TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 329, Number 2, February 1992

# ADAMS' COBAR EQUIVALENCE

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ABSTRACT. Let $F$ be the homotopy fibre of a continuous map $Y \xrightarrow{\omega} X$, with $X$ simply connected. We modify and extend a construction of Adams to obtain equivalences of DGA's and DGA modules,

$$\Omega C_*(X) \xrightarrow{\simeq} CU_*(\Omega X),$$

and

$$\Omega(C_*^{\omega}(Y);\, C_*(X)) \xrightarrow{\simeq} CU_*(F),$$

where on the left-hand side $\Omega(-)$ denotes the cobar construction. Our equivalences are natural in $X$ and $\omega$. Using this result we show how to read off the algebra $H_*(\Omega X;\, R)$ and the $H_*(\Omega X;\, R)$ module, $H_*(F;\, R)$, from free models for the singular cochain algebras $CS^*(X)$ and $CS^*(Y)$; here we assume $R$ is a principal ideal domain and $X$ and $Y$ are of finite $R$ type.

## 1. INTRODUCTION

In this article we consider modules, chain complexes, homology, ... defined over some arbitrary commutative ring, $R$. We denote $\otimes_R$ and $\mathrm{Hom}_R$ simply by $\otimes$ and Hom. If $V$ is a graded $R$ module then $sV$ is defined by $(sV)_i = V_{i-1}$.

Consider a continuous map $\omega: Y \to X$, in which $(X, x_0)$ is a simply-connected pointed space. By slightly altering ordinary singular chains (which we denote by $CS_*(-)$) one obtains from $CS_*(\omega)$ an equivalent morphism,

$$C_*(\omega): C_*^{\omega}(Y) \to C_*(X),$$

of differential graded coalgebras (DGC), with $C_*(X)$ itself one-connected.

A well-known theorem of Eilenberg and Moore [4, Theorem 12.1] asserts that the homology of the homotopy fibre, $F$, of $\omega$ can be computed from $C_*(\omega)$ via the formula

$$H_*(F) \cong \mathrm{Cotor}^{C_*(X)}(C_*^{\omega}(Y);\, R). \tag{1.1}$$

Received by the editors July 21, 1989 and, in revised form, September 18, 1989.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 55P35, 55R20.
*Key words and phrases.* Cobar construction, homotopy fiber, free model, Eilenberg-Moore spectral sequence.
The first author is Maître de recherche au FNRS.
Research of the second author was partially supported by an NSERC operating grant.
Research of the third author was supported by U.R.A. au CNRS 751.
Research of the authors was partially supported by a NATO travel grant.

When $\omega$ is the inclusion $\{x_0\} \to X$ then $F = \Omega X$ is the space of Moore loops (of variable length) in $X$. In this case Adams [1] had already obtained an isomorphism of the form (1.1), which he constructed by a passage to homology from a differential graded algebra (DGA) morphism of the form

$$\phi\colon \Omega C_*(X) \xrightarrow{\simeq} CU_*(\Omega X).$$

Here $CU_*(-)$ denotes singular cubical chains modulo degeneracies as in [11, p. 439]. $CU_*(-)$ carries the natural DGA structure induced from loop composition in $\Omega X$ via the natural map $CU_k(-) \otimes CU_l(-) \to CU_{k+l}(- \times -)$, which converts the tensor product of singular cubes to the geometric product. And $\Omega C_*(X)$ is Adams' *cobar construction* [1] on the DGC, $C_*(X)$; we recall the definition in §2.

As Adams observes in [1], his isomorphism $H(\phi)$ is natural in $X$. Our first objective here is, to modify Adams' construction, to obtain naturality at the chain level.

