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SZEGÖ'S THEOREM ON A BIDISC

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ABSTRACT. G. Szegö showed that

$$\inf \int_0^{2\pi} |1-f|^2 w \, d\theta/2\pi = \exp \int_0^{2\pi} \log w \, d\theta/2\pi$$

where f ranges over analytic polynomials with mean value zeros. We study extensions of the Szegö's theorem on the disc to the bidisc. We show that the quantity is a mixed form of an arithmetic mean and a geometric one of w in some special cases.

1. INTRODUCTION

Let m be the Haar measure of the torus T^2, the distinguished boundary of the unit bidisc U^2 in the space of 2-complex variables (z_1, z_2). Let Z be the set of all integers, Z_+ the set of all nonnegative integers, Z^2 the set of all $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_i \in Z$ and Z_+^2 the set of all $\alpha \in Z^2$ with $\alpha_i \in Z_+$ for $i = 1, 2$. For $1 \le p \le \infty$, $L^p = L^p(T^2, m)$ denotes the Lebesgue space and $H^p = H^p(T^2, m) = \{f \in L^p;\ \hat{f}(\alpha) = 0 \text{ if } \alpha \notin Z_+^2\}$, that is, H^p denotes the usual Hardy spaces on the bidisc. Let $H_0^p = \{f \in H^p;\ \int f\,dm = 0\}$ and $K_0^p = \{f \in L^p;\ \hat{f}(\alpha) = 0 \text{ if } -\alpha \in Z_+^2\}$.

Let $\mathscr{P}$ be a set of all analytic polynomials z_1, z_2 and $\mathscr{P}_0 = \{f \in \mathscr{P};\ \int f\,dm = 0\}$. For each nonnegative function $w \in L^1$ we study the following quantity:

$$S(w) = \inf_{f \in \mathscr{P}_0} \int |1-f|^2 w \, dm.$$

In the case of one complex variable, G. Szegö [6] showed that

$$S(w) = \exp \int \log w \, dm.$$

In the case of two complex variables, this quantity has been studied by A. G. Miamee [1] under some strong condition and then $S(w) = \exp \int \log w \, dm$.

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However it is easy to see that there exists a nonzero function $w \in L^1$ such that $S(w) \neq \exp \int \log w \, dm$. Even if w is zero on some positive measure on T^2 it is possible that $S(w) > 0$.

In §2 several means are defined to estimate $S(w)$ in the latter sections. In §3 we give expressions in terms of w of the $L^2(w\,dm)$-distances between 1 and subalgebras of L^∞ which contain $\mathscr{P}_0$ properly. This follows from the theory of an abstract Hardy space [2]. In §4 we estimate $S(w)$ from the above and the below by means which are defined using conditional expectations. In §5 for special weights we give an expression in terms of w of $S(w)$. In §6 we study the $L^2(w\,dm)$-distance between 1 and K_0^∞ which is a dual version of $S(w)$. In §7 we study relations between $S(w)$ and an invariant subspace defined by w.

2. Various means

Three typical means of w are the following:

$$\int w\,dm \geq \exp \int \log w\,dm \geq \left(\int w^{-1}\,dm \right)^{-1}.$$

$\int w\,dm$, $\exp \int \log w\,dm$ or $(\int w^{-1}\,dm)^{-1}$ is called an arithmetic mean, a geometric mean or a harmonic mean of w, respectively. We would like to define new means in which the means above are mixed. Put $|\alpha|_r = \alpha_1 - r\alpha_2$ where r is a real number. For $1 \leq p \leq \infty$ $\mathscr{L}_r^p$ denotes the space of all $f \in L^p$ whose Fourier coefficients $\hat{f}(\alpha) = 0$, $\alpha \in Z^2$ with $|\alpha|_r = 0$. If r is irrational then $\mathscr{L}_r^p$ is trivial but if r is rational then $\mathscr{L}_r^p$ is nontrivial. Moreover $\mathscr{L}_{-\infty} = \mathscr{L}_\infty$. $\mathscr{L}_r^\infty$ is a commutative von Neumann algebra and hence $\mathscr{L}_r^p = L^p(T^2, \mathscr{B}_r, m)$ where $\mathscr{B}_r$ is the σ-algebra of subsets E of T^2 for which the characteristic function χ_E lies in $\mathscr{L}_r^\infty$. Let $\mathscr{A}$ be the σ-algebra of measurable sets with respect to m. Then $\mathscr{B}_r$ is a sub-σ-algebra of $\mathscr{A}$. Let $\mathscr{E}^r$ denote the conditional expectation for sub-σ-algebra $\mathscr{B}_r$ of $\mathscr{A}$. Define $\mathscr{E}^r(\log w)$ by $\lim_{\varepsilon \to 0} \mathscr{E}^r\{\log(w+\varepsilon)\}$. We consider the following three new means for each r,

