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A DAMPED HYPERBOLIC EQUATION ON THIN DOMAINS

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ABSTRACT. For a damped hyperbolic equation in a thin domain in $\mathbf{R}^3$ over a bounded smooth domain in $\mathbf{R}^2$, it is proved that the global attractors are upper semicontinuous. It is shown also that a global attractor exists in the case of the critical Sobolev exponent.

1. INTRODUCTION

Let $\Omega \subset R^n$ for $n \leq 2$, be a bounded domain; let $Q_\varepsilon \subset R^{n+1}$, with $\varepsilon > 0$, be a bounded domain which converges in some sense to Ω as $\varepsilon \to 0$ and consider a damped hyperbolic equation on Q_ε with some boundary conditions. If Q_ε is to be regarded as a thin domain in R^{n+1}, then the dynamics on Q_ε should be determined from the dynamics of some appropriate hyperbolic equation on the n-dimensional domain Ω.

One objective in this paper is to extend our previous work [12] on thin domains for a reaction-diffusion equation to a damped hyperbolic equation; in particular, we consider the upper semicontinuity of the attractors for $n = 1, 2$. In addition, for thin domains in R^3, we prove the existence of attractors in the critical case where the growth rate of the nonlinearity is cubic. Existence in the general case in R^3 remains an open problem.

To describe the results, we first define carefully the domains Q_ε. We assume always that Ω is at least a C^2-polygonal domain; that is, a bounded open set in R^n with $\partial\Omega$ a curvilinear polygon of class C^1 [9, Definition 1.4.5.1]. Suppose that ε_0 is a positive number and $g: \overline{\Omega} \times [0, \varepsilon_0] \to R$ is a function of class C^3 satisfying

$$(1.1)\qquad \begin{aligned} &g(X, 0) = 0, \qquad g_0(X) = \frac{\partial g}{\partial \varepsilon}(X, 0) > 0 \quad \text{for } X \in \overline{\Omega}, \\ &g(X, \varepsilon) > 0 \quad \text{for } X \in \overline{\Omega}, \ \varepsilon \in (0, \varepsilon_0]. \end{aligned}$$

For $0 < \varepsilon \leq \varepsilon_0$, let Q_ε be the domain

$$(1.2)\qquad Q_\varepsilon = \{(X, Y) \in R^{n+1};\ 0 < Y < g(X, \varepsilon), X \in \Omega\}$$

and denote by ν_ε the outward normal to ∂Q_ε. Choose $\delta > 0$ so that $\widetilde{Q} = \Omega \times (0, \delta)$ contains Q_ε for $0 < \varepsilon \leq \varepsilon_0$.

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For α a positive constant and G a function belonging to $W^{1,\infty}(\tilde{Q})$, we consider the equation in variational form

$$(1.3)_\varepsilon \qquad (u_{tt}+\beta u_t+\alpha u\,,\,v)+(\nabla u\,,\,\nabla v)=(-f(u)-G\,,\,v)\quad \forall v\in H^1(Q_\varepsilon)$$

where $(\cdot\,,\,\cdot)$ is the inner product in $L^2(Q_\varepsilon)$. The initial values for a solution $(u\,,\,u_t)$ of $(1.3)_\varepsilon$ are in $H^1(Q_\varepsilon)\times L^2(Q_\varepsilon)$. The function $f\colon R\to R$ is a C^1-function satisfying

$$(1.4) \qquad \limsup_{|x|\to+\infty}\frac{-f(x)}{x}\le 0\,,$$

$$(1.5) \qquad |f'(x)|\le c(1+|x|^{\tilde{\gamma}})\quad \text{for } x\in R$$

where $0\le\tilde{\gamma}<+\infty$ if $n=1$, $0\le\tilde{\gamma}\le 2$ if $n=2$.

If the initial values are sufficiently regular, then equation $(1.3)_\varepsilon$ is equivalent to

$$(1.6)_\varepsilon \qquad \begin{aligned} u_{tt}+\beta u_t-\Delta u+\alpha u &= -f(u)-G \quad \text{in } Q_\varepsilon\,,\\ \partial u/\partial\nu_\varepsilon &=0 \quad \text{in } \partial Q_\varepsilon\,. \end{aligned}$$

To describe the results and, at the same time, to provide motivation for the equation on Ω, we make the change of variables

$$(1.7) \qquad X=x\,,\qquad Y=g(x\,,\,\varepsilon)y$$

which takes Q_ε into the fixed domain $Q=\Omega\times(0\,,\,1)$.

For $0<\varepsilon\le\varepsilon_0$, let X_ε be the space $L^2(Q)$ endowed with the norm $\|\cdot\|_{X_\varepsilon}$ induced by the inner product

$$(v\,,\,w)_{X_\varepsilon}=\int_Q\frac{g}{\varepsilon}vw\,dx\,dy\,.$$

This is an equivalent norm in $L^2(Q)$ since (1.1) implies that there are positive constants c_1, C_1 such that $c_1\varepsilon\le g(x\,,\,\varepsilon)\le C_1\varepsilon$ for $x\in\Omega$, $0<\varepsilon\le\varepsilon_0$.

To rewrite equation $(1.3)_\varepsilon$, we need the bilinear form $a_\varepsilon(\cdot\,,\,\cdot)$ on $(H^1(Q))^2$ (which is derived from the form: $(u_1\,,\,u_2)\mapsto\int_{Q_\varepsilon}(\nabla u_1\nabla u_2+\alpha u_1u_2)\,dX\,dY$ by the change of variables (1.7)):

$$a_\varepsilon(v\,,\,w)=(\mathscr{L}_\varepsilon^{1/2}v\,,\,\mathscr{L}_\varepsilon^{1/2}w)_{X_\varepsilon}+\alpha(v\,,\,w)_{X_\varepsilon}$$

where $\mathscr{L}_\varepsilon^{1/2}$ is the gradient operator on $H^1(Q)$,

$$\mathscr{L}_\varepsilon^{1/2}w=\left(w_{x_1}-\frac{g_{x_1}}{g}yw_y\,,\,w_{x_2}-\frac{g_{x_2}}{g}yw_y\,,\,\frac{1}{g}w_y\right)\,.$$

If we use this notation and let

$$(1.8) \qquad G_\varepsilon(x\,,\,y)=G(x\,,\,g(x\,,\,\varepsilon)y)\,,$$

then equation $(1.3)_\varepsilon$ is equivalent to

$$(1.9)_\varepsilon \qquad (u_{tt}+\beta u_t\,,\,v)_{X_\varepsilon}+a_\varepsilon(u\,,\,v)=(-f(u)-G_\varepsilon\,,\,v)_{X_\varepsilon}\quad\forall v\in H^1(Q)\,.$$

If the initial data are sufficiently regular, then $(1.9)_\varepsilon$ is equivalent to

$$(1.10)_\varepsilon \qquad u_{tt}+\beta u_t+L_\varepsilon u+\alpha u=-f(u)-G_\varepsilon \quad \text{in } Q$$

with the boundary conditions

$$(1.11)_\varepsilon \qquad \partial u/\partial \nu_{B_\varepsilon} \equiv B_\varepsilon u \cdot \nu = 0 \quad \text{on } \partial Q$$

where ν is the unit outward normal to ∂Q and L_ε is the operator:

$$(1.12) \qquad L_\varepsilon = -\frac{1}{g} \operatorname{div} B_\varepsilon u$$

where

$$(1.13) \qquad B_\varepsilon u = \begin{bmatrix} g u_{x_1} - g_{x_1} y u_y \\ g u_{x_2} - g_{x_2} y u_y \\ -g_{x_1} y u_{x_1} - g_{x_2} y u_{x_2} + \frac{1}{g}(1 + (g_{x_1} y)^2 + (g_{x_2} y)^2) u_y \end{bmatrix}.$$

We also need to write equation $(1.9)_\varepsilon$ as an abstract evolutionary equation. For notation, we let $\|\cdot\|_{0,Q}$, $\|\cdot\|_{1,Q}$ and $\|\cdot\|_{2,Q}$ denote respectively the classical norms in $L^2(Q)$, $H^1(Q)$ and $H^2(Q)$. Relation (1.1) implies that there are constants c_2 and ε_1, $0 < \varepsilon_1 \le \varepsilon_0$, such that, for $0 < \varepsilon \le \varepsilon_1$, $x \in \overline{\Omega}$, we have

$$(1.14) \qquad \begin{cases} \text{(i)} \left|\frac{g_{x_1}}{g}\right| + \left|\frac{g_{x_2}}{g}\right| \le c_2, \qquad c_1 \le \frac{g}{\varepsilon} \le c_2, \\ \text{(ii)} \frac{1}{g^2} - \left|\frac{g_{x_1}}{g}\right|^2 - \left|\frac{g_{x_2}}{g}\right|^2 \ge \frac{1}{2c_2^2 \varepsilon^2}. \end{cases}$$

According to [12], $a_\varepsilon(\cdot, \cdot)$ defines an unbounded linear operator A_ε on $H^1(Q)$ which is selfadjoint, positive, $A_\varepsilon = L_\varepsilon + \alpha I$ with Neumann boundary conditions, and $\mathscr{D}(A_\varepsilon^{1/2}) \cong H^1(Q)$. By the definition of $A_\varepsilon^{1/2}$, we have, for all $u \in H^1(Q)$, the following relation: $[a_\varepsilon(u, u)]^{1/2} = \|A_\varepsilon^{1/2} u\|_{X_\varepsilon}$. Furthermore,

(1.15)(i)

$$c_3 \left(\|u\|_{1,Q}^2 + \frac{1}{\varepsilon^2} \|u_y\|_{0,Q}^2 \right)^{1/2} \le \|A_\varepsilon^{1/2} u\|_{X_\varepsilon} \le c_4 \left(\|u\|_{1,Q}^2 + \frac{1}{\varepsilon^2} \|u_y\|_{0,Q}^2 \right)^{1/2}.$$

For $s = 1, 2$, let X_ε^s be the space $H^1(Q)$ endowed with the norm $\|u\|_{X_\varepsilon^s} = \|A_\varepsilon^{s/2} u\|_{X_\varepsilon}$ and let $Y_\varepsilon^s = \mathscr{D}(A_\varepsilon^{s/2}) \times \mathscr{D}(A_\varepsilon^{(s-1)/2})$ endowed with the norm $\|(\varphi, \psi)\|_{Y_\varepsilon^s} = (\|\varphi\|_{X_\varepsilon^s}^2 + \|\psi\|_{X_\varepsilon^{s-1}}^2)^{1/2}$. Clearly, Y_ε^1 is isomorphic to $H^1(Q) \times L^2(Q)$. Let us point out that, if the following hypothesis

(H) $\quad\Omega$ is a bounded domain which is a curvilinear polygon of class C^2 whose angles are all convex [9, Definition 1.3.4.1]

holds, then the regularity results of [4, 6, 9] (see also [12, Appendix A]) imply that

$$\mathscr{D}(A_\varepsilon) = \{u \in H^2(Q) \colon \partial u/\partial \nu_{B_\varepsilon} = 0 \text{ in } \partial Q\}$$

and Y_ε^2 is isomorphic to $\{u \in H^2(Q) \colon \partial u/\partial \nu_{B_\varepsilon} = 0 \text{ in } \partial Q\} \times H^1(Q)$. Furthermore, by Theorem A.2 of the Appendix, we have the more precise inequalities

(1.15)(ii)

$$\tilde{c}_3 \left(\|u\|_{2,Q}^2 + \frac{1}{\varepsilon^2} \|u_y\|_{0,Q}^2 + \frac{1}{\varepsilon^2} \sum_{i=1}^{n} \|u_{x_i y}\|_{0,Q}^2 + \frac{1}{\varepsilon^4} \|u_{yy}\|_{0,Q}^2 \right)^{1/2} \le \|A_\varepsilon u\|_{X_\varepsilon},$$

$$\|A_\varepsilon u\|_{X_\varepsilon} \le \tilde{c}_4 \left(\|u\|^2_{2,Q} + \frac{1}{\varepsilon^2}\|u_y\|^2_{0,Q} + \frac{1}{\varepsilon^2}\sum_{i=1}^{n} \|u_{x_i y}\|^2_{0,Q} + \frac{1}{\varepsilon^4}\|u_{yy}\|^2_{0,Q} \right)^{1/2} .$$

With this notation, equation $(1.9)_\varepsilon$, with initial data $(\varphi, \psi) \in Y^s_\varepsilon$ is equivalent to the abstract evolutionary equation

$$(1.16)_\varepsilon \qquad u_{tt} + \beta u_t + A_\varepsilon u = -f(u) - G_\varepsilon .$$

To describe the results more precisely, we need more notation. For any Banach space Z and any subsets C, D of Z, let

$$\delta_Z(C, D) = \sup_{c \in C} \inf_{d \in D} \|c - d\|_Z .$$

We say that a semigroup $T(t)$ on Z has a global attractor $\mathscr{A}$ in Z if $\mathscr{A}$ is a compact, invariant set ($T(t)\mathscr{A} = \mathscr{A}$ for $t \ge 0$) and $\delta_Z(T(t)B, \mathscr{A}) \to 0$ as $t \to \infty$ for each bounded set B in Z. By definition, $\mathscr{A}$ is unique in Z. We say that $T(t)$ is bounded dissipative in Z if there is a bounded set $\mathscr{B}_0$ in Z such that, for any bounded set B in Z, there is a $t_0 = t_0(B, \mathscr{B}_0)$ such that $T(t)B \subset \mathscr{B}_0$ for $t \ge t_0$.

We introduce the operator $T_\varepsilon(t)\colon (u_0, u_1) \mapsto (u^\varepsilon(t), u^\varepsilon_t(t)) \in Y^1_\varepsilon$, where $u^\varepsilon(t)$ is the solution of $(1.16)_\varepsilon$ with initial data $(u^\varepsilon(0), u^\varepsilon_t(0)) = (u_0, u_1)$. Under the hypotheses (1.4), (1.5), $T_\varepsilon(t)$ is a C^0-group on Y^1_ε and the positive orbits of bounded sets are bounded (see, for example, [18, 20, 2, 3 or 22]). Moreover, the semigroup $T_\varepsilon(t)$ is bounded dissipative in Y^1_ε and has a global attractor $\mathscr{A}_\varepsilon$ in Y^1_ε if $n = 1$ or $n = 2$ and $\tilde{\gamma} < 2$ (see [10]).

Let us now turn to the limit equation that should correspond to $(1.9)_\varepsilon$ at $\varepsilon = 0$. After some careful consideration, one begins to suspect that the solutions of $(1.9)_\varepsilon$ or, equivalently, for regular initial data, of $(1.10)_\varepsilon$, $(1.11)_\varepsilon$, for ε small, should depend very little upon y. To obtain the variational form of the limit equation, let X_0 be the space $L^2(\Omega)$ endowed with the inner product

$$(v, w)_{X_0} = \int_\Omega g_0 v w \, dx .$$

If we introduce the bilinear form

$$a_0(v, w) = (\nabla_x v, \nabla_x w)_{X_0} + \alpha(v, w)_{X_0},$$

then the variational form for the limit equation is

$$(1.17) \quad (u_{tt} + \beta u_t, v)_{X_0} + a_0(u, v) = (-f(u) - G(x, 0), v)_{X_0} \quad \forall v \in H^1(\Omega).$$

If the initial values are sufficiently regular and if we let $G_0(x) = G(x, 0)$, then equation (1.17) is equivalent to the following equation on Ω,

$$(1.18) \qquad u_{tt} + \beta u_t - \frac{1}{g_0}\sum_{i=1}^{n}(g_0 u_{x_i})_{x_i} + \alpha u = -f(u) - G_0 \quad \text{in } \Omega$$

with the boundary conditions

$$(1.19) \qquad \partial u / \partial n = 0 \quad \text{on } \partial\Omega,$$

where n is the unit outward normal to $\partial\Omega$.

We also need to write (1.17) as an abstract evolutionary equation. The bilinear form a_0 defines a unique unbounded operator A_0 on $H^1(\Omega)$ which is selfadjoint, positive, $A_0 = L_0 + \alpha I$ with Neumann boundary conditions, with

$$L_0 u = -\frac{1}{g_0}\sum_{i=1}^{n}(g_0 u_{x_i})_{x_i}$$

and $\mathscr{D}(A_0^{1/2}) \cong H^1(\Omega)$.

As above, we can define the space $Y_0^s = \mathscr{D}(A_0^{s/2}) \times \mathscr{D}(A_0^{(s-1)/2})$ with the norm $\|(\varphi, \psi)\|_{Y_0^s} = (\|A_0^{s/2}\varphi\|_{X_0}^2 + \|A_0^{(s-1)/2}\psi\|_{X_0}^2)^{1/2}$. Clearly, Y_0^1 is isomorphic to $H^1(\Omega) \times L^2(\Omega)$. If, in addition, hypothesis (H) holds, then

$$\mathscr{D}(A_0) = \{u \in H^2(\Omega) \colon \partial u/\partial n = 0 \text{ on } \partial\Omega\},$$

and Y_0^2 is isomorphic to $\{u \in H^2(\Omega) \colon \partial u/\partial n = 0 \text{ in } \partial\Omega\} \times H^1(\Omega)$. Equation (1.17) is equivalent to the abstract evolutionary equation

$$u_{tt} + \beta u_t + A_0 u = -f(u) - G_0. \tag{1.20}$$

Equation (1.20) has a global attractor $\mathscr{A}_0$ in Y_0^1 if $n = 1$ or $n = 2$ and $\tilde{\gamma} \leq 2$. The attractor is naturally embedded in Y_ε^1.

If we assume that the domain Ω satisfies the hypothesis (H), then the operators $T_\varepsilon(t)$, $t \geq 0$, are C^0-semigroups on Y_ε^2 (see [2, 3, 7, 14]), are bounded dissipative in Y_ε^2 and the global attractor $\mathscr{A}_\varepsilon$ in Y_ε^1 described above is also the global attractor in Y_ε^2 (see [14, 7, 11, 23] for instance). In the case $n = 2$, $\tilde{\gamma} = 2$ for $\varepsilon > 0$, the semigroup $T_\varepsilon(t)$ has a global attractor $\mathscr{A}_\varepsilon^2$ in Y_ε^2 [13] if $f \colon R \to R$ is moreover a C^2-function. If there is an attractor $\mathscr{A}_\varepsilon$ in Y_ε^1 and if the equilibrium points are hyperbolic, then $\mathscr{A}_\varepsilon = \mathscr{A}_\varepsilon^2$ because of the gradient structure (see Remark 4.3).

One of our results is

Theorem 1.1. *Suppose* Ω *satisfies hypothesis* (H).

(i) *If* $n = 1$ *or* $n = 2$ *and* $\tilde{\gamma} < 2$, *the attractors* $\mathscr{A}_\varepsilon$ *are upper semicontinuous at* $\varepsilon = 0$; *that is,*

$$\delta_{Y_\varepsilon^1}(\mathscr{A}_\varepsilon, \mathscr{A}_0) \to 0 \quad \text{as } \varepsilon \to 0.$$

(ii) *If* $n = 2$ *and* $\tilde{\gamma} = 2$, *and if* $f \colon R \to R$ *is a* C^2-*function the attractors* $\mathscr{A}_\varepsilon^2$ *are upper semicontinuous at* $\varepsilon = 0$; *that is,*

$$\delta_{Y_\varepsilon^1}(\mathscr{A}_\varepsilon^2, \mathscr{A}_0) \to 0 \quad \text{as } \varepsilon \to 0.$$

In a subsequent paper, we analyze the lower semicontinuity of the attractors as well as the equivalence of the flows.