**Theorem I.** *For simply-connected pointed spaces* $(X, x_0)$ *there is a natural transformation of* DGA*'s,*

$$\psi_X\colon \Omega C_*(X) \to CU_*(\Omega X),$$

*for which* $H(\psi_X)$ *is an isomorphism.*

Next, we extend this construction to the case of arbitrary continuous maps $\omega\colon Y \to X$, as follows. Recall that a DGA *module* over a DGA, $A$, is an $A$-module $M$ with a differential satisfying $d(m \cdot a) = dm \cdot a + (-1)^{\deg m} m \cdot da$. DGC comodules are defined analogously. If $M$ is a right DGC comodule over the supplemented DGC, $C$ then [9, p. 25] the comodule structure determines a differential in $M \otimes \Omega C$, which makes this into a DGA module over $\Omega C$. We denote this DGA module by $\Omega(M; C)$. (The definition of the differential in $\Omega(M; C)$ will be recalled in §2.) When $C_0 = R$ and $C_1 = 0 = C_{<0}$ ($C$ is 1-*connected*) then $H(\Omega(M; C))$ is exactly $\mathrm{Cotor}^C(M; R)$, cf. [9, p. 25].

Now consider $Y \xrightarrow{\omega} X$. The morphism $C_*(\omega)$ makes $C_*^{\omega}(Y)$ into a $C_*(X)$ comodule, so that we have the DGA module $\Omega(C_*^{\omega}(Y); C_*(X))$ over $\Omega C_*(X)$. On the other hand, the natural right action of $\Omega X$ on the homotopy fibre, $F$, of $\omega$ makes $CU_*(F)$ into a DGA module over $CU_*(\Omega X)$. A model for this action, based on the Adams-Hilton construction, is given in [6]. Here we shall prove

**Theorem II.** *For continuous maps* $\omega\colon Y \to X$, *with* $(X, x_0)$ *a simply-connected pointed space, there is a natural transformation of* DGA *modules,*

$$\psi_\omega\colon \Omega(C_*^{\omega}(Y); C_*(X)) \to CU_*(F),$$

(*over* $\psi_X$) *for which* $H(\psi_\omega)$ *is an isomorphism.*

Notice that the effect of Theorem II is to exhibit the Eilenberg-Moore isomorphism (1.1) in the form $H(\psi_\omega)$, where $\psi_\omega$ is a natural quism of DGA modules. (A *quism* in a differential category is a morphism inducing an isomorphism in homology, and is denoted by $\xrightarrow{\simeq}$.) The usefulness of this kind of observation lies in the fact that equivalence classes of DGA's or of DGA modules, under the equivalence relation generated by quisms, carry a great deal of information which is lost on passage to homology. Indeed (1.1) itself is a striking illustration of this: $H_*(F)$ can be read off from the equivalence class of $C_*(\omega)$ (cf. Remark 2.3 in §2), whereas it cannot be calculated from $H_*(\omega)$.

It can be inconvenient to work with cubical chains. There is, fortunately, a natural quism $\Sigma$: $CU_* \to CN_*$ where $CN_*$ denotes ordinary singular theory modulo degeneracies. It is defined as follows: order the vertices of $I^n$ by setting $(\varepsilon_1, \dots, \varepsilon_n) \le (\delta_1, \dots, \delta_n)$ if each $\varepsilon_i \le \delta_i$ and triangulate $I^n$ by using the linearly ordered subsets as simplices. The signed sum of the $n$ simplices (sign determined by the orientation) is then a chain $c_n$ and $\Sigma$ is defined by the map $\sigma \mapsto CS(\sigma)(c_n)$, $\sigma$ a singular $n$-cube.

It is easy to see that $\Sigma$ converts the natural map $CU_*(-) \otimes CU_*(-) \to CU_*(- \times -)$ into the natural map $CN_*(-) \otimes CN_*(-) \to CN_*(- \times -)$. Thus in Theorems I and II above we may set $\phi_X = \Sigma \circ \psi_X$ and $\phi_\omega = \Sigma \circ \psi_\omega$ to obtain:

**Theorem III.** *Given a pointed simply-connected space $(X, x_0)$ and a continuous map $\omega$: $Y \to X$ with homotopy fiber, $F$, there are quisms, natural in $X$ and $\omega$,*

$$\phi_X : \Omega C_*(X) \xrightarrow{\simeq} CN_*(\Omega X), \text{ and}$$
$$\phi_\omega : \Omega(C_*^\omega(Y); C_*(X)) \xrightarrow{\simeq} CN_*(F),$$

*respectively of* DGA*'s and* DGA *modules.*

The correspondence $X \leftrightarrow \Omega X$ is a well-known equivalence between simply-connected spaces and connected associative $H$-spaces. In [13] Stasheff shows that it extends to an equivalence

$$X \leftrightarrow \Omega X \quad \text{and} \quad (Y \xrightarrow{\omega} X) \leftrightarrow (F \times \Omega X \to F),$$

where $F$ is the homotopy fiber of $\omega$.

