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SYMMETRIC LOCAL ALGEBRAS WITH 5-DIMENSIONAL CENTER

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Dedicated to Hiroyuki Tachikawa on the occasion of his sixtieth birthday

ABSTRACT. We prove that a symmetric split local algebra whose center is 5-dimensional has dimension 5 or 8. This implies that the defect groups of a block of a finite group containing exactly five irreducible Frobenius characters and exactly one irreducible Brauer character have order 5 or are nonabelian of order 8.

Let F be a field, and let A be a finite-dimensional associative unitary F-algebra with center Z and radical J. Then A is called *split local* if $\dim A/J = 1$, and A is called *symmetric* if there is a linear map $\lambda : A \to F$ whose kernel contains all Lie commutators $[x, y] := xy - yx$ $(x, y \in A)$ but no nonzero ideal of A. Suppose now that A is symmetric and split local. In [6] the second author proved that A is necessarily commutative if $\dim Z \leq 4$. This incorporated earlier results by R. Brauer and J. Brandt [1]. In this paper we are dealing with the next case.

Theorem. *Let F be a field, and let A be a symmetric split local F-algebra with center Z. If $\dim Z = 5$ then $\dim A \in \{5, 8\}$.*

The group algebra of a group of order 5 over a field of characteristic 5 is an example for the case $\dim A = 5$, and the group algebra of a nonabelian group of order 8 over a field of characteristic 2 is an example for the case $\dim A = 8$.

Corollary. *Let F be an algebraically closed field, let G be a finite group, let P be an indecomposable projective FG-module, and set $A := \mathrm{End}_{FG}(P)$. If the center of A has dimension 5 then $\dim A \in \{5, 8\}$.*

Proof. We choose a primitive idempotent i in FG such that P is isomorphic to FGi. Then A is isomorphic to $iFGi$. Since FG is a symmetric F-algebra, so are $iFGi$ and A. Since P is indecomposable and F is algebraically closed, A is split local. Hence the corollary follows from the theorem.

We have the following application to block theory.

Proposition. *Let F be an algebraically closed field, let G be a finite group, and let B be a block of FG containing exactly 5 irreducible complex characters*

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and exactly one irreducible Brauer character. Then the defect groups of B *have order* 5 *or are nonabelian of order* 8.

Proof. Let P denote the only indecomposable projective FG-module in B, and set $A := \mathrm{End}_{FG}(P)$. By Lemma B in [4], B is isomorphic to a complete matrix algebra over A; in particular, A and B have isomorphic centers. By $(2G)$ in [2], the dimension of the center of B coincides with the number of irreducible complex characters in B, so the center of A has dimension 5. By the corollary, A has dimension 5 or 8. On the other hand, Lemma B in [4] shows that the dimension of A coincides with the order of a defect group D of B. Hence D has order 5 or 8. Assume now that D is abelian of order 8. Then B cannot be nilpotent in the sense of [3]; for otherwise B would contain 8 irreducible complex characters by the main result of [3]. Thus D must be elementary abelian. But in this case we obtain a contradiction using the results in [7].

The remainder of this paper consists of a proof of the theorem. Let A be a symmetric split local algebra over a field F and denote by Z the center and by J the radical of A. We may and do assume that F is algebraically closed. For a subset X of A, we denote by FX the linear subspace of A spanned by X. The subspace $K := F\{[x, y] : x, y \in A\}$ will be particularly important for us. Since $A = F1 + J$ we have $K = [J, J] \subset J^2$. We fix a linear map $\lambda : A \to F$ the kernel of which contains K but no nonzero ideal of A. Then 0 is the only ideal of A contained in K. For any linear subspace U of A, $U^\perp := \{a \in A : \lambda(aU) = 0\}$ is a linear subspace of A such that $\dim A = \dim U + \dim U^\perp$ and $(U^\perp)^\perp = U$. We have $Z^\perp = K$ (see [5]); in particular, $\dim Z = \dim A/K$. Moreover,

$$I^\perp = \{a \in A : aI = 0\} = \{a \in A : Ia = 0\}$$

for any ideal I of A; in particular, $I^\perp$ is an ideal of A. Furthermore, $\dim J^\perp = \dim A/J = 1$. Hence, if $J^n = 0$ for some positive integer n then $J^{n-1} \subset J^\perp$; in particular, $\dim J^{n-1} \leq \dim J^\perp = 1$. We will often use this fact without special reference.

1. Preliminary results

From now on we suppose that $\dim Z = 5$. We may and will assume that $\dim A \geq 6$; for otherwise we are done.

(1.1) **Lemma.** *We have* $\dim A \geq 8$.

Proof. Assume that $\dim A \leq 7$. Then there are elements $a, b \in A$ such that $A = Z + Fa + Fb$. Therefore $K = F[a, b]$; in particular, $\dim K \cap Z \leq \dim K \leq 1$. Now Lemma D in [6] implies that A is commutative, so $\dim A = \dim Z = 5$, a contradiction.

If $\dim A = 8$, then the theorem is proved, so we may and will assume that $\dim A \geq 9$. We are then looking for a contradiction.

(1.2) **Lemma.** *We have* $\dim A/K + J^3 = 4$, *and one of the following occurs*:

(1.3) $\dim J/J^2 = 2$, $\dim J^2/J^3 = 2$, $\dim J^3/J^4 \geq 2$, $\dim J^4/J^5 \geq 1$, $K + J^3 = K + J^4$;

(1.4) $\dim J/J^2 = 3$, $\dim J^2/J^3 = 2$, $\dim J^3/J^4 \geq 2$, $\dim J^4/J^5 \geq 1$, $J^2 = K + J^3 = K + J^4$;

(1.5) $\dim J/J^2 = 3$, $\dim J^2/J^3 = 3$, $\dim J^3/J^4 \geq 2$, $\dim J^4/J^5 \geq 1$, $J^2 = K + J^3 = K + J^4$.

Proof. Since $\dim J \geq 8$ we have $J^2 \neq 0$. Thus Nakayama's Lemma implies that $J^2 \neq J^3$. Furthermore, $J \not\subset Z$, so $\dim J^2/J^3 \geq 2$ by Lemma G in [6]; in particular, $\dim J/J^2 \geq 2$ by Lemma E in [6], and $J^3 \neq 0$. Hence $J^3 \neq J^4$ by Nakayama's Lemma, and $J^3 \not\subset K$. Thus

$$\dim A/J^2 \leq \dim A/K + J^3 < \dim A/K = \dim Z = 5;$$

in particular, $\dim J/J^2 \in \{2, 3\}$, so $\dim J^2 \geq 5$. This means that $J^2 \not\subset Z$ which implies by Lemma G in [6] that $\dim J^3/J^4 \geq 2$. Hence $J^4 \neq 0$, and $J^4 \neq J^5$ by Nakayama's Lemma again. Moreover, $J^4 \not\subset K$, so $\dim A/K+J^3 \leq \dim A/K + J^4 < \dim A/K = 5$.

Suppose first that $\dim J/J^2 = 2$ and write $J = Fa+Fb+J^2$ with elements $a, b \in J$. Then $A = F\{1, a, b\} + J^2$ and $K \subset F[a, b] + J^3$; in particular, $\dim K+J^3/J^3 \leq 1$, so $\dim A/J^3 \leq 5$ and $\dim J^2/J^3 = 2$. Thus $\dim A/J^3 = 5$ and $\dim A/K + J^3 = 4$. Hence also $\dim A/K + J^4 = 4$.

Finally, suppose that $\dim J/J^2 = 3$ and write $J = F\{a, b, c\} + J^2$ with elements $a, b, c \in J$. Then $K \subset F\{[a, b], [a, c], [b, c]\} + J^3$; in particular, $\dim K + J^3/J^3 \leq 3$. Thus $\dim A/J^3 \leq 7$ and $\dim J^2/J^3 \in \{2, 3\}$. Since $4 = \dim A/J^2 \leq \dim A/K + J^3 \leq \dim A/K + J^4 \leq 4$ the result follows.

We will deal with these cases in §§2, 3 and 4, respectively. The following results will be useful later on.

(1.6) **Lemma.** *There is an element $x \in J$ such that $x^2 \notin J^3$.*

Proof. By (1.2) we have $\dim J/J^2 \leq 3$. We write $J = F\{a, b, c\} + J^2$ with elements $a, b, c \in J$. If $x^2 \in J^3$ for $x \in J$ then $ab+ba = (a+b)^2 - a^2 - b^2 \in J^3$. Thus $ba \equiv -ab \pmod{J^3}$. Similarly, $ca \equiv -ac \pmod{J^3}$ and $cb \equiv -bc \pmod{J^3}$. Therefore

$$J^2 = F\{a^2, ab, ac, ba, b^2, bc, ca, cb, c^2\} + J^3 = F\{ab, ac, bc\} + J^3;$$

in particular, $\dim J^2/J^3 \leq 3$. Now we apply Lemma E in [6] to obtain

$$J^3 = F\{a^2b, a^2c, abc, bab, bac, b^2c\} + J^4 = Fabc + J^4$$

and $J^4 = Fa^2bc + J^5 = J^5$ contradicting (1.2).

(1.7) **Lemma.** *There are elements $a, b \in J$ such that $a^2 + J^3, ab + J^3$ or $a^2 + J^3, ba + J^3$ are linearly independent in J^2/J^3.*

Proof. By (1.6), there is an element $a \in J$ such that $a^2 \notin J^3$; in particular, $a \notin J^2$. By (1.2) there are therefore elements $b, c \in J$ such that $J = F\{a, b, c\} + J^2$. We may assume that $ab, ba, ac, ca \in Fa^2 + J^3$; for otherwise the result is proved. Then $K + J^3 = F\{[a, b], [a, c], [b, c]\} + J^3 \subset F\{a^2, [b, c]\} + J^3$; in particular, $\dim K + J^3/J^3 \leq 2$. Hence, by (1.2), $\dim J^2/J^3 = 2$.