$$\int \exp \mathscr{E}^r(\log w)\,dm, \qquad \left(\int \exp \mathscr{E}^r(\log w^{-1})\,dm \right)^{-1},$$

and

$$\exp \int \log \mathscr{E}^r(w)\,dm.$$

Lemma 1. *For any r and any nonnegative w in L^1, the following inequalities are valid.*

$$\int w\,dm \geq \int \exp \mathscr{E}^r(\log w)\,dm \geq \exp \int \log w\,dm$$
$$\geq \left(\int \exp \mathscr{E}^r(\log w^{-1})\,dm \right)^{-1} \geq \left(\int w^{-1}\,dm \right)^{-1}$$

and

$$\int w\,dm \geq \exp\int \log \mathscr{E}^r(w)\,dm \geq \exp\int \log w\,dm.$$

Proof. We can show the inequality of arithmetic and geometric means for conditional expectation. That is, if v is a real function in L^1 and $\exp v \in L^1$, then $\exp \mathscr{E}^r(v) \leq \mathscr{E}^r(\exp v)$ a.e. Hence

$$\mathscr{E}^r(w) \geq \exp \mathscr{E}^r(\log w) \quad \text{a.e.}$$

and

$$\mathscr{E}^r(w^{-1}) \geq \exp \mathscr{E}^r(\log w^{-1}) \quad \text{a.e.}$$

This implies the first part of the lemma. For the second part, apply to $\mathscr{E}^r(w)$ the classical inequality of arithmetic and geometric means.

For $1 \leq j \leq n < \infty$, let λ_j be a nonnegative number with $\sum_{j=1}^n \lambda_j = 1$ and put $\mathscr{E}^j = \mathscr{E}^{r_j}$ where r_j is a real number. The following lemma gives the inequality for $\sum_{j=1}^n \lambda_j \mathscr{E}^j$.

Lemma 2. *If w is a nonnegative function in L^1, then*

$$\sum_{j=1}^n \lambda_j \mathscr{E}^j(w) \geq \prod_{j=1}^n \mathscr{E}^j(w)^{\lambda_j} \geq \exp \sum_{j=1}^n \lambda_j \mathscr{E}^j(\log w)$$

and hence

$$\begin{aligned}\int w\,dm &\geq \exp\int \log \prod_{j=1}^n \mathscr{E}^j(w)^{\lambda_j}\,dm \\ &\geq \int \exp\sum_{j=1}^n \lambda_j \mathscr{E}^j(\log w)\,dm \geq \exp\int \log w\,dm \\ &\geq \left(\int \exp\sum_{j=1}^n \lambda_j \mathscr{E}_j(\log w^{-1})\,dm\right)^{-1} \geq \left(\int w^{-1}\,dm\right)^{-1}.\end{aligned}$$

Let $L^p_+ = \{f \in L^p ; \hat{f}(\alpha) = 0 \text{ if } \alpha = (\alpha_1, \alpha_2) \text{ and } \alpha_1\alpha_2 < 0\}$ and $L^p_- = \{f \in L^p ; \hat{f}(\alpha) = 0 \text{ if } \alpha = (\alpha_1, \alpha_2) \neq (0, 0) \text{ and } \alpha_1\alpha_2 \geq 0\}$. $L^1_+ + L^1_-$ is a dense subspace in L^1 and $L^1_+ \cap L^1_-$ consists of constant functions. Let P be a projection from $L^1_+ + L^1_-$ to L^1_-. We can define a mean using P,

$$\int \exp P(\log w)\,dm.$$

This mean will be used in §5.