On the other hand, Moore [10], [8, Chapter II] defines adjoint equivalences

$$C \leftrightarrow \Omega C \quad \text{and} \quad (N; C) \leftrightarrow (\Omega(N; C); \Omega C),$$

between appropriate DGC and DGA categories; we call this *Moore duality.* Thus Theorem III asserts that the geometric equivalences above are transformed by the chain DGC into Moore duality.

Another consequence of this machine is the deduction, as an immediate corollary of Theorem III, of the dual result. We use $B(-)$ to denote the bar construction and consider, as in Theorems II and III, a simply-connected pointed space $(X, x_0)$ and a continuous map $\omega$: $Y \to X$ with homotopy fibre, $F$.

**Theorem IV.** *Let $(X, x_0)$, $\omega$, $F$ be as in Theorem* III. *There are then natural sequences of quisms of* DGC*'s and* DGC *comodules connecting*

$$CS_*(X) \simeq B[CS_*(\Omega X)],$$

*and*

$$CS_*(Y) \simeq B[CS_*(F); CS_*(\Omega X)].$$

*Proof.* We construct quisms as indicated in the diagram below.

$$\begin{array}{ccc} CS_*(Y)\ ;\ CS_*(X) & & B(CS_*(F); CS_*(\Omega X))\ ;\ BCS_*(\Omega X) \\ \uparrow \simeq & & \uparrow \simeq \\ C_*^\omega(Y)\ ;\ CS_*^1(X) & \xrightarrow{\simeq} & B(CN_*(F); CN_*(\Omega X))\ ;\ BCN_*(\Omega X). \end{array}$$

Proposition 2.14 will give quisms

$$C_*^\omega(Y)\,;\ C_*(X) \xrightarrow{\simeq} B(\Omega(C_*^\omega Y\,;\, C_*(X))\,;\, \Omega C_*(X))\,;\ B\Omega C_*(X)\,.$$

Compose these with $B(\psi_\omega\,;\,\psi_X)\,;\ B(\psi_X)$ which are quisms by Remark 2.3, to get the horizontal quisms. The others are obvious, once one knows (2.3) that $B(-)$ preserves quisms. Q.E.D.

Theorem IV gives at once "the other" Eilenberg-Moore isomorphism [9]

$$H_*(Y) \cong \mathrm{Tor}^{CS_*(\Omega X)}(CS_*(F)\,;\, R)\,. \tag{1.2}$$

This reinforces our observation about differential objects; whereas Theorem IV follows easily from Theorem III, (1.2) is not a corollary of (1.1). Note that by Stasheff [12, §7] (1.2) also reflects a geometrical construction of $Y$ from the action of $\Omega X$ on $F$.

The proof of Theorems I and II is in §3. In §2 we discuss some properties of DGA's that are tensor algebras and DGC's that are tensor coalgebras. These include the bar and cobar constructions, whose definitions we recall. Finally, we review the adjoint equivalences between appropriate DGA and DGC categories provided by these constructions, as described by Moore [10], [8, Chapter II]. We point out that, *except for the definition of the cobar construction*, §3 *is almost independent of* §2.

We restrict ourselves now to rings $R$ that are *principal ideal domains*, and we say a space $X$ has *finite $R$ type* if its homology groups are all finitely generated $R$ modules. If $X$ is simply connected and has finite $R$ type then we can construct (cf. §4) directly from the cochain DGA, $CS^*(X)$, a DGA quism

$$(F(V)\,,\, d) \xrightarrow{\simeq} CS^*(X)\,,$$

in which $F(V)$ is the tensor algebra on $V = \{V^i\}_{i\geq 2}$, and the $V^i$ are free, finitely generated $R$ modules. This is a *free model* for $X$.

If, in addition, $\omega\colon Y \to X$ is a continuous map from a space $Y$ of finite $R$ type, then $CS^*(\omega)$ makes $CS^*(Y)$ into a DGA module over $CS^*(X)$ and we can construct directly a DGA-module quism of the form

$$(F(V)\otimes W\,,\, d) \xrightarrow{\simeq} CS^*(Y)\,,$$

where $W = \{W^i\}_{i\geq 0}$, each $W^i$ is a free, finitely-generated $R$ module, and the module structure in $F(V)\otimes W$ is by multiplication on the left.

