

MAXIMAL ENTROPY ODD ORBIT TYPES

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ABSTRACT. A periodic orbit of a continuous map of an interval induces in a natural way a cyclic permutation, called its type. We consider a family of orbit types of period n congruent to 1 (mod 4) introduced recently by Misiurewicz and Nitecki. We prove that the Misiurewicz-Nitecki orbit types and their natural generalizations to the remaining odd periods n have maximal entropy among all orbit types of period n , and even among all n -permutations.

1. INTRODUCTION

In this paper we study periodic orbit types with maximal entropy for interval maps.

A periodic orbit for a map f of an interval determines a permutation, in fact a cycle, in a very natural way, via the order on the interval: if the orbit consists of $p_1 < p_2 < \cdots < p_n$, then we get the cycle θ on n letters, where $f(p_i) = p_{\theta(i)}$. We call this cycle the *type* of the orbit. Two orbits of interval maps have the same type when there is an orientation-preserving homeomorphism of the intervals conjugating one orbit to the other. Orbit types have been studied by a number of authors in the last few years, for example [Ba, Be, BH, and J] in addition to [MN]. A good survey of topological dynamics on the interval is [N].

Topological entropy, defined by Adler, Konheim, and McAndrew [AKM], and characterized by Bowen [Bow], gives a measure of the complexity of a dynamical system. We can associate an entropy with an orbit type by taking the infimum of the topological entropies of all systems having an orbit of the given type. This quantity is positive for most orbit types, including all those whose period is not a power of two [BowF, BGM]. We address here the question of which orbit types of a given period have the largest entropy.

The question of which orbit types have *least* entropy has been answered: work of Block and Coppel [BCop] shows that these are exactly the strongly simple orbit types defined in their paper.

Recently, Misiurewicz and Nitecki [MN] have obtained a beautiful asymptotic result on maximal entropy orbit types. They show that the exponential of the maximal entropy is asymptotically $2n/\pi$. In proving this, they make use of an interesting new family of orbit types of period congruent to 1 (mod 4), and

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they prove that these orbit types approach maximal entropy in the limit. They leave open the question of which orbit types attain maximal entropy.

We prove here that the orbit types of Misiurewicz and Nitecki, and their natural generalizations to the remaining odd periods, have maximal entropy. In fact, we prove more generally that these types have maximal entropy even among all permutations associated to unions of periodic orbits.

Briefly, the contents of the paper are as follows.

In §2 we give some basic definitions and results on orbit types and their entropies.

In §3 we define the generalized Misiurewicz-Nitecki orbit types, state the main result, and obtain some necessary results on maximal entropy types which are maximal in the natural partial preorder on permutations.

In §4 we define for each odd n a cone in \mathbf{R}^{n-1} that is used to estimate the entropy of various orbit types of period n , and prove the invariance of this cone under the square of the matrix corresponding to the Misiurewicz-Nitecki orbit type of period n .

In §5 we exploit this invariance together with other properties of the cone to conclude the proof of the main theorem.

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2. PRELIMINARIES

We begin by fixing some notation and terminology. A *map* for us will be a continuous function f from a compact interval I into itself. The *orbit* of a point $x_0 \in I$ is the sequence $\mathcal{O} = \mathcal{O}(x_0) := \{x_i\}_{i=0}^\infty$, where $x_i := f^i(x_0)$ and f^0 is the identity map. Let P_n be the set of permutations on n letters, $P_n := \{\theta: \{1, \dots, n\} \rightarrow \{1, \dots, n\} | \theta \text{ is bijective}\}$, and let $P := \bigcup_{n \geq 1} P_n$. The set of cycles of length (or period) n is $C_n := \{\theta \in P_n | \theta^k(1) \neq 1 \text{ for } 0 < k < n\}$. We let $C := \bigcup_{n \geq 1} C_n$. The *dual* of a permutation $\theta \in P_n$ is the permutation $\bar{\theta} \in P_n$, $\bar{\theta}(i) := n + 1 - \theta(n + 1 - i)$, so that $\bar{\theta}$ is θ conjugated by a reversal of orientation. We will sometimes use cycle notation, so that $\theta \in C_n$ will be written as $(1\theta(1)\theta^2(1)\dots\theta^{n-1}(1))$.

