1}(Q_1 \times \{0\})) \sim \left(\prod C_P(\mu^n)\right)_c$$

and

$$C_P(f_n^{-1}(Q_1 \times \{0\})) \sim C_P(\mu^n).$$

Thus,

$$\begin{aligned} C_P(T^*) &\sim C_P(\mu^n) \times \left(\prod C_P(\mu^n)\right)_c \sim \left(\prod C_P(\mu^n)\right)_c \\ &\sim \left(\prod C_P(\mu^n)\right)_c \times \left(\prod C_P(\mu^n)\right)_c \sim \left(\prod C_P(\mu^n)\right)_c \times C_P(T^*). \end{aligned} \tag{7}$$

Finally,

$$\begin{aligned} C_P(\mu^n) &\sim C_P(T^*) \times C_P(\mu^n; T^*) \quad \text{by (6)} \\ &\sim \left(\prod C_P(\mu^n)\right)_c \times C_P(T^*) \times C_P(\mu^n; T^*) \quad \text{by (7)} \\ &\sim \left(\prod C_P(\mu^n)\right)_c \times C_P(\mu^n) \sim \left(\prod C_P(\mu^n)\right)_c. \end{aligned}$$

2.10 **Theorem.** *Let X be a metric space and τ be an infinite cardinal. Suppose Y is an l^*-embedded subspace of the product X^τ and Y contains an l^*-embedded copy of X^τ. Then $C_P(Y) \sim C_P(X^\tau)$.*

Proof. By Corollary 2.4 and Lemma 2.7(ii), it is enough to show that $C_P(X^\tau) \sim (\prod C_P(X^\tau))_c$. Since τ is infinite we have $X^\tau = (X^\omega)^\tau$. So we can suppose that X is not discrete. Thus, there exists a nontrivial converging sequence $\{x_n\}_{n \in N}$ in X with $\lim x_n = x_0$. Let $T = \{x_0, x_n; n \in N\}$. By Lemma 2.1, $X^\tau \times T$ is l-embedded in $X^\tau \times X$. Therefore,

$$C_P(X^\tau) \sim C_P(X^\tau \times T) \times C_P(X^\tau \times X; X^\tau \times T).$$

But $C_P(X^\tau \times T) \sim C_P(X^\tau \times \{x_0\}) \times C_P(X^\tau \times T; X^\tau \times \{x_0\})$ because $X^\tau \times \{x_0\}$ is also l-embedded in $X^\tau \times T$. Since $X^\tau \times T$ is a k_R-space [N] we have $C_P(X^\tau \times T; X^\tau \times \{x_0\}) \sim (\prod C_P(X^\tau))_c$. Hence,

$$\begin{aligned} C_P(X^\tau \times T) &\sim C_P(X^\tau \times \{x_0\}) \times \left(\prod C_P(X^\tau)\right)_c \sim \left(\prod C_P(X^\tau)\right)_c \\ &\sim \left(\prod C_P(X^\tau)\right)_c \times \left(\prod C_P(X^\tau)\right)_c \sim C_P(X^\tau \times T) \times \left(\prod C_P(X^\tau)\right)_c. \end{aligned}$$

Then

$$\begin{aligned} C_P(X^\tau) &\sim C_P(X^\tau \times T) \times C_P(X^\tau \times X; X^\tau \times T) \\ &\sim \left(\prod C_P(X^\tau)\right)_c \times C_P(X^\tau \times T) \times C_P(X^\tau \times X; X^\tau \times T) \\ &\sim \left(\prod C_P(X^\tau)\right)_c \times C_P(X^\tau) \sim \left(\prod C_P(X^\tau)\right)_c. \end{aligned}$$

2.11 **Corollary.** *Let X be a separable metric space and $\tau > \omega$. Then $C_p(X^\tau) \sim C_p(Y)$ for every closed G_δ-subset Y of X^τ.*

Proof. Suppose Y is a closed G_δ-subset of X^τ. It is well known (see for example [PP]) that modulo a permutation of the coordinates, $Y = Z \times X^{\tau-\omega}$, where Z is a closed subset of X^ω. Thus, by Lemma 2.1, Y is l^*-embedded in X^τ. On the other hand $\{z\} \times X^{\tau-\omega}$ is an l^*-embedded copy of X^τ in Y for each $z \in Z$. Now, Theorem 2.10 completes the proof.

2.12 **Corollary.** *Let U be a functionally open subset of R^τ, $\tau \geq \omega$. Then $C_p(U) \sim C_p(R^\tau)$.*

Proof. Modulo a permutation of the coordinates, $U = V \times R^{\tau-\omega}$, where V is open in R^ω. Obviously, U contains an l^*-embedded copy of R^τ. Since there is an embedding of V in R^ω as a closed subset, by Lemma 2.1, U can be l^*-embedded in R^τ. Thus, by Theorem 2.10, $C_p(U) \sim C_p(R^\tau)$.