Under the general hypotheses (1.4), (1.5) in the case $n = 2$, $\tilde{\gamma} = 2$, it is not known if a global attractor exists for $(1.3)_\varepsilon$ on a general domain in R^3. If the nonlinear function f satisfies some additional conditions (see Remark 1.6 below), we can use the arguments in Babin and Vishik [24, Chapter II, §6] to show that the global attractor does exist on a general smooth domain. For the thin domains Q_ε, we prove that the additional restrictions on f of Babin and Vishik are unnecessary to obtain the existence of a global attractor provided that ε is sufficiently small. When the domain Q_ε is $\Omega \times (0, \varepsilon)$, we can say even more. Precise statements are contained in the following theorems.

Theorem 1.2. *If $n = 2$, $\tilde{\gamma} = 2$, then, for any $\beta_0 > 0$, there is an $\varepsilon_1 = \varepsilon_1(\beta_0)$, such that, for $0 < \varepsilon \leq \varepsilon_1$, $\beta \geq \beta_0$, there is a global attractor $\mathscr{A}_\varepsilon$ in Y_ε^1 for $T_\varepsilon(t)$.*

Let us point out that, if Ω satisfies the hypothesis (H), then $\mathscr{A}_\varepsilon = \mathscr{A}_\varepsilon^2$ (see Remark 4.3), and, by Theorem 1.1, the attractors $\mathscr{A}_\varepsilon$ are upper semicontinuous at $\varepsilon = 0$.

Theorem 1.3. *If $n = 1$ or $n = 2$, $\tilde{\gamma} \leq 2$ and $g(x, \varepsilon) = \varepsilon$, then the attractors $\mathscr{A}_\varepsilon$ are upper semicontinuous at $\varepsilon = 0$; moreover, if $G(X, Y) = G_0(X)$, then, for any $\beta_0 > 0$, there exists a positive constant $\varepsilon_1 = \varepsilon_1(\beta_0)$ such that, for $0 \leq \varepsilon \leq \varepsilon_1$, $\beta \geq \beta_0$, we have $\mathscr{A}_\varepsilon = \mathscr{A}_0$.*

In the case where $g(x, \varepsilon) = \varepsilon$ and $G(X, Y)$ is independent of Y, Theorem 1.3 asserts that, on the cylindrical domain $\Omega \times (0, \varepsilon)$, the flow defined by $(1.3)_\varepsilon$ is equivalent to the flow defined by the same equation on the n-dimensional domain Ω.

It is possible to consider other boundary conditions. The extension of the above results to periodic boundary conditions is made in an obvious way. We also can study mixed boundary conditions or Dirichlet ones. Let us denote by $\Gamma_{j,\varepsilon}$ (respectively Γ_j), $j = 0, 1, 2$, the portions of the boundary of Q_ε (respectively Q) given by

$$\begin{aligned}
&\Gamma_{0,\varepsilon} = \Omega \times \{0\} \quad (\text{resp. } \Gamma_0 = \Omega \times \{0\}), \\
&\Gamma_{1,\varepsilon} = \{(X, Y) \in R^{n+1};\, X \in \Omega,\, Y = g(X, \varepsilon)\} \quad (\text{resp.} \Gamma_1 = \Omega \times \{1\}), \\
&\Gamma_{2,\varepsilon} = \{(X, Y) \in R^{n+1};\, X \in \partial\Omega,\, 0 < Y < g(X, \varepsilon)\} \\
&\qquad\qquad\qquad\qquad\qquad\qquad (\text{resp.} \Gamma_2 = \partial\Omega \times (0, 1)).
\end{aligned}$$

We may define the corresponding unit outward normals $\nu_{j,\varepsilon}$ on $\Gamma_{j,\varepsilon}$ (resp. ν_j on Γ_j).

The mixed problem that we consider is homogeneous Neumann conditions on $\Gamma_{j,\varepsilon}$, $j = 0, 1$ and Dirichlet conditions on $\Gamma_{2,\varepsilon}$. To avoid excessive notation, we do not formulate the variational form of the equation. If the initial data are sufficiently regular, the equation is

$$(1.6\text{bis})_\varepsilon \qquad \begin{aligned}
u_{tt} + \beta u_t - \Delta u + \alpha u &= -f(u) - G &&\text{in } Q_\varepsilon, \\
u &= 0 &&\text{in } \Gamma_{2,\varepsilon}, \\
\partial u/\partial \nu_{j,\varepsilon} &= 0 &&\text{in } \Gamma_{j,\varepsilon},\ j = 0, 1.
\end{aligned}$$

In the new variables (x, y) of the fixed domain Q, this boundary value problem is

$$(1.10)_\varepsilon \qquad u_{tt} + \beta u_t + L_\varepsilon u + \alpha u = -f(u) - G_\varepsilon \quad \text{in } Q$$

with the boundary conditions

$$(1.11\text{bis})_\varepsilon \qquad \begin{aligned}
&u = 0 \quad \text{in } \Gamma_2, \\
&\partial u/\partial \nu_{j,B_\varepsilon} \equiv B_\varepsilon u \cdot \nu_j = 0 \quad \text{in } \Gamma_j,\ j = 0, 1,
\end{aligned}$$

where L_ε and B_ε are defined, respectively, by (1.12) and (1.13). For ε small, the solutions of $(1.10)_\varepsilon$, $(1.11\text{bis})_\varepsilon$ can be compared with those of the equation (1.18) on Ω with the boundary conditions

$$(1.19\text{bis}) \qquad u = 0 \quad \text{in } \partial\Omega.$$

Let V_0 be the subspace $\{v \in H^1(Q)\,;\, v = 0 \text{ in } \Gamma_2\}$ and let A_ε be the unique selfadjoint unbounded linear operator on V_0 defined by the form $a_\varepsilon(\cdot\,,\,\cdot)$ and the space X_ε. We remark that $Y_\varepsilon^1 = D(A_\varepsilon^{1/2}) \times X_\varepsilon$ is isomorphic to $V_0 \times L^2(Q)$. We denote by $T_\varepsilon(t)$ the C^0-group on Y_ε^1 generated by the abstract equation associated with $(1.10)_\varepsilon$, $(1.11\text{bis})_\varepsilon$. Likewise, we denote by A_0 the operator $L_0 + \alpha I$ with Dirichlet boundary conditions on $\partial\Omega$ and by $T_0(t)$ the C^0-group generated on $Y_0^1 = D(A_0^{1/2}) \times X_0$ (Y_0^1 is isomorphic to $H_0^1(\Omega) \times L^2(\Omega)$) by the abstract equation associated with (1.18) and (1.19bis). We still denote by $\mathscr{A}_\varepsilon$, $0 < \varepsilon \le \varepsilon_0$, and by $\mathscr{A}_0$ the global attractors of $T_\varepsilon(t)$ and $T_0(t)$ in Y_ε^1 and Y_0^1, respectively. Then Theorems 1.2, 1.3, as well as Theorem 3.1 and Corollary 3.2 below, hold for the case of mixed boundary conditions. Let us now assume that the domains Ω and Q_ε satisfy the stronger hypothesis $(\widetilde{\mathrm{H}})$:

Hypothesis $(\widetilde{\mathrm{H}})$:

> Ω is a bounded domain which is a curvilinear polygon of class C^2 whose maximal angle ω satisfies $\omega < \pi/2$. If Θ is the maximum of the dihedral angles determined by $\Gamma_{2,\varepsilon}$ and $\Gamma_{1,\varepsilon}$, we suppose that $\Theta < \pi/2$.

Then $T_\varepsilon(t)$ and $T_0(t)$ are C^0-groups on $Y_\varepsilon^2 \equiv D(A_\varepsilon) \times D(A_\varepsilon^{1/2})$ and on $Y_0^2 \equiv D(A_0) \times D(A_0^{1/2})$, respectively. Moreover, Theorems 1.1 as well as Theorem 3.4 and Corollary 3.5 below are true for mixed boundary conditions. We remark that Y_ε^2 and Y_0^2 are isomorphic to $\{u \in H^2(Q) \colon u = 0 \text{ in } \Gamma_2,\, \partial u/\partial \nu_{j,B_\varepsilon} = 0 \text{ in } \Gamma_j,\, j = 0, 1\} \times V_0$ and to $(H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$, respectively.

Let us now turn to homogeneous Dirichlet conditions; that is, after having made the change of variables (1.7), we consider the equation $(1.10)_\varepsilon$ with boundary conditions

$$(1.11\text{ter})_\varepsilon \qquad\qquad u = 0 \quad \text{in } \partial Q.$$

Let A_ε be the unique selfadjoint unbounded linear operator on $H_0^1(Q)$ defined by the form $a_\varepsilon(\cdot\,,\,\cdot)$ and the space X_ε. We remark that $Y_\varepsilon^1 = D(A_\varepsilon^{1/2}) \times X_\varepsilon$ is isomorphic to $H_0^1(Q) \times L^2(Q)$. We still denote by $T_\varepsilon(t)$ the C^0-group on Y_ε^1 generated by the abstract equation associated with $(1.10)_\varepsilon$, $(1.11\text{ter})_\varepsilon$. It is well known that the attractors $\mathscr{A}_\varepsilon$ exist in Y_ε^1 if $n = 1$ or $n = 2$, $\tilde{\gamma} < 2$ (see [10]). In the case $n = 2$, $\tilde{\gamma} = 2$, there is a partial answer to the question of the existence of the attractor due to Babin and Vishik (see [24] and Remark 1.6). Here, we prove

Theorem 1.4. *If $n = 2$, $\tilde{\gamma} = 2$, then, for any $\beta_0 > 0$, there exists a positive number $\varepsilon_1 = \varepsilon_1(\beta_0) > 0$ such that, for $\beta \ge \beta_0$, $0 < \varepsilon \le \varepsilon_1$, the semigroup $T_\varepsilon(t)$ has a global attractor $\mathscr{A}_\varepsilon$ in Y_ε^1.*

For the Dirichlet case, the attractors are very small if ε is small as stated in the following result.

Theorem 1.5. (i) *The attractors $\mathscr{A}_\varepsilon$ of $T_\varepsilon(t)$ are upper semicontinuous at $\varepsilon = 0$; that is,*

$$\delta_{Y_\varepsilon^1}(\mathscr{A}_\varepsilon, 0) \to 0 \quad as\ \varepsilon \to 0.$$

(ii) *Moreover, if* $G_\varepsilon + f(0) = 0$ *for a positive number* ε *with* $0 < \varepsilon \le \varepsilon_1(\beta_0)$, $\beta \ge \beta_0$, *then* $\mathscr{A}_\varepsilon = 0$.

In §7, we indicate some generalizations of the above results to systems of Sine-Gordon equations which, with Dirichlet boundary conditions, have been used as models for Josephson Junctions. Also, we remark that our proofs do not rely on the gradient structure and can be applied to equations considered by [7].

It is possible to replace the Laplacian operator by a more general selfadjoint operator. Also, the theory can be adapted to other types of thin domains; for example, the domain could be a cylinder with a thin wall. These topics will be discussed in a subsequent paper.

In the sequel, the proofs of the results will be given mainly in the case $n = 2$, since the case $n = 1$ is simpler.

Remark 1.6. After this paper had been written, we became aware of the recently published book of Babin and Vishik [24, Chapter II, §6] in which they have proved the following result.

Let $\mathscr{O}$ be a smooth three-dimensional domain. Assume that the nonlinear function f can be written as $f = f_0 + f_1$, were f_0, f_1 are C^1-functions with f_1 satisfying the growth condition (1.5) with $\tilde{\gamma} < 2$ and f_0 satisfying

$$f_0(0) = 0, \quad f_0'(0) = 0, \quad f_0'(u) \ge 0 \quad \text{for } u \in R,$$
$$|f_0'(u_1) - f_0'(u_2)| \le c|u_1 - u_2|(1 + |u_1| + |u_2|).$$

Under these assumptions and for any G in $L^2(\mathscr{O})$, the equation

$$\begin{aligned} u_{tt} + \beta u_t - \Delta u + \alpha u &= - f(u) - G \quad &&\text{in } \mathscr{O}, \\ u &= 0 &&\text{in } \partial\mathscr{O} \end{aligned}$$

has a global attractor in $H_0^1(\mathscr{O}) \times L^2(\mathscr{O})$ which is compact in $(H_0^1(\mathscr{O}) \cap H^2(\mathscr{O})) \times H_0^1(\mathscr{O})$. Although this result is interesting, not every function satisfying the general hypotheses (1.4), (1.5) with $\tilde{\gamma} = 2$ can be written in the above form. For example, we can choose a C^1-function $f(u)$ satisfying

$$f(u) = u^3(1 + \cos(\log(u/3k))) \quad \text{for } |u| \text{ large}$$

which does not have the above decomposition if $k > 2$. Therefore, in the general case, the question of the existence of a global attractor in $H_0^1(\mathscr{O}) \times L^2(\mathscr{O})$ remains open.

2. Background material

Let A_ε be the operator in $(1.16)_\varepsilon$, suppose that $h \in C([0, \infty); L^2(Q))$ and consider the nonhomogeneous linear equation

$$(2.1)_\varepsilon \qquad u_{tt} + \beta u_t + A_\varepsilon u = h(t).$$

In this section, we derive some inequalities which will yield estimates for the solutions of $(2.1)_\varepsilon$. Let $\varepsilon \ge 0$ and let $\lambda_{1,\varepsilon} > 0$ be the first eigenvalue of A_ε. Arguing as in §4.1 of [12], there is a positive constant $\varepsilon_0 > 0$ such that, for $0 < \varepsilon \le \varepsilon_0$, we have

$$(2.2) \qquad 0 < \tfrac{3}{4}\lambda_{1,0} \le \lambda_{1,\varepsilon} \le \tfrac{5}{4}\lambda_{1,0},$$

where $\lambda_{1,0}$ is the first eigenvalue of the operator A_0 given in (1.20). Throughout the remainder of the paper, ε_0 will be chosen so that (2.2) is satisfied.

Lemma 2.1. *If $\beta > 0$ and b is a nonnegative real number satisfying*

$$b \leq \min\left(\frac{\beta}{8}, \frac{\lambda_{1,\varepsilon}}{4\beta}, \frac{\sqrt{\lambda_{1,\varepsilon}}}{4}\right), \tag{2.3}$$

then the following inequalities hold for $(\varphi, \psi) \in Y_\varepsilon^1$,

$$\tfrac{1}{4}\|(\varphi, \psi)\|_{Y_\varepsilon^1}^2 \leq \tfrac{1}{2}\|\psi\|_{X_\varepsilon}^2 + 2b(\varphi, \psi)_{X_\varepsilon} + \tfrac{1}{2}\|\varphi\|_{X_\varepsilon^1}^2 \leq \tfrac{3}{4}\|(\varphi, \psi)\|_{Y_\varepsilon^1}^2 \tag{2.4}$$

and

$$(\beta - 2b)\|\psi\|_{X_\varepsilon}^2 + 2b\beta(\varphi, \psi)_{X_\varepsilon} + 2b\|\varphi\|_{X_\varepsilon}^2 \geq \tfrac{\beta}{2}\|\psi\|_{X_\varepsilon}^2 + b\|\varphi\|_{X_\varepsilon^1}^2 . \tag{2.5}$$

If we observe that $\lambda_{1,\varepsilon}\|\varphi\|_{X_\varepsilon}^2 \leq \|\varphi\|_{X_\varepsilon^1}^2$, the proof of this lemma is obvious.

Our estimates for the solutions of $(2.1)_\varepsilon$ will be obtained from the following energy functional on Y_ε^1,

$$V_\varepsilon(\varphi, \psi) = \tfrac{1}{2}\|\varphi\|_{X_\varepsilon^1}^2 + 2b(\varphi, \psi)_{X_\varepsilon} + \tfrac{1}{2}\|\psi\|_{X_\varepsilon}^2 \tag{2.6}$$

with b satisfying (2.3). From (2.4), $(V_\varepsilon(\varphi, \psi))^{1/2}$ is equivalent to the norm in Y_ε^1.

Lemma 2.2. *Suppose that $0 \leq \varepsilon \leq \varepsilon_0$, $\beta > 0$, b satisfies (2.5), and let $(u(t), u_t(t))$ be a solution of $(2.1)_\varepsilon$. Then, for $t \geq 0$,*

$$\begin{aligned} \frac{d}{dt}V_\varepsilon(u(t), u_t(t)) \leq & -\frac{\beta}{2}\|u_t(t)\|_{X_\varepsilon}^2 - b\|u(t)\|_{X_\varepsilon^1}^2 \\ & + \|h(t)\|_{X_\varepsilon}(2b\|u(t)\|_{X_\varepsilon} + \|u_t(t)\|_{X_\varepsilon}). \end{aligned} \tag{2.7}$$

Proof. Let us recall that, if u is a function such that (u, u_t) belongs to $L^2((0, T); Y_\varepsilon^1)$ and $u_{tt}+A_\varepsilon u$ belongs to $L^2((0, T); X_\varepsilon)$, where T is a positive constant, then

$$\frac{1}{2}\frac{\partial}{\partial t}(\|u_t\|_{X_\varepsilon}^2 + \|A_\varepsilon^{1/2}u\|_{X_\varepsilon}^2) = (u_{tt} + A_\varepsilon u, u_t)_{X_\varepsilon} \tag{2.8}$$

(see [7 or 23, Chapter II, Lemma 4.1]). Arguing as in [23, Chapter II, Lemma 4.1] (see also [11, §4.8]), and using the identity (2.8) and a density argument, one shows that, if the initial data $(u(0), u_t(0))$ belongs to Y_ε^1, then, for $t \geq 0$, we have

$$\begin{aligned} \frac{d}{dt}V_\varepsilon(u(t), u_t(t)) \leq & -(\beta - 2b)\|u_t\|_{X_\varepsilon}^2 - 2b\|u\|_{X_\varepsilon^1}^2 - 2b\beta(u, u_t)_{X_\varepsilon} \\ & + \|h(t)\|_{X_\varepsilon}(2b\|u(t)\|_{X_\varepsilon} + \|u_t(t)\|_{X_\varepsilon}). \end{aligned}$$

This inequality together with (2.5) implies (2.7) and the proof is complete.

It is worthwhile to remark that, if $h \equiv 0$, then (2.7), (2.3) and (2.4) yield an estimate for the solutions of the linear damped wave equation. In fact, in this case, they imply that $dV_\varepsilon(u(t), u_t(t))/dt \leq -\frac{4}{3}bV(u(t), u_t(t))$. Integrating this relation and using (2.4) again, we obtain, for $t \geq 0$,

$$\|(u(t), u_t(t))\|_{Y_\varepsilon^1} \leq \sqrt{3}e^{-2bt/3}\|(u(0), u_t(0))\|_{Y_\varepsilon^1} .$$

3. Uniform bounded dissipativeness

The results of this section are concerned with bounded dissipativeness of $(1.16)_\varepsilon$ uniform with respect to β and ε. The first result is concerned with Y_ε^1. We give the proofs in the case $n = 2$.