If an orbit \mathcal{O} of a map f is periodic, then we can write it as $\mathcal{O} = \{p_1, \dots, p_n\}$ where $p_1 < \dots < p_n$. In this case the restriction of f to \mathcal{O} induces, in a very natural way, a cycle $\theta \in C_n$, which we call the *orbit type* of \mathcal{O} . Specifically, we define $\theta(i) := j$ if $f(p_i) = p_j$.

More generally, if $S = \{p_1, \dots, p_n\}$, $p_1 < \dots < p_n$, and $f(S) = S$, we can define the *type* of the finite invariant set S to be the permutation $\theta \in P_n$ given as before by $\theta(i) := j$ if $f(p_i) = p_j$.

The (topological) *entropy* of a map f on a compact interval I [AKM] is

$$h(f) := \sup_{\mathcal{U}} \lim_{n \rightarrow \infty} (1/n) \log M_n(\mathcal{U}),$$

where for any open cover \mathcal{U} of I , $M_n(\mathcal{U})$ is the minimum cardinality of a

subcover of

$$\mathcal{U}_n := \mathcal{U} \vee f^{-1}\mathcal{U} \vee \dots \vee f^{-(n-1)}\mathcal{U} \quad \text{and} \quad \mathcal{U} \vee \mathcal{V} := \{U \cap V : U \in \mathcal{U}, V \in \mathcal{V}\}.$$

Equivalently [Bow],

$$h(f) = \lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} (1/n) \log S_n(\varepsilon),$$

where $S_n(\varepsilon)$ is the maximum cardinality of an (n, ε) separated set. Here a set $E \subset I$ is (n, ε) separated if for all distinct x and y in E , there is some k , $0 \leq k < n$, such that $|f^k(x) - f^k(y)| > \varepsilon$.

Definition 1. The entropy of an orbit type $\theta \in C$ is

$$h(\theta) := \inf\{h(f) : f \text{ is a map with an orbit of type } \theta\}.$$

More generally, if $\theta \in P$ we define $h(\theta) := \inf\{h(f) : f \text{ has an invariant set of type } \theta\}$. It is easy to see that for all $\theta \in P$, $h(\bar{\theta}) = h(\theta)$.

If θ is an orbit type, or more generally an invariant set type in P_n , we define the *primitive function* f_θ to be the piecewise linear interpolation of the graph of θ , $f_\theta(x) = \{x\}\theta(\lceil x \rceil) + (1 - \{x\})\theta(\lfloor x \rfloor)$, $1 \leq x \leq n$. Here $\{x\}$ is the fractional part of x and $\lceil x \rceil$ and $\lfloor x \rfloor$ represent the least integer as large as x and the largest integer as small as x , respectively. $\theta \in P_n$ (or f_θ) is *maximodal* if each $f_\theta(i)$ is a (local) extremum of f_θ , for $1 \leq i \leq n$.

Proposition 1. For $\theta \in P$, $h(\theta) = h(f_\theta)$.

Define the *Markov graph* of $\theta \in P_n$ to be the directed graph with $n - 1$ vertices $\{I_1, \dots, I_{n-1}\}$ and an edge from I_i to I_j if $f_\theta([i, i+1]) \supset [j, j+1]$. The *induced matrix* $M(\theta)$ of $\theta \in P_n$ is the $(n-1) \times (n-1)$ zero-one adjacency matrix of the Markov graph of θ , so that the (i, j) th entry of $M(\theta)$ is 1 if $f_\theta([i, i+1]) \supset [j, j+1]$ and 0 otherwise.

Proposition 2. For $\theta \in P$, $h(\theta) = \log \rho(M(\theta))$, the spectral radius of the induced matrix of θ .

Propositions 1 and 2 are well known [MS, BGMV].

The entropy of a map is the supremum of the entropies of its orbit types [T, BCov, MN].

If $\theta, \eta \in P$, we say that θ *forces* η , or write $\theta \vdash \eta$, if every map which has an invariant set of type θ also has an invariant set of type η .

Proposition 3. \vdash is a partial preorder on P , and a partial order on C .

That is, \vdash is a reflexive, transitive relation on P and in addition [Ba] is antisymmetric on C .

θ is *forcing-maximal* in some subset S of P if $\theta \in S$, and for all $\eta \in S$, $\eta \vdash \theta$ implies $\theta \vdash \eta$. It is clear from the definition that if $\theta, \eta \in P$ and θ forces η then $h(\theta) \geq h(\eta)$. Misiurewicz and Nitecki [MN] have determined when one can conclude from $\theta \vdash \eta$ that the inequality is strict.