Now consider the case where $b^2 \notin Fa^2 + J^3$; in particular, $b^2 \notin J^3$. Then we can interchange the roles of a and b and therefore assume that

$ab, ba, bc, cb \in Fb^2 + J^3$. Since $a^2 + J^3$ and $b^2 + J^3$ form a basis of J^2/J^3 this implies that $ab, ba \in J^3$. Thus $(a+b)^2 + J^3 = a^2 + b^2 + J^3$ and $(a+b)b + J^3 = b^2 + J^3$ are linearly independent, and the result follows in this case.

Therefore we may also assume that $b^2 \in Fa^2 + J^3$ and, similarly, $c^2 \in Fa^2 + J^3$. Then

$$J^2 = F\{a^2, ab, ac, ba, b^2, bc, ca, cb, c^2\} + J^3 = F\{a^2, bc, cb\} + J^3.$$

Thus $J^2 = F\{a^2, bc\} + J^3$ or $J^2 = F\{a^2, cb\} + J^3$; we may assume that $J^2 = F\{a^2, bc\} + J^3$. Then Lemma E in [6] implies that

$$J^3 = F\{a^3, abc, ba^2, b^2c\} + J^4 = F\{a^3, a^2c\} + J^4 = Fa^3 + J^4;$$

in particular, $\dim J^3/J^4 \leq 1$ contradicting (1.2).

We now choose elements $a, b \in J$ as in (1.7). By symmetry we may assume that $ab \notin Fa^2 + J^3$; in particular, $a \notin J^2$ and $b \notin Fa + J^2$. Thus $a + J^2, b + J^2$ are linearly independent in J/J^2. By (1.2), we can find an element $c \in J$ such that $J = F\{a, b, c\} + J^2$.

2. The case (1.3)

In this section we use the same hypothesis and notation as before, but we assume in addition that (1.3) holds. Then $J = Fa + Fb + J^2$ and $J^2 = Fa^2 + Fab + J^3$. Thus Lemma E in [6] implies that $J^3 = Fa^3 + Fa^2b + J^4$, $J^4 = Fa^4 + Fa^3b + J^5$ and $J^5 = Fa^5 + Fa^4b + J^6$; in particular, $\dim J^3/J^4 = 2$. Thus $a^3 + J^4$ and $a^2b + J^4$ form a basis of J^3/J^4. Moreover, $\dim J^4 \geq 2$; in particular, $J^5 \neq 0$. Thus $J^5 \not\subset K$ and $4 = \dim A/K + J^4 \leq \dim A/K + J^5 < \dim A/K = \dim Z = 5$ by (1.2). We conclude that $\dim A/K + J^5 = 4$. Furthermore, $A = F\{1, a, b, a^2, ab, a^3, a^2b\} + J^4$, so

$$K \subset F\{[a, b], [a, ab], [a, a^2b], [b, a^2], [b, ab], [b, a^3], [b, a^2b], [a^2, ab]\} + J^5.$$

Since $J^2 = F\{a^2, ab, a^3, a^2b\} + J^4$ there are elements $\alpha_i, \beta_i, \gamma_i, \delta_i \in F$ $(i = 1, 2)$ such that

$$ba \equiv \alpha_1 a^2 + \beta_1 ab + \gamma_1 a^3 + \delta_1 a^2 b \pmod{J^4},$$
$$b^2 \equiv \alpha_2 a^2 + \beta_2 ab + \gamma_2 a^3 + \delta_2 a^2 b \pmod{J^4}.$$

We have to distinguish between two cases.

Case 1. $\beta_1 \neq 1$. In this case we set $\xi := \alpha_1/(1-\beta_1)$ and $b' := b - \xi a$. Then $J = Fa + Fb' + J^2$, $J^2 = Fa^2 + Fab' + J^3$ and

$$\begin{aligned} b'a &\equiv ba - \xi a^2 \equiv (\alpha_1 - \xi)a^2 + \beta_1 ab \\ &\equiv (\alpha_1 - \xi + \beta_1\xi)a^2 + \beta_1 ab' \equiv \beta_1 ab' \pmod{J^3}. \end{aligned}$$

Thus we may replace b by b' and therefore assume that $\alpha_1 = 0$. Then

$$\begin{aligned} 0 &\equiv (b^2)a - b(ba) \equiv \alpha_2 a^3 + \beta_2 aba - \beta_1 bab \equiv \alpha_2 a^3 + \beta_1\beta_2 a^2 b - \beta_1^2 ab^2 \\ &\equiv (\alpha_2 - \alpha_2\beta_1^2)a^3 + (\beta_1\beta_2 - \beta_1^2\beta_2)a^2 b \pmod{J^4} \end{aligned}$$

and, similarly,

$$0 \equiv (b^2)b - b(b^2) \equiv (\alpha_2\beta_2 - \alpha_2\beta_1\beta_2)a^3 + (\alpha_2 + \beta_2^2 - \alpha_2\beta_1^2 - \beta_1\beta_2^2)a^2b \pmod{J^4}.$$

Since $a^3 + J^4$ and $a^2b + J^4$ form a basis of J^3/J^4 we conclude that

(2.1) $0 = \alpha_2 - \alpha_2\beta_1^2$, (2.2) $0 = \beta_1\beta_2 - \beta_1^2\beta_2$,

(2.3) $0 = \alpha_2\beta_2 - \alpha_2\beta_1\beta_2$, (2.4) $0 = \alpha_2 + \beta_2^2 - \alpha_2\beta_1^2 - \beta_1\beta_2^2$.

Subtracting (2.1) from (2.4) we obtain $\beta_2^2 = \beta_1\beta_2^2$. Since $\beta_1 \neq 1$ this implies $\beta_2 = 0$. From (2.1) we also conclude that $\alpha_2 = 0$ or $\beta_1^2 = 1$. We assume first that $\alpha_2 = 0$. Then

$$[a, ab] \equiv a^2b - aba \equiv (1 - \beta_1)a^2b \pmod{J^4},$$
$$[b, a^2] \equiv ba^2 - a^2b \equiv \beta_1 aba - a^2b \equiv (\beta_1^2 - 1)a^2b \pmod{J^4},$$
$$[b, ab] \equiv bab - ab^2 \equiv \beta_1 ab^2 \equiv 0 \pmod{J^4}.$$

This shows that $K \subset F[a, b] + Fa^2b + J^4$; in particular, $\dim K + J^4/J^4 \leq 2$. Thus $\dim A/J^4 \leq 6$ by (1.2), a contradiction.

Hence we must have $\alpha_2 \neq 0$ and $\beta_1^2 = 1$. Since $\beta_1 \neq 1$ this implies $\beta_1 = -1$ and $\operatorname{char} F \neq 2$. It is now easy to check that

$$[a, a^2b] \equiv 2a^3b \pmod{J^5}, \qquad [b, a^2] \equiv -2\delta_1 a^3b \pmod{J^5},$$
$$[b, a^3] \equiv -2a^3b \pmod{J^5}, \qquad [b, a^2b] \equiv [a^2, ab] \equiv 0 \pmod{J^5}.$$

Thus $K \subset F\{[a, b], [a, ab], [b, ab], a^3b\} + J^5$; in particular, $\dim K + J^5/J^5 \leq 4$. Hence $\dim A/J^5 \leq 8$ and $\dim J^4/J^5 = 1$. By Lemma G in [6], this implies that $J^3 \subset Z$; in particular, $a^2b \in Z$. Thus $a^3b \equiv a^2ba \equiv -a^3b \pmod{J^5}$. Since $\operatorname{char} F \neq 2$ this implies $a^3b \in J^5$. Therefore

$$K \subset F\{[a, b], [a, ab], [b, ab]\} + J^5;$$

in particular, $\dim K + J^5/J^5 \leq 3$. Hence $\dim A/J^5 \leq 7$, a contradiction.

Case 2. $\beta_1 = 1$. Assume first that $\alpha_1 = 0$. Then $[a, b] \in J^3$ and $K \subset J^3$, so $\dim A/K + J^3 = \dim A/J^3 = 5$ contradicting (1.2). Thus we must have $\alpha_1 \neq 0$. Now we set $a' := \alpha_1 a$. Then $J = Fa' + Fb + J^2$, $J^2 = F(a')^2 + Fa'b + J^3$ and

$$ba' \equiv \alpha_1 ba \equiv \alpha_1^2 a^2 + \alpha_1 ab \equiv (a')^2 + a'b \pmod{J^3}.$$

Hence we may replace a by a' and therefore assume that $\alpha_1 = 1$. As in Case 1 we compute

$$0 \equiv (b^2)a - b(ba) \equiv (\beta_2 - 2)a^3 - 2a^2b \pmod{J^4}.$$

Since $a^3 + J^4$ and $a^2b + J^4$ form a basis of J^3/J^4 this implies that $\operatorname{char} F = 2$ and $\beta_2 = 0$. Hence

$$[a, a^2b] \equiv a^4 \pmod{J^5}, \qquad [b, a^2] \equiv \delta_1 a^4 \pmod{J^5},$$
$$[b, a^3] \equiv a^4 \pmod{J^5}, \qquad [b, a^2b] \equiv [a^2, ab] \equiv 0 \pmod{J^5}.$$

Therefore $K \subset F\{[a, b], [a, ab], [b, ab], a^4\} + J^5$; in particular, $\dim K + J^5/J^5 \leq 4$. Hence $\dim A/J^5 \leq 8$ and $\dim J^4/J^5 = 1$. By Lemma G in [6], this implies that $J^3 \subset Z$; in particular, $a^2b \in Z$. Thus $a^3b \equiv a^2ba \equiv a^4 + a^3b$

$(\mathrm{mod}\, J^5)$. Therefore $a^4 \in J^5$ and $J^5 = Fa^5 + Fa^4b + J^6 = J^6$. Hence $J^5 = 0$ by Nakayama's Lemma, a contradiction.