Let f be a mapping from a space X onto a space Y. Recall that a continuous linear operator $u: C_p(X) \to C_p(Y)$ is said to be an averaging operator for f if $u(h \circ f) = h$ for every $h \in C_p(Y)$. If f admits a regular averaging operator $u: C_p(X) \to C_p(Y)$ we can define a mapping $r: Y \to P_\infty(X)$ by the formula $r(y)(g) = u(g)(y)$. The mapping r has the following property [Dr1]: $\mathrm{supp}(r(y))$ is contained in $f^{-1}(y)$ for each $y \in Y$. Conversely, if there is a mapping $r: Y \to P_\infty(X)$ such that $\mathrm{supp}(r(y)) \subset f^{-1}(y)$ for every $y \in Y$, then the formula $u(g)(y) = r(y)(g)$ defines a regular averaging operator u for f. It is easily seen that if u is a regular averaging operator for f the mapping $v(g) = (u(g), g - u(g) \circ f)$ is a linear homeomorphism from $C_p(X)$ onto $C_p(Y) \times E$, where $E = \{g - u(g) \circ f: g \in C_p(X)\}$. Dranishnikov proved [Dr1, Theorem 9] that $C_p(R^n) \sim C_p(U)$ for every open subset U of R^n. The same arguments are used in the proof of Proposition 2.13 below.

2.13 **Proposition.** *Let $\{U_i: i \in N\}$ be an infinite locally finite functionally open cover of a space X. Suppose there is a space Y with $C_p(\mathrm{cl}_X(U_i)) \sim C_p(Y)$ for each $i \in N$. Then $C_p(X) \sim C_p(Y)^\omega$ provided X contains an l-embedded copy of a topological sum $\sum_{i=1}^\infty F_i$ such that $C_p(F_i) \sim C_p(Y)$ for every $i \in N$.*

Proof. For every $i \in N$ take an $f_i \in C_p(X)$ such that $f_i^{-1}(0) = X - U_i$ and $f_i \geq 0$. Without loss of generality we can suppose that $\sum_{i=1}^\infty f_i = 1$. Let $f \in C_p(\sum \mathrm{cl}_X(U_i))$ such that $f|\mathrm{cl}_X(U_i) = f_i|\mathrm{cl}_X(U_i)$. Consider the natural mapping $p: \sum \mathrm{cl}_X(U_i) \to X$ with all preimages finite. Let $r: X \to P_\infty(\sum \mathrm{cl}_X(U_i))$ be defined by $r(x) = \sum\{f(y)\cdot\delta_y: y \in p^{-1}(x)\}$. It is easily seen that r is continuous and $\mathrm{supp}(r(x)) \subset p^{-1}(x)$ for every $x \in X$. Thus, there is a regular averaging operator $u: C_p(\sum \mathrm{cl}_X(U_i)) \to C_p(X)$ for p. Hence, $C_p(\sum \mathrm{cl}_X(U_i))$ is linearly homeomorphic to $C_p(X) \times E$, where E is a linear subspace of $C_p(\sum \mathrm{cl}_X(U_i))$. Since $\sum F_i$ is l-embedded in X we have $C_p(X) \sim C_p(\sum F_i) \times C_p(X; \sum F_i)$. Observe that

$$C_p\left(\sum \mathrm{cl}_X(U_i)\right) \sim \prod_{i=1}^{\infty} C_p(\mathrm{cl}_X(U_i)) \sim C_p(Y)^\omega \sim C_p\left(\sum F_i\right).$$

Now, using the technique of Pelczynski [P] and Bessaga [B] we have

$$\begin{aligned} C_P(X) &\sim C_P\left(\sum F_i\right) \times C_P\left(X; \sum F_i\right) \sim C_P(Y)^\omega \times C_P\left(X; \sum F_i\right) \\ &\sim (C_P(Y)^\omega \times \cdots \times C_P(Y)^\omega \times \cdots) \times C_P(Y)^\omega \times C_P\left(X; \sum F_i\right) \\ &\sim (C_P(Y)^\omega \times \cdots \times C_P(Y)^\omega \times \cdots) \times C_P(X) \\ &\sim (C_P(X) \times E \times \cdots \times C_P(X) \times E \times \cdots) \times C_P(X) \\ &\sim C_P(X)^\omega \times E^\omega \sim (C_P(X) \times E)^\omega \sim C_P\left(\sum \mathrm{cl}_X(U_i)\right)^\omega \sim C_P(Y)^\omega. \end{aligned}$$

2.14 **Theorem.** *Let Y be a noncompact separable metric space and X be one of the spaces Q, I^n, μ^n, l_2. Then $C_P(Y) \sim C_P(X)^\omega$ provided Y is an X-manifold.*

Proof. Let $\{U_i: i \in N\}$ be an infinite locally finite open cover of Y such that each $\mathrm{cl}_Y(U_i)$ is regularly closed subset of X. It is clear that a topological sum $\sum F_i$ of infinitely many regularly closed subsets F_i of X is contained in Y as a closed subset. Since each of the sets $\mathrm{cl}_Y(U_i)$ and F_i, $i \in N$, contains a closed copy of X, it follows from Theorem 2.8, Theorem 2.9 and Theorem 2.10 that $C_P(\mathrm{cl}_Y(U_i)) \sim C_P(F_i) \sim C_P(X)$ for every $i \in N$. Hence, by Proposition 2.13, $C_P(Y) \sim C_P(X)^\omega$.