3. MISIUREWICZ-NITECKI ORBIT TYPES AND FORCING MAXIMALITY

For n odd, define $l := \lfloor (n-1)/4 \rfloor$ so that if $n \equiv 1 \pmod{4}$ then $n = 4l + 1$ and if $n \equiv 3 \pmod{4}$ then $n = 4l + 3$. Define the orbit type θ_n of period n

by

$$j \mapsto \begin{cases} n - 2l - j & \text{if } 1 \leq j < n - 2l \text{ and } j \text{ odd;} \\ j - n + 2l + 1 & \text{if } n - 2l \leq j \leq n \text{ and } j \text{ odd;} \\ n - 2l + j - 1 & \text{if } 1 \leq j \leq 2l \text{ and } j \text{ even;} \\ n + 2l - j + 2 & \text{if } 2l < j \leq n \text{ and } j \text{ even.} \end{cases}$$

Thus, for example, $\theta_3 = (123)$, $\theta_5 = (12453)$, $\theta_7 = (1473265)$, and $\theta_9 = (148732695)$. It is easy to see that θ_n is maximodal and that f_{θ_n} has a (local) minimum at $x = 1$. One can also verify easily that θ_n is indeed a cycle of period n . (See [MN] for the case $n \equiv 1 \pmod{4}$; the argument is similar for $n \equiv 3 \pmod{4}$.)

For $n \equiv 1 \pmod{4}$, θ_n was defined by Misiurewicz and Nitecki [MN] and played a part in their result characterizing the asymptotic growth rate of the maximal entropy of n -cycles. It follows from their results that θ_n (for $n \equiv 1 \pmod{4}$) must at least have near-maximal entropy asymptotically. The main result of this paper is that for n odd, θ_n actually *attains* maximal entropy in P_n , the permutations of period n , and so in particular in C_n , the cycles of period n .

Theorem 1. *For n odd, the cycle θ_n and its dual $\bar{\theta}_n$ have maximal entropy among all permutations of period n .*

Corollary 1. *For n odd, θ_n and $\bar{\theta}_n$ have maximal entropy among cycles of period n .*

The proof builds on an idea of Misiurewicz and Nitecki [MN]. We need to show that no other permutation of period n has an induced matrix of larger spectral radius than that of θ_n . Using properties of the partial preorder on permutations, it will suffice to focus our attention on a certain class of maximodal permutations. The difficult part of the proof is coming up with a cone in \mathbf{R}^{n-1} which has a particular invariance property and furthermore allows us to demonstrate that the spectral radius of the induced matrix of θ_n dominates that of its competitors.

Since entropy respects the partial preorder on permutations, i.e. if θ forces η then $h(\theta) \geq h(\eta)$, we have that if θ_n does not have maximal entropy among permutations of length n , then there is a permutation of length n which is forcing-maximal in P_n and has maximal entropy in P_n and which satisfies $h(\theta) > h(\theta_n)$.

The following result is in [MN]; an analogue (without the entropy condition) was first proved for cycles in [J].

Proposition 4. *If $\theta \in P_n$ is forcing-maximal and has maximal entropy among permutations P_n , then θ is maximodal and f_θ has all maximum values above all minimum values, that is,*

$$\begin{aligned} \min\{\theta(j) : j \in [1, n] \text{ is a maximum of } f_\theta\} \\ < \max\{\theta(j) : j \in [1, n] \text{ is a minimum of } f_\theta\}. \end{aligned}$$

If M is an $m \times m$ matrix, we write $M^{(j)}$ for the j th column of M , $1 \leq j \leq m$, so $M^{(j)} \in \mathbf{R}^m$. For $r = (r_1, \dots, r_m) \in \mathbf{R}^m$, we let $|r| := |r|_1 = \sum_{i=1}^m |r_i|$.