Theorem 3.1. *Fix* $\varepsilon_0 > 0$, $\beta_0 > 0$. *For* $0 < \varepsilon \le \varepsilon_0$, $\beta \ge \beta_0$, *the system* $(1.16)_\varepsilon$ *is uniformly bounded dissipative in* Y_ε^1; *that is, there is a constant* $K_0 = K_0(\varepsilon_0, \beta_0)$ *such that, for any* $\beta \ge \beta_0$ *and any* $r_0 > 0$, *there is a constant* $t_0 = t_0(r_0, \beta_0)$ *such that, for* $0 < \varepsilon \le \varepsilon_0$, *any solution* $U(t) = (u(t), u_t(t))$ *of* $(1.16)_\varepsilon$ *with* $\|U(0)\|_{Y_\varepsilon^1} \le r_0$, *the following estimate holds*

$$\|U(t)\|_{Y_\varepsilon^1} \le K_0 \quad \textit{for } t \ge t_0. \tag{3.1}$$

From Theorem 3.1 and the invariance of the global attractor $\mathscr{A}_\varepsilon$ (if it exists), we deduce at once the following result.

Corollary 3.2. *For fixed* $\varepsilon_0 > 0$, $\beta_0 > 0$, *there is a constant* $K_1 > 0$ *such that, for* $0 \le \varepsilon \le \varepsilon_0$, $\beta \ge \beta_0$, *if the global attractor* $\mathscr{A}_\varepsilon$ *exists, then*

$$\|(\varphi, \psi)\|_{Y_\varepsilon^1} \le K_1 \quad \textit{for all } (\varphi, \psi) \in \mathscr{A}_\varepsilon.$$

Proof of Theorem 3.1. The proof follows closely the one of Theorem 2.2 of [13] for a corresponding result on singularly perturbed hyperbolic equations. We introduce the following energy functional on Y_ε^1,

$$V_\varepsilon^b(\varphi, \psi) = \tfrac{1}{2}\|\psi\|_{X_\varepsilon}^2 + 2b(\varphi, \psi)_{X_\varepsilon} + \tfrac{1}{2}\|\varphi\|_{X_\varepsilon^1}^2 + (G_\varepsilon, \varphi)_{X_\varepsilon} + (F(\varphi), 1)_{X_\varepsilon}$$

where $F(u) = \int_0^u f(s)\,ds$, 1 is the constant function one, and

$$0 < b < \min(\beta/8, 5\lambda_{1,0}/16\beta, \sqrt{5\lambda_{1,0}/8}). \tag{3.2}$$

To simplify the notation, c with or without any subscripts will denote a positive constant independent of b, ε, β, with $0 < \varepsilon \le \varepsilon_0$, $\beta \ge \beta_0$. Arguing as in [23] and in the proof of Lemma 2.2, by using the identity (2.8) and a density argument, one shows that, if $U(0) = (u(0), u_t(0))$ belongs to Y_ε^1, then, for $t \ge 0$,

$$\begin{aligned} \frac{d}{dt}V_\varepsilon^b(U(t)) = &-(\beta - 2b)\|u_t\|_{X_\varepsilon}^2 - 2b\|u\|_{X_\varepsilon^1}^2 - 2b\beta(u, u_t)_{X_\varepsilon} \\ &- 2b(f(u), u)_{X_\varepsilon} - 2b(G_\varepsilon, u)_{X_\varepsilon} \\ \le &-\frac{\beta}{2}\|u_t\|_{X_\varepsilon}^2 - b\|u\|_{X_\varepsilon^1}^2 - 2b(f(u), u)_{X_\varepsilon} + 2b|(G_\varepsilon, u)_{X_\varepsilon}|, \end{aligned} \tag{3.3}$$

by (2.5).

From hypothesis (1.4), for any $\eta > 0$, there is a constant $c_\eta > 0$ such that, for $v \in R$,

$$\begin{cases} \text{(i)} & -f(v)v \le \eta v^2 + c_\eta, \\ \text{(ii)} & -F(v) \le \eta v^2 + c_\eta, \end{cases} \tag{3.4}$$

(see [15], for example). Using the inequalities (3.4)(i), (2.3) and (3.3) and letting $\eta = 3\lambda_{1,0}/32$, we obtain

$$\begin{aligned} \frac{d}{dt}V_\varepsilon^b(U(t)) \le &-b\|U(t)\|_{Y_\varepsilon^1}^2 + b\left(\frac{2\eta}{\lambda_{1,\varepsilon}}\|u\|_{X_\varepsilon^1}^2 + 2c_\eta\|1\|_{X_\varepsilon}\right) \\ &+ b\left(\frac{2\eta}{\lambda_{1,\varepsilon}}\|u\|_{X_\varepsilon^1}^2 + \frac{1}{2\eta}\|G\|_{C^0(\widetilde{Q})}^2\|1\|_{X_\varepsilon}^2\right) \end{aligned}$$

or,

$$\frac{d}{dt}V_\varepsilon^b(U(t)) \le -\frac{b}{2}\|U(t)\|_{Y_\varepsilon^1}^2 + c_1 b \quad \text{for } t \ge 0, \tag{3.5}$$

from the definition of η and (2.2). From the inequalities (2.4) and (3.4), the definition of η and by the property (2.2), we see that

$$V_\varepsilon^b(U(t)) \geq \tfrac{1}{8}\|U(t)\|_{Y_\varepsilon^1}^2 - c_2 . \tag{3.6}$$

On the other hand, if condition (1.5) is satisfied, then

$$|F(s)| < c_3(|s|^4 + 1) \quad \text{for } s \in R . \tag{3.7}$$

Thanks to the fact that $H^1(Q)$ is continuously embedded in $L^p(Q)$ for $1 \leq p \leq 6$, it follows from (3.7) that

$$|(F(\varphi), 1)_{X_\varepsilon}| \leq c_4(\|\varphi\|_{X_\varepsilon^1}^4 + 1) \quad \text{for } \varphi \in X_\varepsilon^1 . \tag{3.8}$$

Using (3.8) and (2.4), we have

$$\begin{aligned} V_\varepsilon^b(U(t)) &\leq \tfrac{3}{4}\|U(t)\|_{Y_\varepsilon^1}^2 + c_4\|u(t)\|_{X_\varepsilon^1}^4 + c_4 \\ &\quad + \left(\frac{2\eta}{\lambda_{1,\varepsilon}}\|u\|_{X_\varepsilon^1}^2 + \frac{1}{2\eta}\|G\|_{C^0(\widetilde{Q})}^2\|1\|_{X_\varepsilon}^2\right) \\ &\leq \|U(t)\|_{Y_\varepsilon^1}^2 + c_4\|u(t)\|_{X_\varepsilon^1}^4 + c_5 . \end{aligned}$$

If we use (3.6) and let $\widetilde{V}(\varphi, \psi) = V_\varepsilon^b(\varphi, \psi) + c_2$, then the last inequality implies that

$$\|U(t)\|_{Y_\varepsilon^1}^2 \geq c_6(\widetilde{V}(U(t)))^{1/2} - c_7 \tag{3.9}$$

Using (3.5) and (3.9), we obtain

$$\frac{d}{dt}\widetilde{V}(U(t)) \leq -bc_8(\widetilde{V}(U(t)))^{1/2} + bc_9$$

A simple exercise in differential inequalities (see, for example, [13]) shows that there is a constant K_0 independent of ε, β, b such that for any $r_0 > 0$, there is a $t_0 = t_0(r_0, b)$ such that $\widetilde{V}(U(t)) \leq K_0$ for $t \geq t_0(r_0, b)$. Since $\widetilde{V}(U(t)) \geq \frac{1}{8}\|U(t)\|_{Y_\varepsilon^1}^2$, Theorem 3.1 is proved.

We also need the following result.

Lemma 3.3. *Fix $\varepsilon_0 > 0$, $\beta_0 > 0$. For any $r_0 > 0$, there is a constant $c_0(r_0)$ such that, for $0 < \varepsilon \leq \varepsilon_0$, $\beta \geq \beta_0$, the solution $U(t) = (u(t), u_t(t))$ of $(1.16)_\varepsilon$, with $\|U(0)\|_{Y_\varepsilon^1} \leq r_0$, satisfies*

$$\int_0^\infty \|u_t(s)\|_{X_\varepsilon}^2\, ds \leq c_0(r_0), \tag{3.10}$$

$$\|U(t)\|_{Y_\varepsilon^1} \leq c_0(r_0) \quad \textit{for } t \geq 0 . \tag{3.11}$$

Proof. Using the classical Liapunov function $V_\varepsilon^0(\varphi, \psi)$ and arguing in the same way as in the proof of Theorem 3.2, one shows that

$$V_\varepsilon^0(U(t)) \leq V_\varepsilon^0(U(0)) \quad \text{for } t \geq 0,$$

$$\int_0^\infty \|u_t(s)\|_{X_\varepsilon}^2\, ds \leq \frac{2}{\beta}V_\varepsilon^0(U(0)),$$

and, for any $(\varphi, \psi) \in Y_\varepsilon^1$,

$$V_\varepsilon^0(\varphi, \psi) \leq \|(\varphi, \psi)\|_{Y_\varepsilon^1}^2 + c(\|\varphi\|_{X_\varepsilon^1}^4 + 1),$$

$$V_\varepsilon^0(\varphi, \psi) \geq \tfrac{1}{4}\|(\varphi, \psi)\|_{Y_\varepsilon^1}^2 - c .$$

These inequalities prove the lemma.

If we assume that Ω satisfies hypothesis (H), then we know that the global attractor $\mathscr{A}_\varepsilon$ belongs to the space Y_ε^2 if $n = 1$ or $n = 2$ with $\tilde{\gamma} < 2$ (see [14 and 7]). Using a proof following the lines of Theorem 2.5 in [13], we prove that $\mathscr{A}_\varepsilon$ is uniformly bounded with respect to ε in the space Y_ε^2.

Theorem 3.4. *Assume that Ω satisfies the hypothesis* (H) *and fix $\varepsilon_0 > 0$. Then there exist a constant $K > 0$ and, for any $r_1 > 0$, $r_2 > 0$, two positive constants $K_1^*(r_1)$, $K_2^*(r_1, r_2)$ such that, for $0 \leq \varepsilon \leq \varepsilon_0$, any solution $U(t) = (u(t), u_t(t))$ of* $(1.16)_\varepsilon$ *with $\|U(0)\|_{Y_\varepsilon^i} \leq r_i$, $i = 1, 2$, satisfies the following estimate for $t \geq 0$,*

$$\|u_{tt}\|_{X_\varepsilon}^2 + \|U(t)\|_{Y_\varepsilon^2}^2 \leq K_1^*(r_1) + K_2^*(r_1, r_2)e^{-Kt}. \tag{3.12}$$

In particular, the system $(1.16)_\varepsilon$ *is bounded dissipative in Y_ε^2 uniformly in ε, in the sense that there is a constant $K_3 > 0$ such that, for any bounded set B in Y_ε^2, there is a constant $t_0^\varepsilon = t_0(B, \varepsilon)$ such that*

$$\|T_\varepsilon(t)U_0\|_{Y_\varepsilon^2} \leq K_3 \quad \text{for } t \geq t_0^\varepsilon, \ U_0 \in B.$$

Using the invariance property of the attractors $\mathscr{A}_\varepsilon$ (or $\mathscr{A}_\varepsilon^2$), we deduce from Theorem 3.4 the following result

Corollary 3.5. *Fix $\varepsilon_0 > 0$. For $0 \leq \varepsilon \leq \varepsilon_0$, the following estimates hold,*

$$\|(\varphi, \psi)\|_{Y_\varepsilon^2} \leq K_3 \quad \text{for } (\varphi, \psi) \in \mathscr{A}_\varepsilon \text{ if } n = 1 \text{ or } n = 2 \text{ and } \tilde{\gamma} < 2, \tag{3.13}$$

$$\|(\varphi, \psi)\|_{Y_\varepsilon^2} \leq K_3 \quad \text{for } (\varphi, \psi) \in \mathscr{A}_\varepsilon^2 \text{ if } n = 2, \ \tilde{\gamma} = 2. \tag{3.14}$$

Proof of Theorem 3.4. By hypotheses (1.4) and (1.5), the mapping $f\colon w \in X_\varepsilon^1 \mapsto f(w) \in X_\varepsilon$ is a C^1-mapping. Moreover, we show below that, for $w \in X_\varepsilon^2$,

$$\|f'(w)\|_{\mathscr{L}(X_\varepsilon; X_\varepsilon)} \leq c(1 + \|w\|_{X_\varepsilon^1}^{\tilde{\gamma}/2}\|w\|_{X_\varepsilon^2}^{\tilde{\gamma}/2}). \tag{3.15}$$

In fact, $\|f'(w)\|_{\mathscr{L}(X_\varepsilon, X_\varepsilon)} \leq c(1 + \|w\|_{L^\infty(Q)}^{\tilde{\gamma}})$. Using the Gagliardo-Nirenberg inequality and (1.15), we have

$$\|w\|_{L^\infty(Q)} \leq c\|w\|_{H^1(Q)}^{1/2}\|w\|_{H^2(Q)}^{1/2} \leq c\|w\|_{X_\varepsilon^1}^{1/2}\|w\|_{X_\varepsilon^2}^{1/2},$$

which gives (3.15).

If $U_0 = (u_0, u_1)$, $\|U_0\|_{Y_\varepsilon^i} \leq r_i$, $i = 1, 2$, then the solution $(u(t), u_t(t)) = T_\varepsilon(t)U_0$ belongs to $C^0([0, \infty); Y_\varepsilon^2)$, the function $f'(u)u_t$ belongs to $C^0([0, \infty); X_\varepsilon)$ and one may consider the following linear hyperbolic equation

$$z_{tt} + \beta z_t + A_\varepsilon z = -f'(u)u_t \tag{3.16}$$

with $z(0) = u_1$, $z_t(0) = -f(u_0) - G_\varepsilon - u_1 - A_\varepsilon u_0$. There is a unique solution $Z(t) = (z(t), z_t(t))$ of (3.16) which belongs to $C^0([0, \infty), Y_\varepsilon^1)$ (see, for example, [19, Chapter 3, §8.4 or 23, Chapter 3]). It is easily seen that $Z(t) \equiv (z(t), z_t(t)) = (u_t, u_{tt})$.

Our next objective is to obtain a bound on $\|f'(u)u_t\|_{X_\varepsilon}$ using (3.15) applied to $w = u(t)$. To estimate $\|u(t)\|_{X_\varepsilon^2}$, we use the fact that $A_\varepsilon u = -f(u) - G_\varepsilon - \beta u_t - u_{tt}$ to obtain

$$\|u(t)\|_{X_\varepsilon^2} \leq c(\|f(u(t))\|_{X_\varepsilon}^2 + \|G_\varepsilon\|_{X_\varepsilon}^2 + \beta\|z(t)\|_{X_\varepsilon}^2 + \|z_t(t)\|_{X_\varepsilon}^2)^{1/2}. \tag{3.17}$$

Using (1.5) and the continuous imbedding of $H^1(Q)$ into $L^6(Q)$, we have, for $t \geq 0$,

$$\|f(u(t))\|_{X_\varepsilon}^2 \leq c(1 + \|u(t)\|_{X_\varepsilon}^6). \tag{3.18}$$

If we recall that $z(t) = u_t(t)$, then property (3.15) with $w = u(t)$ together with the properties (3.11), (3.17) and (3.18) imply, for $t \geq 0$,

$$\|f'(u(t))z(t)\|_{X_\varepsilon}^2 \leq K_3^*(r_1)\|z(t)\|_{X_\varepsilon}^2(1 + \|z(t)\|_{X_\varepsilon}^2 + \|z_t(t)\|_{X_\varepsilon}^2) \tag{3.19}$$

where $K_3^*(r_1)$ is a constant depending only on r_1, β_0.

Let $V_\varepsilon(\varphi, \psi)$ be the energy functional on Y_ε^1 defined by (2.6) with b satisfying (3.2). Since $(z(0), z_t(0))$ belongs to Y_ε^1, the Lemma 2.2 implies that, for $t \geq 0$,

$$\begin{aligned}\frac{d}{dt}V_\varepsilon(z, z_t) &\leq -\frac{\beta}{2}\|z_t\|_{X_\varepsilon}^2 - b\|z\|_{X_\varepsilon^1}^2 + \|f'(u)z\|_{X_\varepsilon}(2b\|z\|_{X_\varepsilon} + \|z_t\|_{X_\varepsilon})\\ &\leq -\frac{b}{2}(\|z_t\|_{X_\varepsilon}^2 + \|z\|_{X_\varepsilon^1}^2) + \left(\frac{1}{\beta} + \frac{8b}{3\lambda_{1,0}}\right)\|f'(u)z\|_{X_\varepsilon}^2,\end{aligned}$$

where all functions are evaluated at t. This inequality together with (3.19) and (2.4) gives

$$\frac{d}{dt}V_\varepsilon(z, z_t) \leq \left(-\frac{2b}{3} + K_4^*(r_1)\|z\|_{X_\varepsilon}^2\right)V_\varepsilon(z, z_t) + K_4^*(r_1)\|z\|_{X_\varepsilon}^2,$$

where $K_4^*(r_1)$ is a positive constant depending only on r_1 and β_0. Integrating this differential inequality, we obtain, for $t \geq 0$,

$$\begin{aligned}V_\varepsilon(z(t), z_t(t)) &\leq e^{-2bt/3}e^{\int_0^\infty K_4^*(r_1)\|z(s)\|_{X_\varepsilon}^2\,ds}V_\varepsilon(z(0), z_t(0))\\ &\quad + K_4^*(r_1)e^{\int_0^\infty K_4^*(r_1)\|z(s)\|_{X_\varepsilon}^2\,ds}\cdot\int_0^\infty \|z(s)\|_{X_\varepsilon}^2\,ds.\end{aligned}$$

Since $\int_0^\infty \|z(s)\|_{X_\varepsilon}^2\,ds < \infty$ by (3.10), the above inequality together with Lemma 2.1 implies that

$$\|Z(t)\|_{Y_\varepsilon^1} \leq K_5^*(r_1)[\|Z(0)\|_{Y_\varepsilon^1}e^{-2bt/3} + 1],$$

where $K_5^*(r_1)$ is a positive constant depending only on r_1 and β_0. Using this inequality as well as (3.17) and (3.18) and the definition of $Z(0)$, we infer that, for $t \geq 0$,

$$\begin{aligned}&\|u_{tt}\|_{X_\varepsilon}^2 + \|U(t)\|_{Y_\varepsilon^2}^2 \leq K_6^*(r_1)\\ &\quad + K_7^*(r_1)e^{-2bt/3}[\|u_1\|_{X_\varepsilon^1}^2 + \|G_\varepsilon\|_{X_\varepsilon}^2 + \|u_1\|_{X_\varepsilon}^2 + \|A_\varepsilon u_0\|_{X_\varepsilon}^2 + \|f(u_0)\|_{X_\varepsilon}^2],\end{aligned} \tag{3.20}$$

where $K_6^*(r_1)$ and $K_7^*(r_1)$ are positive constants depending only on r_1 and β_0. Now inequality (3.12) is a direct consequence of (3.20). The bounded dissipativeness in Y_ε^2 is a consequence of (3.12) and Theorem 3.1. This completes the proof of Theorem 3.4.