2.15 **Theorem.** *Let U be a functionally open subset of I^τ and τ be an uncountable cardinal. Then $C_P(U) \sim C_P(I^\tau)^\omega$.*

Proof. There exists a projection p from I^τ onto a countable face of I^τ such that $p^{-1}(p(U)) = U$ (see [PP]). Take a locally finite open cover $\{U_i: i \in N\}$ of $p(U)$ such that $\mathrm{cl}_{I^\tau}(p^{-1}(U_i)) \subset U$ for every $i \in N$. Since each $\mathrm{cl}_{I^\tau}(p^{-1}(U_i))$ is a closed G_δ-subset of I^τ, by Corollary 2.11, $C_P(\mathrm{cl}_{I^\tau}(p^{-1}(U_i))) \sim C_P(I^\tau)$.

Now, let $\{x_i: i \in N\}$ be a closed discrete infinite subset of $p(U)$. So, the topological sum $\sum p^{-1}(x_i)$ is l-embedded in U (by Lemma 2.1) and obviously, each $p^{-1}(x_i)$ is homeomorphic to I^τ. Thus, by Proposition 2.13, $C_P(U) \sim C_P(I^\tau)^\omega$.

2.16 **Theorem.** *Let X be one of the spaces Q, I^n, μ^n, and Y be a locally compact subset of an X-manifold. Then $C_P(Y) \sim C_P(X)^\omega$ if and only if Y contains a closed copy of the topological sum $\sum X$ of infinitely many copies of X.*

Proof. The proof of the part "if" is based on a Dranishnikov's idea from [Dr1, Theorem 9′], where it is shown that $C_P(P) \sim C_P(R^n)$ for every locally compact subset P of R^n with $\mathrm{cl}_{R^n}(\mathrm{Int}(P)) \cap (R^n - P) \neq \varnothing$.

Suppose Y is a locally compact subspace of an X-manifold Z and contains a closed copy of the topological sum $\sum X$. Then $C_P(Y) \sim C_P(\sum X) \times C_P(Y; \sum X)$. Next, take a locally finite open cover $\{V_i: i \in N\}$ of Y such that each $\mathrm{cl}_Y(V_i)$ is compact. For every $i \in N$ there exists an open subset U_i

of Z such that $V_i = U_i \cap Y = U_i \cap \mathrm{cl}_Y(V_i)$. Since every set V_i is closed in U_i, $\sum V_i$ is closed in $\sum U_i$. Thus, $C_P(\sum U_i) \sim C_P(\sum V_i) \times C_P(\sum U_i; \sum V_i)$. Let $\{f_i : i \in N\}$ be a partition of unity subordinated to the cover $\{V_i : i \in N\}$. Define a continuous mapping $r: Y \to P_\infty(\sum V_i)$ as in the proof of Proposition 2.13 and by the same arguments we get that $C_P(\sum V_i)$ is linearly homeomorphic to $C_P(Y) \times E$, where E is a linear subspace of $C_P(\sum V_i)$. It follows from Theorem 2.14 that $C_P(U_i) \sim C_P(X)^\omega$ for every $i \in N$. Hence

$$C_P(X)^\omega \sim C_P\left(\sum U_i\right) \sim C_P\left(\sum V_i\right) \times C_P\left(\sum U_i; \sum V_i\right)$$
$$\sim C_P(Y) \times E \times C_P\left(\sum U_i; \sum V_i\right).$$

Now, using the scheme of Pelczynski and Bessaga we get $C_P(Y) \sim C_P(X)^\omega$.

Suppose there is a linear homeomorphism θ from $C_P(\sum X) = C_P(X)^\omega$ onto $C_P(Y)$. Let K be the set $\{y \in Y;$ every neighborhood of y in Y contains a copy of $X\}$. We use the following property of X (for Q and I^n this is obvious, and for μ^n see [Bt]):

$(*)$ Every open subset of X contains a copy of X.

Now we show that K is nonempty. Indeed, by [Pv], Y contains an open subset of $\sum X$. So, by $(*)$, Y contains a copy F of X and $F \subset K$. Obviously K is closed in Y and it follows also from $(*)$ that $Y - K$ does not contain a copy of X. Next, assume K is compact. Consider the set

$$L = \mathrm{cl}\left(\bigcup\{\mathrm{supp}(\theta^*(\delta_y)) : y \in K\}\right),$$