Proposition 5. *If $\theta \in P_n$ is a permutation of length n , then its induced matrix $M = M(\theta)$ has j th column sum $|M^{(j)}| \leq \min\{2j, 2(n-j)\}$.*

Proof [MN]. Let $I_i := [i, i+1] \subset \mathbf{R}$, $i = 1, \dots, n-1$, and let $f = f_\theta$. Then if $i_1 := f^{-1}(1)$, we have that $f(I_i) \not\supset I_1$ if $i \neq i_1, i_1-1$, since any interval covering I_1 must send one of its endpoints to the left of I_1 . Similarly, if $i_2 := f^{-1}(2)$ then $f(I_i) \not\supset I_2$ for $i \notin \{i_1, i_1-1, i_2, i_2-1\}$. In general, if we let $i_j := f^{-1}(j)$ then $f(I_i) \not\supset I_j$ and hence $M_{ij} = 0$ for $i \notin \{i_1, i_1-1, \dots, i_j, i_j-1\}$, and this set has cardinality at most $2j$. Analogously, but working from the right, we have that $f(I_i) \not\supset I_j$ for $i \notin \{i_{j+1}, i_{j+1}+1, \dots, i_n, i_n+1\}$, and the cardinality of this set is at most $2(n-j)$. This establishes Proposition 5. \square

We say that a maximodal $\theta \in P_n$, n odd, is *normalized* if $x = 1$ is a minimum of f_θ (rather than a maximum). Note that if a maximodal $\theta \in P_n$, n odd, is not normalized then its dual $\bar{\theta}$ is. Note also that if $\theta \in P_n$ is a permutation of odd length n , $f := f_\theta$ is maximodal with all maximum values above all minimum values, and θ is normalized, then the k th column $B^{(k)}$ of the induced matrix $B := M(\theta)$ contains a 0, where $k = (n-1)/2$. For otherwise none of the (local) minimum values of f could exceed k (nor could the maximum values be less than $k+1$), which is impossible since f has $k+1$ minima, each with distinct image.

Together with Proposition 4 and Proposition 5, this establishes

Proposition 6. *Let n be odd and $k = (n-1)/2$. If $\theta \in P_n$ is normalized, forcing maximal, and entropy maximal among permutations of length n , with induced matrix $B = M(\theta)$, then the j th column sum $|B^{(j)}|$ of B satisfies*

$$|B^{(j)}| \leq \begin{cases} 2j & \text{for } j < k; \\ 2k-1 & \text{for } j = k; \\ 2(2k-j+1) & \text{for } j > k. \end{cases}$$

If n is odd, we will write $A = M(\theta_n)$, suppressing the dependence of A on n in the interest of readability. This should cause no confusion.

Proposition 7. *Let n be odd, $k = (n-1)/2$, and $A = M(\theta_n)$. If k is odd then*

$$A_{jr} = 1 \quad \text{iff} \quad \begin{cases} k-j+2 \leq r \leq k+j+1 & \text{for odd } j \leq k-2; \\ k-j+1 \leq r \leq k+j & \text{for even } j \leq k-1; \\ 2 \leq r \leq 2k & \text{for } j = k; \\ j-k-1 \leq r \leq 3k-j & \text{for odd } j \geq k+2; \\ j-k \leq r \leq 3k-j+1 & \text{for even } j \geq k+1. \end{cases}$$

If k is even then

$$A_{jr} = 1 \quad \text{iff} \quad \begin{cases} k-j+1 \leq r \leq k+j & \text{for odd } j \leq k-1; \\ k-j \leq r \leq k+j-1 & \text{for even } j \leq k-2; \\ 1 \leq r \leq 2k-1 & \text{for } j = k; \\ j-k \leq r \leq 3k-j+1 & \text{for odd } j \geq k+1; \\ j-k+1 \leq r \leq 3k-j+2 & \text{for even } j \geq k+2. \end{cases}$$

In both cases A is symmetric.

The proof is a routine verification in the various cases; it is omitted.

In the case when k is odd (i.e., when $n \equiv 3 \pmod{4}$), the matrix A has the following form:

$$A = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 1 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 1 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 1 & 1 & \cdots & 1 & 1 & 0 \\ 0 & 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & 1 & 0 \\ 0 & 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 1 & 1 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 1 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 1 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 1 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}.$$

Here row and column $k+1$ are those containing all ones.

Corollary 2.

$$|A^{(j)}| = \begin{cases} 2k & \text{for } j < k; \\ 2k - 1 & \text{for } j = k; \\ 2(2k - j + 1) & \text{for } j > k. \end{cases}$$

We call a vector $t = (t_1, \dots, t_{2k}) \in \mathbf{R}^{2k}$ *even* if both $\min\{i: t_i \neq 0\}$ and $\min\{i: t_{n-i} \neq 0\}$ are even. We say t is *odd* if both of these indices are odd. It is easily checked that $A^{(j)}$ is even for $j \leq k-1$ and $A^{(j)}$ is odd for $j \geq k+1$. ($A^{(k)}$ is neither even nor odd.)