4. The attractor for critical exponents

In this section, we prove Theorem 1.2 about the existence and properties of the attractor in the critical case $n = 2$, $\tilde{\gamma} = 2$. For any $u \in L^2(Q)$, let

$$Mu = \int_0^1 u(x, y)\,dy. \tag{4.1}$$

Lemma 4.1 [12, Lemma 3.1]. *If u belongs to $H^j(Q)$, $j \geq 0$, then Mu belongs to $H^j(\Omega)$ and*

$$\|Mu\|_{H^j(\Omega)} \leq \|u\|_{H^j(Q)}. \tag{4.2}$$

Moreover, there is a positive constant C such that, for each $\varepsilon \geq 0$, we have
(i) *for any $u \in H^1(Q)$,*

$$\|u - Mu\|_{X_\varepsilon} \leq C\varepsilon \|u\|_{X_\varepsilon^1} \tag{4.3}$$

(ii) *for any $u \in H^2(Q)$ with $\partial u(x, 0)/\partial y = 0$,*

$$\|u - Mu\|_{X_\varepsilon} + \varepsilon \|u - Mu\|_{X_\varepsilon^1} \leq C\varepsilon^2 \|u\|_{X_\varepsilon^2}. \tag{4.4}$$

We will need the following interesting result showing that the embedding constant for the space $\{w \in X_\varepsilon^1 : Mw = 0\}$ into $L^6(Q)$ approaches zero as $\varepsilon \to 0$.

Proposition 4.2. *If $n = 2$, there is a constant $c > 0$, independent of ε such that, for any $w \in X_\varepsilon^1$ with $Mw = 0$, we have*

$$\|w\|_{L^6(Q)} \leq c\varepsilon^{1/3} \|w\|_{X_\varepsilon^1}. \tag{4.5}$$

Proof. We follow the proof of the Sobolev embedding theorem given in [1, Chapter V]. We denote by $x = (x_1, x_2, x_3)$ the points of Q (instead of $x = (x_1, x_2, y)$ as before). Since Ω has the cone property, by Theorem 4.8 of [1], Ω may be expressed as a union of finitely many subdomains each of which has the strong local Lipschitz property (and therefore the segment property) and each of which is itself a union of parallel translates of a corresponding parallelepiped. As we want, at first, to show that, for any $u \in H^1(Q)$,

$$\|u\|_{L^6(Q)} \leq c \left(\|u\|_{L^2(Q)} + \left\| \frac{\partial u}{\partial x_3} \right\|_{L^2(Q)} \right)^{1/3} \|u\|_{H^1(Q)}^{2/3} \tag{4.6}$$

it is sufficient to assume that Ω is one of these subdomains. By Theorem 3.35 of [1] and a suitable nonsingular linear transformation, we may assume that the parallelepiped involved is, in fact, a square S having edge length 1 unit and having edges parallel to the coordinates axes e_1, e_2. Accordingly, we assume hereafter that $\Omega = \bigcup_{(x_1, x_2) \in B} ((x_1, x_2) + S)$ where B is a subset of Ω and that Ω has the segment property. Therefore, we have

$$Q = \bigcup_{x \in B \times (0,1)} (x + S \times (0, 1)).$$

We point out that we have not made any change of variables in the x_3 direction. Of course, it is sufficient to establish (4.6) for u in $C^\infty(\overline{Q})$. For $x \in Q$, let $w_i(x)$ denote the intersection of Q with the straight line through x parallel to the x_i coordinate axis. Clearly, $w_i(x)$ contains a segment of length $\frac{1}{2}$ with one endpoint at x, say the segment $x + te_i$, $0 \leq t < \frac{1}{2}$, where e_i is a unit vector along the x_i-axis. Integration by parts gives, for $u \in C^\infty(\overline{Q})$,

$$\int_0^{1/2} u\left(x + \left(\frac{1}{2} - t\right) e_i\right)^4 dt$$
$$= \frac{1}{2}|u(x)|^4 - 4 \int_0^{1/2} t \left(u\left(x + \left(\frac{1}{2} - t\right) e_i\right)\right)^3 \frac{d}{dt}\left(u\left(x + \left(\frac{1}{2} - t\right) e_i\right)\right) dt.$$

If we let $\hat{x}_1 = (x_2, x_3)$, $\hat{x}_2 = (x_1, x_3)$, $\hat{x}_3 = (x_1, x_2)$ and set

$$F_i(\hat{x}_i) = \sup_{z \in w_i(x)} |u(z)|^2,$$

then we obtain from the last inequality:

$$|F_i(\hat{x}_i)|^2 \leq 2\int_{w_i(x)} |u(x)|^4\,dx_i + 4\int_{w_i(x)} |u(x)|^3|D_i u(x)|dx_i.$$

Integrating over Q_i, the projection of Q onto the plane $x_i = 0$, now leads to

$$\int_{Q_i} |F_i(\hat{x})|^2\,d\hat{x} \leq 2\int_Q |u(x)|^4\,dx + 4\int_Q |u(x)|^3|D_i u(x)|\,dx.$$

An application of Hölder's inequality gives

$$\int_{Q_i} |F_i(\hat{x})|^2\,dx \leq 4\left[\int_Q (|u(x)| + |D_i u(x)|)^2\,dx\right]^{1/2} \times \left[\int_Q |u(x)|^6\,dx\right]^{1/2}.$$

By Lemma 5.9 of [1], we can write

$$\begin{aligned}\|u\|^6_{L^6(Q)} &= \int_Q |u(x)|^6\,dx \leq \int_Q \prod_{i=1}^3 F_i(\hat{x}_i)\,dx \leq \prod_{i=1}^3 \|F_i\|_{L^2(Q_i)} \\ &\leq 8 \times 2^{3/4} \prod_{i=1}^3 \left(\int_Q (|u(x)|^2 + |D_i u(x)|^2)\,dx\right)^{1/4} \|u\|^{9/2}_{L^6(Q)}\end{aligned}$$

which becomes

$$\|u\|_{L^6(Q)} \leq 8\left(\|u\|^2_{L^2(Q)} + \|\partial u/\partial x_3\|^2_{L^2(Q)}\right)^{1/2 \times 1/3} \|u\|^{2/3}_{H^1(Q)} \tag{4.7}$$

which gives the estimate (4.6).

Let now consider an element $w \in X^1_\varepsilon$ such that $Mw = 0$. By Lemma 4.1, we have

$$\|w\|_{L^2(Q)} + \|\partial w/\partial x_3\|_{L^2(Q)} \leq C\varepsilon\|w\|_{X^1_\varepsilon}. \tag{4.8}$$

The proposition is a direct consequence of (4.6) and (4.8).

Proof of Theorem 1.2. To simplify notation, we let c denote a generic constant independent of ε, β, $0 < \varepsilon \leq \varepsilon_0$, $\beta \geq \beta_0$ where ε_0, β_0 are given positive constants. Let V^0_ε be the energy function used in the proof of Lemma 3.3. Let K_0 be as in Theorem 3.1 and choose R_1 so large that the set

$$\mathscr{U}_1 = \{(\varphi, \psi) \in Y^1_\varepsilon : V^0_\varepsilon(\varphi, \psi) \leq R_1\} = (V^0_\varepsilon)^{-1}(R_1)$$

contains the ball $B_{K_0} \equiv \{(\varphi, \psi) : \|(\varphi, \psi)\|_{Y^1_\varepsilon} \leq K_0\}$. The set $\mathscr{U}_1$ is positively invariant and is contained in a ball $B_{R_2} \subset Y^1_\varepsilon$.

We show that the global attractor $\mathscr{A}_\varepsilon$ exists by showing that $T_\varepsilon(t)$ is an α-contraction on $\mathscr{U}_1$ and then $\mathscr{A}_\varepsilon$ is the ω-limit set of $\mathscr{U}_1$ (see [11]). We use the method of [21] (see also [11]) to show that $T_\varepsilon(t)$ is an α-contraction.

We first estimate the norm of $f(u + \delta u) - f(u)$ in X_ε for $\|u + \delta u\|_{X^1_\varepsilon} \leq R_2$, $\|u\|_{X^1_\varepsilon} \leq R_2$. If $u = v + w$, $\delta u = \delta v + \delta w$, with $v, \delta v \in MX^1_\varepsilon$, w,

$\delta w \in (I-M)X_\varepsilon^1$, then using a Taylor formula, hypothesis (1.5) with $\tilde{\gamma} = 2$ and Hölder inequalities, we obtain

$$\begin{aligned}\|f(u+\delta u) - f(u)\|_{X_\varepsilon}^2 &= \left\| \int_0^1 f'(u+s\delta u)\delta u \, ds \right\|_{X_\varepsilon}^2 \\ &\leq c \int_Q (1 + |u+\delta u|^4 + |u|^4)(\delta u)^2 \, dx \, dy \\ &\leq c[\|\delta u\|_{L^2(Q)}^2 + (\|w+\delta w\|_{L^6(Q)}^4 + \|w\|_{L^6(Q)}^4)\|\delta u\|_{L^6(Q)}^2 \\ &\quad + (\|v+\delta v\|_{L^{12}(\Omega)}^4 + \|v\|_{L^{12}(\Omega)}^4)\|\delta u\|_{L^6(Q)}\|\delta u\|_{L^2(Q)}].\end{aligned}$$

Using the continuous embedding of $H^1(Q)$ into $L^6(Q)$ and $H^1(\Omega)$ into $L^{12}(\Omega)$ together with Proposition 4.2 and the fact that $\|u+\delta u\|_{X_\varepsilon^1} \leq R_2$ and $\|u\|_{X_\varepsilon^1} \leq R_2$, we prove the existence of constants c and $C^*(R_2)$ such that

$$\begin{aligned}&\|f(u+\delta u) - f(u)\|_{X_\varepsilon} \\ &\qquad \leq c\|\delta u\|_{X_\varepsilon} + C^*(R_2)[\varepsilon^{2/3}\|\delta u\|_{X_\varepsilon^1} + \|\delta u\|_{X_\varepsilon^1}^{1/2}\|\delta u\|_{X_\varepsilon}^{1/2}].\end{aligned} \tag{4.9}$$

Let $V_\varepsilon(\varphi, \psi)$ be defined by (2.6) and choose b as in the proof of Theorem 3.1 satisfying (3.2). Let $U(t) + \delta U(t)$ and $U(t)$ be solutions of $(1.16)_\varepsilon$ in $\mathscr{U}_1$, with initial data $U_0 + \delta U_0$ and U_0, respectively. The function $\delta u(t)$ satisfies the equation $(2.1)_\varepsilon$ with $h = -(f(u+\delta u) - f(u))$. To apply Lemma 2.2, we use (4.9) to obtain the following estimate

(4.10)

$$\begin{aligned}&\|f(u+\delta u) - f(u)\|_{X_\varepsilon}(2b\|\delta u\|_{X_\varepsilon} + \|(\delta u)_t\|_{X_\varepsilon}) \\ &\quad \leq \left(\frac{3b}{8} + 3\frac{C^*(R_2)^2\varepsilon^{4/3}}{\beta}\right)\|\delta u\|_{X_\varepsilon^1}^2 + \frac{\beta}{4}\|(\delta u)_t\|_{X_\varepsilon}^2 \\ &\qquad + \left(2bc + \frac{3c^2}{\beta} + C^*(R_2)^2\left(8b\varepsilon^{4/3} + \frac{16b}{\sqrt{3\lambda_{1,0}}} + \frac{36C^*(R_2)^2}{\beta^2 b}\right)\right)\|\delta u\|_{X_\varepsilon}^2.\end{aligned}$$

We now choose ε_1 so that

$$\varepsilon_1^{4/3} \leq \frac{b\beta}{24C^*(R_2)^2} \tag{4.11}$$

We note that we can always choose b so that $b\beta \geq c(\beta_0)$ for all $\beta \geq \beta_0$, where $c(\beta_0)$ is a positive constant depending only on β_0 and $\lambda_{1,0}$. Therefore, (4.11) can be satisfied by $\varepsilon_1 = \varepsilon_1(\beta_0)$. Likewise, we can show that the coefficient of $\|\delta u\|_{X_\varepsilon}^2$ in (4.10) can be bounded, for $\beta \geq \beta_0$, by a constant $c_1(\beta_0)$ depending only on β_0. If we now apply Lemma 2.2, taking into account the estimate (4.10), we have, for $0 < \varepsilon \leq \varepsilon_1(\beta_0)$, for $\beta \geq \beta_0$, and for $t \geq 0$, the following inequality

$$\frac{d}{dt}V_\varepsilon(\delta u, (\delta u)_t) \leq -\frac{\beta}{4}\|(\delta u)_t\|_{X_\varepsilon}^2 - \frac{b}{2}\|\delta u\|_{X_\varepsilon^1}^2 + bc_1(\beta_0)\|\delta u\|_{X_\varepsilon}^2.$$

Using (2.3) and the inequality (2.4) of Lemma 2.1, this implies that

$$\frac{d}{dt}V_\varepsilon(\delta u, (\delta u)_t) \leq -\frac{2b}{3}V_\varepsilon(\delta u, (\delta u)_t) + bc_1(\beta_0)\|\delta u\|_{X_\varepsilon}^2.$$

Integrating this inequality and using (2.4) of Lemma 2.1 again, we infer that, for $t \geq 0$, for $\beta \geq \beta_0$, and for $0 < \varepsilon \leq \varepsilon_1(\beta_0)$,

$$\|\delta U(t)\|_{Y_\varepsilon^1}^2 \leq 4e^{-2bt/3}\|\delta U_0\|_{Y_\varepsilon^1}^2 + \rho^t(U + \delta U, U)$$

where

$$\rho^t(U + \delta U, U) = \sup_{0 \leq s \leq t} (6c_1(\beta_0)\|\delta u(s)\|_{X_\varepsilon}^2).$$

We now show that $\rho^t(\cdot, \cdot)$ is a compact pseudo-metric on Y_ε^1. It is obviously a pseudo-metric. To show that it is compact, suppose that U_{n0} is a bounded sequence in Y_ε^1 and let $U_n(t) = T_\varepsilon(t)U_{n0}$. Since positive orbits of bounded sets are bounded, the sequence $U_n(t)$ is bounded in Y_ε^1 uniformly in t. This implies that $\bigcup_{t \in R^+} \bigcup_{n \geq 0} u_n(t)$ is precompact in X_ε and the family of mappings $u_n(\cdot) \in C^0(R^+; X_\varepsilon)$, $n \geq 0$, is equicontinuous from R^+ into X_ε. This is enough to imply by the Arzela-Ascoli theorem that ρ^t is precompact.

If we choose t_1 so that $2e^{-bt_1/3} < 1$, then $T_\varepsilon(t)|\mathscr{U}_1$ is an α-contraction for $t \geq t_1$ (see [21] or [11, p. 16]). This completes the proof of Theorem 1.2.

Remark 4.3. As we have remarked in the introduction, if Ω satisfies hypothesis (H) and if f is a C^2-function, then, in the case $n = 2$, $\tilde{\gamma} = 2$, there is a global attractor $\mathscr{A}_\varepsilon^2$ in Y_ε^2 for $T_\varepsilon(t)$. Obviously, $\mathscr{A}_\varepsilon^2 \subset \mathscr{A}_\varepsilon$. Since $T_\varepsilon(t)$ is a gradient system, if all of the equilibrium points are hyperbolic, then they are finite in number, say N_0, and bounded in Y_ε^2 (since (H) holds). Moreover,

$$\mathscr{A}_\varepsilon = \bigcup_{1 \leq i \leq N_0} W_\varepsilon^u(\varphi_i, 0)$$

where $W_\varepsilon^u(\varphi_i, 0)$ is the unstable manifold of the equilibrium point $(\varphi_i, 0)$. Using Lemma 6.7 of [2], one easily shows that $W_\varepsilon^u(\varphi_i, 0) \subset Y_\varepsilon^2$ and hence $\mathscr{A}_\varepsilon$ is bounded in Y_ε^2. Therefore $\mathscr{A}_\varepsilon \subset \mathscr{A}_\varepsilon = \mathscr{A}_\varepsilon^2$.

Proof of Theorem 1.3. Assume that $g(x, \varepsilon) = \varepsilon$. If $u = v + w$ in $(1.16)_\varepsilon$, $v = Mu$, $w = (I - M)u$, then v, w satisfy the equations

$$\text{(4.12)} \quad \begin{cases} \text{(i) } v_{tt} + \beta v_t + A_0 v = -f(v) - M[f(v + w) - f(v)] - MG_\varepsilon, \\ \text{(ii) } w_{tt} + \beta w_t + A_\varepsilon w = -(I - M)[f(v + w) - f(v)] - (I - M)G_\varepsilon, \end{cases}$$

with initial data given, respectively, by $(v_0, v_1) = (Mu_0, Mu_1)$ and $(w_0, w_1) = ((I - M)u_0, (I - M)u_1)$.