3. The case (1.4)

In this section we assume hypothesis and notation from §1. In addition, we assume that (1.4) holds. Then $J^2 = Fa^2 + Fab + J^3$ and $J^3 = Fa^3 + Fa^2b + J^4$ by Lemma E in [6]; in particular, $\dim J^3/J^4 = 2$. Hence $a^3 + J^4$, $a^2b + J^4$ form a basis of J^3/J^4. There are elements $\alpha, \beta \in F$ such that $ac \equiv \alpha a^2 + \beta ab$ $(\mathrm{mod}\, J^3)$. Setting $c' := c - \alpha a - \beta b$ we then have $J = F\{a, b, c'\} + J^2$ and $ac' \equiv ac - \alpha a^2 - \beta ab \equiv 0 \pmod{J^3}$. Hence we may replace c by c' and therefore assume that $ac \in J^3$. We choose elements $\alpha_i, \beta_i \in F$ $(i = 1, 2, 3, 4)$ such that

$$bc \equiv \alpha_1 a^2 + \beta_1 ab \pmod{J^3}, \qquad ca \equiv \alpha_2 a^2 + \beta_2 ab \pmod{J^3},$$
$$cb \equiv \alpha_3 a^2 + \beta_3 ab \pmod{J^3}, \qquad c^2 \equiv \alpha_4 a^2 + \beta_4 ab \pmod{J^3}.$$

Then

$$0 \equiv (ac)a \equiv a(ca) \equiv \alpha_2 a^3 + \beta_2 a^2 b \pmod{J^4},$$
$$0 \equiv (ac)b \equiv a(cb) \equiv \alpha_3 a^3 + \beta_3 a^2 b \pmod{J^4},$$
$$0 \equiv (ac)c \equiv a(c^2) \equiv \alpha_4 a^3 + \beta_4 a^2 b \pmod{J^4}.$$

Hence $\alpha_2 = \beta_2 = \alpha_3 = \beta_3 = \alpha_4 = \beta_4 = 0$; in particular, $ca, cb, c^2 \in J^3$. Thus

$$0 \equiv b(c^2) \equiv (bc)c \equiv \alpha_1 a^2 c + \beta_1 abc \equiv \alpha_1\beta_1 a^3 + \beta_1^2 a^2 b \pmod{J^4},$$

and we obtain $\beta_1 = 0$. Thus $0 \equiv b(cb) \equiv (bc)b \equiv \alpha_1 a^2 b \pmod{J^4}$. Therefore $\alpha_1 = 0$; in particular, $bc \in J^3$. Thus $[a, c], [b, c] \in J^3$ and $K \subset F\{[a, b], [a, c], [b, c]\} + J^3 \subset F[a, b] + J^3$; in particular, $\dim K + J^3/J^3 \leq 1$. Thus $\dim A/J^3 \leq 5$ by (1.2), a contradiction.

4. The case (1.5)

In this section we assume hypothesis and notation from §1. In addition, we assume that (1.5) holds. Since $J = F\{a, b, c\} + J^2$ we have $J^2 = F\{a^2, ab, ac, ba, b^2, bc, ca, cb, c^2\} + J^3$. Since $\dim J^2/J^3 = 3$ we must have $J^2 = F\{a^2, ab, d\} + J^3$ for some element $d \in \{ac, ba, b^2, bc, ca, cb, c^2\}$. Since $J^2 = K + J^4$ we obtain

$$J^2 = F\{[a, b], [a, c], [a, ab], [a, d], [b, c], [b, a^2], [b, ab], [b, d], [c, a^2], [c, ab], [c, d]\} + J^4.$$

We choose elements $\alpha_i, \beta_i, \gamma_i \in F$ $(i = 1, 2, \dots, 7)$ such that

$$ac \equiv \alpha_1 a^2 + \beta_1 ab + \gamma_1 d \pmod{J^3}, \qquad ba \equiv \alpha_2 a^2 + \beta_2 ab + \gamma_2 d \pmod{J^3},$$
$$b^2 \equiv \alpha_3 a^2 + \beta_3 ab + \gamma_3 d \pmod{J^3}, \qquad bc \equiv \alpha_4 a^2 + \beta_4 ab + \gamma_4 d \pmod{J^3},$$
$$ca \equiv \alpha_5 a^2 + \beta_5 ab + \gamma_5 d \pmod{J^3}, \qquad cb \equiv \alpha_6 a^2 + \beta_6 ab + \gamma_6 d \pmod{J^3},$$
$$c^2 \equiv \alpha_7 a^2 + \beta_7 ab + \gamma_7 d \pmod{J^3}.$$

(4.1) **Lemma.** *We may assume that $d = ac$ or $d = ba$.*

Proof. Case 1. $d = ac$. In this case there is nothing to prove.

Case 2. $d = ba$. In this case there is nothing to prove either.

Case 3. $d = b^2$. In this case we may assume that $ba \in Fa^2 + Fab + J^3$; for otherwise we are in Case 2. Similarly, we may assume that $ba \in Fb^2+Fab+J^3$; for otherwise we interchange a and b and are in Case 2 again. Hence $ba \in Fab + J^3$, and we may write $ba \equiv \alpha ab \pmod{J^3}$ for some element $\alpha \in F$. Now we set $b' := a + b$. Then we have

$$ab' = a^2 + ab, \qquad (b')^2 \equiv a^2 + (1+\alpha)ab + b^2 \pmod{J^3};$$

in particular, $J = F\{a, b', c\} + J^2$ and $J^2 = F\{a^2, ab', (b')^2\} + J^3$. Hence we may similarly assume that $b'a \in Fab' + J^3$. We write $b'a \equiv \beta ab' \pmod{J^3}$ with some element $\beta \in F$. Then

$$\beta a^2 + \beta ab \equiv \beta ab' \equiv b'a \equiv (a+b)a \equiv a^2 + ba \equiv a^2 + \alpha ab \pmod{J^3}.$$

Since a^2+J^3 and $ab+J^3$ are linearly independent this means that $\alpha = \beta = 1$; in particular, $[a, b] \in J^3$, and $J^2 = F[a, c] + F[b, c] + J^3$ contradicting the fact that $\dim J^2/J^3 = 3$.

Case 4. $d = bc$. In this case we may assume that $ac, ba, b^2 \in Fa^2+Fab+J^3$; for otherwise we are in Cases 1, 2 or 3 again. Then we replace c by $c-\alpha_1 a-\beta_1 b$ and may therefore assume that $0 = \alpha_1 = \beta_1$. Moreover, we may assume that

$$\begin{aligned} J^2 \neq (a+\xi b)J + J^3 &= F\{(a+\xi b)a, (a+\xi b)b, (a+\xi b)c\} + J^3 \\ &= F\{(1+\alpha_2\xi)a^2 + \beta_2\xi ab, \alpha_3\xi a^2 + (1+\beta_3\xi)ab, \xi bc\} + J^3 \end{aligned}$$

for $\xi \in F$; for otherwise we replace a by $a+\xi b$ and are in Case 1. Since $a^2 + J^3, ab + J^3, bc + J^3$ form a basis of J^2/J^3 this implies that

$$0 = \begin{vmatrix} 1+\alpha_2\xi & \beta_2\xi & 0 \\ \alpha_3\xi & 1+\beta_3\xi & 0 \\ 0 & 0 & \xi \end{vmatrix} = \xi + (\alpha_2+\beta_3)\xi^2 + (\alpha_2\beta_3 - \alpha_3\beta_2)\xi^3$$

for $\xi \in F$. Since F is infinite this is impossible.

Case 5. $d = ca$, i.e., $\alpha_5 = \beta_5 = 0, \gamma_5 = 1$. We may assume that $\gamma_i = 0$ for $i = 1, 2, 3, 4$; for otherwise we are in Cases 1, 2, 3, 4, respectively. Then we replace c by $c - \alpha_1 a - \beta_1 b$ and may therefore assume that $0 = \alpha_1 = \beta_1$. Moreover, $\beta_2 = 0$; for otherwise we are in Case 1. Similarly, we may assume that $\alpha_3 = 0$; for otherwise we interchange a and b and are then in Case 4 for the opposite algebra of A. Now we replace b by $b - \gamma_6 a$ and may then assume that $\gamma_6 = 0$. Furthermore, we may assume that $\alpha_7 = 0$; for otherwise we replace (a, b, c) by (b, c, a) and are then in Case 4 again. Finally, we may assume that $\beta_7 = 0$; for otherwise we interchange b and c and are then in Case 3 for the opposite algebra of A. As in Case 4, we may assume

$$\begin{aligned} J^2 \neq F\{&(\xi a + \eta b + c)a, (\xi a + \eta b + c)b, (\xi a + \eta b + c)c\} + J^3 \\ = F\{&(\xi + \alpha_2\eta)a^2 + ca, \alpha_6 a^2 + (\xi + \beta_3\eta + \beta_6)ab, \\ &\alpha_4\eta a^2 + \beta_4\eta ab + \gamma_7 ca\} + J^3 \end{aligned}$$

for $\xi, \eta \in F$. Since $a^2 + J^3, ab + J^3, ca + J^3$ form a basis of J^2/J^3 we may compute the corresponding determinant and obtain

$$\begin{aligned} 0 = \gamma_7\xi^2 &+ (\beta_3\gamma_7 + \alpha_2\gamma_7 - \alpha_4)\xi\eta + \beta_6\gamma_7\xi + (\alpha_2\beta_3\gamma_7 - \alpha_4\beta_3)\eta^2 \\ &+ (\alpha_2\beta_6\gamma_7 + \alpha_6\beta_4 - \alpha_4\beta_6)\eta \end{aligned}$$