where $\theta^*: L_P(Y) \to L_P(\sum X)$ is the dual homeomorphism of θ. By a result of Arhangel′skii [A2], L is a compact subset of $\sum X$. Therefore, there is a $k \in N$ such that $L \subset \sum_{i=1}^k X_i$. Let $P = \sum_{i=1}^k X_i$, $f \in C_P(\sum X; P)$ and $y \in K$. We have $\theta^*(\delta_y)(f) = \delta_y(\theta(f)) = \theta(f)(y)$. But $\theta^*(\delta_y)(f) = 0$ because $\mathrm{supp}(\theta^*(\delta_y)) \subset P$. Thus, $\theta(f)$ belongs to $C_P(Y; K)$ for every $f \in C_P(\sum X; P)$. Let p be the linear projection from $C_P(\sum X) = C_P(P) \times C_P(\sum X; P)$ onto $C_P(\sum X; P)$. Then $\theta \circ p \circ \theta^{-1}: C_p(Y; K) \to \theta(C_P(\sum X; P))$ is a continuous linear retraction. This means that there is a closed linear subspace E of $C_P(Y; K)$ such that $C_P(Y; K)$ is linearly homeomorphic to $C_P(\sum X; P) \times E$. Clearly, $C_P(Y; K) \sim C_P(Y/K; (K))$, where (K) is the identification point of K in the quotient space Y/K. Analogously, $C_P(\sum X; P) \sim C_P((\sum X)/P; (P))$. Since $C_P(Y/K) \sim R \times C_P(Y/K; (K))$ and

$$C_P\left(\left(\sum X\right)/P; (P)\right) \times R \sim C_P\left(\left(\sum X\right)/P\right),$$

we get that $C_P(Y/K) \sim C_P((\sum X)/P) \times E$. Now, we need the following result of Dranishnikov [Dr1, Theorem 6]: Let X_1 and X_2 be compact metric spaces and $C_P(X_1)$ be linearly homeomorphic to a product $C_P(X_2) \times E_1$. Then $\dim X_2 \le \dim X_1$. Actually, it is proved that X_2 is a union of countably many compact subsets which are embeddable in X_1. It follows from Dranishnikov's arguments that the last statement remains valid if X_1 and X_2 are separable locally compact metric spaces. Hence, there is a countable family $\{F_i: i \in N\}$ of compact subsets of $(\sum X)/P$ such that $(\sum X)/P = \bigcup\{F_i: i \in N\}$ and each F_i can be embedded in Y/K. Since $(\sum X)/P$ has the Baire property, there exists an $i_0 \in N$ with $\mathrm{Int}(F_{i_0}) \neq \varnothing$. Then the set $\mathrm{Int}(F_{i_0}) - \{(P)\}$ is both open in $\sum X$ and embeddable in Y/K. Thus, by $(*)$, Y/K contains a copy of X. So $Y-K$ contains also a copy of X. But we have already seen that this is not possible. Therefore K is not compact.

Take a countable infinite discrete family $\{W_i: i \in N\}$ in K consisting of open subsets of K. Let W_i^* be an open subspace of Y with $W_i^* \cap K = W_i$ for each $i \in N$. For every $i \in N$ there is a copy X_i of X such that $X_i \subset W_i^*$. It follows from $(*)$ that $X_i \subset K$ because $Y - K$ does not contain a copy of X. Hence, $X_i \subset W_i$ for every $i \in N$. So $\{X_i: i \in N\}$ is a discrete family in K. Thus, $\sum X_i$ is a closed subset of Y.

2.17 **Corollary.** *Let X be a locally compact (n-dimensional) separable metric space. Then $C_P(X) \sim C_P(Q)^\omega$ (resp., $C_P(X) \sim C_P(\mu^n)^\omega$) if and only if X contains a closed copy of the topological sum $\sum Q$ (resp., $\sum \mu^n$).*

Proof. Since X can be embedded in Q (resp., in μ^n), the proof follows from Theorem 2.16.

3. Linear topological classifications of $C_P^*(X)$

The proofs of the Lemmas 3.1–3.4 below are similar to the proofs of the corresponding lemmas from §2.

3.1 **Lemma.** *Let A be an l^*-embedded subset of a space X. Then $(\prod C_P^*(X))_b^* \sim (\prod C_P^*(A))_b^* \times (\prod C_P^*(X; A))_b^*$.*

3.2 **Lemma.** *Let A be an l^*-embedded subset of a space X. If every closed bounded subset of A is compact then $(\prod C_P^*(X \times Y))_c^* \sim (\prod C_P^*(A \times Y))_c^* \times (\prod C_P^*(X \times Y; A \times Y))_c^*$ for any space Y.*

3.3 **Corollary.** *Let A be an l^*-embedded subset of a product X of metric spaces. Then*

$$\left(\prod C_P^*(X)\right)_c^* \sim \left(\prod C_P^*(A)\right)_c^* \times \left(\prod C_P^*(X; A)\right)_c^*.$$

3.4 **Lemma.** *Suppose X is a space such that both $X \times T$ and $X \times I$ are k_R-spaces, where $T = \{0, 1/n: n \in N\}$. Then we have $C_P^*(X \times I) \sim (\prod C_P^*(X \times I))_c^*$.*

3.5 **Corollary.** *Let $X = \sum I^\tau$ be a topological sum of infinitely many copies of I^τ, $\tau \geq 1$. Then $C_P^*(X) \sim (\prod C_P^*(X))_c^*$.*

3.6 **Lemma.** *Suppose a space X contains an l^*-embedded copy F_1 of a space Y and Y contains an l^*-embedded copy F_2 of X. Then:*

(i) $C_P^*(X) \sim (\prod C_P^*(X))_b^* \sim C_P^*(Y)$ *if* $C_P^*(Y) \sim (\prod C_P^*(Y))_b^*$;

(ii) $C_P^*(X) \sim (\prod C_P^*(X))_c^* \sim C_P^*(Y)$ *if* $C_P^*(Y) \sim (\prod C_P^*(Y))_c^* \sim (\prod C_P^*(F_2))_c^* \times (\prod C_P^*(Y; F_2))_c^*$.