We recall that $A_\varepsilon v = A_0 v$. There is positive constant k_3 such that the first eigenvalue $\nu_{1,\varepsilon}$ of $A_\varepsilon|(I - M)$ satisfies

$$\text{(4.13)} \qquad \nu_{1,\varepsilon} \geq k_3/\varepsilon^2.$$

We use the same notation as in the proof of Theorem 1.2, restricting the solution $U(t)$ to $\mathscr{U}_1$ and selecting the ball B_{R_2} in Y_ε^1 so that $\mathscr{U}_1 \subset B_{R_2}$. Let $V_\varepsilon(\varphi, \psi)$ be defined by (2.6) and choose b as in the proof of Theorem 3.1. If $(u_0, u_1) \in \mathscr{U}_1$, then $(u(t), u_t(t)) \in \mathscr{U}_1 \subset B_{R_2}$ and $(w(t), w_t(t))$ belongs to B_{R_2} and is a solution of (4.12)(ii). To apply Lemma 2.2 to equation (4.12)(ii), we must estimate the quantity

$$P(t) \equiv [\|(I - M)[f(v + w) - f(v)]\|_{X_\varepsilon} + \|(I - M)G_\varepsilon\|_{X_\varepsilon}](2b\|w\|_{X_\varepsilon} + \|w_t\|_{X_\varepsilon}).$$

If we use (4.9), (4.11), (4.13) and considerations as in the proof of Theorem 1.2 (see (4.10)), we obtain the following estimate

$$\begin{aligned} P(t) \le & \left(\frac{3b}{8} + \frac{3C^*(R_2)^2\varepsilon^{4/3}}{\beta}\right)\|w\|^2_{X^1_\varepsilon} + \frac{\beta}{4}\|w_t\|^2_{X_\varepsilon} \\ & + \left(2bc + \frac{3c^2}{\beta} + C^*(R_2)^2\left(8b\varepsilon^{4/3} + \frac{16b}{\sqrt{3\lambda_{1,0}}} + \frac{36C^*(R_2)^2}{\beta^2 b}\right)\right)\|w\|^2_{X_\varepsilon} \\ & + \frac{b}{8}\|w\|^2_{X^1_\varepsilon} + \frac{\beta}{8}\|w_t\|^2_{X_\varepsilon} + \left(\frac{2}{\beta} + \frac{16b\varepsilon^2}{k_3}\right)\|(I-M)G_\varepsilon\|^2_{X_\varepsilon}. \end{aligned}$$

In the proof of Theorem 1.2, we have already remarked that we can choose b satisfying (3.2) such that $b\beta \ge c(\beta_0) > 0$. Taking into account this remark, we infer from the above inequality that, for $0 < \varepsilon \le \varepsilon_1(\beta_0)$,

(4.14)
$$\begin{aligned} P(t) \le & \frac{5b}{8}\|w\|^2_{X^1_\varepsilon} + \frac{3\beta}{8}\|w_t\|^2_{X_\varepsilon} \\ & + b\left(2c + \frac{3c^2}{c(\beta_0)} + C^*(R_2)^2\left(8\varepsilon^{4/3} + \frac{16}{\sqrt{3\lambda_{1,0}}} + \frac{36C^*(R_2)^2}{c(\beta_0)^2}\right)\right)\|w\|^2_{X_\varepsilon} \\ & + \left(\frac{2}{\beta} + \frac{16b\varepsilon^2}{k_3}\right)\|(I-M)G_\varepsilon\|^2_{X_\varepsilon}. \end{aligned}$$

From (4.13), it follows that $\|w\|^2_{X_\varepsilon} \le (\varepsilon^2/k_3)\|w\|^2_{X^1_\varepsilon}$. From this inequality and (4.14), we deduce that there exists a positive constant $\varepsilon_2(\beta_0) \le \varepsilon_1(\beta_0)$ such that, for $\beta \ge \beta_0$, for $0 < \varepsilon \le \varepsilon_2(\beta_0)$, and for $t \ge 0$, we have,

$$P(t) \le \frac{3b}{4}\|w\|^2_{X^1_\varepsilon} + \frac{3\beta}{8}\|w_t\|^2_{X_\varepsilon} + \frac{(1+\varepsilon^2)c}{\beta}\|(I-M)G_\varepsilon\|^2_{X_\varepsilon}. \tag{4.15}$$

If we now apply Lemma 2.2, making use of (4.15) and Lemma 2.1, we deduce that, for $t \ge 0$,

$$\frac{d}{dt}V_\varepsilon(w, w_t) \le -\tfrac{1}{3}bV_\varepsilon(w, w_t) + \frac{(1+\varepsilon^2)c}{\beta}\|(I-M)G_\varepsilon\|^2_{X_\varepsilon}. \tag{4.16}$$

Since $(I-M)G_\varepsilon = (I-M)(G_\varepsilon - G_0)$ and

$$G_\varepsilon(x, y) - G_0(x, y) = yg(x, \varepsilon)\int_0^1 \frac{\partial G}{\partial Y}(x, sg(x, \varepsilon)y)\,ds,$$

we conclude that

$$\|(I-M)G_\varepsilon\|_{X_\varepsilon} \le c\varepsilon. \tag{4.17}$$

Integrating the inequality (4.16) from 0 to t and using (4.17) together with (2.4) of Lemma 2.1, we obtain, for $0 < \varepsilon \le \varepsilon_2(\beta_0)$, for $t \ge 0$,

$$\|W(t)\|^2_{Y^1_\varepsilon} \le 3e^{-bt/3}\|W(0)\|^2_{X^1_\varepsilon} + c_2(\beta_0)\varepsilon^2 \tag{4.18}$$

where $W(t) = (w(t), w_t(t))$.

From (4.18) and the invariance of the attractor $\mathscr{A}_\varepsilon$, it follows that

$$\|(I-M)(\varphi, \psi)\|^2_{Y^1_\varepsilon} \le c_2(\beta_0)\varepsilon^2 \quad \text{if } (\varphi, \psi) \in \mathscr{A}_\varepsilon. \tag{4.19}$$

Let us now suppose that $U(t) = T_\varepsilon(t)U_0 = V(t) + W(t)$ belongs to $\mathscr{U}_1$ and that $\|W(t)\|^2_{Y^1_\varepsilon} \leq c_2(\beta_0)\varepsilon^2$ for all t. Then, by (4.18),

$$\|(I - M)U(t)\|^2_{Y^1_\varepsilon} \leq 4c_2(\beta_0)\varepsilon^2. \tag{4.20}$$

Let T be a positive constant. We now want to estimate the term

$$\|T_\varepsilon(t)U_0 - T_0(t)MU_0\|_{Y^1_\varepsilon} \quad \text{for } 0 \leq t \leq T.$$

If $T_0(t)MU_0 = V_0(t) \equiv (v_0(t), v_{0t}(t))$ and $Z(t) \equiv (z(t), z_t(t)) = V(t) - V_0(t)$, then $z(t)$ is a solution of the equation

$$z_{tt} + \beta z_t + A_0 z = -M(f(u) - f(v_0)) - M(G_\varepsilon - G_0) \tag{4.21}$$

Taking the inner product of (4.21) by z_t, using the equality (2.8) and arguing as in [23, Chapter IV, §1], we prove that, for $t \geq 0$,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|z_t\|^2_{L^2(\Omega)} + \beta\|z_t\|^2_{L^2(\Omega)} + \frac{1}{2}\frac{d}{dt}\|z\|^2_{H^1(\Omega)} \\ = -(M(f(u) - f(v_0)) + M(G_\varepsilon - G_0), z_t)_{X_\varepsilon}.\end{aligned}$$

Using (4.9) and (4.20), together with the fact that $\|G_\varepsilon - G_0\|_{X_\varepsilon} \leq c\varepsilon$, we deduce that, for $T \geq 0$,

$$\frac{d}{dt}\|z\|^2_{L^2(\Omega)} + \frac{d}{dt}\|z\|^2_{H^1(\Omega)} \leq \frac{\widetilde{C}(R_2)}{\beta}[\|z\|^2_{H^1(\Omega)} + c_3(\beta_0)\varepsilon^2] + \frac{c_3(\beta_0)\varepsilon^2}{\beta}.$$

Integrating this inequality from 0 to t and using (4.20) again, we see that there is a constant $K(R_2, T, \beta_0)$ that depends only on R_2, T and β_0, such that, for $0 \leq t \leq T$,

$$\|T_\varepsilon(t)U_0 - T_0(t)MU_0\|^2_{Y^1_\varepsilon} \leq \frac{K(R_2, T)\varepsilon^2}{\beta} + 4c_2(\beta_0)\varepsilon^2. \tag{4.22}$$

Using the same type of argument as in [12] (see also the proof of Theorem 1.1), the upper semicontinuity of $\mathscr{A}_\varepsilon$ at $\varepsilon = 0$ follows from the attractivity property of $\mathscr{A}_0$ and the estimate (4.22).

Now suppose that $G(X, Y) = G_0(X)$. Then $(I - M)G_\varepsilon = 0$ and the inequality (4.16) implies that $\|W(t)\|_{Y^1_\varepsilon}$ approaches zero exponentially as $t \to \infty$. Thus, $(I - M)(\varphi, \psi) = 0$ for all $(\varphi, \psi) \in \mathscr{A}_\varepsilon$. Thus, $\mathscr{A}_\varepsilon \subset \mathscr{A}_0$. Since $\mathscr{A}_0$ is contained in $\mathscr{A}_\varepsilon$, we have $\mathscr{A}_\varepsilon = \mathscr{A}_0$ and Theorem 1.3 is proved.

5. Upper semicontinuity of the attractors

In this section, we prove Theorem 1.1. We need the following result.

Proposition 5.1. *Let $0 < \varepsilon \leq \varepsilon_0$, $\beta > 0$ and assume that Ω satisfies hypothesis* (H). *For any $r > 0$, there is a constant $k(r) > 0$ such that, for any solution $U^\varepsilon(t) = (u^\varepsilon(t), u^\varepsilon_t(t))$ of $(1.16)_\varepsilon$, $U^\varepsilon(0) = U_0$, with $\|U_0\|_{Y^2_\varepsilon} \leq r$, we have, for $t \geq 0$,*

$$\|U^\varepsilon(t) - U^0(t)\|^2_{Y^1_\varepsilon} \leq \varepsilon k(r)e^{k(r)t} \tag{5.1}$$

where $U^0(t) = T_0(t)MU_0$.

Proof. Let $U^0(t) = (u^0(t), u^0_t(t))$. The function $u^0(t)$ satisfies, for all $v_1 \in H^1(\Omega)$,

$$(u^0_{tt}, v_1)_{X_0} + \beta(u^0_t, v_1)_{X_0} + a_0(u^0, v_1) = (-f(u^0) - G_0, v_1)_{X_0}.$$

If $v_1 = gv/\varepsilon g_0$ with v in $H^1(\Omega)$, then

$$\int_\Omega \frac{g}{\varepsilon} u_{tt}^0 v\,dx + \beta \int_\Omega \frac{g}{\varepsilon} u_t^0 v\,dx + \int_\Omega \frac{g}{\varepsilon} \nabla_x u^0 \cdot \nabla_x v\,dx + \alpha \int_\Omega \frac{g}{\varepsilon} u^0 v\,dx$$
$$= \int_\Omega \frac{g}{\varepsilon}(-f(u^0) - G_0)v\,dx - \int_\Omega \frac{g}{\varepsilon} \sum_{i=1}^{2} \left(\frac{g_{x_i}}{g} - \frac{g_{0x_i}}{g_0}\right) u_{x_i}^0 v\,dx.$$

Since Mz belongs to $H^1(\Omega)$ if z is in $H^1(Q)$, and u^0 is independent of y, this equality becomes, for any $z \in H^1(Q)$,

$$(5.2)\qquad \begin{aligned}(u_{tt}^0, z)_{X_\varepsilon} + \beta(u_t^0, z)_{X_\varepsilon} + a_\varepsilon(u^0, z) = (-f(u^0) - G_0, z)_{X_\varepsilon} \\ - \sum_{i=1}^{n} \left(\left(\frac{g_{x_i}}{g} - \frac{g_{0x_i}}{g_0}\right) u_{x_i}^0, z\right)_{X_\varepsilon} - \sum_{i=1}^{n} \left(\frac{g_{x_i}}{g} u_{x_i}^0, y z_y\right)_{X_\varepsilon}.\end{aligned}$$

If we let $z(t) = u^\varepsilon(t) - u^0(t)$, then z_t belongs to $H^1(Q)$ and (5.2) implies, for $t \geq 0$,

$$\begin{aligned}&(z_{tt}, z_t)_{X_\varepsilon} + (\beta z_t, z_t)_{X_\varepsilon} + a_\varepsilon(z, z_t) \\ &\quad = -(f(z + u^0) - f(u^0, z^t)_{X_\varepsilon} - (G_\varepsilon - G_0, z_t)_{X_\varepsilon} \\ &\qquad + \sum_{i=1}^{n} \left(\left(\frac{g_{x_i}}{g} - \frac{g_{0x_i}}{g_0}\right) u_{x_i}^0, z_t\right)_{X_\varepsilon} + \sum_{i=1}^{n} \left(\frac{g_{x_i}}{g} u_{x_i}^0, y z_{yt}\right)_{X_\varepsilon}.\end{aligned}$$

If we use inequalities (4.17), (1.14) and the facts that $G \in W^{1,\infty}(\tilde{Q})$ and $g \in C^3(\overline{\Omega} \times [0, \varepsilon_0]; R)$, we obtain, for $t \geq 0$,

$$(5.3)\qquad \begin{aligned}&\frac{d}{dt}\|z_t\|_{X_\varepsilon}^2 + \beta\|z_t\|_{X_\varepsilon}^2 + \frac{d}{dt}\|z\|_{X_\varepsilon^1}^2 \\ &\quad \leq c[\|f(z + u^0) - f(u^0)\|_{X_\varepsilon}^2 + \varepsilon^2 + \varepsilon^2\|u^0\|_{X_0^1}^2 + \|u^0\|_{X_0^1}\|u_{yt}^\varepsilon\|_{L^2(Q)}].\end{aligned}$$

From Lemma 4.1, we have $\|MU_0\|_{Y_0^2} \leq c\|U_0\|_{Y_\varepsilon^2}$. By Theorem 3.4, this implies that there is a constant $k_1(r)$ such that, for $t \geq 0$, $i = 1, 2$,

$$(5.4)\qquad \|U^\varepsilon(t)\|_{Y_\varepsilon^i} + \|U^0(t)\|_{Y_0^i} \leq k_1(r).$$

The inequality (5.4) also implies that, for $t \geq 0$,

$$(5.5)\qquad \|u_{yt}^\varepsilon(t)\|_{L^2(Q)} \leq c\varepsilon k_1(r).$$

Arguing as in the proof of (4.9) and using (5.4), one shows that there is a constant $k_2(r)$ such that, for $t \geq 0$,

$$(5.6)\qquad \|f(z(t) + u^0(t)) - f(u^0(t))\|_{X_\varepsilon}^2 \leq k_2(r)\|z(t)\|_{X_\varepsilon^1}^2.$$

Integrating (5.3) from 0 to t, and taking into account (5.4) to (5.6), we deduce that there is a constant $k_3(r)$ such that, for $t \geq 0$,

$$(5.7)\qquad \|z_t(t)\|_{X_\varepsilon}^2 + \|z(t)\|_{X_\varepsilon^1}^2 \leq k_3(r)\left[\int_0^t \|z(s)\|_{X_\varepsilon^1}^2\,ds + \|(I - M)U_0\|_{Y_\varepsilon^1}^2 + \varepsilon\right].$$

Remarking that, by (4.3), $\|(I - M)U_0\|_{Y_\varepsilon^1} \leq c\varepsilon$ and applying Gronwall's inequality to (5.7), we obtain (5.1).

Proof of Theorem 1.1. Assume that $n = 1$ or $n = 2$ and $\tilde{\gamma} < 2$. From Corollary 3.5 and the equivalence of norms (1.15), we deduce that there is a constant R such that, for $0 \le \varepsilon \le \varepsilon_0$ and any $(\varphi, \psi) \in \mathscr{A}_\varepsilon$,

$$\begin{aligned} &\|\varphi\|_{H^2(Q)} + \frac{1}{\varepsilon}\|\varphi_y\|_{L^2(Q)} + \frac{1}{\varepsilon}\sum_{i=1}^{n}\|\varphi_{x_i y}\|_{L^2(Q)} + \frac{1}{\varepsilon^2}\|\varphi_{yy}\|_{L^2(Q)} \\ &\quad + \|\psi\|_{H^1(Q)} + \frac{1}{\varepsilon}\|\psi_y\|_{L^2(Q)} \le R. \end{aligned} \tag{5.8}$$

Let $\mathscr{B}_0 = \{(\varphi, \psi) \in Y_0^2 : (\varphi, \psi)$ satisfies (5.8)$\}$. If $(\varphi, \psi) \in \mathscr{A}_\varepsilon$, then, by Lemma 4.1, $(M\varphi, M\psi)$ belongs to $\mathscr{B}_0$. Since $\mathscr{A}_0$ is the attractor of $T_0(t)$ and the norms $\|\cdot\|_{Y_0^1}$ and $\|\cdot\|_{Y_\varepsilon^1}$ are equivalent on Y_0^1 with constants of equivalence independent of ε, for any $\eta > 0$, there is a $\tau_\eta > 0$ such that $T_0(\tau_\eta)\mathscr{B}_0 \subset \mathscr{N}_{Y_\varepsilon^1}(\mathscr{A}_0, \eta/2)$, the $\eta/2$ neighborhood of $\mathscr{A}_0$. If $(\varphi_\varepsilon, \psi_\varepsilon) = T_\varepsilon(\tau_\eta)(\varphi_0, \psi_0)$ belongs to $\mathscr{A}_\varepsilon$, then, due to Proposition 5.1, we obtain

$$\|T_\varepsilon(\tau_\eta)(\varphi_0, \psi_0) - T_0(\tau_\eta)(M\varphi_0, M\psi_0)\|_{Y_\varepsilon^1} \le \varepsilon k(r) e^{k(r)\tau_\eta} \le \eta/2$$

if $0 < \varepsilon \le \varepsilon_1$, with ε_1 small enough. Thus, for $0 < \varepsilon \le \varepsilon_1$, we have $(\varphi_\varepsilon, \psi_\varepsilon) \in \mathscr{N}_{Y_\varepsilon^1}(\mathscr{A}_0, \eta)$ and upper semicontinuity is proved.

The proof is the same in the case $n = 2$, $\tilde{\gamma} = 2$.

6. Other boundary conditions

We do not prove the results stated in the Introduction concerning the problem $(1.10)_\varepsilon$, $(1.11\text{bis})_\varepsilon$ for mixed boundary conditions since they are so similar to the Neumann case. We do point out that by [12, §4] the property (2.2) of the first eigenvalue of A_ε is still true. Likewise, the first eigenvalue $\nu_{1,\varepsilon}$ of the operator $A_\varepsilon|(I - M)\mathscr{D}(\mathscr{A}_\varepsilon)$ satisfies the inequality (4.13).

For the Dirichlet boundary conditions $(1.11\text{ter})_\varepsilon$, Theorem 3.1 and Corollary 3.2 hold and, if Ω satisfies (H), then Theorem 3.4 and Corollary 3.5 hold. The proofs are the same as the ones for the Neumann case with minor modifications if one observes (see [12]) that there is a positive constant k such that

$$\lambda_{1,\varepsilon} \ge k/\varepsilon^2 \ge k/\varepsilon_0^2 \quad \text{for } 0 < \varepsilon \le \varepsilon_0 \tag{6.1}$$

and then replace the condition for b in (2.3) by

$$b < \inf\left(\frac{\beta}{8}, \frac{k}{4\beta\varepsilon_0^2}, \frac{\sqrt{k}}{4\varepsilon_0}\right). \tag{6.2}$$

To prove Theorems 1.4 and 1.5, we need the following Sobolev inequality.

Lemma 6.1. *Fix* $\varepsilon_0 > 0$. *There exists a positive constant* C *such that, for* $0 < \varepsilon \le \varepsilon_0$ *and any* $u \in X_\varepsilon^1$, *we have*

$$\|u\|_{L^6(Q)} \le C\varepsilon^{1/3}\|u\|_{X_\varepsilon^1}. \tag{6.3}$$

Proof. Let us denote by Q_ε^* the open set

$$Q_\varepsilon^* = \{(x, Y) \in R^{n+1} ; x \in \Omega, 0 < Y < \varepsilon\}.$$

If u belongs to X_ε^1, then $\overline{u}(x, Y) = u(x, Y/\varepsilon) = u(x, y)$ belongs to the space $H_0^1(Q_\varepsilon^*)$ and we have

$$\overline{u}_Y = \tfrac{1}{\varepsilon}u_y, \qquad \overline{u}_x = u_x. \tag{6.4}$$

Since Q_ε^* has a Lipschitzian boundary, we can extend the function $\overline{u}$ by 0 to the open set $Q_1^* = Q = \{(x, Y) \in R^{n+1}; x \in \Omega, 0 < Y < 1\}$. We denote by $\tilde{u}$ this extension of $\overline{u}$. On Q_1^*, we have the Sobolev inequality

$$\|\tilde{u}\|_{L^6(Q_1^*)} \leq c\|\tilde{u}\|_{H^1(Q_1^*)}$$

which becomes, by restriction to Q_ε^*,

$$\|\overline{u}\|_{L^6(Q_\varepsilon^*)} \leq c\|\overline{u}\|_{H^1(Q_\varepsilon^*)}. \tag{6.5}$$

Now notice that, by (6.4) and the definition of the norm in X_ε^1, we have

$$\|\overline{u}\|_{L^6(Q_\varepsilon^*)} = \varepsilon^{1/6}\|u\|_{L^6(Q)} \quad \text{and} \quad \|\overline{u}\|_{H^1(Q_\varepsilon^*)} \leq c\varepsilon^{1/2}\|u\|_{X_\varepsilon^1}.$$

These relations and (6.5) imply (6.3).