Our objective, which we carry out in §4, is to show how $(F(V)\,,\, d)$ and $(F(V)\otimes W\,,\, d)$ can be used to read off the homology algebra $H_*(\Omega X)$ and its module $H_*(F)$, where $F$ denotes the homotopy fibre of $\omega$.

To do so, we dualize $(F(V)\,,\, d)$ and $(F(V)\otimes W\,,\, d)$ to obtain respectively a DGC $(T(V^\vee)\,,\, d^\vee)$ and a DGC comodule, $(W^\vee \otimes T(V^\vee)\,,\, d^\vee)$. Here $T(-)$ denotes tensor coalgebra and $\mathrm{Hom}(-\,;\,R)$ is denoted by $(-)^\vee$. Denote by $T^k \subset T$ the $k$th tensor power.

Then this DGC and DGC comodule are filtered by the $R$ submodules

$$T^{\leq k}(V^\vee) \quad \text{and} \quad W^\vee \otimes T^{\leq k}(V^\vee)\,,$$

and so their differentials restrict, respectively to differentials $d_V^\vee$ in $V^\vee$ and $d_W^\vee$ in $W^\vee$. In the respective spectral sequences we have the obvious identifications

$$(E^0_{k,*}, d^0) = \bigotimes^k (V^\vee, d_V^\vee) \quad \text{and}$$
$$(E^0_{k,*}, d^0) = (W^\vee, d_W^\vee) \otimes \bigotimes^k (V^\vee, d_V^\vee), \quad k \geq 0.$$

These in turn define inclusions

(1.3) $T^k(H(V^\vee)) \to E^1_{k,*}$ and $H(W^\vee) \otimes T^k(H(V^\vee)) \to E^1_{k,*}$, $k \geq 0$,

which are isomorphisms if $H(V^\vee)$ has no $R$ torsion.

We can now state the main result of §4:

**Theorem V.** *Suppose $X$ is simply connected, $X$ and $Y$ have finite $R$ type and $\omega\colon X \to Y$ is a continuous map with homotopy fibre $F$. Then, with the terminology and constructions above,*

(i) *$H(V^\vee, d_V^\vee) \cong sH_+(\Omega X)$ and $H(W^\vee, d_W^\vee) \cong H_*(F)$.*

(ii) *The inclusions* (1.3) *are respectively a* DGC *morphism and a* DGC-*comodule morphism*:

$$B(H_*(\Omega X)) \to (E^1, d^1) \quad \textit{and} \quad B(H_*(F); H_*(\Omega X)) \to (E^1, d^1).$$

*In particular, they commute with the differentials.*

**Corollary 1.** *If $H(\Omega X)$ has no $R$ torsion then the morphisms of Theorem* V(ii) *are isomorphisms. In this case the multiplication and module maps*

$$H_+(\Omega X) \otimes H_+(\Omega X) \to H_+(\Omega X) \quad \textit{and} \quad H_*(F) \otimes H_+(\Omega X) \to H_*(F),$$

*are given, with a shift of degrees, respectively by the differentials $d^1\colon E^1_{2,*} \to E^1_{1,*}$ and by $d^1\colon E^1_{1,*} \to E^1_{0,*}$.*

**Corollary 2.** *A simply-connected space $X$ of finite $R$ type admits a free model $(F(V), d)$ in which*

$$\operatorname{Im} d \subset F(V)^+ \cdot F(V)^+,$$

*if and only if $H(\Omega X)$ has no $R$ torsion.*

*Proof*. If $d$ satisfies the condition of the corollary then $sH_+(\Omega X) \cong V$ and so it is $R$ free. The converse is the standard minimalization argument for free DGA's: if $H(\Omega X)$ is $R$ free then we have $V^\vee = H^\vee \oplus C^\vee \oplus D^\vee$ with $d_V^\vee = 0$ in $H^\vee$ and $D^\vee$ and $d_V^\vee\colon C^\vee \xrightarrow{\cong} D^\vee$.