Proof. Let $C_P^*(Y) \sim (\prod C_P^*(Y))_b^*$. Using the same arguments as in the proof of Lemma 2.7(i), one can show that $C_P^*(X) \sim C_P^*(Y)$. Next, by Lemma 3.1, we have

$$\left(\prod C_P^*(X)\right)_b^* \sim \left(\prod C_P^*(F_1)\right)_b^* \times \left(\prod C_P^*(X; F_1)\right)_b^*$$

and

$$\left(\prod C_P^*(Y)\right)_b^* \sim \left(\prod C_P^*(F_2)\right)_b^* \times \left(\prod C_P^*(Y; F_2)\right)_b^*.$$

Thus,

$$\begin{aligned}
\left(\prod C_P^*(X)\right)_b^* &\sim \left(\prod C_P^*(F_1)\right)_b^* \times \left(\prod C_P^*(X; F_1)\right)_b^* \\
&\sim \left(\prod C_P^*(F_1)\right)_b^* \times \left(\prod C_P^*(F_1)\right)_b^* \times \left(\prod C_P^*(X; F_1)\right)_b^* \\
&\sim \left(\prod C_P^*(F_1)\right)_b^* \times \left(\prod C_P^*(X)\right)_b^* \\
&\sim \left(\prod C_P^*(Y)\right)_b^* \times \left(\prod C_P^*(X)\right)_b^* \\
&\sim \left(\prod C_P^*(F_2)\right)_b^* \times \left(\prod C_P^*(Y; F_2)\right)_b^* \times \left(\prod C_P^*(X)\right)_b^* \\
&\sim \left(\prod C_P^*(F_2)\right)_b^* \times \left(\prod C_P^*(Y; F_2)\right)_b^* \times \left(\prod C_P^*(F_2)\right)_b^* \\
&\sim \left(\prod C_P^*(F_2)\right)_b^* \times \left(\prod C_P^*(Y; F_2)\right)_b^* \\
&\sim \left(\prod C_P^*(Y)\right)_b^* \sim C_P^*(Y) \sim C_P^*(X).
\end{aligned}$$

Using the same arguments we can prove that $(\prod C_P^*(X))_c^* \sim C_P^*(X) \sim C_P^*(Y)$ if $C_P^*(Y) \sim (\prod C_P^*(F_2))_c^* \times (\prod C_P^*(Y; F_2))_c^* \sim (\prod C_P^*(Y))_c^*$.

3.7 **Corollary.** *Let $\{X_i : i \in N\}$ be an infinite family of spaces such that each X_i is strongly l-embedded in a space Y and contains a strongly l-embedded copy Y_i of Y. Then $C_P^*(\sum Y_i) \sim (\prod C_P^*(\sum X_i))_b^* \sim C_P^*(\sum X_i)$ if $C_P^*(\sum Y_i) \sim (\prod C_P^*(\sum Y_i))_b^*$.*

Proof. Let for each i $u_i : C_P(X_i) \to C_P(Y)$ be a regular extension operator. Then the mapping $u : C_P(\sum X_i) \to C_P(\sum Y_i)$, defined by $u(f) = \sum u_i(f|X_i)$ is also a regular extension operator. Thus, $\sum X_i$ is l^*-embedded in $\sum Y_i$. Analogously, $\sum Y_i$ is l^*-embedded in $\sum X_i$. Now the proof follows from Lemma 3.6(i).

3.8 **Theorem.** *Let X be a metric space and τ be an infinite cardinal. Suppose Y is an l^*-embedded subspace of the product X^τ and Y contains an l^*-embedded copy of X^τ. Then $C_P^*(Y) \sim C_P^*(X^\tau) \sim (\prod C_P^*(X^\tau))_c^*$.*

Proof. By Corollary 3.3 and Lemma 3.6(ii), it is enough to show that $C_P^*(X^\tau) \sim (\prod C_P^*(X^\tau))_c^*$. The last can be proved using the same arguments as in the proof of Theorem 2.10.

3.9 **Corollary.** *Let X be a separable metric space and $\tau > \omega$. Then $C_P^*(X^\tau) \sim C_P^*(Y)$ for every closed G_δ-subset Y of X^τ.*

3.10 **Corollary.** *Let U be a functionally open subset of R^τ, $\tau \geq \omega$. Then $C_P^*(R^\tau) \sim C_P^*(U)$.*

The proofs of Corollaries 3.9 and 3.10 are similar respectively to the proofs of Corollaries 2.11 and 2.12.