Proof of Theorem 1.4. As in the proof of Theorem 1.2, we confine our attention to the set

$$\mathscr{U}_1 = \{(\varphi, \psi) \in Y_\varepsilon^1 : V_\varepsilon^0(\varphi, \psi) \leq R_1\}$$

and choose R_2 so that $\mathscr{U}_1 \subset B_{R_2}$, the ball in Y_ε^1 of center zero and radius R_2. As in the proof of Theorem 1.2, we only need to show that $T_\varepsilon(t)|\mathscr{U}_1$ is an α-contraction by using the method of [21] (see also [11]).

We first estimate $\|f(u_1) - f(u_2)\|_{X_\varepsilon}$ when $\|u_i\|_{X_\varepsilon^1} \leq R_2$, $i = 1, 2$. Arguing as in the proof of inequality (4.9) and using a Hölder inequality, we have

$$\begin{aligned}\|f(u_1) - f(u_2)\|_{X_\varepsilon} &\leq c\left(\int_Q (1 + |u_1|^4 + |u_2|^4)(u_1 - u_2)^2\,dx\,dy\right)^{1/2} \\ &\leq c[\|u_1 - u_2\|_{X_\varepsilon} + (\|u_1\|_{L^6(Q)}^2 + \|u_2\|_{L^6(Q)}^1)\|u_1 - u_2\|_{L^6(Q)}].\end{aligned}$$

Thanks to Lemma 6.1 and (6.1), we have

$$\|f(u_1) - f(u_2)\|_{X_\varepsilon} \leq c(1 + K^*(R_2))\varepsilon\|u_1 - u_2\|_{X_\varepsilon^1}. \tag{6.6}$$

Let $V_\varepsilon(\varphi, \psi)$ be defined by (2.6) and let b be a positive number satisfying (6.2). Let $U(t) + \delta U(t)$, $U(t)$ be two solutions of $(1.10)_\varepsilon$, $(1.11\text{ter})_\varepsilon$ which belong to $\mathscr{U}_1$, with initial data $U_0 + \delta U_0$ and U_0, respectively. The function δu satisfies the equation $(2.1)_\varepsilon$ with $h = -(f(u+\delta u) - f(u))$. To apply Lemma 2.2, we use (6.6) to obtain the following estimate

$$\begin{aligned}&\|f(u + \delta u) - f(u)\|_{X_\varepsilon}(2b\|\delta u\|_{X_\varepsilon} + \|(\delta u)_t\|_{X_\varepsilon}) \\ &\quad\leq \|f(u + \delta u) - f(u)\|_{X_\varepsilon}\left(\frac{2b}{\sqrt{\lambda_{1,\varepsilon}}}\|\delta u\|_{X_\varepsilon^1} + \|(\delta u)_t\|_{X_\varepsilon}\right) \\ &\quad\leq \frac{2bc}{\sqrt{k}}\varepsilon^2(1 + K^*(R_2))\|\delta u\|_{X_\varepsilon^1}^2 \\ &\qquad + \frac{c}{2}\varepsilon(1 + K^*(R_2))(\|\delta u\|_{X_\varepsilon^1}^2 + \|(\delta u)_t\|_{X_\varepsilon}^2).\end{aligned} \tag{6.7}$$

For $\beta \geq \beta_0$, there is a positive number $\varepsilon_1 = \varepsilon_1(\beta_0) \leq \varepsilon_0$, such that, for $0 < \varepsilon \leq \varepsilon_1$, we have

$$\begin{aligned}\frac{2bc}{\sqrt{k}}\varepsilon^2(1 + K^*(R_2)) + \frac{c}{2}\varepsilon(1 + K^*(R_2)) &\leq \frac{b}{2}, \\ \frac{c}{2}\varepsilon(1 + K^*(R_2)) &\leq \frac{\beta}{4}.\end{aligned} \tag{6.8}$$

If we now apply Lemma 2.2, take into account (6.7), (6.8), integrate the resulting inequality and use (2.4) of Lemma 2.1, we obtain, for $t \geq 0$, $0 < \varepsilon \leq \varepsilon_1$,

$$\|\delta U(t)\| \leq \sqrt{3} R_2 e^{-bt/3}.$$

Thus, $T_\varepsilon(t)$ is an α-contraction for t large enough. The remainder of the argument is the same as the one in the proof of Theorem 1.2.

Proof of Theorem 1.5. As in the proof of Theorem 1.3 and Theorem 1.4, we consider $T_\varepsilon(t)|\mathscr{U}_1$. Fix $\beta_0 > 0$ and choose $0 < \varepsilon \leq \varepsilon_1$ with ε_1 as in Theorem 1.4. Let b satisfy (6.2) and introduce the functional on $\mathscr{U}_1$

$$V_\varepsilon^*(\varphi, \psi) = \tfrac{1}{2}(\|\psi\|_{X_\varepsilon}^2 + \|\varphi\|_{X_\varepsilon^1}^2) + 2b(\varphi, \psi)_{X_\varepsilon} + (G_\varepsilon + f(0), \varphi)_{X_\varepsilon}.$$

Let $U(t) = (u(t), u_t(t))$ be a solution of $(1.10)_\varepsilon$, $(1.11\text{ter})_\varepsilon$ with initial condition $U_0 = (u_0, u_1) \in \mathscr{U}_1$. Arguing as in the proof of Theorems 3.1. and 1.4, we prove that, for $t \geq 0$,

$$\frac{d}{dt} V_\varepsilon^*(u, u_t) \leq -\frac{\beta}{2}\|u_t\|_{X_\varepsilon}^2 - b\|u\|_{X_\varepsilon^1}^2 + \widetilde{P}(t), \tag{6.9}$$

where

$$\begin{aligned} \widetilde{P}(t) = \|f(u) - f(0)\|_{X_\varepsilon} \left(\frac{2b}{\sqrt{k}}\varepsilon\|u\|_{X_\varepsilon^1} + \|u_t\|_{X_\varepsilon}\right) \\ + \frac{2b}{\sqrt{k}}\varepsilon\|G_\varepsilon + f(0)\|_{X_\varepsilon}\|u\|_{X_\varepsilon^1}. \end{aligned} \tag{6.10}$$

Therefore, for $0 < \varepsilon \leq \varepsilon_1(\beta_0)$, we deduce from (6.7), (6.8), (6.9), (6.10) and (6.2) that, for $t \geq 0$,

$$\frac{d}{dt} V_\varepsilon^*(u, u_t) \leq -\frac{b}{4}(\|u_t\|_{X_\varepsilon}^2 + \|u\|_{X_\varepsilon^1}^2) + \frac{8b}{k}\varepsilon^2\|G_\varepsilon + f(0)\|_{X_\varepsilon}^2. \tag{6.11}$$

If we integrate (6.11) from 0 to t and use (2.4) of Lemma 2.1, we easily deduce the following inequality

$$\|U(t)\|_{Y_\varepsilon^1} \leq \sqrt{3} R_2 e^{-bt/2} + c\varepsilon\|G_\varepsilon + f(0)\|_{X_\varepsilon^1}. \tag{6.12}$$

Inequality (6.12) and the invariance of the attractor implies the first statement in Theorem 1.5. If there is an ε such that $G_\varepsilon + f(0) = 0$, then the same reasoning implies that $\mathscr{A}_\varepsilon = 0$.

7. Further generalizations

The equation $(1.3)_\varepsilon$ was a model equation. It can be replaced by more general equations or even systems. For instance, $(1.6)_\varepsilon$ can be replaced by a system of Sine-Gordon equations on Q_ε, where $k \geq 0$,

$$\begin{cases} \frac{\partial^2 u_1}{\partial t^2} + \beta \frac{\partial u_1}{\partial t} - \Delta u_1 = -\sin u_1 - k(u_1 - u_2) - G_1, \\ \frac{\partial^2 u_2}{\partial t^2} + \beta \frac{\partial u_2}{\partial t} - \Delta u_2 = -\sin u_2 - k(u_2 - u_1) - G_2, \end{cases} \tag{7.1}$$

with Neumann or Dirichlet boundary conditions. In the case of Dirichlet boundary conditions, we have the above results. In the case of Neumann boundary conditions, the above results still hold, the limit equation on Ω being

$$\begin{cases} \frac{\partial^2 v_1}{\partial t^2} + \beta \frac{\partial v_1}{\partial t} - \frac{1}{g_0}\left(\sum_{i=1}^n \frac{\partial}{\partial x_i}\left(g_0 \frac{\partial v_1}{\partial x_i}\right)\right) = -\sin\ v_1 - k(v_1 - v_2) - G_{10}, \\ \frac{\partial^2 v_2}{\partial t^2} + \beta \frac{\partial v_2}{\partial t} - \frac{1}{g_0}\left(\sum_{i=1}^n \frac{\partial}{\partial x_i}\left(g_0 \frac{\partial v_2}{\partial x_i}\right)\right) = -\sin\ v_2 - k(v_2 - v_1) - G_{20}, \end{cases} \tag{7.2}$$

with Neumann boundary conditions, where

$$G_{10}(x) = G_1(x, 0), \qquad G_{20}(x) = G_2(x, 0).$$

In the case of Dirichlet boundary conditions, this is a system occurring in Josephson junctions. Other examples are given in [7] or [23, Chapter IV].

In the proofs of Theorems 3.1 and 3.4, we used the fact that the equation was a gradient system (that is $(F(u))' = f(u)$). However, this property is not essential, and at least in the case $\tilde{\gamma} < 2$, we can generalize the above results to the case where the more general hypotheses of [7, §2.2] hold (see also Example 5.4 of [7]).

The results above in the case of Neumann boundary conditions are generalized to the case of periodic boundary conditions in an obvious way.

Appendix

We recall that, for $0 < \varepsilon \leq \varepsilon_0$, Q_ε denotes the domain

$$Q_\varepsilon = \{(\xi_1, \dots, \xi_{n+1}) \in R^{n+1}; 0 < \xi_{n+1} < g(\xi_1, \dots, \xi_n, \varepsilon), (\xi_1, \dots, \xi_n) \in \Omega\}$$

where Ω is a C^2-polygonal domain in R^n, $n = 1$ or 2, and the function $g: \overline{\Omega} \times [0, \varepsilon_0] \to \mathbf{R}$ is a function of class C^3 satisfying the conditions (1.1). The boundary ∂Q_ε of Q_ε can be written as

$$\partial Q_\varepsilon = \overline{\Gamma}_{0,\varepsilon} \cup \overline{\Gamma}_{1,\varepsilon} \cup \overline{\Gamma}_{2,\varepsilon}$$

where

$$\Gamma_{0,\varepsilon} = \Omega \times \{0\},$$
$$\Gamma_{1,\varepsilon} = \{(\xi_1, \dots, \xi_{n+1}) \in R^{n+1}; (\xi_1, \dots, \xi_n) \in \Omega, \xi_{n+1} = g(\xi_1, \dots, \xi_n, \varepsilon)\},$$
$$\Gamma_{2,\varepsilon} = \{(\xi_1, \dots, \xi_{n+1}) \in R^{n+1}; (\xi_1, \dots, \xi_n) \in \partial\Omega, \\ 0 < \xi_{n+1} < g(\xi_1, \dots, \xi_n, \varepsilon)\}.$$

Given $H \in L^2(Q_\varepsilon)$, we are interested in the following problems:

$(1)_N$ Find $U \in H^1(Q_\varepsilon)$ such that, for any $W \in H^1(Q_\varepsilon)$,

$$\int_{Q_\varepsilon} (\nabla U \nabla W + \alpha U W)\, d\xi = \int_{Q_\varepsilon} H W \, d\xi,$$

$(1)_D$ Find $U \in H_0^1(Q_\varepsilon)$ such that, for any $W \in H_0^1(Q_\varepsilon)$,

$$\int_{Q_\varepsilon} (\nabla U \nabla W + \alpha U W)\, d\xi = \int_{Q_\varepsilon} H W \, d\xi,$$

$(1)_M$ Find $U \in V_0^\varepsilon \equiv \{W \in H^1(Q_\varepsilon); W = 0 \text{ in } \Gamma_{2,\varepsilon}\}$ such that, for any $W \in V_0^\varepsilon$,

$$\int_{Q_\varepsilon} (\nabla U \nabla W + \alpha U W)\, d\xi = \int_{Q_\varepsilon} H W \, d\xi.$$

In the case $n = 2$, the following regularity result is proved in [6]. In the case $n = 1$, one can prove this regularity result, by arguing as in [9, Chapter V] and using the regularity results contained in [9, Chapter IV].

Theorem A1. (i) *If Ω satisfies the condition* (H), *then there exists a real number $p_0 > 2$, such that, for any function $H \in L^p(Q_\varepsilon)$, $2 \le p \le p_0$, the unique solution U of the problem $(1)_N$ (resp. $(1)_D$) belongs to $W^{2,p}(Q_\varepsilon)$.*

(ii) *If Q_ε satisfies the hypothesis* $(\widetilde{\mathrm{H}})$, *then there exists a real number $p_\varepsilon > 2$, such that, for any function $H \in L^p(Q_\varepsilon)$, $2 \le p \le p_\varepsilon$, the unique solution U of the problem $(1)_M$ belongs to $W^{2,p}(Q_\varepsilon)$.*

The change of variables

$$(2) \qquad \xi_i = x_i, \qquad 1 \le i \le n, \qquad \xi_{n+1} = g(x_1, \dots, x_n, \varepsilon)x_{n+1}$$

takes Q_ε into the fixed domain $Q = \Omega \times (0, 1)$. The boundary ∂Q of Q can be written as

$$\partial Q = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2$$

where $\Gamma_0 = \Omega \times \{0\}$, $\Gamma_1 = \Omega \times \{1\}$, $\Gamma_2 = \partial\Omega \times (0, 1)$.

If we define $h(x_1, \dots, x_{n+1}) = H(x_1, \dots, x_n, g(x_1, \dots, x_n, \varepsilon)x_{n+1})$, then the problems $(1)_N$, $(1)_D$ and $(1)_M$ become: given $h \in L^2(Q)$,

$(3)_N$ find $u \in H^1(Q)$ such that, for any $w \in H^1(Q)$,

$$a_\varepsilon(u, w) = (h, w)_{X_\varepsilon},$$

$(3)_D$ find $u \in H_0^1(Q)$ such that, for any $w \in H_0^1(Q)$,

$$a_\varepsilon(u, w) = (h, w)_{X_\varepsilon},$$

$(3)_M$ find $u \in V_0 \equiv \{w \in H^1(Q);\, w = 0 \text{ in } \Gamma_2\}$ such that, for any $w \in V_0$,

$$a_\varepsilon(u, w) = (h, w)_{X_\varepsilon}.$$

According to [12, §2], the solution u of $(3)_N$, $(3)_D$ or $(3)_M$ satisfies the inequalities (1.15)(i), i.e.,

$$(4) \qquad c_3\left(\|u\|_{1,Q}^2 + \frac{1}{\varepsilon^2}\|u_{x_{n+1}}\|_{0,Q}^2\right)^{1/2} \le \|h\|_{X_\varepsilon}.$$

Moreover, by Theorem A.1, if the hypothesis (H) (resp. $(\widetilde{\mathrm{H}})$) holds, the solution u of $(3)_N$ or $(3)_D$ (resp. $(3)_M$) belongs to $H^2(Q)$ and the problems $(3)_N$, $(3)_D$ (resp. $(3)_M$) are equivalent to

$$(5)_N \qquad \begin{cases} L_\varepsilon u + \alpha u = h & \text{in } Q, \\ \frac{\partial u}{\partial \nu_{B_\varepsilon}} \equiv B_\varepsilon u \cdot \nu = 0 & \text{in } \partial Q. \end{cases}$$

$$(5)_D \qquad \begin{cases} L_\varepsilon u + \alpha u = h & \text{in } Q, \\ u = 0 & \text{in } \partial Q. \end{cases}$$

resp.

$$(5)_M \qquad \begin{cases} L_\varepsilon u + \alpha u = h & \text{in } Q, \\ u = 0 & \text{in } \Gamma_2, \\ \frac{\partial u}{\partial \nu_{B_\varepsilon}} = 0 & \text{in } \Gamma_0 \cup \Gamma_1, \end{cases}$$

where

$$L_\varepsilon u = -\tfrac{1}{g}\operatorname{div} B_\varepsilon u$$

and

$$B_\varepsilon u = \begin{bmatrix} g u_{x_1} - g_{x_1} x_3 \mu_{x_3} \\ g u_{x_2} - g_{x_2} x_3 u_{x_3} \\ - g_{x_1} x_3 u_{x_1} - g_{x_2} x_3 u_{x_2} + \frac{1}{g}(1 + (g_{x_1} x_3)^2 + (g_{x_2} x_3)^2) u_{x_3} \end{bmatrix}$$

if $n = 2$, for instance.

In this appendix, we want to prove the following result.

Theorem A.2. *If the hypothesis* (H) (*resp.* ($\widetilde{\mathrm{H}}$)) *holds, then there exist three positive constants* ε_0, $\tilde{c}_3$, $\tilde{c}_4$ *such that, for* $0 < \varepsilon \leq \varepsilon_0$, *for any* $h \in L^2(Q)$, *the solution* u *of* $(3)_N$, $(3)_D$ (*resp.* $(3)_M$) *satisfies*

$$\text{(6)(i)} \qquad \|h\|^2_{X_\varepsilon} \geq \tilde{c}_3 \Big(\|u\|^2_{2,Q} + \frac{1}{\varepsilon^2} \|u_{x_{n+1}}\|^2_{0,Q} + \frac{1}{\varepsilon^2} \sum_{i=1}^{n} \|u_{x_i x_{n+1}}\|^2_{0,Q} + \frac{1}{\varepsilon^4} \|u_{x_{n+1} x_{n+1}}\|^2_{0,Q} \Big)$$

and

$$\text{(6)(ii)} \qquad \|h\|^2_{X_\varepsilon} \leq \tilde{c}_4 \Big(\|u\|^2_{2,Q} + \frac{1}{\varepsilon^2} \|u_{x_{n+1}}\|^2_{0,Q} + \frac{1}{\varepsilon^2} \sum_{i=1}^{n} \|u_{x_i x_{n+1}}\|^2_{0,Q} + \frac{1}{\varepsilon^4} \|u_{x_{n+1} x_{n+1}}\|^2_{0,Q} \Big).$$

Due to the equivalence of the problems (3) and (5), the estimate (6)(ii) is a straightforward consequence of the properties (1.1) of g. We shall prove the estimate (6)(i) in the case of Neumann or Dirichlet boundary conditions. The proof in the case of mixed boundary conditions is very similar and is left to the reader. Also, in order to simplify the notation, we shall consider the case $n = 2$. The proof in the case $n = 1$ is the same and can even be simplified.