Dualizing, we write $V = H \oplus C \oplus D$ with $d\colon D \xrightarrow{\cong} C \bmod [F^+(V)]^2$. It follows that $F(V) = F(H \oplus D \oplus d(D))$ and that the ideal generated by $D$ and $d(D)$ is acyclic. Hence we obtain a surjective quism of the form

$$(F(H \oplus D \oplus d(D)), d) \xrightarrow{\cong} (F(H), \overline{d}),$$

which splits by Proposition 4.4 in §4 below. Note that $\operatorname{Im} \overline{d} \subset [F(H)^+]^2$. Q.E.D.

In the same way we have

**Corollary 3.** *The free model* $F(V)\otimes W \xrightarrow{\simeq} C^*(Y)$ *can be chosen so that* $d\colon W\to F(V)^+\otimes W$ *if and only if* $H_*(F)$ *has no* $R$ *torsion.*

*Remark.* When $H_*(\Omega X)$ has no $R$ torsion then, as we shall see in §4, the spectral sequences above are respectively isomorphic with the Eilenberg-Moore spectral sequences for $B(CS_*^0(\Omega X))$ and for $B(CS_*(F)\,;\,CS_*^0(\Omega X))$. We have emphasized the $E^1$ isomorphism in Theorem V, because this is what permits us to read off the algebra $H_*(\Omega X)$ and the $H_*(\Omega X)$ module, $H_*(F)$, from the free models, rather than just $\mathrm{Tor}^{H_*(\Omega X)}(R\,;\,R)$ and $\mathrm{Tor}^{H_*(\Omega X)}(H_*(F),\,R)$, which is what one finds at the $E^2$ term.

This is another illustration of a principle stressed by Sullivan in his approach to rational homotopy; free models contain easily computable geometric information which is lost on passage to homology.

## 2. Free DGA's and DGC's

In this section all modules, algebras etc. are defined over a fixed, but arbitrary ring $R$. Graded objects will be graded over $\mathbb{Z}$; i.e., we permit elements of positive and negative degrees. Degrees are written as subscripts and differentials (usually denoted by $d$) have degree $-1$. If $V$ is a graded (differential) module then for $r\in\mathbb{Z}$, $\mathfrak{s}^rV$ is defined by $(\mathfrak{s}^rV)_k=V_{k-r}$, and the induced differential is defined by $d\mathfrak{s}^r=(-1)^r\mathfrak{s}^r d$.

We begin by recalling the *canonical filtrations* associated with supplemented graded algebras $A=\overline{A}\oplus R$ and their modules, $M$ and with supplemented graded coalgebras, $C=\overline{C}\oplus R$ and their comodules $N$. Indeed for $A$ and $M$ these are defined by

$$\mathscr{F}^k(A)=(\overline{A})^k \quad\text{and}\quad \mathscr{F}^k(M)=M\cdot(\overline{A})^k\,;$$

here we have a *decreasing* filtration, stable under the differential.

For coalgebras we recall that the *reduced coproducts* $\overline{\Delta}_C\colon\overline{C}\to\overline{C}\otimes\overline{C}$, $\overline{\Delta}_N\colon\overline{C}\to N\otimes\overline{C}$ are defined by $\overline{\Delta}_Cx=\Delta_Cx-(x\otimes 1+1\otimes x)$ and $\overline{\Delta}_Ny=\Delta_Ny-y\otimes 1$, and are coassociative. Hence they can be iterated unambiguously to produce $\overline{\Delta}_C^{(k)}\colon\overline{C}\to\bigotimes^{k+1}\overline{C}$ and $\overline{\Delta}_N^{(k)}\colon N\to N\otimes\bigotimes^k\overline{C}$ (we put $\overline{\Delta}_C^{(0)}=\mathrm{id}_{\overline{C}}$ and $\overline{\Delta}_N^{(0)}=\mathrm{id}_N$) and then the canonical filtrations are the *increasing* filtrations defined by

$$\mathscr{F}_k(C)=\ker\overline{\Delta}_C^{(k)}\oplus R \quad\text{and}\quad \mathscr{F}_k(N)=\ker\overline{\Delta}_N^{(k)}.$$

For DGC's and their comodules these are stable under the differentials.