for $\xi, \eta \in F$. Since F is infinite this implies that all coefficients on the right-hand side vanish; in particular, $0 = \gamma_7 = \alpha_4$. Then, similarly, we may assume that

$$\begin{aligned} J^2 &\neq F\{a(a+\eta b+c), b(a+\eta b+c), c(a+\eta b+c)\} + J^3 \\ &= F\{a^2+\eta ab, \alpha_2 a^2 + (\beta_3\eta+\beta_4)ab, \alpha_6\eta a^2 + \beta_6\eta ab + ca\} + J^3 \end{aligned}$$

for $\eta \in F$. Computing the corresponding determinant we obtain $0 = (\beta_3 - \alpha_2)\eta + \beta_4$ for $\eta \in F$. As before this implies that $\beta_3 = \alpha_2$ and $\beta_4 = 0$. Finally, we may assume that

$$\begin{aligned} J^2 &\neq F\{(\xi a+b+c)^2, (\xi a+b+c)a, a(\xi a+b+c)\} + J^3 \\ &= F\{(\xi^2+\alpha_2\xi+\alpha_6)a^2 + (\xi+\alpha_2+\beta_6)ab + \xi ca, \\ &\qquad (\xi+\alpha_2)a^2 + ca, \xi a^2 + ab\} + J^3 \end{aligned}$$

for $\xi \in F$; for otherwise we replace (a, b) by $(\xi a + b + c, a)$ and are in Case 2 again. Computing the corresponding determinant we obtain $0 = \xi^2 + (\alpha_2 + \beta_6)\xi - \alpha_6$ for $\xi \in F$ which is impossible.

Case 6. $d = cb$, i.e. $\alpha_6 = \beta_6 = 0, \gamma_6 = 1$. We may assume that $\gamma_i = 0$ for $i = 1, 2, \ldots, 5$; for otherwise we are in Cases $1, 2, \ldots, 5$, respectively. Then we replace c by $c - \alpha_1 a - \beta_1 b$ and may therefore assume that $0 = \alpha_1 = \beta_1$. We may also assume that $\beta_2 = 0$; for otherwise we are in Case 4 for the opposite algebra of A. Similarly, we may assume that $\alpha_3 = 0$; for otherwise we interchange a and b and are then in Case 1. As in the previous cases we may assume

$$\begin{aligned} J^2 &\neq F\{(\xi a+\eta b+c)a, (\xi a+\eta b+c)b, (\xi a+\eta b+c)c\} + J^3 \\ &= F\{(\xi+\alpha_2\eta+\alpha_5)a^2 + \beta_5 ab, (\xi+\beta_3\eta)ab + cb, \\ &\qquad (\alpha_4\eta+\alpha_7)a^2 + (\beta_4\eta+\beta_7)ab + \gamma_7 cb\} + J^3 \end{aligned}$$

for $\xi, \eta \in F$. We work out the corresponding determinant and obtain

$$\begin{aligned} 0 = \gamma_7\xi^2 + (\beta_3\gamma_7 + \alpha_2\gamma_7 - \beta_4)\xi\eta + (\alpha_5\gamma_7 - \beta_7)\xi + (\alpha_2\beta_3\gamma_7 - \alpha_2\beta_4)\eta^2 \\ + (\alpha_5\beta_3\gamma_7 - \alpha_2\beta_7 - \alpha_5\beta_4 + \alpha_4\beta_5)\eta + (\alpha_7\beta_5 - \alpha_5\beta_7) \end{aligned}$$

for $\xi, \eta \in F$. Therefore all coefficients on the right-hand side vanish; in particular, $0 = \gamma_7 = \beta_4 = \beta_7$. Similarly, we have

$$\begin{aligned} J^2 &\neq F\{a(\xi a+b+c), b(\xi a+b+c), c(\xi a+b+c)\} + J^3 \\ &= F\{\xi a^2+ab, (\alpha_2\xi+\alpha_4)a^2 + \beta_3 ab, (\alpha_5\xi+\alpha_7)a^2 + \beta_5\xi ab + cb\} + J^3 \end{aligned}$$

for $\xi \in F$. Computing the corresponding determinant we obtain $0 = (\beta_3 - \alpha_2)\xi - \alpha_4$ for $\xi \in F$ which again implies that $\beta_3 = \alpha_2$ and $\alpha_4 = 0$. We may also assume that

$$\begin{aligned} J^2 &\neq F\{(\xi a+\eta b+c)^2, (\xi a+\eta b+c)a, a(\xi a+\eta b+c)\} + J^3 \\ &= F\{(\xi^2+\alpha_2\xi\eta+\alpha_5\xi+\alpha_7)a^2 + (\xi\eta+\beta_5\xi+\alpha_2\eta^2)ab + \eta cb, \\ &\qquad (\xi+\alpha_2\eta+\alpha_5)a^2 + \beta_5 ab, \xi a^2 + \eta ab\} + J^3 \end{aligned}$$

for $\xi, \eta \in F$; for otherwise we replace (a, b, c) by $(\xi a + \eta b + c, a, b)$ and are then in Case 2 again. Working out the corresponding determinant we obtain $0 = \xi\eta^2 + \alpha_2\eta^3 - \beta_5\xi\eta + \alpha_5\eta^2$ for $\xi, \eta \in F$ which is impossible.

Case 7. $d = c^2$. In this case we may assume that $ac, ba, b^2, bc, ca, cb \in Fa^2 + Fab + J^3$; for otherwise we are in Cases $1, 2, \ldots, 6$, respectively. Then $J^2 = F\{[a,b],[a,c],[b,c]\} + J^3 \subset Fa^2 + Fab + J^3$; in particular, $\dim J^2/J^3 \leq 2$ contradicting (1.5).

(4.2) **Lemma.** *We may assume that* $d = ac$.

Proof. We assume the contrary. Then we may assume that $d = ba$, by (4.1); in particular, $\alpha_2 = \beta_2 = 0$, $\gamma_2 = 1$. We have $\gamma_1 = 0$. After replacing c by $c - \alpha_1 a - \beta_1 b$ we may even assume $0 = \alpha_1 = \beta_1$. Similarly, we may assume $\beta_5 = 0$. Moreover, after replacing b by $b - \gamma_3 a$ we may also assume that $\gamma_3 = 0$. We then have

$$\begin{aligned} J^2 &\neq (\xi a + \eta b + \zeta c)J + J^3 \\ &= F\{(\xi a + \eta b + \zeta c)a, (\xi a + \eta b + \zeta c)b, (\xi a + \eta b + \zeta c)c\} + J^3 \\ &= F\{(\xi + \alpha_5\zeta)a^2 + (\eta + \gamma_5\zeta)ba, (\alpha_3\eta + \alpha_6\zeta)a^2 + (\xi + \beta_3\eta + \beta_6\zeta)ab \\ &\qquad + \gamma_6\zeta ba, (\alpha_4\eta + \alpha_7\zeta)a^2 + (\beta_4\eta + \beta_7\zeta)ab + (\gamma_4\eta + \gamma_7\zeta)ba\} + J^3 \end{aligned}$$

for $\xi, \eta, \zeta \in F$. Since $a^2 + J^3$, $ab + J^3$, $ba + J^3$ form a basis of J^2/J^3 this implies that

$$\begin{aligned} 0 &= \begin{vmatrix} \xi + \alpha_5\zeta & 0 & \eta + \gamma_5\zeta \\ \alpha_3\eta + \alpha_6\zeta & \xi + \beta_3\eta + \beta_6\zeta & \gamma_6\zeta \\ \alpha_4\eta + \alpha_7\zeta & \beta_4\eta + \beta_7\zeta & \gamma_4\eta + \gamma_7\zeta \end{vmatrix} \\ &= \gamma_4\xi^2\eta + \gamma_7\xi^2\zeta + (\beta_3\gamma_4 - \alpha_4)\xi\eta^2 \\ &\quad + (\beta_3\gamma_7 + \beta_6\gamma_4 + \alpha_5\gamma_4 - \beta_4\gamma_6 - \alpha_7 - \alpha_4\gamma_5)\xi\eta\zeta \\ &\quad + (\beta_6\gamma_7 + \alpha_5\gamma_7 - \beta_7\gamma_6 - \alpha_7\gamma_5)\xi\zeta^2 + (\alpha_3\beta_4 - \alpha_4\beta_3)\eta^3 \\ &\quad + (\alpha_5\beta_3\gamma_4 + \alpha_3\beta_7 + \alpha_6\beta_4 + \alpha_3\beta_4\gamma_5 - \alpha_4\beta_6 - \alpha_7\beta_3 - \alpha_4\beta_3\gamma_5)\eta^2\zeta \\ &\quad + (\alpha_5\beta_3\gamma_7 + \alpha_5\beta_6\gamma_4 - \alpha_5\beta_4\gamma_6 + \alpha_6\beta_7 + \alpha_3\beta_7\gamma_5 + \alpha_6\beta_4\gamma_5 \\ &\qquad\qquad - \alpha_7\beta_6 - \alpha_4\beta_6\gamma_5 - \alpha_7\beta_3\gamma_5)\eta\zeta^2 \\ &\quad + (\alpha_5\beta_6\gamma_7 - \alpha_5\beta_7\gamma_6 + \alpha_6\beta_7\gamma_5 - \alpha_7\beta_6\gamma_5)\zeta^3 \end{aligned}$$

for $\xi, \eta, \zeta \in F$. Since F is infinite this implies that all coefficients on the right-hand side have to vanish; in particular, $0 = \gamma_4 = \gamma_7 = \alpha_4 = \alpha_3\beta_4$ and $\alpha_7 = -\beta_4\gamma_6$. Then, similarly, we have