We would like to calculate the means for some special functions w. Let w_j be a nonnegative function in L^1 with $\log w_1 \in L^1_+$ and $\log w_2 \in L^1_-$. If $w = w_1 w_2 \in L^1$ then

$$\int \exp P(\log w)\,dm = \exp\int \log w_1\,dm \int w_2\,dm.$$

Let w_r (or w_s) be a nonnegative function in $\mathscr{L}_r^1$ (or $\mathscr{L}_s^1$) and $r \neq s$. If $w = w_r w_s \in L^1$ then

$$\int \exp \mathscr{E}^r(\log w)\, dm = \exp \int \log w_s\, dm \int w_r\, dm,$$

$$\left(\int \exp \mathscr{E}^r(\log w^{-1})\, dm\right)^{-1} = \exp \int \log w_s\, dm \left(\int w_r^{-1}\, dm\right)^{-1},$$

$$\exp \int \log \mathscr{E}^r(w)\, dm = \exp \int \log w_r\, dm \int w_s\, dm,$$

$$\int \exp \frac{\mathscr{E}^r + \mathscr{E}^s}{2}(\log w)\, dm = \int w^{1/2}\, dm \exp \int \log w^{1/2}\, dm,$$

and

$$\left(\int \exp \frac{\mathscr{E}^r + \mathscr{E}^s}{2}(\log w^{-1})\, dm\right)^{-1} = (w^{-1/2}\, dm)^{-1} \exp \int \log w^{1/2}\, dm.$$

The results above shows that the means with respect to three operators: $\mathscr{E}^r$, $\sum \lambda_j \mathscr{E}^j$ and P, are mixed ones of arithmetic and geometric means.

3. Extended weak-* Dirichlet algebra

Let $\mathbf{H}_r = \{f \in L^\infty;\ \hat{f}(\alpha) = 0 \text{ if } \alpha \in Z^2 \text{ and } |\alpha|_r < 0\}$, then $\mathbf{H}_r \cap \overline{\mathbf{H}}_r = \mathscr{L}_r^\infty$, $\mathscr{E}^r$ is multiplicative on $\mathbf{H}_r$ and $\mathbf{H}_r + \overline{\mathbf{H}}_r$ is weak-* dense in L^∞. That is, $\mathbf{H}_r$ is an extended weak-* Dirichlet algebra with respect to $\mathscr{E}^r$ [2]. If r is irrational then $\mathbf{H}_r$ is a weak-* Dirichlet algebra [5]. Let $I_r = \{f \in L^\infty;\ \hat{f}(\alpha) = 0 \text{ if } \alpha \in Z^2 \text{ and } |\alpha|_r \leq 0\}$ then $I_r = \{f \in \mathbf{H}_r;\ \mathscr{E}^r(f) = 0\}$. $\mathbf{H}_r = \mathscr{L}_r^\infty + I_r$. Let $\mathscr{H}_r^\infty = \{f \in \mathscr{L}_r^\infty;\ \hat{f}(\alpha) = 0 \text{ if } \alpha_2 < 0\}$ and $\mathscr{H}_{r,0}^\infty = \{f \in \mathscr{H}_r^\infty;\ \int f\, dm = 0\}$. $\mathscr{H}_r^\infty + I_r$ is a weak-* Dirichlet algebra. Putting $\mathbf{H}_{r,0} = \{f \in \mathbf{H}_r;\ \int f\, dm = 0\}$, for any $r < 0$ and $r \neq -\infty$,

$$K_0^\infty \supset \mathbf{H}_{r,0} \supset \mathscr{H}_{r,0}^\infty + I_r \supset I_r \supset H_0^\infty.$$

The following lemma and proposition are essentially known [2].

Lemma 3. *Let w be a nonnegative function in L^1. For any $v \in \mathscr{L}_r^\infty$ and any r,*

$$\mathbf{S}(v, r) = \inf\left\{\int |v - f|^2 w\, dm;\ f \in I_r\right\}$$
$$= \int \exp \mathscr{E}^r(\log w)|v|\, dm.$$

Hence

$$\inf\left\{\mathbf{S}(v, r);\ \int \log|v|\, dm \geq 0\right\} = \exp \int \log w\, dm$$

and

$$\inf\left\{\mathbf{S}(v, r);\ \int v\, dm = 1\right\} = \left(\int \exp \mathscr{E}^r(\log w^{-1})\, dm\right)^{-1}.$$

Proposition 1. *Let* w *be a nonnegative function in* L^1. *Then, for any* r

(1) $$\inf_{f\in\mathscr{H}^\infty_{r,0}}\int |1-f|^2 w\,dm = \exp\int \log\mathscr{E}^r(w)\,dm,$$