Before proving the estimate (6)(i), we need to recall some properties related with the curvature of the boundary of a domain (see [9, Chapter 3, §3.1]). We consider a bounded domain O of R^m, $m = 2, 3$ with a C^2 boundary $\Gamma = \partial O$ and denote by ν the unit outward normal to ∂O. We denote by $\mathscr{B}$ the second fundamental quadratic form of ∂O. An elementary definition of $\mathscr{B}$ is recalled in [9, Chapter 3, p. 133]; if P is a point of Γ, then we have, for any tangent vectors ξ and η to Γ at P,

$$\text{(7)} \qquad \mathscr{B}_P(\xi, \eta) = -\partial\nu/\partial\xi \cdot \eta$$

where $\partial/\partial\xi$ denotes differentiation in the direction of ξ. Following [9], another possible local definition is the following. If P is a point of Γ, we consider related new orthogonal coordinates $\{y_1, \dots, y_m\}$ with origin at P as follows: there exist a hypercube $V = \{(y_1, \dots, y_m); -a_j < y_j < a_j, 1 \leq j \leq m\}$ and a function φ of class C^2 in V' where $V' = \{(y_1, \dots, y_{m-1}); -a_j < y_j < a_j, 1 \leq j \leq m-1\}$ such that $|\varphi(y')| \leq a_m/2$ for every $y' \in V'$, $O \cap V = \{y = (y', y_m) \in V; y_m < \varphi(y')\}$, $\Gamma \cap V = \{y = (y', y_m) \in V; y_m = \varphi(y')\}$. We can even choose the new coordinates so that the hyperplane $y_m = 0$ is tangent to Γ at P, which implies that $\nabla\varphi(0) = 0$. Then, if ξ

and η are tangent vectors to Γ at P with components $(\xi_1, \dots, \xi_{m-1})$ and $(\eta_1, \dots, \eta_{m-1})$ in the direction of $\{y_1, \dots, y_{m-1}\}$, we have

$$\mathscr{B}_P(\xi, \eta) = \sum_{j,k=1}^{m-1} \frac{\partial^2 \varphi}{\partial y_k \partial y_j}(0)\xi_k \eta_j . \tag{8}$$

Hereafter, we shall drop the subscript P. We remark that, when O is convex, the function φ is convex and the form $\mathscr{B}$ is nonpositive. Also, if the domain O has a C^2 boundary, the form $\mathscr{B}$ is uniformly bounded on Γ, i.e., there exists a positive constant K such that

$$|\mathscr{B}_P(\xi, \eta)| \leq K|\xi|\,|\eta|, \quad \text{for all } P \in \Gamma, \tag{9}$$

for any tangent vectors ξ and η to Γ at P. We need the following notation. Let $\mathbf{v}$ be any vector field on Γ; we denote by $v_\nu \equiv \mathbf{v} \cdot \nu$ the component of $\mathbf{v}$ in the direction of ν and by $\mathbf{v}_T \equiv \mathbf{v} - v_\nu \nu$ the projection of $\mathbf{v}$ on the tangent hyperplane to Γ and we set: $\operatorname{div}_T \mathbf{v} = \operatorname{div} v - \partial \mathbf{v}/\partial \nu \cdot \nu$. We also introduce the notation

$$\nabla_T u = \nabla u - \partial u/\partial \nu \cdot \nu .$$

Finally we denote by $\operatorname{tr}\mathscr{B}$ the trace of the form $\mathscr{B}$.

Let us now consider less regular domains O of R^m. We say that the domain O of R^m with a Lipschitz boundary Γ has a piecewise C^2 boundary if $\Gamma = \widetilde{\Gamma} \cup \bigcup_{j=1}^{l} \Gamma^j$, where

(i) $\widetilde{\Gamma}$ has zero measure (for the surface measure $d\sigma$).

(ii) Γ^j is open in Γ and each point P of Γ^j has the property described above with a function φ of class C^2.

Arguing as in [9, Chapter 3, Theorem 3.1.1.2], we prove the following result.

Theorem A.3. *Let O be a bounded domain of R^m with a Lipschitz boundary Γ. Assume, in addition, that Γ is piecewise C^2 and that each Γ^j in the above decomposition has a Lipschitz boundary $\partial\Gamma^j$. Then, for all $\mathbf{v} \in H^s(O)^m$, $s > 1$, or for all $\mathbf{v} \in W^{1,p}(O)^m$, $p > 2$, we have*:

$$\begin{aligned} \int_O |\operatorname{div} \mathbf{v}|^2 \, d\xi - \sum_{i,j=1}^{m} \int_O \frac{\partial v_i}{\partial \xi_j} \frac{\partial v_j}{\partial \xi_i} \, d\xi \\ = \sum_{j=1}^{l} \int_{\Gamma^j} \{\operatorname{div}_T(v_\nu \mathbf{v}_T) - 2\mathbf{v_T} \cdot \nabla_\mathbf{T} v_\nu\} \, d\sigma \\ - \sum_{j=1}^{l} \int_{\Gamma^j} \{(\operatorname{tr}\mathscr{B}) v_\nu^2 + \mathscr{B}(\mathbf{v}_T ; \mathbf{v}_T)\} \, d\sigma . \end{aligned} \tag{10}$$

Remark A1. One proves Theorem A.3 by showing at first that the identity (10) holds for $\mathbf{v} \in C^2(\overline{O})^m$ and then by extending it to $\mathbf{v} \in H^s(O)^m$ or $W^{1,p}(O)^m$. Since, in both cases, the bracket $\mathbf{v}_T \cdot \nabla_T v_\nu$ has a meaning on Γ^j, this gives a sense to $\int_{\Gamma^j} \operatorname{div}_T(v_\nu \mathbf{v}_T)\, d\sigma$ by density. Note that, in general, one cannot extend the equality (10) to $\mathbf{v} \in H^1(0)^m$ since $\mathbf{v}_T$. $\nabla_T v_\nu$ has no meaning on Γ^j. Let us just show that, if $p > 2$, $\mathbf{v}_T \cdot \nabla_T v_\nu$ has a sense on Γ^j. Indeed, if $p > 2$, then $q < 2$ where $q = p/(p-1)$ and $\mathbf{v_T}|\Gamma^j$ belongs to the space $W^{1-1/p,p}(\Gamma^j)^m$ and thus to the space $W^{1/2,q}(\Gamma^j)^m$ which coincides with $W_0^{1/2,q}(\Gamma^j)^m$. On the other hand, $\nabla_T v_\nu|\Gamma^j$ belongs to the space $W^{-1/2,p}(\Gamma^j)^m$ which is the dual space of $W_0^{1/2,q}(\Gamma^j)^m$. Therefore $\mathbf{v}_T \cdot \nabla_T v_\nu$ has a meaning on Γ^j.

From Theorem A.3, we at once deduce the following result.

Corollary A.4. *Assume that the hypotheses of Theorem* A.3 *hold. Let* $\mathbf{v}$ *be an element of* $H^s(O)^m$, $s > 1$, *or of* $W^{1,p}(O)^m$, $p > 2$.

(i) *If* $v_\nu|\Gamma^j \equiv 0$, $1 \le j \le l$, *we have*

$$(11)\qquad \int_O |\operatorname{div} \mathbf{v}|^2 \, d\xi - \sum_{i,j=1}^{m} \int_O \frac{\partial v_i}{\partial \xi_j}\frac{\partial v_j}{\partial \xi_i} \, d\xi = -\sum_{j=1}^{l} \int_{\Gamma^j} \mathscr{B}(\mathbf{v}_T; \mathbf{v}_T)\, d\sigma;$$

(ii) *If* $\mathbf{v}_T|\Gamma^j \equiv 0$, $1 \le j \le l$, *we have*

$$(12)\qquad \int_O |\operatorname{div} \mathbf{v}|^2 \, d\xi - \sum_{i,j=1}^{m} \int_O \frac{\partial v_i}{\partial \xi_j}\frac{\partial v_j}{\partial \xi_i} \, d\xi = -\sum_{j=1}^{l} \int_{\Gamma^j} (\operatorname{tr}\mathscr{B}) v_\nu^2 \, d\sigma.$$

We now come back to our problem. As Ω has piecewise C^2 boundary, we can write $\partial\Omega$ as $\partial\Omega = \bigcup_{i=1}^{l} (\partial\Omega)_i \cup \widetilde{\partial\Omega}$ where $\widetilde{\partial\Omega}$ has zero surface measure and $(\partial\Omega)_i$ is of class C^2, $1 \le i \le l$. Thus, the domain Q_ε has a piecewise C^2 boundary and

$$\partial Q_\varepsilon = \Gamma_{0,\varepsilon} \cup \Gamma_{1,\varepsilon} \cup \widetilde{\Gamma}_\varepsilon \cup \left(\bigcup_{i=1}^{l} \Gamma_{2,\varepsilon}^i \right)$$

where $\widetilde{\Gamma}_\varepsilon$ has zero surface measure and $\Gamma_{2,\varepsilon}^i$ is the face "generated by $(\partial\Omega)_i$".

Thanks to the regularity Theorem A.1, we can deduce the following result from the Corollary A.4, by using a density argument.

Proposition A.5. *Assume that the hypothesis* (H) *holds.*

(1) *if* U *is the solution of the problem* $(1)_N$, *then*

$$(13)\qquad \begin{aligned} &\int_{Q_\varepsilon} |\Delta U|^2 \, d\xi - \sum_{i,j=1}^{3} \int_{Q_\varepsilon} \left| \frac{\partial^2 U}{\partial \xi_i \partial \xi_j} \right|^2 d\xi \\ &\quad = -\sum_{i=1}^{l} \int_{\Gamma_{2,\varepsilon}^i} \mathscr{B}(\nabla U, \nabla U)\, d\sigma - \int_{\Gamma_{1,\varepsilon}} \mathscr{B}(\nabla U, \nabla U)\, d\sigma. \end{aligned}$$

(2) *If* U *is the solution of the problem* $(1)_D$, *then*

$$(14)\qquad \begin{aligned} &\int_{Q_\varepsilon} |\Delta U|^2 \, d\xi - \sum_{i,j=1}^{3} \int_{Q_\varepsilon} \left| \frac{\partial^2 U}{\partial \xi_i \partial \xi_j} \right|^2 d\xi \\ &\quad = -\sum_{i=1}^{l} \int_{\Gamma_{2,\varepsilon}^i} (\operatorname{tr}\mathscr{B})(\nabla U \cdot \nu_\varepsilon)^2 \, d\sigma - \int_{\Gamma_{1,\varepsilon}} (\operatorname{tr}\mathscr{B})(\nabla U \cdot \nu_\varepsilon)^2 \, d\sigma. \end{aligned}$$

Proof. (1) We recall that the problem $(1)_N$ is equivalent to

$$(15)_N\qquad \begin{cases} -\Delta U + \alpha U = H & \text{in } Q_\varepsilon, \\ \frac{\partial U}{\partial \nu_\varepsilon} = 0 & \text{in } \Gamma_{0,\varepsilon} \cup \Gamma_{1,\varepsilon} \cup \left(\bigcup_{i=1}^{l} \Gamma_{2,\varepsilon}^i \right). \end{cases}$$

For any function H in $L^2(Q_\varepsilon)$, there exists a sequence of functions $H_n \in L^{p_0}(Q_\varepsilon)$ such that H_n converges to H as n goes to infinity, where $p_0 > 2$ is given in Theorem A.1. Hence, by Theorem A.1 and the open mapping theorem, there exists a sequence of functions U_n in $W^{2,p_0}(Q_\varepsilon)$ such that U_n converges to U in $H^2(Q_\varepsilon)$ as n goes to infinity and U_n is the solution of $(1)_N$ (or $(15)_N$) with H replaced by H_n.

Let us set $\mathbf{v}_n = \nabla U_n$. Since $\mathbf{v}_n \cdot \nu_\varepsilon \equiv 0$ in $\Gamma_{0,\varepsilon} \cup \Gamma_{1,\varepsilon} \cup (\bigcup_{i=1}^l \Gamma_{2,\varepsilon}^i)$, we can apply the formula (11) of Corollary A.4 to $\mathbf{v}_n$. Using the local definition (8) of $\mathscr{B}_P$, we at once see that $\mathscr{B}$ vanishes identically on $\Gamma_{0,\varepsilon}$. Therefore, the equality (11) becomes

$$
\begin{aligned}
&\int_{Q_\varepsilon} |\Delta U_n|^2 \, d\xi - \sum_{i,j=1}^{3} \int_{Q_\varepsilon} \left| \frac{\partial^2 U_n}{\partial \xi_i \partial \xi_j} \right|^2 d\xi \\
&\quad = -\sum_{i=1}^{l} \int_{\Gamma_{2,\varepsilon}^i} \mathscr{B}(\nabla U_n, \nabla U_n)\, d\sigma - \int_{\Gamma_{1,\varepsilon}} \mathscr{B}(\nabla U_n, \nabla U_n)\, d\sigma .
\end{aligned} \tag{16}
$$

Now, passing to the limit in (16), we obtain the equality (13).

(2) The proof is similar in the case of the problem $(1)_D$. We only remark that the problem $(1)_D$ is equivalent to

$$
(15)_D \qquad \begin{cases} -\Delta U + \alpha U = H & \text{in } Q_\varepsilon, \\ U = 0 & \text{in } \Gamma_{0,\varepsilon} \cup \Gamma_{1,\varepsilon} \cup \left(\bigcup_{i=1}^l \Gamma_{2,\varepsilon}^i\right). \end{cases}
$$

Again there exists a sequence of functions U_n in $W^{2,p_0}(Q_\varepsilon)$ such that U_n converges to U in $H^2(Q_\varepsilon)$ as n tends to infinity and U_n is the solution of $(1)_D$ (or $(15)_D$) with H replaced by H_n. Let us set $\mathbf{v}_n = \nabla U_n$. Now $\mathbf{v}_{n_T} = \nabla_T U_n$ vanishes in $\Gamma_{0,\varepsilon} \cup \Gamma_{1,\varepsilon} \cup (\bigcup_{i=1}^l \Gamma_{2,\varepsilon}^i)$ and one can apply the formula (12) of Corollary A.4 to $\mathbf{v}_n$. One then finishes the proof as above.

We are now able to prove Theorem A.2.

Proof of Theorem A.2 *in the case of Neumann boundary conditions.* The proof of the estimate (6)(i) will be done in three steps. By Proposition A.5, the solution U of $(1)_N$ satisfies

$$
\begin{aligned}
\sum_{i,j=1}^{3} \int_{Q_\varepsilon} \left| \frac{\partial^2 U}{\partial \xi_i \partial \xi_j} \right|^2 d\xi \le \int_{Q_\varepsilon} |\Delta U|^2 \, d\xi &+ \sum_{i=1}^{l} \int_{\Gamma_{2,\varepsilon}^i} \mathscr{B}(\nabla U, \nabla U)\, d\sigma \\
&+ \int_{\Gamma_{1,\varepsilon}} \mathscr{B}(\nabla U, \nabla U)\, d\sigma .
\end{aligned} \tag{17}
$$

(1) Our next objective is to estimate the integrals $\sum_{i=1}^l \int_{\Gamma_{2,\varepsilon}^i} \mathscr{B}(\nabla U, \nabla U)\, d\sigma$ and $\int_{\Gamma_{1,\varepsilon}} \mathscr{B}(\nabla U, \nabla U)\, d\sigma$.

Let (ξ_1^0, ξ_2^0) be a point of $(\partial\Omega)_i$. Since $(\partial\Omega)_i$ is of class C^2, we define new orthogonal coordinates $\{z_1, z_2\}$ with origin at (ξ_1^0, ξ_2^0) as follows. There exist a rectangle $V_{i,0} = \{(z_1, z_2)\colon -a_j < z_j < a_j,\ j = 1, 2\}$ and a function $\psi_{i,0}$ of class C^2 in $V'_{i,0}$, where $V'_{i,0} = \{z_1\colon -a_1 < z_1 < a_1\}$ such that $|\psi_{i,0}(z_1)| \le a_2/2$, for all $z_1 \in V'_{i,0}$, $\Omega \cap V_{i,0} = \{(z_1, z_2) \in V_{i,0}\colon z_2 < \psi_{i,0}(z_1)\}$, $(\partial\Omega)_i \cap V_{i,0} = \partial\Omega \cap V_{i,0} = \{(z_1, z_2) \in V_{i,0}\colon z_2 = \psi_{i,0}(z_1)\}$ and $\nabla\psi_{i,0}(0) = 0$. Let now $P = (\xi_1^0, \xi_2^0, \xi_3^0)$ be a point of $\Gamma_{2,\varepsilon}^i$. Thanks to the above property, we may consider the new coordinates $\{z_1, z_2, z_3 = \xi_3 - \xi_3^0\}$; indeed, there exists a positive number a_3 (which depends on ε) such that, if $V_i = \{(z_1, z_2, z_3): -a_j < z_j < a_j, 1 \leq j \leq 3\}$ and $V_i' = \{(z_1, z_2): -a_j < z_j < a_j, j = 1, 2\}$, then $Q_\varepsilon \cap V_i = \{(z_1, z_2, z_3) \in V_i: z_2 < \psi_{i,0}(z_1)\}$, $\Gamma^i_{2,\varepsilon} = \{(z_1, z_2, z_3) \in V_i: z_2 = \psi_{i,0}(z_1)\}$. Therefore, by (8), we have, for any tangent vectors $\mathbf{v}_T$ and $\tilde{\mathbf{v}}_T$ to $\Gamma^i_{2,\varepsilon}$ at P,

$$\mathscr{B}_P(\mathbf{v}_T, \tilde{\mathbf{v}}_T) = \frac{\partial^2 \psi_{i,0}}{\partial z_1^2}(0)\eta_1 \cdot \tilde{\eta}_1 \tag{18}$$

where η_1, $\tilde{\eta}_1$ are respectively the first components of $\mathbf{v}_T$ and $\tilde{\mathbf{v}}_T$ in the new coordinates system (z_1, z_3) and are independent of ε. From this property as well as from property (18), we deduce that there exists a positive constant K_i, independent of ε, such that

$$\left| \int_{\Gamma^i_{2,\varepsilon}} \mathscr{B}(\nabla U, \nabla U)\, d\sigma \right| \leq K_i \int_{\Gamma^i_{2,\varepsilon}} |\nabla U|^2\, d\sigma. \tag{19}$$

Likewise, we derive from (18) that

$$|\operatorname{tr} \mathscr{B}_P| \leq K_i, \quad \text{for any } P \in \Gamma^i_{2,\varepsilon}. \tag{20}$$

Let now $P_0 \equiv (\xi_1^0, \xi_2^0, \xi_3^0 \equiv g(\xi_1^0, \xi_2^0, \varepsilon))$ be a point of $\Gamma_{1,\varepsilon}$. By the change of variables $\zeta_i = \xi_i - \xi_i^0$, the new origin is at P_0 and we have

$$\zeta_3 = \tilde{g}(\zeta_1, \zeta_2, \varepsilon) \equiv g(\zeta_1 + \xi_1^0, \zeta_2 + \xi_2^0, \varepsilon) - g(\xi_1^0, \xi_2^0, \varepsilon). \tag{21}$$