3.11 **Proposition.** *Let $\sum \mu_i^n$ be a topological sum of infinitely many copies of the n-dimensional Menger compactum. Then $C_P^*(\sum \mu_i^n) \sim (\prod C_P^*(\sum \mu_i^n))_c^*$.*

Proof. For each $i \in N$ take a mapping f_n^i from μ_i^n onto a copy Q_i of the Hilbert cube Q such that $(f_n^i)^{-1}(P)$ is homeomorphic to μ^n for every $LC^{n-1}\&C^{n-1}$-compact subspace P of Q_i (see [Dr2]). Define $f_n: \sum \mu_i^n \to \sum Q_i$ by $f_n|\mu_i^n = f_n^i$. Consider Q_i as a product $Q_i^1 \times I$, where Q_i^1 is a copy of Q. Let $T_i = Q_i^1 \times \{0, 1/k : k \in N\}$ and $T = f_n^{-1}(\sum T_i)$. Then we have

$$C_P^*\left(\sum \mu_i^n\right) \sim C_P^*(T) \times C_P^*\left(\sum \mu_i^n; T\right)$$

and

$$C_P^*(T) \sim C_P^*\left(f_n^{-1}\left(\sum (Q_i^1 \times \{0\})\right)\right) \times C_P^*\left(T; f_n^{-1}\left(\sum (Q_i^1 \times \{0\})\right)\right).$$

Since each of the sets $f_n^{-1}(\sum(Q_i^1 \times \{0\}))$ and $f_n^{-1}(\sum(Q_i^1 \times \{1/k\}))$ for $k \in N$ is homeomorphic to $\sum \mu_i^n$, the following holds

$$C_P^*\left(f_n^{-1}\left(\sum (Q_i^1 \times \{0\})\right)\right) \sim C_P^*\left(\sum \mu_i^n\right)$$

and

$$C_P^*\left(T; f_n^{-1}\left(\sum (Q_i^1 \times \{0\})\right)\right) \sim \left(\prod C_P^*\left(\sum \mu_i^n\right)\right)_c^*.$$

Thus,

$$\begin{aligned} C_P^*(T) &\sim C_P^*\left(\sum \mu_i^n\right) \times \left(\prod C_P^*\left(\sum \mu_i^n\right)\right)_c^* \sim \left(\prod C_P^*\left(\sum \mu_i^n\right)\right)_c^* \\ &\sim \left(\prod C_P^*\left(\sum \mu_i^n\right)\right)_c^* \times \left(\prod C_P^*\left(\sum \mu_i^n\right)\right)_c^* \\ &\sim \left(\prod C_P^*\left(\sum \mu_i^n\right)\right)_c^* \times C_P^*(T). \end{aligned}$$

Finally we get

$$\begin{aligned} C_P^*\left(\sum \mu_i^n\right) &\sim C_P^*(T) \times C_P^*\left(\sum \mu_i^n; T\right) \\ &\sim \left(\prod C_P^*\left(\sum \mu_i^n\right)\right)_c^* \times C_P^*(T) \times C_P^*\left(\sum \mu_i^n; T\right) \\ &\sim \left(\prod C_P^*\left(\sum \mu_i^n\right)\right)_c^* \times C_P^*\left(\sum \mu_i^n\right) \sim \left(\prod C_P^*\left(\sum \mu_i^n\right)\right)_c^*. \end{aligned}$$

3.12 **Lemma.** *Suppose p is a mapping from a space X onto a space Y such that for every compact subset K of Y the preimage $p^{-1}(K)$ is also compact.*

Let p admit a regular averaging operator $u\colon C_p(X) \to C_p(Y)$. Then $C_p^*(X) \sim C_p^*(Y) \times E_1$ and $(\prod C_p^*(X))_c^* \sim (\prod C_p^*(Y))_c^* \times (\prod E_1)_c^*$, where $E_1 = \{g - u(g) \circ p\colon g \in C_p^*(X)\}$.

Proof. Consider the mapping $r\colon Y \to P_\infty(X)$ defined by $r(y)(g) = u(g)(y)$ for all $g \in C_p(X)$. We have $\operatorname{supp}(r(y)) \subset p^{-1}(y)$ for each $y \in Y$. The last implies that $\|u(g)\|_K \le \|g\|_{p^{-1}(K)}$ for every $g \in C_p^*(X)$ and $K \subset Y$. Hence, $u(C_p^*(X)) = C_p^*(Y)$ and the mapping $v(g) = (u(g), g - u(g) \circ p)$ is a linear homeomorphism from $C_p^*(X)$ onto $C_p^*(Y) \times E_1$. Next, let $(g_1, \dots, g_n, \dots) \in (\prod C_p^*(X))_c^*$ and K be a compact subset of Y. Since, $\|u(g_n)\|_K \le \|g_n\|_{p^{-1}(K)}$ and $p^{-1}(K)$ is compact, we have $(u(g_1), \dots, u(g_n), \dots) \in (\prod C_p^*(Y))_c^*$ and $(g_1 - u(g_1)\circ p, \dots, g_n - u(g_n)\circ p, \dots) \in (\prod E_1)_c^*$. Obviously, $(g_1, \dots, g_n, \dots) \in (\prod C_p^*(X))_c^*$ if $(u(g_1), \dots, u(g_n), \dots) \in (\prod C_p^*(Y))_c^*$ and $(g_1 - u(g_1) \circ p, \dots, g_n - u(g_n) \circ p, \dots) \in (\prod E_1)_c^*$. Thus, the mapping