$$\begin{aligned} J^2 &\neq F\{a(\xi a + \eta b + \zeta c), b(\xi a + \eta b + \zeta c), c(\xi a + \eta b + \zeta c)\} + J^3 \\ &= F\{\xi a^2 + \eta ab, \alpha_3\eta a^2 + (\beta_3\eta + \beta_4\zeta)ab + \xi ba, \\ &\qquad (\alpha_5\xi + \alpha_6\eta + \alpha_7\zeta)a^2 + (\beta_6\eta + \beta_7\zeta)ab + (\gamma_5\xi + \gamma_6\eta)ba\} + J^3 \end{aligned}$$

for $\xi, \eta, \zeta \in F$. As before, we work out the corresponding determinant and obtain

$$0 = (\beta_3\gamma_5 - \beta_6 + \alpha_5)\xi^2\eta + (\beta_4\gamma_5 - \beta_7)\xi^2\zeta + (\beta_3\gamma_6 - \alpha_3\gamma_5 + \alpha_6)\xi\eta^2 - \alpha_3\gamma_6\eta^3$$

for $\xi, \eta, \zeta \in F$. Again, this implies that $\beta_6 = \beta_3\gamma_5 + \alpha_5$, $\beta_7 = \beta_4\gamma_5$, $\alpha_6 = \alpha_3\gamma_5 - \beta_3\gamma_6$, $0 = \alpha_3\gamma_6$. On the other hand,

$$\begin{aligned} J^2 &= F\{[a,b],[a,c],[b,c]\} + J^3 \\ &= F\{ab - ba, \alpha_5 a^2 + \gamma_5 ba, (\beta_3\gamma_6 - \alpha_3\gamma_5)a^2 \\ &\qquad\qquad + (\beta_4 - \alpha_5 - \beta_3\gamma_5)ab - \gamma_6 ba\} + J^3. \end{aligned}$$

Since a^2+J^3, $ab+J^3$ and $ba+J^3$ form a basis of J^2/J^3 a computation of the corresponding determinant yields

$$0 \neq \alpha_5\gamma_6 + \beta_3\gamma_5\gamma_6 - \alpha_3\gamma_5^2 - \alpha_5\beta_4 + \alpha_5^2 + \alpha_5\beta_3\gamma_5 .$$

Moreover, since $J^2 = F\{a^2, ab, ba\} + J^3$, Lemma E in [6] implies that

$$J^3 = F\{a^3, a^2b, aba, ba^2, bab, b^2a\}+J^4 = F\{a^3, a^2b, aba, ba^2, bab\}+J^4 .$$

Now we distinguish two cases.

Case 1. $\alpha_5 \neq 0$. In this case we replace a by $\alpha_5 a$ and may then assume that $\alpha_5 = 1$. Thus

$$0 \equiv a(ca) - (ac)a \equiv a^3 + \gamma_5 aba \pmod{J^4},$$
$$0 \equiv b(ca) - (bc)a \equiv (\beta_3\gamma_5 - \alpha_3\gamma_5^2 - \beta_4)aba + ba^2 \pmod{J^4}.$$

Now we distinguish two more cases.

Case 1.1. $\beta_4 \neq 0$. In this case we have $\alpha_3 = 0$ since $0 = \alpha_3\beta_4$. Moreover,

$$0 \equiv (b^2)c - b(bc) \equiv \beta_3\beta_4 a^2b - \beta_4 bab \pmod{J^4},$$
$$0 \equiv a(c^2) - (ac)c \equiv \beta_4\gamma_5 a^2b + \beta_4\gamma_5\gamma_6 aba \pmod{J^4};$$

in particular, $J^3 = Fa^2b + Faba + J^4$. Hence $a^2b + J^4$ and $aba + J^4$ are linearly independent. Then $\gamma_5 = 0$, and we obtain the contradiction

$$0 \equiv a(cb) - (ac)b \equiv a^2b + \gamma_6 aba \pmod{J^4}.$$

Case 1.2. $\beta_4 = 0$. Here we have to distinguish two more cases.

Case 1.2.1. $\beta_3 \neq 0$. In this case we replace b by $\beta_3^{-1}b$ and may then assume that $\beta_3 = 1$. Then

$$0 \equiv (b^2)b - b(b^2) \equiv (1+\alpha_3)a^2b - \alpha_3^2\gamma_5^2 aba - bab \pmod{J^4},$$
$$0 \equiv a(cb) - (ac)b \equiv (1+\gamma_5)a^2b + (\gamma_5\gamma_6 - \alpha_3\gamma_5^2 + \gamma_6)aba \pmod{J^4};$$

in particular, $J^3 = Fa^2b+Faba+J^4$. Thus a^2b+J^4 and $aba+J^4$ are linearly independent. Then $\gamma_5 = -1$ and $\alpha_3 = 0$. But now we obtain the contradiction

$$\alpha_5\gamma_6 + \beta_3\gamma_5\gamma_6 - \alpha_3\gamma_5^2 - \alpha_5\beta_4 + \alpha_5^2 + \alpha_5\beta_3\gamma_5 = 0.$$

Case 1.2.2. $\beta_3 = 0$. Here we have

$$0 \equiv a(cb) - (ac)b \equiv a^2b + (\gamma_6 - \alpha_3\gamma_5^2)aba \pmod{J^4},$$
$$0 \equiv b(cb) - (bc)b \equiv \alpha_3^2\gamma_5^3 aba + bab \pmod{J^4};$$

in particular, $J^3 = Faba + J^4$, a contradiction.

Case 2. $\alpha_5 = 0$. Then

$$0 \neq \alpha_5\gamma_6 + \beta_3\gamma_5\gamma_6 - \alpha_3\gamma_5^2 - \alpha_5\beta_4 + \alpha_5^2 + \alpha_5\beta_3\gamma_5 = \beta_3\gamma_5\gamma_6 - \alpha_3\gamma_5^2;$$

in particular, $\gamma_5 \neq 0$. Now we replace b by $\gamma_5 b$ and may therefore assume that $\gamma_5 = 1$. Hence

$$0 \equiv a(ca) - (ac)a \equiv aba \pmod{J^4}.$$

We distinguish two more cases.

Case 2.1. $\alpha_3 \neq 0$. In this case $\beta_4 = \gamma_6 = 0$ since $0 = \alpha_3\beta_4 = \alpha_3\gamma_6$. We now replace a by $\sqrt{\alpha_3}a$ and may therefore assume that $\alpha_3 = 1$. Then

$$\begin{aligned} 0 &\equiv b(ca) - (bc)a \equiv a^3 \pmod{J^4}, \\ 0 &\equiv b(cb) - (bc)b \equiv ba^2 + \beta_3 bab \pmod{J^4}, \\ 0 &\equiv a(cb) - (ac)b \equiv \beta_3 a^2 b \pmod{J^4}, \\ 0 &\equiv (b^2)b - b(b^2) \equiv a^2 b \pmod{J^4}; \end{aligned}$$

in particular, $J^3 = Fbab + J^4$, a contradiction.

Case 2.2. $\alpha_3 = 0$. In this case we have $0 \neq \beta_3\gamma_6 - \alpha_3 = \beta_3\gamma_6$, i.e. $\beta_3 \neq 0 \neq \gamma_6$. We now replace a by $\beta_3 a$ and may then assume that $\beta_3 = 1$. We compute

$$\begin{aligned} 0 &\equiv a(cb) - (ac)b \equiv a^2 b - \gamma_6 a^3 \pmod{J^4}, \\ 0 &\equiv (b^2)b - b(b^2) \equiv \gamma_6 a^3 - bab \pmod{J^4}, \\ 0 &\equiv (bc)c - b(c^2) \equiv (\beta_4^2\gamma_6 - \beta_4\gamma_6)a^3 + \beta_4\gamma_6 ba^2 \pmod{J^4}; \end{aligned}$$

in particular, $J^3 = Fa^3 + Fba^2 + J^4$. Thus $a^3 + J^4$ and $ba^2 + J^4$ are linearly independent. Then $\beta_4 = 0$ since $\gamma_6 \neq 0$. But now we obtain the contradiction

$$0 \equiv b(cb) - (bc)b \equiv \gamma_6 a^3 - \gamma_6 ba^2 \pmod{J^4}.$$

In the remainder of this paper we may and will assume that $J^2 = F\{a^2, ab, ac\} + J^3$. Then $J^3 = F\{a^3, a^2b, a^2c\} + J^4$ and $J^4 = F\{a^4, a^3b, a^3c\} + J^5$ by Lemma E in [6]; in particular, $\dim J^3/J^4 \in \{2, 3\}$. Since $J^4 \neq J^5$ we have $a^3 \notin J^4$.

(4.3) **Lemma.** *The elements a, b, c can be chosen such that one of the following holds*:

(4.4) $0 = \alpha_2 = \beta_2 = \alpha_5$, $\gamma_2 = 1$, $\alpha_6 = \alpha_4 - 1$; $\beta_5 + \gamma_5 \neq 1$;
(4.5) $0 = \alpha_2 = \beta_2$, $\gamma_2 = \alpha_5 = 1$, $\gamma_5 = 1 - \beta_5$, $\beta_6 - \beta_4 + \gamma_6 - \gamma_4 \neq 0$;
(4.6) $0 = \alpha_2 = \gamma_2 = \alpha_5 = \beta_5$, $\gamma_5 = \beta_2 \neq 1$, $\alpha_4 = 1 \neq \alpha_6$.

Proof. We distinguish between two cases.

Case 1. $\gamma_2 \neq 0$. In this case we replace c by $\alpha_2 a + \beta_2 b + \gamma_2 c$ and may therefore assume that $0 = \alpha_2 = \beta_2$ and $\gamma_2 = 1$. Now we distinguish two more cases.