In the differential categories these filtrations lead to spectral sequences which are natural with respect to morphisms. We therefore make:

**Definition 2.1.** The spectral sequences arising from the canonical filtrations will be called the *canonical spectral sequences.*

In this section we are particularly interested in DGA's and DGC's whose underlying algebra (coalgebra) is the tensor algebra (coalgebra) on a module $V$. We therefore adopt the

*Convention.* The *tensor algebra* on a graded $R$ module $V$ will be denoted $F(V)$; the $k$th tensor power of $V$ will be written $F^k(V)\subset F(V)$, and if $v_i\in V$ we write $v_1\otimes\cdots\otimes v_k\in F^k(V)$.

The *tensor coalgebra* on $V$ will be written $T(V)$; the $k$th tensor power is then denoted by $T^k(V)$ and we write $[v_1|\cdots|v_k] \in T^k(V)$ for the tensor product of elements $v_i \in V$. In particular the reduced coproduct for $T(V)$ is given by

$$\overline{\Delta}[v_1|\cdots|v_k] = \sum_{i=1}^{k-1}[v_1|\cdots|v_i]\otimes[v_{i+1}|\cdots|v_k].$$

**Example 2.2.** (i) If $A$ is a DGA of the form $(F(V), d)$ and if $M$ is an $A$ module of the form $(W\otimes F(V), d)$ then the canonical filtrations are given by

$$\mathcal{F}^k(A) = F^{\geq k}(V) \quad \text{and} \quad \mathcal{F}^k(M) = W\otimes F^{\geq k}(V).$$

The projections $F^{\geq 1}(V)\to V$ and $W\otimes F(V)\to W$ induce differentials $d_V$ in $V$ and $d_W$ in $W$ and the 0th term of the canonical spectral sequence is given by $E^0_{-p,*}=0$, $p<0$, and

$$(E^0_{-p,*}, d^0) = \bigotimes^p(V, d_V) \quad \text{and}$$
$$(E^0_{-p,*}, d^0) = (W, d_W)\otimes\bigotimes^p(V, d_V), \quad p\geq 0.$$

(ii) If $N=(W\otimes T(V), d)$ is a comodule over the DGC, $C=(T(V), d)$, then the canonical filtrations are given by

$$\mathcal{F}_k(C) = T^{\leq k}(V) \quad \text{and} \quad \mathcal{F}_k(N) = W\otimes T^{\leq k}(V).$$

In particular, $d$ restricts to differentials $d_V$ and $d_W$ in $V$ and $W$ and the 0th term of the canonical spectral sequence is given by $E^0_{p,*}=0$, $p<0$, and

$$(E^0_{p,*}, d^0) = \bigotimes^p(V, d_V) \quad \text{and}$$
$$(E^0_{p,*}, d^0) = (W, d_W)\otimes\bigotimes^p(V, d_V), \quad p\geq 0.$$

The prime examples of DGA's and DGC's considered in Example 2.2 arise in the bar construction of Eilenberg-Mac Lane and the cobar construction of Adams. We recall the definitions now. The *bar construction* on a supplemented DGA, $A=\overline{A}\oplus R$ (resp., on a right $A$ module, $M$) is a DGC $BA$ (resp., a $BA$ comodule $B(M;A)$). Explicitly, $BA=(T(s\overline{A}), D)$ and $B(M;A)=(M\otimes T(s\overline{A}), D)$, and in both cases $D=D_1+D_2$, where the operators $D_i$ are given by the formulae immediately below (the formulae are given for $B(M;A)$; for $BA$ simply set $M=R$):

$$D_1 m = dm,$$
$$D_1 m[sa_1|\cdots|sa_k] = dm[sa_1|\cdots|sa_k] - \sum_{i=1}^{k}(-1)^{\varepsilon_i}m[sa_1|\cdots|s\,da_i|\cdots|sa_k],$$
$$D_2 m = 0,$$
$$D_2 m[sa_1|\cdots|sa_k] = (-1)^{\deg m}ma_1[sa_2|\cdots|sa_k] + \sum_{i=2}^{k}(-1)^{\varepsilon_i}m[sa_1|\cdots|sa_{i-1}a_i|\cdots|sa_k];$$

here $\varepsilon_i = \deg m + \sum_{j<i}\deg sa_j$.