Proposition 8. *Let $n = 2k + 1$ be odd, $A = M(\theta_n)$, and let $B = M(\theta)$, where $\theta \in P_n$ is normalized, forcing maximal, and entropy maximal among permutations of length n . Then for $j \neq k$, $1 \leq j \leq 2k$, $B^{(j)}$ has the same parity as $A^{(j)}$. That is, $B^{(j)}$ is even for $j \leq k-1$ and odd for $j \geq k+1$.*

Proof. Consider first some $j \leq k-1$. Now let $i_- = i_-(j)$ be the first point where f_θ does not exceed j , $i_- := \min\{i: f_\theta(i) \leq j\} \in \{1, \dots, n\}$. Then i_- must be a minimum for f_θ , since otherwise by maximodality it would be a maximum, hence different from 1 by normalization, and so we would have $f_\theta(i_- - 1) < f_\theta(i_-) \leq j$, contradicting the definition of i_- . But then maximodality tells us that i_- is odd. A similar argument shows that $i_+ := \max\{i: f_\theta(i) \leq j\}$ must also be a minimum and hence also odd. But this implies that $B^{(j)}$ is even, since if $B^{(j)} = (t_1, \dots, t_{2k})$ then $\min\{i: t_i \neq 0\} = i_- - 1$ and $\min\{i: t_{n-i} \neq 0\} = n - i_+$, both of which are even.

An analogous argument shows that $i^- := \min\{i: f_\theta(i) \geq j+1\}$ and $i^+ := \max\{i: f_\theta(i) \geq j+1\}$ are maxima and hence $B^{(j)}$ is odd for $j \geq k+1$. \square

4. INVARIANCE OF THE CONE \mathcal{P} UNDER A^2

Let n be odd, $n = 2k + 1$. Consider again the Euclidean space $\mathbf{R}^{n-1} = \mathbf{R}^{2k}$. A vector r in this space will now be denoted as

$$r = (u, v) = (u_1, \dots, u_k, v_k, \dots, v_1).$$

This notation is different from the one used in the previous section; it turns out to be more convenient to describe the following class of “peaked” vectors.

Definition 2. We say that r belongs to the class of vectors $\mathcal{P} \subset \mathbf{R}_+^{2k} = \{x \in \mathbf{R}^{2k} : x \geq 0\}$ if the following two conditions hold:

(1.e) $v_k \geq u_k \geq u_{k-1} \geq v_{k-1} \geq v_{k-2} \geq \dots \geq u_2 \geq u_1 \geq v_1$, if k is even (i.e., if $n \equiv 1 \pmod{4}$).

(1.o) $v_k \geq u_k \geq u_{k-1} \geq v_{k-1} \geq v_{k-2} \geq \dots \geq v_2 \geq v_1 \geq u_1$, if k is odd (i.e., if $n \equiv 3 \pmod{4}$).

(2) $v_k - u_k \geq u_{k-1} - v_{k-1} \geq v_{k-2} - u_{k-2} \geq \dots \geq (-1)^k(u_1 - v_1) \geq 0$, or, separating in cases:

(2.e) $v_k - u_k \geq u_{k-1} - v_{k-1} \geq v_{k-2} - u_{k-2} \geq \dots \geq v_2 - u_2 \geq u_1 - v_1 \geq 0$, if k is even ($n \equiv 1 \pmod{4}$);

(2.o) $v_k - u_k \geq u_{k-1} - v_{k-1} \geq v_{k-2} - u_{k-2} \geq \dots \geq u_2 - v_2 \geq v_1 - u_1 \geq 0$, if k is odd ($n \equiv 3 \pmod{4}$).

Note that here, as in the definition of A , we suppress the dependence of \mathcal{P} on n .

Remark. Condition (2) implies the following “half” of (1):

$$v_k \geq u_k, \quad u_{k-1} \geq v_{k-1}, \quad v_{k-2} \geq u_{k-2}, \dots$$

Therefore, we need only assume in (1) the remaining half:

$$u_k \geq u_{k-1}, \quad v_{k-1} \geq v_{k-2}, \dots$$

Consequently, if we assume condition (2) then condition (1) is equivalent to the following:

$$\min(v_{j+1}, u_{j+1}) \geq \max(v_j, u_j) \quad \text{for all } j = 1, 2, \dots, k-1.$$

In what follows we shall assume that k is odd (i.e., $n \equiv 3 \pmod{4}$). The arguments for the case when k is even are analogous.