Case 1.1. $\beta_5 + \gamma_5 \neq 1$. In this case we set $\xi := \alpha_5/(\beta_5 + \gamma_5 - 1)$ and replace b by $b + \xi a$ and c by $c + \xi a$. Then we have $\alpha_5 = 0$. Hence

$$\begin{aligned} J^2 &= F\{[a, b], [a, c], [b, c]\} + J^3 \\ &= F\{ab - ac, \beta_5 ab + (\gamma_5 - 1)ac, (\alpha_4 - \alpha_6)a^2 \\ &\qquad + (\beta_4 - \beta_6)ab + (\gamma_4 - \gamma_6)ac\} + J^3; \end{aligned}$$

in particular, $\alpha_4 \neq \alpha_6$. Now we replace a by $(\alpha_4 - \alpha_6)^{1/2}a$ and may then assume that $\alpha_6 = \alpha_4 - 1$.

Case 1.2. $\beta_5 + \gamma_5 = 1$. In this case we have

$$\begin{aligned} J^2 &= F\{[a, b], [a, c], [b, c]\} + J^3 \\ &= F\{ab - ac, \alpha_5 a^2 + \beta_5 ab - \beta_5 ac, (\alpha_4 - \alpha_6)a^2 \\ &\qquad + (\beta_4 - \beta_6)ab + (\gamma_4 - \gamma_6)ac\} + J^3. \end{aligned}$$

Since $a^2 + J^3, ab + J^3, ac + J^3$ form a basis of J^2/J^3 we work out the corresponding determinant and obtain $0 \neq \alpha_5(\beta_6 - \beta_4 + \gamma_6 - \gamma_4)$, so $\beta_6 - \beta_4 + \gamma_6 - \gamma_4 \neq 0 \neq \alpha_5$. Then we replace a by $\alpha_5 a$ and may therefore assume that $\alpha_5 = 1$.

Case 2. $\gamma_2 = 0$. In this case we may assume that $\beta_5 = 0$; for otherwise we interchange b and c and are then in Case 1 again. Similarly, we may assume that $\gamma_5 = \beta_2$; otherwise we replace b by $b + c$ and are then in Case 1 again. Hence

$$\begin{aligned} J^2 &= F\{[a, b], [a, c], [b, c]\} + J^3 \\ &= F\{\alpha_2 a^2 + (\beta_2 - 1)ab, \alpha_5 a^2 + (\beta_2 - 1)ac, (\alpha_4 - \alpha_6)a^2 \\ &\qquad + (\beta_4 - \beta_6)ab + (\gamma_4 - \gamma_6)ac\} + J^3. \end{aligned}$$

Since $\dim J^2/J^3 = 3$ this implies that $\beta_2 \neq 1$. Now we replace b by $b + \alpha_2(\beta_2 - 1)^{-1}a$ and c by $c + \alpha_5(\beta_2 - 1)^{-1}a$ and may then assume that $0 = \alpha_2 = \alpha_5$. In this situation we have $\alpha_4 \neq 0$ or $\alpha_6 \neq 0$. If necessary, we interchange b and c and may then assume that $\alpha_4 \neq 0$. Finally we replace b by $\alpha_4^{-1} b$ and may therefore assume that $\alpha_4 = 1$.

Now we treat the cases above separately.

(4.7) **Lemma.** *The case* (4.4) *does not occur.*

Proof. We assume the contrary and distinguish two cases.

Case 1. $\dim J^3/J^4 = 3$. In this case the elements $a^3 + J^4$, $a^2 b + J^4$, $a^2 c + J^4$ form a basis of J^3/J^4. Since

$$\begin{aligned} 0 \equiv (b^2)a - b(ba) &\equiv (\alpha_3 - \alpha_7)a^3 + (\beta_5\gamma_3 - \beta_7)a^2 b \\ &\quad + (\beta_3 + \gamma_3\gamma_5 - \gamma_7)a^2 c \pmod{J^4} \end{aligned}$$

we conclude that $\alpha_7 = \alpha_3$, $\beta_7 = \beta_5\gamma_3$ and $\gamma_7 = \beta_3 + \gamma_3\gamma_5$. Similarly, using the fact that $0 = (bc)a - b(ca) + c(ba) - (cb)a + J^4$ we obtain $\beta_5 = -1$, so $\gamma_5 \neq 2$. This also shows that $\gamma_6 = \beta_4 - \beta_6 + \gamma_4$ and $0 = (2 - \gamma_5)(\beta_4 - \beta_6)$. Since $\gamma_5 \neq 2$ this implies that $\beta_6 = \beta_4$ and $\gamma_6 = \gamma_4$. Then, using the fact that $0 = (b^2)b - b(b^2) + J^4$ and $0 = (bc)b - b(cb) + J^4$ we see that $0 = (\alpha_3 - \alpha_4 + 1)(\beta_3 - \gamma_3) = (\alpha_3 - \alpha_4 + 1)(\beta_4 - \gamma_4)$. Now we distinguish two cases.

Case 1.1. $\alpha_4 \neq \alpha_3 + 1$. Then $\gamma_3 = \beta_3$ and $\gamma_4 = \beta_4$. Moreover, the fact that $0 = (bc)a - b(ca) + J^4$ implies that $0 = \beta_3\gamma_5$. We distinguish two more cases.

Case 1.1.1. $\gamma_5 \neq 0$, so $\beta_3 = 0$. In this case we use the fact that $0 = (bc)b - b(cb) + J^4$ to obtain $0 = \gamma_5(1 - \alpha_4)$, so $\alpha_4 = 1$. But this leads to a contradiction using the fact that $0 = (bc)b - b(cb) + J^4$ again.

Case 1.1.2. $\gamma_5 = 0$. In this case we use the fact that $0 = (bc)b - b(cb) + J^4$ to obtain $2\alpha_4 = 1$; in particular, $\operatorname{char} F \neq 2$. Then we use the fact that $0 = (bc)a - b(ca) + J^4$ to conclude that $\beta_4 = 0$, we use the fact that $0 = (c^2)a - c(ca) + J^4$ to see that $\beta_3 = 0$, and we use the fact that $0 = (bc)c - b(c^2) + J^4$ to show that $\alpha_4 = 0$. But this contradicts the fact that $0 = (bc)b - b(cb) + J^4$.

Case 1.2. $\alpha_4 = \alpha_3 + 1$. In this case the fact that $0 = (bc)a - b(ca) + J^4$ implies that $0 = 2\alpha_3 + 1 - \alpha_3\gamma_5$. Thus

$$\begin{aligned} J^2 &= F\{[a, b], [a, c], [b, c], [a, ab], [a, ac], [b, a^2], \\ &\qquad [b, ab], [b, ac], [c, a^2], [c, ab], [c, ac]\} + J^4 \\ &\subset F\{[a, b], [a, c], [b, c], a^2b, a^2c\} + J^4 \end{aligned}$$

as is easily checked. But this is a contradiction since $\dim J^2/J^4 = 6$.

Case 2. $\dim J^3/J^4 = 2$. Here we distinguish two more cases.

Case 2.1. $a^2b \in Fa^3 + J^4$. In this case we have $J^3 = F\{a^3, a^2b, a^2c\} + J^4 = F\{a^3, a^2c\} + J^4$ and write $a^2b \equiv \delta a^3 \pmod{J^4}$ with some element $\delta \in F$. Then $a^3c \equiv a^2ba \equiv \delta a^4 \pmod{J^5}$, so $J^4 = F\{a^4, a^3c\} + J^5 = Fa^4 + J^5$. Since $J^4 \neq J^5$ this implies that $\dim J^4/J^5 = 1$. By Lemma G in [6], $J^3 \subset Z$; in particular, $a^2c \in Z$. Hence

$$0 \equiv (a^2c)a - a(a^2c) \equiv a^2(ca) - a^3c \equiv (\beta_5 + \gamma_5 - 1)\delta a^3 \pmod{J^4}.$$

Since $\beta_5 + \gamma_5 \neq 1$ and $a^3 \notin J^4$ we conclude that $\delta = 0$. But now

$$\begin{aligned} &0 \equiv a^2(bc) - (a^2b)c \equiv \alpha_4 a^4 \pmod{J^5}, \\ &0 \equiv a^2(b^2) - (a^2b)b \equiv \alpha_3 a^4 \pmod{J^5}, \\ &0 \equiv (b^2)a - b(ba) \equiv (\alpha_3 - \alpha_7)a^3 + (\beta_3 + \gamma_3\gamma_5 - \gamma_7)a^2c \pmod{J^4}, \\ &0 \equiv (cb)a - c(ba) \equiv (\alpha_4 - 1 - \alpha_4\beta_5 - \alpha_7\gamma_5)a^3 \\ &\qquad + (\beta_6 + \gamma_5\gamma_6 - \beta_5\gamma_4 - \gamma_5\gamma_7)a^2c \pmod{J^4}. \end{aligned}$$

This leads to the contradiction $0 = \alpha_4 = \alpha_3 = \alpha_7 = -1$.

Case 2.2. $a^2b \notin Fa^3 + J^4$. Since $a^3 \notin J^4$ and $\dim J^3/J^4 = 2$ the elements $a^3 + J^4$ and $a^2b + J^4$ form a basis of J^3/J^4 in this case. We write $a^2c \equiv \delta a^3 + \varepsilon a^2b \pmod{J^4}$ with elements $\delta, \varepsilon \in F$. Since $J^4 = Fa^4 + Fa^3b + J^5$ and $J^4 \neq J^5$ we have $\dim J^4/J^5 \in \{1, 2\}$. Let us distinguish the corresponding cases.