$$\begin{aligned} &v_0(g_1, \dots, g_n, \dots) \\ &\quad = ((u(g_1), \dots, u(g_n), \dots), (g_1 - u(g_1) \circ p, \dots, g_n - u(g_n) \circ p, \dots)) \end{aligned}$$

is a linear homeomorphism from $(\prod C_p^*(X))_c^*$ onto $(\prod C_p^*(Y))_c^* \times (\prod E_1)_c^*$.

3.13 **Proposition.** *Let $\{U_i\colon i \in N\}$ be an infinite locally finite functionally open cover of a space X. Suppose there is a space Y such that $C_p^*(Y) \sim C_p^*(\sum \operatorname{cl}_X(U_i)) \sim (\prod C_p^*(\sum \operatorname{cl}_X(U_i)))_c^*$. Then $C_p^*(X) \sim C_p^*(Y)$ if X contains an l^*-embedded copy of Y.*

Proof. There exists a natural mapping p from $\sum \operatorname{cl}_X(U_i)$ onto X such that $p^{-1}(K)$ is compact for every compact subset K of X. As in the proof of Proposition 2.13 we conclude that p admits a regular averaging operator

$$u\colon C_p\left(\sum \operatorname{cl}_X(U_i)\right) \to C_p(X).$$

By Lemma 3.12, $(\prod C_p^*(\sum \operatorname{cl}_X(U_i)))_c^* \sim (\prod C_p^*(X))_c^* \times (\prod E_1)_c^*$, where $E_1 = \{g - u(g) \circ p\colon g \in C_p^*(\sum \operatorname{cl}_X(U_i))\}$. Since Y is l^*-embedded in X, $C_p^*(X) \sim C_p^*(Y) \times C_p^*(X; Y)$. Then we have

$$\begin{aligned} C_p^*(X) &\sim C_p^*(Y) \times C_p^*(X; Y) \sim \left(\prod C_p^*\left(\sum \operatorname{cl}_X(U_i)\right)\right)_c^* \times C_p^*(X; Y) \\ &\sim \left(\prod C_p^*\left(\sum \operatorname{cl}_X(U_i)\right)\right)_c^* \times C_p^*\left(\sum \operatorname{cl}_X(U_i)\right) \times C_p^*(X; Y) \\ &\sim \left(\prod C_p^*\left(\sum \operatorname{cl}_X(U_i)\right)\right)_c^* \times C_p^*(Y) \times C_p^*(X; Y) \\ &\sim \left(\prod C_p^*\left(\sum \operatorname{cl}_X(U_i)\right)\right)_c^* \times C_p^*(X) \\ &\sim \left(\prod C_p^*(X)\right)_c^* \times \left(\prod E_1\right)_c^* \times C_p^*(X) \sim \left(\prod C_p^*(X)\right)_c^* \times \left(\prod E_1\right)_c^* \\ &\sim \left(\prod C_p^*\left(\sum \operatorname{cl}_X(U_i)\right)\right)_c^* \sim C_p^*(Y). \end{aligned}$$

3.14 **Theorem.** *Suppose X is a noncompact Y-manifold, where Y is one of the spaces Q, I^n, μ^n, l_2. Then $C_P^*(X) \sim C_P^*(\sum Y)$.*

Proof. Let $\{U_i; i \in N\}$ be an infinite locally finite open cover of X such that each $\mathrm{cl}_X(U_i)$ is regularly closed subset of Y. By Corollary 3.5, Proposition 3.11 and Theorem 3.8 we have $C_P^*(\sum Y) \sim (\prod C_P^*(\sum Y))_c^*$. Since each set $\mathrm{cl}_X(U_i)$ is closed in Y and contains a closed copy of Y, it follows from Corollary 3.7 that $(\prod C_P^*(\sum \mathrm{cl}_X(U_i)))_c^* \sim C_P^*(\sum Y)$. Obviously X contains a closed copy of $\sum Y$. Thus, by Proposition 3.13, $C_P^*(X) \sim C_P^*(\sum Y)$.