(2) $$\inf_{f\in I_r}\int |1-f|^2 w\,dm = \int \exp\mathscr{E}^r(\log w)\,dm,$$

(3) $$\inf_{f\in\mathscr{H}^\infty_{r,0}}\int |1-f|^2 w\,dm = \exp\int \log w\,dm,$$

(4) $$\inf_{f\in\mathbf{H}_{r,0}}\int |1-f|^2 w\,dm = \left(\int \exp\mathscr{E}^r(\log w^{-1})\,dm\right)^{-1}.$$

4. $S(w)$ AND MEANS WITH RESPECT TO $\mathscr{E}^r$

In this section we will improve the following known inequality:

$$\int w\,dm \geq S(w) \geq \exp\int \log w\,dm.$$

Theorem 2. *Let* w *be a nonnegative function in* L^1.

(1) *If* $0\leq s\leq\infty$ *and* $-\infty<r<0$, *then*

$$\exp\int\log\mathscr{E}^s(w)\,dm \geq S(w) \geq \int \exp\mathscr{E}^r(\log w)\,dm,$$

(2) *Suppose* $0>r_j>-\infty$, $\lambda_j\geq 0$, $\sum_{j=1}^n\lambda_j=1$ *and* $n<\infty$, *and* $0\leq s_j\leq\infty$, $\gamma_j\geq 0$, $\sum_{j=1}^l\gamma_j=1$ *and* $l<\infty$. *Then*

$$\exp\int\log\prod_{j=1}^l\mathscr{E}^j(w)^{\gamma_j}\,dm \geq S(w) \geq \int\exp\sum_{j=1}^n\lambda_j\mathscr{E}^j(\log w)\,dm.$$

Proof. (1) Since $-\infty<r<0$, $H_0^\infty\subset I_r$ and hence by (2) of Proposition 1

$$S(w)\geq \inf_{f\in I_r}\int|1-f|^2w\,dm = \int\exp\mathscr{E}^r(\log w)\,dm.$$

Since $0\leq r\leq\infty$, $\mathscr{H}^\infty_{r,0}\subset H_0^\infty$ and hence by (1) of Proposition 1

$$S(w)\leq \inf_{f\in\mathscr{H}^\infty_{r,0}}\int|1-f|^2w\,dm = \exp\int\log\mathscr{E}^r(w)\,dm.$$

(2) Since $-\infty<r_j<0$, if $f\in H_0^\infty$ then $f\in\bigcap_{j=1}^n I_{r_j}$ and hence by the first part of Lemma 2,

$$\int|1-f|^2w\,dm \geq \int\exp\sum_{j=1}^n\lambda_j\mathscr{E}^j(\log|1-f|^2w)\,dm$$

$$\geq \int\exp\sum_{j=1}^n\lambda_j\mathscr{E}^j(\log w)\,dm.$$

Since $0 \le s_j \le \infty$, by (1)

$$\exp \int \log \prod_{j=1}^{l} \mathscr{E}^{j}(w)^{\lambda_j}\, dm = \prod_{j=1}^{l} \left(\exp \int \log \mathscr{E}^{j}(w)\, dm \right)^{\lambda_j} \ge S(w).$$

Corollary 1. *If* $w = w_t w_l \in L^1$ *where* $w_t \in \mathscr{L}_t^1$, $w_l \in \mathscr{L}_l^1$, $-\infty < t < 0$ *and* $0 \le l \le \infty$, *then*

$$\exp \int \log \mathscr{E}^{l}(w)\, dm = S(w) = \int \exp \mathscr{E}^{t}(\log w)\, dm$$

and hence

$$S(w) = \int w_t\, dm \exp \int \log w_l\, dm.$$

Proof. It is easy to see that

$$\begin{aligned}\exp \int \log \mathscr{E}^{l}(w)\, dm &= \int w_t\, dm \exp \int \log w_l\, dm \\ &= \int \exp \mathscr{E}^{t}(\log w)\, dm.\end{aligned}$$

Hence (1) of Theorem 2 implies the corollary.