Case 2.2.1. $\dim J^4/J^5 = 1$. In this case $J^3 \subset Z$ by Lemma G in [6]; in particular, $a^2b, a^2c \in Z$. Hence

$$\begin{aligned} 0 &\equiv (a^2b)a - a(a^2b) \equiv a^2(ba) - a^3b \\ &\equiv a^3c - a^3b \equiv \delta a^4 + (\varepsilon - 1)a^3b \pmod{J^5}, \\ 0 &\equiv (a^2c)a - a(a^2c) \equiv a^2(ca) - a^3c \equiv (\gamma_5 - 1)\delta a^4 + (\beta_5 + \gamma_5\varepsilon - \varepsilon)a^3b \\ &\equiv (\beta_5 + \gamma_5 - 1)a^3b \pmod{J^5}. \end{aligned}$$

Since $\beta_5 + \gamma_5 \neq 1$ this implies that $a^3b \in J^5$. Hence $J^4 = Fa^4 + J^5$ and $\delta a^4 \in J^5$. Since $\dim J^4/J^5 = 1$ we must have $\delta = 0$. Therefore

$$\begin{aligned} 0 &\equiv (b^2)a - b(ba) \equiv (\alpha_3 - \alpha_7)a^3 \\ &\quad + (\beta_5\gamma_3 - \beta_7 + \beta_3\varepsilon + \gamma_3\gamma_5\varepsilon - \gamma_7\varepsilon)a^2b \pmod{J^4}; \end{aligned}$$

in particular, $\alpha_7 = \alpha_3$. Similarly, using the fact that $0 = (bc)a - b(ca) + c(ba) - (cb)a + J^4$ we see that $\beta_5 = -1$; in particular, $\gamma_5 \neq 2$. Hence

$$\begin{aligned} &0 \equiv a^2(cb) - (a^2c)b \equiv (\alpha_4 - 1 - \alpha_3\varepsilon)a^4 \pmod{J^5}, \\ &0 \equiv a^2(c^2) - (a^2c)c \equiv (\alpha_3 - \alpha_4\varepsilon)a^4 \pmod{J^5}. \end{aligned}$$

Since $a^4 \notin J^5$ this implies that $\alpha_3 = \alpha_4\varepsilon$ and $\alpha_4 - \alpha_4\varepsilon^2 = 1$; in particular, $\alpha_4 \neq 0$ and $\varepsilon^2 \neq 1$. But since $a^2c \in Z$ we have

$$\begin{aligned} 0 &\equiv (a^2c)b - b(a^2c) \equiv (a^2c)b - (ba)(ac) \\ &\equiv (\varepsilon^2 + 1 - \gamma_5\varepsilon)\alpha_4 a^4 \pmod{J^5}, \\ 0 &\equiv (a^2c)b - b(a^2c) \equiv (2 - \gamma_5\varepsilon)\alpha_4\varepsilon^2 a^4 \pmod{J^5}. \end{aligned}$$

Hence $\gamma_5\varepsilon = 2$ and $\varepsilon^2 = 1$, a contradiction.

Case 2.2.2. $\dim J^4/J^5 = 2$. In this case

$$\begin{aligned} 0 &\equiv (a^2c)a - a^2(ca) \equiv (\delta + \delta\varepsilon - \gamma_5\delta)a^4 + (\varepsilon^2 - \beta_5 - \gamma_5\varepsilon)a^3b \pmod{J^5}, \\ 0 &\equiv (a^2c)b - a^2(cb) \equiv (\alpha_3\varepsilon + \gamma_3\delta\varepsilon - \alpha_4 + 1 - \gamma_6\delta)a^4 \\ &\quad + (\delta + \beta_3\varepsilon + \gamma_3\varepsilon^2 - \beta_6 - \gamma_6\varepsilon)a^3b \pmod{J^5}, \\ 0 &\equiv (a^2c)c - a^2c^2 \equiv (\alpha_4\varepsilon + \gamma_4\delta\varepsilon - \alpha_7 - \gamma_7\delta + \delta^2)a^4 \\ &\quad + (\delta\varepsilon + \beta_4\varepsilon + \gamma_4\varepsilon^2 - \beta_7 - \gamma_7\varepsilon)a^3b \pmod{J^5}. \end{aligned}$$

Since $a^4 + J^5$ and $a^3b + J^5$ form a basis of J^4/J^5 this implies that all coefficients on the right-hand side vanish; in particular, $0 = \delta + \delta\varepsilon - \gamma_5\delta$. Assume that $\delta \neq 0$. Then $\varepsilon = \gamma_5 - 1$ and we obtain the contradiction $0 = \varepsilon^2 - \beta_5 - \gamma_5\varepsilon = 1 - \beta_5 - \gamma_5$. Hence we must have $\delta = 0$. Therefore

$$\begin{aligned} 0 &\equiv (b^2)a - b(ba) \equiv (\alpha_3 - \alpha_7)a^3 \\ &\quad + (\beta_5\gamma_3 - \beta_7 + \beta_3\varepsilon + \gamma_3\gamma_5\varepsilon - \gamma_7\varepsilon)a^2b \pmod{J^4}; \end{aligned}$$

in particular, $\alpha_7 = \alpha_3$. Similarly, using the fact that $0 = (bc)a - b(ca) + c(ba) - (cb)a + J^4$ we see that $\beta_5 = -1$. Hence $\varepsilon^2 - \gamma_5\varepsilon = -1$; in particular, $\varepsilon \neq 0$. Therefore $0 = \alpha_3\varepsilon + \gamma_3\delta\varepsilon - \alpha_4 + 1 - \gamma_6\delta = \alpha_3\varepsilon - \alpha_4 + 1$, and $\alpha_4 = \alpha_3\varepsilon + 1$. Hence $0 = \alpha_4\varepsilon + \gamma_4\delta\varepsilon - \alpha_7 - \gamma_7\delta + \delta^2 = \alpha_3\varepsilon^2 + \varepsilon - \alpha_3$; in particular, $\alpha_3 \neq 0$ and $\varepsilon^2 \neq 1$. But this leads to the contradiction

$$\begin{aligned} 0 &\equiv b(a^2c) - (ba)ac \\ &\equiv -\varepsilon^2a^4 + (\beta_6\gamma_5\varepsilon - \beta_3\varepsilon + \gamma_5\gamma_6\varepsilon^2 - \gamma_3\varepsilon^2 - \beta_7\gamma_5 + \beta_4 - \gamma_5\gamma_7\varepsilon + \gamma_4\varepsilon)a^3b \\ &\qquad \pmod{J^5}. \end{aligned}$$

(4.8) **Lemma.** *The case* (4.5) *does not occur.*

Proof. We assume the contrary and distinguish two cases.

Case 1. $\dim J^3/J^4 = 3$. In this case we have

$$\begin{aligned} 0 &\equiv (bc)a - b(ca) \\ &\equiv (\alpha_4 + \gamma_4 - 1 - \alpha_6\beta_5 - \alpha_7 + \alpha_7\beta_5)a^3 + (\beta_5\gamma_4 - \beta_5 - \beta_5\beta_6 - \beta_7 + \beta_5\beta_7)a^2b \\ &\quad + (\beta_4 + \gamma_4 - \beta_5\gamma_4 - 1 + \beta_5 - \beta_5\gamma_6 - \gamma_7 + \beta_5\gamma_7)a^2c \pmod{J^4}. \end{aligned}$$

Since $a^3 + J^4$, $a^2b + J^4$, $a^2c + J^4$ form a basis of J^3/J^4 we obtain

$$0 = \alpha_4 + \gamma_4 - 1 - \alpha_6\beta_5 - \alpha_7 + \alpha_7\beta_5, \tag{4.9}$$

$$0 = \beta_5\gamma_4 - \beta_5 - \beta_5\beta_6 - \beta_7 + \beta_5\beta_7, \tag{4.10}$$

$$0 = \beta_4 + \gamma_4 - \beta_5\gamma_4 - 1 + \beta_5 - \beta_5\gamma_6 - \gamma_7 + \beta_5\gamma_7. \tag{4.11}$$

Similarly, using the fact that $0 \equiv (ca)b - c(ab) \pmod{J^4}$ we obtain the following equations:

$$(4.12)\qquad 0 = \alpha_6 + \gamma_6 - \alpha_4\beta_5 - \alpha_7 + \alpha_7\beta_5,$$
$$(4.13)\qquad 0 = \beta_5\gamma_6 - \beta_4\beta_5 - \beta_7 + \beta_5\beta_7,$$
$$(4.14)\qquad 0 = \beta_6 + \gamma_6 - \beta_5\gamma_6 - 1 - \beta_5\gamma_4 - \gamma_7 + \beta_5\gamma_7.$$

Now we add (4.10) and (4.11) and subtract (4.13) and (4.14) from the result to obtain $0 = (\beta_5+1)(\beta_4-\beta_6+\gamma_4-\gamma_6)$. Hence $\beta_5 = -1$. Then we subtract (4.12) from (4.9) and obtain $0 = \gamma_4 - \gamma_6 - 1$. Hence $\gamma_6 = \gamma_4 - 1$. Next we subtract (4.14) from (4.11) and obtain $0 = \beta_4 - \beta_6$. Hence $\beta_6 = \beta_4$. Then we use the fact that $b(ba) \equiv (b^2)a \pmod{J^4}$ to obtain that $\alpha_7 = \alpha_3 + \gamma_3$, $\beta_7 = -\gamma_3$ and $\gamma_7 = \beta_3 + 2\gamma_3$. Now (4.10) implies that $\beta_4 = \gamma_4 - 1 - 2\gamma_3$. Using the fact that $0 \equiv (c^2)a - c(ca) \pmod{J^4}$ we obtain the following equations:

$$(4.15)\qquad 0 = \beta_3 - \gamma_3 - 3 - 4\alpha_3 + 2\alpha_4 + 2\alpha_6,$$
$$(4.16)\qquad 0 = 4\gamma_4 - 2\beta_3 - 1 - 6\gamma_3;$$

in particular, $\operatorname{char} F \neq 2$. Now (4.11) forces $0 = 4 + 2\beta_3 + 6\gamma_3 - 4\gamma_4$, so $\beta_3 = 2\gamma_4 - 3\gamma_3 - 2$. Next we multiply (4.9) by 2 and subtract (4.15) to obtain $0 = 3$, so $\operatorname{char} F = 3$. Thus

$$\begin{aligned} J^2 = F\{&[a,b],[a,c],[b,c],[a,ab],[a,ac],[b,a^2],\\ &[b,ab],[b,ac],[c,a^2],[c,ab],[c,ac]\} + J^4 \\ \subset F\{&[a,b],[a,c],[b,c],a^3,a^2b-a^2c\} + J^4 \end{aligned}$$

as is easily checked. But this contradicts the fact that $\dim J^2/J^4 = 6$.