Dually, if $N$ is a comodule over the supplemented DGC, $C = \overline{C} \oplus R$, the *cobar constructions* $\Omega C = (F(\mathfrak{s}^{-1}\overline{C}), D)$ and $\Omega(N; C) = (N \otimes F(\mathfrak{s}^{-1}\overline{C}), D)$ are respectively the DGA and DGA module whose differentials are given by the following formulae: write the reduced coproducts in the form $\overline{\Delta}_C x = \sum x_i \otimes y_i$ and $\overline{\Delta}_N a = \sum a_i \otimes z_i$; then

$$D\mathfrak{s}^{-1}x = -\mathfrak{s}^{-1}dx + \sum (-1)^{\deg x_i} \mathfrak{s}^{-1}x_i \otimes \mathfrak{s}^{-1}y_i, \qquad x \in \overline{C},$$

$$Da = da - \sum (-1)^{\deg a_i} a_i \otimes \mathfrak{s}^{-1}z_i, \qquad a \in N.$$

*Remark* 2.3. The canonical spectral sequences for $BA$ and $B(M; A)$ and for $\Omega C$ and $\Omega(N; C)$ are the well-known *Eilenberg-Moore spectral sequences.* In the case of DGA's these spectral sequences (for $BA$ and $B(M; A)$) converge, because the filtrations are increasing. In particular, if $\phi: A \to A'$ is a quism of supplemented $R$-projective DGA's then $B\phi$ is also a quism; it suffices to apply Example 2.2 and remark that the tensor product of quisms between $R$-projectives is a quism. Similarly if $\psi: M \to M'$ is a quism of $R$-projective DGA modules (with respect to $\phi$) then $B(\psi; \phi)$ is also a quism.

In the case of quisms of supplemented $R$-projective DGC's $\alpha: C \to C'$ and of $R$-projective comodules $\beta: N \to N'$ we have again that $\Omega\alpha$ and $\Omega(\beta; \alpha)$ are quisms *provided that* for some integer $n$ either

$$\overline{C}_{\leq 1} = \overline{C}'_{\leq 1} = N_{\leq -n} = N'_{\leq -n} = 0 \quad \text{or else} \quad \overline{C}_{\geq 1} = \overline{C}'_{\geq 1} = N_{\geq n} = N'_{\geq n} = 0.$$

The additional hypothesis on degrees, which forces the spectral sequence to converge, is essential. Indeed the inclusion $\lambda: R \to CS_*(pt)$ is a DGC quism but $H_0(\Omega R) = R$ whereas $H_0(\Omega CS_*(pt)) = R \oplus R$! An even easier example is provided by the inclusion $R \to C = R \cdot 1 \oplus Rx \oplus Ry$ with $x$ a primitive cycle of degree 1, $dy = x$ and $\overline{\Delta} y = x \otimes x$; again

$$H_0(\Omega R) = R \quad \text{and} \quad H_0(\Omega C) = R \oplus R.$$

Note that, as in [2, 9, 10 and 8, II, Proposition 5.2] we have

**Proposition 2.4.** *For any supplemented* DGA, $A$ *and supplemented* DGC, $C$ *the obvious maps* $B(A; A) \to R$ *and* $R \to \Omega(C; C)$ *are quisms.*

*Proof*. Let $\varepsilon$ denote the augmentations. In the case of $B(A; A)$ an explicit homotopy is given by $h(a[\mathfrak{s}a_1|\cdots|\mathfrak{s}a_k]) = 1[\mathfrak{s}(a - \varepsilon a)|\mathfrak{s}a_1|\cdots|\mathfrak{s}a_k]$. In the case of $\Omega(C; C)$ an explicit homotopy $h$ is given by $h(c) = 0$ and

$$h(c \otimes \mathfrak{s}^{-1}c_1 \otimes \cdots \otimes \mathfrak{s}^{-1}c_k) = \varepsilon(c)c_1 \otimes \mathfrak{s}^{-1}c_2 \otimes \cdots \otimes \mathfrak{s}^{-1}c_k. \quad \text{Q.E.D.}$$

Our next objective is to recall the calculation of the homology of the bar construction for DGA's of the form $(F(V), d)$ and for modules of the form $(W \otimes F(V), d)$, and of the cobar construction for DGC's $(T(V), d)$ and for comodules $(W \otimes T(V), d)$. The $E^0$ term of the canonical spectral sequence will be denoted by $(F(V), d_V)$ and $(W \otimes F(V), d_W)$ (resp., by $(T(V), d_V)$ and $(W \otimes T(V), d_W)$): for the precise formulae see Example 2.2.