3.15 **Theorem.** *Let U be a functionally open subset of I^τ and τ be an uncountable cardinal. Then $C_P^*(U) \sim C_P^*(\sum I^\tau)$.*

Proof. Take a projection p from I^τ onto a countable face I^ω of I^τ such that $p^{-1}(p(U)) = U$ (for the existence of a such projection see [PP]). Now, let $\{U_i; i \in N\}$ be a locally finite open cover of $p(U)$ such that $\mathrm{cl}_{I^\omega}(U_i) \subset p(U)$ for each $i \in N$. Then $\{p^{-1}(U_i): i \in N\}$ is an infinite locally finite functionally open cover of U with $\mathrm{cl}_{I^\tau}(p^{-1}(U_i)) \subset U$ for every $i \in N$. Since p is an open mapping we have $\mathrm{cl}_{I^\tau}(p^{-1}(U_i)) = p^{-1}(\mathrm{cl}_{I^\omega}(U_i))$. Thus, by Lemma 2.1, each set $\mathrm{cl}_{I^\tau}(p^{-1}(U_i))$ is strongly l-embedded in I^τ and contains a strongly l-embedded copy of I^τ. Hence, it follows from Corollary 3.5 and Corollary 3.7 that $C_P^*(\sum \mathrm{cl}_{I^\tau}(p^{-1}(U_i))) \sim C_P^*(\sum I^\tau)$. On the other hand U contains an l^*-embedded copy of $\sum I^\tau$ (see the proof of Theorem 2.15). Therefore, by Proposition 3.13, $C_P^*(U) \sim C_P^*(\sum I^\tau)$.

3.16 **Theorem.** *Let Y be one of the spaces Q, I^n, μ^n and X be a locally compact subset of a Y-manifold. Then $C_P^*(X) \sim C_P^*(\sum Y)$ if X contains a closed copy of $\sum Y$.*

Proof. Let X be a locally compact subspace of a Y-manifold Z and let X contain a closed copy of $\sum Y$. Then $C_P^*(X) \sim C_P^*(\sum Y) \times C_P^*(X; \sum Y)$. Take an infinite locally finite open cover $\{V_i; i \in N\}$ of X such that each set $\mathrm{cl}_X(V_i)$ is compact and $\mathrm{cl}_X(V_i) \subset U_i$, where U_i is an open subset of Y. Thus, each $\mathrm{cl}_X(V_i)$ is contained in a copy Y_i of Y. Let $u: C_P(\sum \mathrm{cl}_X(V_i)) \to C_P(X)$ be a regular averaging operator for the natural mapping $p: \sum \mathrm{cl}_X(V_i) \to X$. As in the proof of Proposition 3.13, we get $(\prod C_P^*(\sum \mathrm{cl}_X(V_i)))_c^* \sim (\prod C_P^*(X))_c^* \times (\prod E)_c^*$, where E is a linear subspace of $C_P^*(\sum \mathrm{cl}_X(V_i))$. Since $\sum \mathrm{cl}_X(V_i)$ is a closed subset of $\sum Y_i$, by Corollary 3.3 we have $(\prod C_P^*(\sum Y_i))_c^* \sim (\prod C_P^*(\sum \mathrm{cl}_X(V_i)))_c^* \times (\prod G)_c^*$, where $G = C_P^*(\sum Y_i; \sum \mathrm{cl}_X(V_i))$. Thus,

$$\left(\prod C_P^*\left(\sum Y_i\right)\right)_c^* \sim \left(\prod C_P^*(X)\right)_c^* \times \left(\prod E\right)_c^* \times \left(\prod G\right)_c^*.$$

Then

$$\begin{aligned} C_P^*(X) &\sim C_P^*\left(\sum Y\right) \times C_P^*\left(X; \sum Y\right) \\ &\sim \left(\prod C_P^*\left(\sum Y\right)\right)_c^* \times C_P^*\left(X; \sum Y\right) \end{aligned}$$

because $C_P^*(\sum Y) \sim (\prod C_P^*(\sum Y))_c^*$ (see Corollary 3.5 and Proposition 3.11). Hence

$$\begin{aligned} C_P^*(X) &\sim \left(\prod C_P^*\left(\sum Y\right)\right)_c^* \times C_P^*\left(X; \sum Y\right) \\ &\sim \left(\prod C_P^*\left(\sum Y\right)\right)_c^* \times C_P^*\left(\sum Y\right) \times C_P^*\left(X; \sum Y\right) \\ &\sim \left(\prod C_P^*\left(\sum Y\right)\right)_c^* \times C_P^*(X) \\ &\sim C_P^*(X) \times \left(\prod C_P^*(X)\right)_c^* \times \left(\prod E\right)_c^* \times \left(\prod G\right)_c^* \\ &\sim \left(\prod C_P^*(X)\right)_c^* \times \left(\prod E\right)_c^* \times \left(\prod G\right)_c^* \\ &\sim \left(\prod C_P^*\left(\sum Y\right)\right)_c^* \sim C_P^*\left(\sum Y\right). \end{aligned}$$

Added in proof. After this paper was submitted for publication Arhangel'skii [A4] introduced the notion of an S-stable space. A space X is S-stable if $C_P(X) \sim C_P(X \times S)$, where $S = \{0, 1/n, n \in N\}$. Obviously, if $X \times S$ is a k_R-space, then X is S-stable iff $(\prod C_P(X))_c \sim C_P(X)$. An elementary proof of the S-stability of μ^n (without using Dranishnikov's results, see the proof of this fact in our Theorem 2.9) is given in [A4]. Arhangel'skii [A4] generalized our Theorem 2.8(ii) by proving that if a compact metric space X contains a subspace Y with $C_P(Y) \sim C_P(Q)$ then $C_P(X) \sim C_P(Q)$.

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