We can ask whether if

$$\inf_{0\le s\le\infty} \exp \int \log \mathscr{E}^{s}(w)\, dm = S(w) = \sup_{-\infty<r<0} \int \exp \mathscr{E}^{r}(\log w)\, dm.$$

Unfortunately this equality does not hold for some w. Suppose $w = w_t w_l \in L^1$, and both $w_t \in \mathscr{L}_t^1$ and $w_l \in \mathscr{L}_l^1$ are nonconstant functions.

If $-\infty < t$, $l < 0$ and $w \in L_-^1$ then

$$\begin{aligned}\inf_{0\le s\le\infty} \exp \int \log \mathscr{E}^{s}(w)\, dm &= S(w) \\ &\not\ge \sup_{-\infty<r<0} \int \exp \mathscr{E}^{r}(\log w)\, dm.\end{aligned}$$

In fact,

$$\inf_{0\le s\le\infty} \exp \int \log \mathscr{E}^{s}(w)\, dm = \int w\, dm$$

and

$$\begin{aligned}&\sup_{-\infty<r<0} \int \exp \mathscr{E}^{r}(\log w)\, dm \\ &\quad = \max \left\{ \exp \int \log w\, dm,\ \int w_t\, dm \exp \int \log w_l\, dm, \right. \\ &\qquad\qquad \left. \exp \int \log w_t\, dm \int w_l\, dm \right\}.\end{aligned}$$

By (1) of Theorem 4, $S(w) = \int w\, dm$.

If $0 \le t,\ l \le \infty$ and $w \in L^1_+$ then

$$\inf_{0\le s\le\infty} \exp \int \log \mathscr{E}^s(w)\, dm \ngeq S(w)$$
$$= \sup_{-\infty<r<0} \int \exp \mathscr{E}^r(\log w)\, dm .$$

In fact,

$$\inf_{0\le s\le\infty} \exp \int \log \mathscr{E}^s(w)\, dm$$
$$= \min\left\{ \int w\, dm ,\ \exp \int \log w_t\, dm \int w_l\, dm ,\ \int w_t\, dm \exp \int \log w_l\, dm \right\}$$

and

$$\sup_{-\infty<r<0} \int \exp \mathscr{E}^r(\log w)\, dm = \exp \int \log w\, dm .$$

While by (2) of Theorem 4 if $w_l \in \mathscr{L}_l^\infty S(w) = \exp\int \log w\, dm$. Moreover we can ask whether if $S(w) = \inf_{0\le s\le\infty} \exp\int \log \mathscr{E}^s(w)\, dm$ or $S(w) = \sup_{-\infty<r<0} \int \exp \mathscr{E}^r(\log w)\, dm$. However this is also not true. For such an example, suppose $w = w_t w_l w_k$ where $w_j \in \mathscr{L}_j^\infty$ $(j = t, l, k)$, $-\infty < t < 0$ and $0 \le l,\ s \le \infty$.

There does not exist a universal finite constant γ_0 such that

$$S(w) \le \gamma_0 \exp \int \log w\, dm$$

for all w in L^∞ with $w^{-1} \in L^\infty$. Let $D_r^\infty = \mathscr{H}_r^\infty + I_r$ and $D_r^2 = \mathscr{H}_r^2 + [I_r]$. For $-\infty \le r \le 0$, let γ_r be the norm of the orthogonal projection form K^2 onto D_r^2 in $L^2(w^{-1}\, dm)$. If both w and w^{-1} are in L^∞ then for each r γ_r is finite.

Theorem 3. *Let w be a nonnegative function in L^1. If $w^{-1} \in L^\infty$ then*

$$S(w) \le (\gamma_0 + \gamma_\infty) \exp \int \log w\, dm .$$

Proof. By the duality

$$S(w) = \sup \left| \int g w\, dm \right|$$

where g ranges over the unit ball of $(H_0^\infty)^\perp \cap L^2(w\, dm)$. If $w^{-1} \in L^\infty$ then

$$(H_0^\infty)^\perp \cap L^2(w\, dm) = w^{-1} \overline{K}^2 .$$

Since

$$K^2 = D_0^2 + D_\infty^2 = D_0^2 \oplus \{D_\infty^2 \ominus [K^2 \ominus D_0^2]\} ,$$

if $F \in K^2$ then

$$w^{-1} F = w^{-1} F_0 + w^{-1} F_\infty$$

for some $F_0 \in D_0^2$ and $F_\infty \in D_\infty^2 \ominus [K^2 \ominus D_0^2]$. By hypothesis if $\int |w^{-1}F|^2 w\,dm \le 1$ then

$$\int |w^{-1}F_r|^2 w\,dm \le \gamma_r \ (r = 0, \infty).$$

Hence

$$S(w) \le \sum_{r=0,\infty} \gamma_r \sup \left\{ \left| \int hw\,dm \right| ; h \in (D_{r,0}^\infty)^\perp \cap L^2(w\,dm) \right.$$
$$\left. \text{and } \int |h|^2 w\,dm \le 1 \right\}.$$