Consider the matrix $A = M(\theta_n)$ of the orbit type θ_n (see Proposition 7). We now prove the following result.

Proposition 9. The cone \mathcal{P} is invariant under A^2 , i.e., $\mathcal{P}A^2 \subset \mathcal{P}$.

Proof. Take a vector (w, z) in \mathcal{P} , and denote $(u, v) = (w, z)A$, $(x, y) = (u, v)A$. We prove that $(x, y) \in \mathcal{P}$. Note that the vector (u, v) in general does not belong to \mathcal{P} ; \mathcal{P} is not invariant under A . Let us define, for $p = 1, 2, \dots, k$,

$$U(p) = \sum_{j=p}^k u_j \quad \text{and} \quad V(p) = \sum_{j=p}^k v_j.$$

We define also

$$U(k+1) = 0; \quad V(0) = V(1),$$

and we note that U and V are (weakly) decreasing in p . The numbers $W(p)$ and $Z(p)$ are defined similarly, in terms of the w_j and z_j respectively. Equality $(x, y) = (u, v)A$ then becomes

$$\begin{aligned} x_s &= U(k+2-s) + V(k-s), \quad \text{where } s \text{ is odd, } s = 1, 3, \dots, k; \\ x_{s+1} &= U(k-s) + V(k-s), \quad s \text{ odd, } s = 1, 3, \dots, k; \\ y_s &= U(k+1-s) + V(k+1-s), \quad s \text{ odd, } s = 1, 3, \dots, k; \\ y_{s+1} &= U(k-1-s) + V(k+1-s), \quad s \text{ odd, } s = 1, 3, \dots, k. \end{aligned}$$

We have analogous formulas for $(u, v) = (w, z)A$.

Step 1. Proof of condition (1). By our remark above, we need only prove that $x_{k+1-s} \geq x_{k-s}$ and $y_{k-s} \geq y_{k-1-s}$ for s odd, $s = 1, 3, \dots, k-2$. For such s , we have

$$x_{k+1-s} - x_{k-s} = V(s-1) - V(s+1),$$

which is nonnegative since V is decreasing. The second inequality is also true:

$$y_{k-s} - y_{k-1-s} = U(s) - U(s+2) \geq 0$$

for all s odd, $s = 1, 3, \dots, k-2$.

We now proceed to prove (2.o).

Step 2. We first prove the inequalities

$$y_{k+1-s} - x_{k+1-s} \geq x_{k-s} - y_{k-s},$$

or

$$y_{k-s} + y_{k+1-s} - x_{k-s} - x_{k+1-s} \geq 0$$

for s odd, $s = 1, 3, \dots, k-2$. We have

$$\begin{aligned} & y_{k-s} + y_{k+1-s} - x_{k-s} - x_{k+1-s} \\ &= U(s) + V(s+2) + U(s) + V(s) \\ &\quad - [U(s+1) + V(s+1) + U(s+1) + V(s-1)] \\ &= 2u_s - v_{s-1} - v_{s+1}, \end{aligned}$$

for s odd, $s = 3, 5, \dots, k-2$. Recall now that $(u, v) = (w, z)A$. Reasoning as above, using $W(p)$ and $Z(p)$ instead of $U(p)$ and $V(p)$ respectively, we obtain

$$\begin{aligned} 2u_s - v_{s-1} - v_{s+1} &= [z_{k-s} + z_{k+1-s} - w_{k-s} - w_{k+1-s}] \\ &\quad + [z_{k+2-s} - w_{k+1-s}] + [z_{k-s} - w_{k-1-s}], \end{aligned}$$

for s odd, $s = 3, 5, \dots, k-2$. The first expression in brackets is nonnegative by assumption (2) on the vector (w, z) . The last two are also nonnegative by condition (1.o) on (w, z) .

For the special case $s = 1$ we obtain

$$\begin{aligned} y_{k-1} + y_k - x_{k-1} - x_k &= 2u_1 - v_2 \\ &= [z_k + z_{k-1} - w_k - w_{k-1}] + [z_{k-1} - w_{k-2}] \geq 0, \end{aligned}$$

since, again by assumption (2), the expression in the first bracket is nonnegative, and so is that in the second bracket by condition (1.o).