We introduce the following notation

$$g_\varepsilon^0 = g(\xi_1^0, \xi_2^0, \varepsilon), \quad \frac{\partial g_\varepsilon^0}{\partial \xi_i} = \frac{\partial g}{\partial \xi_i}(\xi_1^0, \xi_2^0, \varepsilon), \quad \frac{\partial^2 g_\varepsilon^0}{\partial \xi_i \partial \xi_j} = \frac{\partial^2 g}{\partial \xi_i \partial \xi_j}(\xi_1^0, \xi_2^0, \varepsilon).$$

Now we replace the usual orthonormal basis (e_1, e_2, e_3), where

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

by the orthonormal basis $(\tau_1, \tau_2, \nu_\varepsilon)$ where

$$\tau_1 = \frac{1}{\alpha_0} \begin{bmatrix} 1 + \left(\frac{\partial g_\varepsilon^0}{\partial \xi_2}\right)^2 \\ -\frac{\partial g_\varepsilon^0}{\partial \xi_1} \frac{\partial g_\varepsilon^0}{\partial \xi_2} \\ \frac{\partial g_\varepsilon^0}{\partial \xi_1} \end{bmatrix}, \quad \tau_2 = \frac{1}{\beta_0} \begin{bmatrix} 0 \\ 1 \\ \frac{\partial g_\varepsilon^0}{\partial \xi_2} \end{bmatrix}, \quad \nu_3 = \frac{1}{\gamma_0} \begin{bmatrix} -\frac{\partial g_\varepsilon^0}{\partial \xi_1} \\ -\frac{\partial g_\varepsilon^0}{\partial \xi_2} \\ 1 \end{bmatrix}$$

and

$$\begin{cases} \alpha_0 = \left(\left(1 + \left(\frac{\partial g_\varepsilon^0}{\partial \xi_2}\right)^2\right)^2 + \left(\frac{\partial g_\varepsilon^0}{\partial \xi_1} \frac{\partial g_\varepsilon^0}{\partial \xi_2}\right)^2 + \left(\frac{\partial g_\varepsilon^0}{\partial \xi_1}\right)^2 \right)^{1/2}, \\ \beta_0 = \left(1 + \left(\frac{\partial g_\varepsilon^0}{\partial \xi_2}\right)^2\right)^{1/2}, \quad \gamma_0 = \left(1 + \left(\frac{\partial g_\varepsilon^0}{\partial \xi_1}\right)^2 + \left(\frac{\partial g_\varepsilon^0}{\partial \xi_2}\right)^2\right)^{1/2}. \end{cases}$$

If we denote by (z_1, z_2, z_3) the coordinates in this new basis, then the equation (21) becomes $F(z_1, z_2, z_3, \varepsilon) = 0$, where

$$(22)\qquad \begin{aligned} F(z_1, z_2, z_2, \varepsilon) \equiv{}& \frac{1}{\alpha_0}\frac{\partial g_\varepsilon^0}{\partial \xi_1} z_1 + \frac{1}{\beta_0}\frac{\partial g_\varepsilon^0}{\partial \xi_2} z_2 + \frac{1}{\gamma_0} z_3 \\ & - \tilde{g}\left(\frac{1}{\alpha_0}\left(1 + \left(\frac{\partial g_\varepsilon^0}{\partial \xi_2}\right)^2\right) z_1 - \frac{1}{\gamma_0}\frac{\partial g_\varepsilon^0}{\partial \xi_1} z_3, \right. \\ & \left. - \frac{1}{\alpha_0}\frac{\partial g_\varepsilon^0}{\partial \xi_1}\frac{\partial g_\varepsilon^0}{\partial \xi_2} z_1 + \frac{1}{\beta_0} z_2 - \frac{1}{\gamma_0}\frac{\partial g_\varepsilon^0}{\partial \xi_2} z_3, \varepsilon\right). \end{aligned}$$

Since $\partial F(0, 0, 0, \varepsilon)/\partial z_3 \geq 1$, the implicit function theorem implies that there exist a neighbourhood $V = \{(z_1, z_2, z_3) \colon -a_j < z_j < a_j, 1 \leq j \leq 3\}$ of 0 in R^3 and a function g^* such that

$$(23)\qquad \begin{cases} F(z_1, z_2, g^*(z_1, z_2, \varepsilon), \varepsilon) = 0, \\ g^*(0, 0, \varepsilon) = 0, \end{cases}$$

and $g^* \colon V' \times [0, 1] \to R$ is of class C^3 where $V' = \{(z_1, z_2) \colon -a_j < z_j < a_j, j = 1, 2\}$. We have

$$(24)\qquad \frac{\partial g^*}{\partial z_1}(0, 0, \varepsilon) = \frac{\partial g^*}{\partial z_2}(0, 0, \varepsilon) = 0.$$

Moreover, an easy calculation gives

(25)

$$\begin{aligned} \frac{\partial^2 g^*}{\partial z_1^2}&(0, 0, \varepsilon) \\ &= \frac{1}{\alpha_0^2 \gamma_0}\left[\left(1 + \left(\frac{\partial g_\varepsilon^0}{\partial \xi_2}\right)^2\right)^2 \frac{\partial^2 g_\varepsilon^0}{\partial \xi_1^2}\right. \\ &\qquad \left. -2\frac{\partial g_\varepsilon^0}{\partial \xi_1}\frac{\partial g_\varepsilon^0}{\partial \xi_2}\left(1 + \left(\frac{\partial g_\varepsilon^0}{\partial \xi_2}\right)^2\right)\frac{\partial^2 g_\varepsilon^0}{\partial \xi_1 \partial \xi_2} + \left(\frac{\partial g_\varepsilon^0}{\partial \xi_1}\frac{\partial g_\varepsilon^0}{\partial \xi_2}\right)^2 \frac{\partial^2 g_\varepsilon^0}{\partial \xi_2^2}\right], \end{aligned}$$

$$\frac{\partial^2 g^*}{\partial z_1 \partial z_2}(0, 0, \varepsilon) = \frac{1}{\alpha_0 \beta_0 \gamma_0}\left[\left(1 + \left(\frac{\partial g_\varepsilon^0}{\partial \xi_2}\right)^2\right)^2 \frac{\partial^2 g_\varepsilon^0}{\partial \xi_1 \partial \xi_2} - \frac{\partial g_\varepsilon^0}{\partial \xi_1}\frac{\partial g_\varepsilon^0}{\partial \xi_2}\frac{\partial^2 g_\varepsilon^0}{\partial \xi_2^2}\right],$$

$$\frac{\partial^2 g^*}{\partial z_2^2}(0, 0, \varepsilon) = \frac{1}{\beta_0^2 \gamma_0}\frac{\partial^2 g_\varepsilon^0}{\partial \xi_2^2}.$$

Since, for any tangent vectors $\mathbf{v}_T$, $\tilde{\mathbf{v}}_T$ to $\Gamma_{1,\varepsilon}$ at P_0,

$$(26)\qquad \mathscr{B}_{P_0}(\mathbf{v}_T, \tilde{\mathbf{v}}) = \sum_{i,j=1}^{2} \frac{\partial^2 g^*}{\partial z_i \partial z_j}(0, 0)\eta_i \tilde{\eta}_j$$

where (η_1, η_2), $(\tilde{\eta}_1, \tilde{\eta}_2)$ are the components of $\mathbf{v}_T$ and $\tilde{\mathbf{v}}_T$ in the basis (τ_1, τ_2), and since the above change of coordinates is orthonormal, we at once deduce from (25), (26) as well as from the properties (1.1) of the function g that there exists a positive constant $\mathscr{K}_1$ (independent of ε) such that

$$(27)\qquad \left|\int_{\Gamma_{1,\varepsilon}} \mathscr{B}(\nabla U, \nabla U)\, d\sigma\right| \leq \mathscr{K}_1 \varepsilon \int_{\gamma_{1,\varepsilon}} |\nabla U|^2\, d\sigma.$$

Likewise, we show that

$$|\operatorname{tr}\mathscr{B}_P| \le \mathscr{K}_1\varepsilon, \quad \text{for any } P \in \Gamma_{1,\varepsilon}. \tag{28}$$

From (17), (19) and (27), we finally infer that

$$\begin{aligned}\sum_{i,j=1}^{3} \int_{Q_\varepsilon} \left|\frac{\partial^2 U}{\partial\xi_i\partial\xi_j}\right|^2 d\xi \\ \le \int_{Q_\varepsilon} |\Delta U|^2\, d\xi + K\sum_{i=1}^{l}\int_{\Gamma_{2,\varepsilon}^i} |\nabla U|^2\, d\sigma + K\varepsilon\int_{\Gamma_{1,\varepsilon}} |\nabla U|^2\, d\sigma\end{aligned} \tag{29}$$

where K is a positive constant independent of ε.

(2) We now make the change of variables (2) in the formula (29) and denote by $u(x_1, x_2, x_3)$ the function $U(x_1, x_2, g(x_1, x_2, \varepsilon)x_3)$. We have

$$\begin{aligned}\sum_{i,j=1}^{3}\left(\frac{\partial^2 U}{\partial\xi_i\partial\xi_j}\right)^2 = \sum_{i,j=1}^{2}\Bigg(\frac{\partial}{\partial x_i}\left(\frac{\partial u}{\partial x_j} - \frac{g_{x_j}}{g}x_3\frac{\partial u}{\partial x_3}\right) \\ - \frac{g_{x_i}}{g}x_3\frac{\partial}{\partial x_3}\left(\frac{\partial u}{\partial x_j} - \frac{g_{x_j}}{g}x_3\frac{\partial u}{\partial x_3}\right)\Bigg)^2 \\ + \frac{1}{g^4}\left(\frac{\partial^2 u}{\partial x_3}\right)^2 + \frac{2}{g^2}\sum_{i=1}^{2}\left(\frac{\partial}{\partial x_3}\left(\frac{\partial u}{\partial x_i} - \frac{g_{x_i}}{g}x_3\frac{\partial u}{\partial x_3}\right)\right)^2.\end{aligned} \tag{30}$$

Thanks to the properties (1.1) of the function g, we at once infer from (30) that there exist two positive constants ε_0 and β_0 such that, for $0 < \varepsilon \le \varepsilon_0$,

$$\begin{aligned}\beta_0\left(\sum_{i,j=1}^{2}\left(\frac{\partial u}{\partial x_i\partial x_j}\right)^2 + \frac{1}{\varepsilon^2}\sum_{i=1}^{2}\left(\frac{\partial^2 u}{\partial x_i\partial x_3}\right)^2 + \frac{1}{\varepsilon^4}\left(\frac{\partial^2 u}{\partial x_3^2}\right)^2\right) \\ \le \sum_{i,j=1}^{3}\left(\frac{\partial^2 U}{\partial\xi_i\partial\xi_j}\right)^2 + \frac{1}{\varepsilon^2}\left(\frac{\partial u}{\partial x_3}\right)^2.\end{aligned} \tag{31}$$

Thanks to the estimate (31), we deduce from (29) that, for $0 < \varepsilon \le \varepsilon_0$,

$$\begin{aligned}\int_Q\left(\sum_{i,j=1}^{2}\left(\frac{\partial^2 u}{\partial x_i\partial x_j}\right)^2 + \frac{1}{\varepsilon^2}\sum_{i=1}^{2}\left(\frac{\partial^2 u}{\partial x_i\partial x_3}\right)^2 + \frac{1}{\varepsilon^4}\left(\frac{\partial^2 u}{\partial x_3^2}\right)^2\right)\frac{g}{\varepsilon}\,dx \\ \le \widetilde{K}\Bigg[\int_Q (L_\varepsilon u)^2\frac{g}{\varepsilon}\,dx + \frac{1}{\varepsilon^2}\int_Q\left(\frac{\partial u}{\partial x_3}\right)^2\frac{g}{\varepsilon}\,dx \\ + \int_{\Gamma_1}\left(\left(\frac{\partial u}{\partial x_1}\right)^2 + \left(\frac{\partial u}{\partial x_2}\right)^2 + \frac{1}{\varepsilon^2}\left(\frac{\partial u}{\partial x_3}\right)^2\right)dx_1\,dx_2 \\ + \sum_{i=1}^{l}\int_{\Gamma_2^i}\left(\left(\frac{\partial u}{\partial x_1}\right)^2 + \left(\frac{\partial u}{\partial x_2}\right)^2 + \frac{1}{\varepsilon^2}\left(\frac{\partial u}{\partial x_3}\right)^2\right)\frac{g}{\varepsilon}\,ds\,dx_3\Bigg],\end{aligned} \tag{32}$$

where $\widetilde{K}$ is a positive constant independent of ε (and u) and $\Gamma_2^i = (\partial\Omega)_i \times (0, 1)$.

(3) By [9, Theorem 1, 5.1.10], there exists a positive constant K^* depending only on the domain Q such that

$$\begin{aligned}
&\int_{\Gamma_1}\left(\left(\frac{\partial u}{\partial x_1}\right)^2+\left(\frac{\partial u}{\partial x_2}\right)^2+\frac{1}{\varepsilon^2}\left(\frac{\partial u}{\partial x_3}\right)^2\right)dx_1\,dx_2 \\
&\quad+\sum_{i=1}^{l}\int_{\Gamma_2^i}\left(\left(\frac{\partial u}{\partial x_1}\right)^2+\left(\frac{\partial u}{\partial x_2}\right)^2+\frac{1}{\varepsilon^2}\left(\frac{\partial u}{\partial x_3}\right)^2\right)ds\,dx_3 \\
&\quad\le K^*\left[\eta^{1/2}\int_Q\left(\sum_{i,j=1}^{2}\left(\frac{\partial^2 u}{\partial x_i\partial x_j}\right)^2+\frac{1}{\varepsilon^2}\sum_{i=1}^{2}\left(\frac{\partial^2 u}{\partial x_i\partial x_3}\right)^2\right.\right. \\
&\qquad\qquad\left.+\frac{1}{\varepsilon^2}\left(\frac{\partial^2 u}{\partial x_3^2}\right)^2\right)dx \\
&\qquad\left.+\eta^{-1/2}\int_Q\left(\left(\frac{\partial u}{\partial x_1}\right)^2+\left(\frac{\partial u}{\partial x_2}\right)^2+\frac{1}{\varepsilon^2}\left(\frac{\partial u}{\partial x_3}\right)^2\right)dx\right],
\end{aligned}\tag{33}$$

for any $\eta\in(0,1)$. Since $c_1\le g/\varepsilon\le c_2$ (see (1.14)), we at once deduce from the inequality (32), by applying the estimate (33) with $\eta=c_1^2/4(\widetilde{K}K^*c_2)^2$, that

$$\begin{aligned}
&\int_Q\left(\sum_{i,j=1}^{2}\left(\frac{\partial^2 u}{\partial x_i\partial x_j}\right)^2+\frac{1}{\varepsilon^2}\sum_{i=1}^{2}\left(\frac{\partial^2 u}{\partial x_i\partial x_j}\right)^2+\frac{1}{\varepsilon^4}\left(\frac{\partial^2 u}{\partial x_3^2}\right)^2\right)dx \\
&\quad\le c_0\left[\|L_\varepsilon u\|_{X_\varepsilon}^2+\|u\|_{1,Q}^2+\frac{1}{\varepsilon^2}\left\|\frac{\partial u}{\partial x_3}\right\|_{0,Q}^2\right].
\end{aligned}\tag{34}$$

The estimate (6)(i) is now a direct consequence of (34) and (4).

Proof of Theorem A.2 *in the case of Dirichlet boundary conditions.* By Proposition A.5, the solution U of $(1)_D$ satisfies

$$\begin{aligned}
\sum_{i,j=1}^{3}\int_{Q_\varepsilon}\left(\frac{\partial^2 U}{\partial\xi_i\partial\xi_j}\right)^2 d\xi &\le \int_{Q_\varepsilon}|\Delta U|^2\,d\xi+\sum_{i=1}^{l}\int_{\Gamma_{2,\varepsilon}^i}(\mathrm{tr}\mathscr{B})(\nabla U\cdot\nu_\varepsilon)^2\,d\sigma \\
&\quad+\int_{\Gamma_{1,\varepsilon}}(\mathrm{tr}\mathscr{B})(\nabla U\cdot\nu_\varepsilon)^2\,d\sigma.
\end{aligned}\tag{35}$$

Arguing as in the case of Neumann boundary conditions and using the estimates (20) and (28), one proves that the inequality (29) still holds. The steps 2 and 3 are the same as in the case of Neumann boundary conditions.

We end this Appendix by an estimate of second derivatives in the case of convex C^2 domains. Let O be a domain of R^3 with a C^2 boundary. In [9, Chapter 3, Theorem 3.1.1.1], it is shown that we have, for any $\mathbf{v}\in H^1(O)^3$,

$$\begin{aligned}
\int_O|\operatorname{div}\mathbf{v}|^2\,dx-\sum_{i,j=1}^{3}\int_O\frac{\partial v_i}{\partial x_j}\frac{\partial v_j}{\partial x_i}\,dx &= -2\int_{\partial O}\mathbf{v}_T\cdot\nabla_T(\mathbf{v}\cdot\nu)\,d\sigma \\
&\quad-\int_{\partial O}\{\mathscr{B}(\mathbf{v}_T;\mathbf{v}_T)+(\mathrm{tr}\mathscr{B})[\mathbf{v}\cdot\nu]^2\}\,d\sigma.
\end{aligned}\tag{36}$$

We now consider the following problems: given $h \in L^2(O)$,

$$(37)_N \qquad \text{Find } u \in H^1(O) \text{ such that} \quad \begin{cases} L_\varepsilon u + \alpha u = h & \text{in } O, \\ B_\varepsilon u \cdot \nu = 0 & \text{on } \partial O, \end{cases}$$

where ν is the outward normal to O, and

$$(37)_D \qquad \text{Find } u \in H_0^1(O) \text{ such that} \quad \begin{cases} L_\varepsilon u + \alpha u = h & \text{in } O, \\ u = 0 & \text{on } \partial O. \end{cases}$$

The problems $(37)_N$ (resp. $(37)_D$) have a unique solution u and since O has a C^2 boundary, u belongs to $H^2(O)$. We now assume that O is, in addition, a convex domain. Let us set $\mathbf{v} = B_\varepsilon u$ in the equality (36). If u is the solution of $(37)_N$ (resp. $(37)_D$, then $\mathbf{v} \cdot \nu = 0$ on ∂O (resp. $\mathbf{v}_T = 0$ on ∂O). Using the equality (36) with $\mathbf{v} = B_\varepsilon u$ and remarking that $\mathscr{B}_P$ is nonpositive, one easily shows the following results.

Theorem A.6. *Assume that O is a convex domain with a C^2 boundary. Then there exist three positive constants ε_0, $\tilde{c}_3$, $\tilde{c}_4$ such that, for $0 < \varepsilon \le \varepsilon_0$, for any $h \in L^2(O)$, the solution u of $(37)_N$ (resp. $(37)_D$) satisfies*

$$(38) \quad \tilde{c}_3 \left(\|u\|_{2,O}^2 + \frac{1}{\varepsilon^2}\|u_{x_3}\|_{0,O}^2 + \frac{1}{\varepsilon^2}\sum_{i=1}^{2}\|u_{x_i x_3}\|_{0,O}^2 + \frac{1}{\varepsilon^4}\|u_{x_3 x_3}\|_{0,O}^2 \right) \le \|h\|_{0,O}^2.$$

The inequality (38) has been used in [12, Remark 2.6].

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