Again by the duality

$$S(w) \le \sum_{r=0,\infty} \gamma_r \inf \left\{ \int |1-f|^2 w\,dm \,;\, f \in D_{r,0}^\infty \right\}.$$

Thus by (3) of Proposition 1 the theorem follows.

5. Arithmetic mean and geometric one

The function $g \in H^2$ is called an outer function if

$$\int \log|g|\,dm = \log\left|\int g\,dm\right| > -\infty.$$

The function $g \in H^2$ is called a generator if $[g\mathscr{P}] = H^2$. If $g \in H^2$ is a generator then it is an outer function [4, p. 73]. However there exists an outer function which is not a generator [4, p. 76]. The following lemma is known [4, pp. 73 and 77].

Lemma 4. *Let $w \in L^1$ be a nonnegative function. There exists an outer function $g \in H^2$ such that $w = |g|^2$ if and only if* $\log w \in L_+^1$.

Theorem 4. *Let w be a nonnegative function in L^1.*

(1) $w \in L_-^1$ *if and only if*

$$S(w) = \int w\,dm.$$

(2) $w = |g|^2$ *for some generator* $g \in H^2$ *if and only if*

$$S(w) = \exp \int \log w\,dm.$$

Proof. (1) If w is a nonzero function in L_-^1 and $\int w\,dm = a$ then $w\,dm/a$ is a representing measure of the evaluation at the origin and hence for any f in H_0^∞,

$$\int |1-f|^2 w\,dm/a \ge 1.$$

Thus $S(w) = \int w\,dm$. Conversely if $S(w) = \int w\,dm$ then for any $f \in H_0^\infty$,

$$\int |f|^2 w\,dm \ge 2\,\mathrm{Re}\int fw\,dm.$$

Hence for any $f \in H_0^\infty$ and for any positive number ε,

$$\varepsilon \int |f|^2 w\, dm \geq 2 \left| \int f w\, dm \right| .$$

As $\varepsilon \to 0$,

$$\int f w\, dm = 0$$

and hence $w \in L^1_-$.

(2) If $w = |g|^2$ and g is a generator then

$$S(w) = \left| \int g\, dm \right|^2$$

and by the remark above g is an outer function. Thus $S(w) = \exp \int \log w\, dm$. Conversely if $S(w) = \exp \int \log w\, dm$ then by (1) of Theorem 2 for any r with $-\infty < r < 0$,

$$\int \exp \mathscr{E}^r(\log w)\, dm = \exp \int \log w\, dm$$

and hence

$$\mathscr{E}^r(\log w) = \int \log w\, dm .$$

Assuming $\log w \in L^1$ without loss of the generality, $(\log w)\hat{}(\alpha) = 0$ if $|\alpha|_r = 0$ and $-\infty < r < 0$, and so $\log w \in L^1_+$. By Lemma 4 $w = |g|^2$ for some outer $g \in H^2$. Then g has the decomposition $g = g_0 + g_1$ where $g_0 \in [g\mathscr{P}] \ominus [g\mathscr{P}_0]$ and $g_1 \in [g\mathscr{P}_0]$. Hence since $S(w) = \exp \int \log w\, dm$,

$$S(w) = \int |g_0|^2\, dm = \left| \int g_0\, dm \right|^2 > 0 .$$

Thus g_0 is constant and hence g is a generator.

We want to present a mixed form of (1) and (2) of Theorem 4.