Step 3. The proof of the remainder of (2.o) for (x, y) , that

$$x_{k-s} - y_{k-s} \geq y_{k-s-1} - x_{k-s-1}$$

for s odd, $s = 1, 3, \dots, k-2$, is very similar to the previous step and is therefore omitted.

Step 4. Finally, we must show that $y_1 - x_1 \geq 0$. A direct computation yields

$$y_1 - x_1 = u_k - v_{k-1} = z_2 + (z_1 - w_1) \geq 0,$$

since $z_1 \geq w_1$.

This establishes Proposition 9. \square

5. ENTROPY MAXIMALITY

We introduce some convenient notation. Given a positive integer m (which for us will just be $m = 2k = n - 1$) and $1 \leq j \leq j' \leq m$, we write $\langle j, j' \rangle := \sum_{i=j}^{j'} e^i = (0, \dots, 0, 1, \dots, 1, 0, \dots, 0) \in \mathbf{R}^m$, where e^i is the i th standard basis vector in \mathbf{R}^m . For consistency we write $\langle j \rangle$ for $e^j = \langle j, j \rangle \in \mathbf{R}^m$. We say $\langle j, j' \rangle$ is *centered* if $j' = m + 1 - j$.

Proposition 10. *Let $n = 2k+1$ be odd, $k > 1$, and $A = M(\theta_n)$. Let $B = M(\theta)$, where $\theta \in P_n$ is normalized, forcing-maximal, and has maximal entropy among permutations of length n . If $r \in \mathcal{P}$, then $rA \geq rB$.*

Proof. We consider the case $n \equiv 3 \pmod{4}$, i.e. $k = (n-1)/2$ odd; the case $n \equiv 1 \pmod{4}$ is handled similarly. If A and B are as above and $r \in \mathcal{P}$, we want to show that $(rA)_j \geq (rB)_j$, in other words $rA^{(j)} \geq rB^{(j)}$, for $j = 1, \dots, 2k$.

Note first that from Proposition 6 and the corollary to Proposition 7, we have $|B^{(j)}| \leq |A^{(j)}|$. Thus B has no more ones in the j th column than A does.

It is easiest to see what we want to show when j is even. For in this case, $A^{(j)} = \langle k-j+1, k+j \rangle$ if $j \leq k-1$ and $A^{(j)} = \langle j-k, 3k-j+1 \rangle$ if $j \geq k+1$, both of which are centered, and r is “peaked towards the center” (by (1.o) in the definition of \mathcal{P}). That is to say, if $r = (u_1, \dots, u_k, v_k, \dots, v_1) \in \mathcal{P} \subset \mathbf{R}_+^{2k}$, then $rA^{(j)} = \sum_{i=k-j+1}^k (u_i + v_i)$ if $j \leq k-1$ (and $rA^{(j)} = \sum_{i=j-k}^k (u_i + v_i)$ if $j \geq k+1$) and these are the $2j$ (or $2(2k-j+1)$) largest coordinates of r (by (1.o) in the definition of \mathcal{P}). Thus $rA^{(j)} \geq rB^{(j)}$ for j even.

Now let j be odd, and assume $j \leq k-2$. Then $A^{(j)} = \langle k-j+2, k+1+j \rangle$, which unfortunately is not centered. Assume, for a contradiction, that $rB^{(j)} > rA^{(j)}$.

Claim: $\langle k-j, k+j+1 \rangle B^{(j)} = 2j$. (Note that $\langle k-j, k+j+1 \rangle A^{(j)} = \langle 1, 2k \rangle A^{(j)} = |A^{(j)}| = 2j$.) Otherwise, we would have $\langle k-j, k+j+1 \rangle B^{(j)} \leq 2j-1$. Now from the definition of \mathcal{P} , we know that the $2j+2$ largest components of r are the $2j+2$ central ones, that is, r_{k-j} to r_{k+j+1} . The idea here is that at least three of these are “missed” by $B^{(j)}$, but only the $2j$ th and $(2j+1)$ th largest are missed by $A^{(j)}$. More precisely, referring to part (1.o) of the definition of \mathcal{P} we can write

$$\begin{aligned} r^{(1)} &= r_{k+1} \geq r^{(2)} = r_k \geq r^{(3)} = r_{k-1} \\ &\geq \dots \geq r^{(2k-2)} = r_{2k-1} \geq r^{(2k-1)} = r_{2k} \geq r^{(2k)} = r_1. \end{aligned}$$

Then $rA^{(j)} - rB^{(j)} = r(A^{(j)} - B^{(j)}) \geq r^{(2j+2)} - r^{(i_0)}$, where $i_0 \geq 2j+3$, so the last difference is nonnegative. This establishes the claim.