Case 2. $\dim J^3/J^4 = 2$. We distinguish two more cases.

Case 2.1. $a^2b \in Fa^3 + J^4$. In this case $J^3 = F\{a^3, a^2b, a^2c\} + J^4 = F\{a^3, a^2c\} + J^4$, and $a^2b \equiv \delta a^3 \pmod{J^4}$ for some element $\delta \in F$. Since $a^3c \equiv a^2ba \equiv \delta a^4 \pmod{J^5}$ we see that $J^4 = Fa^4 + Fa^3c + J^5 = Fa^4 + J^5$. Since $J^4 \neq J^5$ this implies that $\dim J^4/J^5 = 1$. Now Lemma G in [6] shows that $J^3 \subset Z$; in particular, $a^2c \in Z$. But this leads to the contradiction $0 \equiv (a^2c)a - a(a^2c) \equiv a^2(ca) - a^3c \equiv a^4 \pmod{J^5}$.

Case 2.2. $a^2b \notin Fa^3 + J^4$. Since $a^3 \notin J^4$ and $\dim J^3/J^4 = 2$ the elements $a^3 + J^4$ and $a^2b + J^4$ form a basis of J^3/J^4 in this case. We write $a^2c \equiv \delta a^3 + \varepsilon a^2b \pmod{J^4}$ with elements $\delta, \varepsilon \in F$. Since $J^4 = Fa^4 + Fa^3b + J^5$ and $J^4 \neq J^5$ we have $\dim J^4/J^5 \in \{1, 2\}$. Let us distinguish the corresponding cases.

Case 2.2.1. $\dim J^4/J^5 = 2$. In this case the elements $a^4 + J^5$ and $a^3b + J^5$ form a basis of J^4/J^5. Since

$$0 \equiv (a^2c)a - a^2(ca) \equiv (\delta\varepsilon + \beta_5\delta - 1)a^4 + (\varepsilon - 1)(\varepsilon + \beta_5)a^3b \pmod{J^5}$$

this implies that $\delta\varepsilon + \beta_5\delta - 1 = 0$ and $(\varepsilon - 1)(\varepsilon + \beta_5) = 0$. The first equation forces $\varepsilon \neq -\beta_5$, so $\varepsilon = 1$ by the second equation. Then, using the fact that $0 \equiv (bc)a - b(ca) + c(ba) - (cb)a \pmod{J^4}$ we obtain the contradiction $0 = (1 + \beta_5)(\beta_4 - \beta_6 + \gamma_4 - \gamma_6)$.

Case 2.2.2. $\dim J^4/J^5 = 1$. In this case $J^3 \subset Z$ by Lemma G in [6]; in particular, $a^2b, a^2c \in Z$. Hence $a^3b \equiv a^2ba \equiv a^3c \pmod{J^5}$ and $a^3b \equiv a^3c \equiv a^2ca \equiv a^4 + a^3b \pmod{J^5}$, so $a^4 \in J^5$ and $J^4 = Fa^3b + J^5$. Furthermore, since $a^3b \equiv a^3c \equiv \varepsilon a^3b \pmod{J^5}$ we must have $\varepsilon = 1$. Using the fact that $0 \equiv (bc)a - b(ca) + c(ba) - (cb)a \pmod{J^4}$ we obtain $0 = (1+\beta_5)(\beta_4 - \beta_6 + \gamma_4 - \gamma_6)$, so $\beta_5 = -1$. Similarly, using the fact that $0 \equiv (b^2)a - b(ba) \pmod{J^4}$ we obtain $\gamma_7 = \beta_3 + \gamma_3 - \beta_7$. Then $0 \equiv (a^2c)b - b(a^2c) \equiv (a^2c)b - (ba)ac \pmod{J^5}$ implies that $\delta = 1 + \beta_3 + \gamma_3 - \beta_4 - \gamma_4$. But now the fact that $0 \equiv a^2(c^2) - (a^2c)c \pmod{J^5}$ leads to a contradiction.

(4.17) **Lemma.** *The case* (4.6) *does not occur.*

Proof. We assume the contrary and distinguish two cases.

Case 1. $\dim J^3/J^4 = 3$. In this case the elements $a^3 + J^4$, $a^2b + J^4$, $a^2c + J^4$ form a basis of J^3/J^4. Since $\beta_2 \neq 1$ and

$$0 \equiv (bc)a - b(ca) \equiv (1 - \beta_2^2)a^3 + \beta_4(\beta_2 - \beta_2^2)a^2b + \gamma_4(\beta_2 - \beta_2^2)a^2c \pmod{J^4}$$

this implies that $\beta_2 = -1$; in particular, $\operatorname{char} F \neq 2$. Hence $0 = 2\beta_4 = 2\gamma_4$, so $0 = \beta_4 = \gamma_4$. Then, using similarly the fact that $0 \equiv c(ba) - (cb)a \pmod{J^4}$ we obtain $0 = \beta_6 = \gamma_6$. But now the fact that $0 \equiv (bc)b - b(cb) \pmod{J^4}$ leads to a contradiction.

Case 2. $\dim J^3/J^4 = 2$. We distinguish two more cases.

Case 2.1. $a^2b \in Fa^3 + J^4$. In this case we have $J^3 = F\{a^3, a^2b, a^2c\} + J^4 = Fa^3 + Fa^2c + J^4$ and $J^4 = Fa^4 + Fa^3c + J^5$. Assume that $a^4 \in J^5$. Then $J^4 = Fa^3c + J^5$; in particular, $\dim J^4/J^5 = 1$ since $J^4 \neq J^5$. Hence Lemma G in [6] implies that $J^3 \subset Z$; in particular, $a^2c \in Z$. But this leads to the contradiction $a^3c \equiv a^2ca \equiv \beta_2a^3c \pmod{J^5}$.

We write $a^2b \equiv \delta a^3 \pmod{J^4}$ with some element $\delta \in F$. Then $\delta a^4 \equiv a^2ba \equiv \beta_2a^3b \equiv \beta_2\delta a^4 \pmod{J^5}$. Since $a^4 \notin J^5$ and $\beta_2 \neq 1$ this implies that $\delta = 0$. As in Case 1, we now use the fact that $0 \equiv (bc)a - b(ca) \pmod{J^4}$ to obtain that $\beta_2 = -1$, $\operatorname{char} F \neq 2$ and $\gamma_4 = 0$. Similarly, using the fact that $0 \equiv (cb)a - c(ba) \equiv (b^2)a - b(ba) \pmod{J^4}$ we obtain $0 = 2\gamma_6 = 2\gamma_3$, so $0 = \gamma_6 = \gamma_3$. But this yields a contradiction using the fact that $0 \equiv (bc)c - b(c^2) \pmod{J^4}$.

Case 2.2. $a^2b \notin Fa^3 + J^4$. Since $a^3 \notin J^4$ and $\dim J^3/J^4 = 2$ the elements $a^3 + J^4$ and $a^2b + J^4$ form a basis of J^3/J^4 in this case, and $J^4 = Fa^4 + Fa^3b + J^5$. Assume that $a^4 \in J^5$. Then $J^4 = Fa^3b + J^5$; in particular, $\dim J^4/J^5 = 1$ since $J^4 \neq J^5$. Hence Lemma G in [6] implies that $J^3 \subset Z$; in particular, $a^2b \in Z$. But now we obtain the contradiction $a^3b \equiv a^2ba \equiv \beta_2a^3b \pmod{J^5}$.

Hence $a^4 \notin J^5$, and we write $a^2c \equiv \delta a^3 + \varepsilon a^2b \pmod{J^4}$ with elements $\delta, \varepsilon \in F$. Then $0 \equiv (a^2c)a - a^2(ca) \equiv (1 - \beta_2)\delta a^4 \pmod{J^5}$, so $\delta = 0$ since $\beta_2 \neq 1$ and $a^4 \notin J^5$. As in Case 1, we now use the fact that $0 \equiv (bc)a - b(ca) \pmod{J^4}$ to obtain $\beta_2 = -1$ and $\operatorname{char} F \neq 2$. Then we distinguish two more cases.

Case 2.2.1. $\dim J^4/J^5 = 2$. In this case the elements $a^4 + J^5$ and $a^3b + J^5$ form a basis of J^4/J^5. Using the fact that $0 \equiv b(a^2c) - (ba)ac \pmod{J^5}$

we obtain $\alpha_3\varepsilon = 1$. But this leads to a contradiction using the fact that $0 \equiv (a^2c)b - a^2(cb) \pmod{J^4}$.

Case 2.2.2. $\dim J^4/J^5 = 1$. In this case we have $J^4 = Fa^4 + J^5$ since $a^4 \notin J^5$. Moreover, Lemma G in [6] implies that $J^3 \subset Z$; in particular, $a^2b \in Z$. Thus $a^3b \equiv a^2ba \equiv -a^3b \pmod{J^5}$, so $a^3b \in J^5$ since $\operatorname{char} F \neq 2$. This, however, leads to a contradiction using the fact that $0 \equiv (a^2c)b - b(a^2c) \equiv a^2(cb) - (ba)ac \pmod{J^5}$.

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