Proposition 5. *Let w_j be a nonnegative function in L^1 and $w = w_1 w_2 \in L^1$.*

(1) *If $\log w_1 \in L^1_+$ and $w_2 \in L^1_-$ then*

$$S(w) \geq \exp \int \log w_1\, dm \int w_2\, dm .$$

(2) *If $w_1 = |g|^2$ for some generator in H^2 and $w_2 \in L^\infty_-$ then*

$$S(w) = \exp \int \log w_1\, dm \int w_2\, dm .$$

Proof. (1) By Lemma 4 $w_1 = |g|^2$ for some outer function $g \in H^2$ and by the hypothesis $w_2 \in L^1_-$. Hence

$$\begin{aligned} S(w) &= \inf_{f \in \mathscr{P}_0} \int |g - gf|^2 w_2 \, dm \\ &\geq \inf_{f \in \mathscr{P}_0} \left| \int g w_2 \, dm - \int g f w_2 \, dm \right|^2 \left(\int w_2 \, dm \right)^{-1} \\ &= \left| \int g w_2 \, dm \right|^2 \left(\int w_2 \, dm \right)^{-1} \\ &= \left| \int g \, dm \right|^2 \int w_2 \, dm \\ &= \exp \int \log w_1 \, dm \int w_2 \, dm . \end{aligned}$$

(2) Since $w_2 \in L^\infty$, H_0^2 is in the closure of $g\mathscr{P}_0$ in $L^2(w_2 \, dm)$ and hence

$$\begin{aligned} \inf_{f \in \mathscr{P}_0} \int |g - gf|^2 w_2 \, dm &\leq \left| \int g \, dm \right|^2 \int w_2 \, dm \\ &= \exp \int \log w_1 \, dm \int w_2 \, dm . \end{aligned}$$

Let $L^\infty_{-+} = \{ f \in L^\infty ; \; \hat{f}(\alpha) = 0 \text{ if } \alpha = (\alpha_1, \alpha_2), \; \alpha_1 \leq 0 \text{ and } \alpha_2 \geq 0 \}$. $L^\infty_{-+} + \overline{L^\infty_{-+}}$ is weak-* dense in L^∞_- and L^∞_{-+} is a weak-* closed algebra. If w satisfies that

$$\log w \in L^1_+ + L^\infty_{-+} + \overline{L^\infty_{-+}}$$

then $w = w_1 w_2$, $\log w_1 \in L^1_+$, $\log w_2 \in L^1_-$ and $w_2 \in L^\infty_-$ and

$$\int \exp P(\log w) \, dm = \exp \int \log w_1 \, dm \int w_2 \, dm .$$

6. The dual version of Szegö's Theorem

$\overline{K_0^\infty}$ is an annihilator of $\mathscr{P}$ (and hence H^∞) in L^∞. Hence we would like to give an expression in terms of w of the following quantity:

$$S^\perp(w) = \inf_{f \in K_0^\infty} \int |1 - f|^2 w \, dm .$$

For any r with $-\infty < r < 0$,

$$H_0^\infty \subset I_r \subset \mathscr{H}^\infty_{r,0} + I_r \subset \mathbf{H}_{r,0} \subset K_0^\infty$$

and hence by (3) of Proposition 1

$$\int w \, dm \geq S(w) \geq \exp \int \log w \, dm \geq S^\perp(w) \geq \left(\int w^{-1} \, dm \right)^{-1}$$

and moreover by (4) of Proposition 1

$$\left(\int \exp \mathscr{E}^r(\log w^{-1})\, dm\right)^{-1} \geq S^{\perp}(w).$$

If both w and w^{-1} are in L^{∞}, by Theorem 1 in [3]

$$S^{\perp}(w) = S(w^{-1})^{-1}.$$

Unfortunately we do not know whether if both w and w^{-1} in L^1 then $S^{\perp}(w) = S(w^{-1})^{-1}$. However by Theorem 1 in [3]

$$\begin{aligned} S(w) &\geq S(w^{-1})^{-1} \geq \exp \int \log w\, dm \\ &\geq S^{\perp}(w) \geq S(w^{-1})^{-1} \geq \left(\int w^{-1}\, dm\right)^{-1}. \end{aligned}$$

Hence by (1) of Theorem 2, for any s with $0 \leq s \leq \infty$,

$$S^{\perp}(w) \geq \left(\exp \int \log \mathscr{E}^s(w^{-1})\, dm\right)^{-1}.$$

If $w = |g|^2$ for some generator $g \in H^2$, by (2) of Theorem 4 then $S^{\perp}(w) = S(w^{-1})^{-1}$. If both w and w^{-1} are in L^1_-, by (1) of Theorem 4 then $S^{\perp}(w) = S(w^{-1})^{-1}$.