Thus we know that $B^{(j)} = \langle k - j, k + j + 1 \rangle - \langle i_1 \rangle - \langle i_2 \rangle$ for some i_1, i_2 satisfying $k - j \leq i_1, i_2 \leq k + j + 1$, so that $B_{i_1}^{(j)} = B_{i_2}^{(j)} = 0$. Then

$$\begin{aligned} 0 < rB^{(j)} - rA^{(j)} &= r_{k-j} + r_{k-j+1} - r_{i_1} - r_{i_2} \\ &= r^{(2j+1)} + r^{(2j)} - r^{(i'_1)} - r^{(i'_2)}, \end{aligned}$$

where $1 \leq i'_1 < i'_2 \leq 2j + 2$. For this positivity claim to hold, we must have $i'_2 = 2j + 2$, so $i_2 = k + j + 1$. Now $B^{(j)}$ is even by Proposition 8, so since $n - (i_2 - 1) = k - j + 1$ is odd, we must have that $i_1 = k + j$. Then the positivity claim boils down to

$$0 < rB^{(j)} - rA^{(j)} = -(r_{k+j} - r_{k-j+1}),$$

but this contradicts part (2) of the definition of \mathcal{P} . This establishes the result for j odd, $j \leq k - 2$.

For j odd, $j \geq k + 2$ we have a similar argument. For $j = k$, we have $rA^{(k)} - rB^{(k)} \geq r_{i_3} - r_1 \geq 0$, $1 \leq i_3 \leq 2k$. Analogous arguments holds for k even. \square

We now are in a position to finish off the proof of the main theorem.

Proof of Theorem 1. Let $w := (1, \dots, 1) \in \mathbf{R}_+^{2k}$, $k = (n - 1)/2$. Then $w \in \mathcal{P}$. Let $A = M(\theta_n)$. Note that wA satisfies $(wA)_j = |A^{(j)}|$, so by the corollary to Proposition 7 we have

$$(wA)_j = \begin{cases} 2j & \text{for } j < k; \\ 2k - 1 & \text{for } j = k; \\ 2(2k - j + 1) & \text{for } j > k, \end{cases}$$

and thus $wA \in \mathcal{P}$. Therefore, since $\mathcal{P}A^2 \subset \mathcal{P}$, it follows that $wA^m \in \mathcal{P}$ for all $m \geq 1$.

Let $\theta \in P_n$ be entropy maximal and forcing maximal among all permutations of length n , and let θ be normalized. We will show that $h(\theta_n) \geq h(\theta)$. Let $B = M(\theta)$.

Claim: $wA^m \geq wB^m$ for all $m \geq 1$. Since $w \in \mathcal{P}$, Proposition 10 implies the claim for the case $m = 1$. To prove the general case, we use induction on m . Assume $wA^m \geq wB^m$. Then, applying Proposition 10 and the fact that $wA^m \in \mathcal{P}$, we get that

$$wA^{m+1} = (wA^m)A \geq (wA^m)B \geq (wB^m)B = wB^{m+1},$$

where for the last inequality we used the induction hypothesis and the fact that B is nonnegative. This establishes the claim.

If M is a nonnegative $2k \times 2k$ matrix we can write $\|M\| := \sum_{i,j} M_{ij} = |wM|$. The spectral radius $\rho(M)$ satisfies $\rho(M) = \lim_{m \rightarrow \infty} \|M^m\|^{1/m}$. We have

$$\begin{aligned} h(\theta_n) &= \log \rho(A) = \log \lim_{m \rightarrow \infty} \|A^m\|^{1/m} = \log \lim_{m \rightarrow \infty} |wA^m|^{1/m} \\ &\geq \log \lim_{m \rightarrow \infty} |wB^m|^{1/m} = \log \lim_{m \rightarrow \infty} \|B^m\|^{1/m} = \log \rho(B) = h(\theta). \end{aligned}$$

So θ_n has maximal entropy and we are done. \square

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