7. Invariant subspace

A closed subspace M of L^2 is said to be invariant if

$$z_j M \subseteq M \quad \text{for } j = 1, 2.$$

In this section we study invariant subspaces M which are singly generated, that is, $M = [vH^{\infty}]$ for some $v \in L^2$. Then we can give an expression in terms of w of $S(w)$.

Let w be a nonnegative function in L^1 and $H^2(w)$ the $L^2(w\, dm)$-closure of $\mathscr{P}$. It is easy to see that

$$w^{1/2} H^2(w) = [w^{1/2} H^{\infty}] \subset L^2$$

and

$$\operatorname{dist}(w^{1/2}, [w^{1/2} H_0^{\infty}]) = S(w)^{1/2}.$$

Let w_2 be a nonnegative function in L^1_-. Then

$$H^2(w_2) = [1] \oplus H_0^2(w_2)$$

where $H_0^2(w_2)$ is the $L^2(w_2\, dm)$-closure of $\mathscr{P}_0$. We can expect that $w_2^{1/2} H^2(w_2) = [w_2^{1/2} \mathscr{P}]$ has a simple structure.

Conjecture. If M is a singly generated invariant subspace and $S_0 = M \ominus [\mathscr{P}_0 M] \neq [0]$, then S_0 is one dimension and contains a cyclic vector.

Proposition 6. *Suppose the conjecture is true for a singly generated invariant subspace M of L^2 with $S_0 \neq [0]$.*

(1) *$M = q w_2^{1/2} H^2(w_2)$ where q is unimodular and $w_2 \in L^1_-$.*

(2) *Let g be a nonzero function in L^2. If M is generated by g then*

$$|g|^2 = |h|^2 w_2$$

where $w_2 \in L^1_-$ and h is a generator for $H^2(w_2)$.

(3) *Let w be a nonnegative function in L^1. If M is generated by $w^{1/2}$ then*

$$S(w) = \left|\int h w_2 \, dm\right|^2 \left(\int w_2 \, dm\right)^{-1}$$

where $w_2 \in L^1_-$, h is a generator for $H^2(w_2)$ and $w = |h|^2 w_2$.

Proof. (1) Suppose $S_0 = [u]$. Since u is orthogonal to $u\mathscr{P}_0$, $|u|^2 \in L^1_-$. Put

$$q(x) = \begin{cases} u(x)/|u(x)| & \text{if } u(x) \neq 0, \\ 1 & \text{if } u(x) = 0, \end{cases}$$

and $w_2 = |u|^2$, then $u = q w_2^{1/2}$. Since M is generated by u,

$$M = [q w_2^{1/2} \mathscr{P}] = q w_2^{1/2} H^2(w_2).$$

(2) is clear by (1), (3). By (2), putting $g = w^{1/2}$ $w = |h|^2 w_2$. Hence

$$S(w) = \inf\left\{\int |h - hf|^2 w_2 \, dm;\ f \in \mathscr{P}_0\right\}$$

$$= \left|\int h w_2 \, dm\right|^2 \left(\int w_2 \, dm\right)^{-1}.$$

In (3) of Proposition 6 if $w_2^{-1} \in L^\infty$ then $S(w) = |\int h \, dm|^2 \int w_2 \, dm$. If w_2 is in L^∞ then putting $w_1 = |h|^2$

$$S(w) = \exp \int \log w_1 \, dm \int w_2 \, dm.$$

References

1. A. G. Miamee, *Extension of three theorems for Fourier series on the disc to the torus*, Bull. Austral. Math. Soc. **33** (1986), 335–350.
2. T. Nakazi, *Extended weak-* Dirichlet algebras*, Pacific J. Math. **81** (1979), 493–513.
3. ——, *Two problems in prediction theory*, Studia Math. **78** (1984), 7–14.
4. W. Rudin, *Function theory in polydiscs*, Benjamin, New York, 1969.
5. T. P. Srinivasan and J. K. Wang, *Weak-* Dirichlet algebras*, Proc. Internat. Sympos. on Function Algebras (Tulane Univ., 1965), Scott-Foresman, Chicago, Ill., 1966, pp. 216–249.
6. G. Szegö, *Beiträge zur Theorie der Toeplitzschen Formen* (*Erste Mitteilung*), Math. Z. **6** (1920), 167–202.

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