The calculation we have in mind does not apply to general DGA's $(F(V), d)$ and their modules, although it does apply in general to DGC's. Thus is $V(i) \subset V$ is a submodule we will (by a serious abuse of notation) use $FV(i)$ to denote the image of $F(V(i))$ in $F(V)$ and make the:

**Definition 2.5.** (i) A DGA $A = (F(V), d)$ is *locally nilpotent* if there exists an increasing filtration $0 = V(0) \subset V(1) \cdots$ of graded $R$ modules such that $V = \bigcup_i V(i)$ and

$$d - d_V : V(i) \to F(V(i-1)).$$

(ii) An $A$ module $M = (W \otimes F(V), d)$ is *locally nilpotent* if there is an increasing filtration $0 = W(0) \subset W(1) \subset \cdots$ such that $W = \bigcup_i W(i)$ and

$$d - d_W : W(i) \to W(i-1) \otimes F(V).$$

(Notice we have again abused notation: $W(i-1) \otimes F(V)$ denotes the image in $W \otimes F(V)$.)

Given a DGA, $A = (F(V), d)$, and an $A$ module $M = (W \otimes F(V), d)$ we define

$$(2.6) \qquad \rho_A : BA \to \mathfrak{s}(V, d_V) \oplus R \quad \text{and} \quad \rho_M : B(M; A) \to (W, d_W),$$

as follows: $\rho_A = 0$ in $T^{\geq 2}(s\overline{A})$ and $\rho_A$ coincides with the projection of Example 2.2 in $T^1(\mathfrak{s}\overline{A}) = \mathfrak{s}F^{\geq 1}(V)$, and $\rho_M$ is the obvious projection $W \otimes F(V) \otimes T(\mathfrak{s}F^{\geq 1}(V)) \to W$. A straightforward check using the formulae of the bar construction shows that $\rho_A$ and $\rho_M$ commute with the differentials.

Dually, given a DGC, $C = (T(V), d)$, and a $C$ comodule $N = (W \otimes T(V), d)$ we define

$$(2.7) \qquad \lambda_C : \mathfrak{s}^{-1}(V, d_V) \oplus R \to \Omega C \quad \text{and} \quad \lambda_N : \mathfrak{s}^{-1}(W, d_W) \to \Omega(N; C)$$

to be the obvious inclusions; again they commute with the differentials.

**Proposition 2.8.** (i) *If* $A = (F(V), d)$ *is a locally nilpotent* DGA *and if* $M = (W \otimes F(V), d)$ *is a locally nilpotent* $A$ *module then the morphisms* $\rho_A$ *and* $\rho_M$ *in* (2.6) *induce isomorphisms of homology.*

(ii) *If* $C = (T(V), d)$ *is a* DGC *and if* $N = (W \otimes T(V), d)$ *is a* $C$ *comodule then the morphisms* $\lambda_C$ *and* $\lambda_N$ *in* (2.7) *induce isomorphisms of homology.*

*Remark* 2.9. When $A$ is itself a bar construction and $C$ is a cobar construction and everything is positively graded and connected, then this is [8, Chapter II, Theorems 4.4, 4.5 and 5.15]. Our proof follows the same general idea.

*Proof of* 2.8. (i) (a) $H(\rho_A)$ *is an isomorphism.* Define an $R$-linear map $k$ in $\ker \rho$ by

$$k : \begin{cases} [\mathfrak{s}v | \cdots] \to 0, & \text{and} \\ [\mathfrak{s}(v_1 \otimes \cdots \otimes v_l) | \cdots] \to [\mathfrak{s}v_1 | \mathfrak{s}(v_2 \otimes \cdots \otimes v_l) | \cdots], & l \geq 2. \end{cases}$$

If $d = d_V$ in $F(V)$ then $hD + Dh = \mathrm{id}$ in $\ker \rho_A$. In general, the filtration of $V$ used to define the local nilpotence induces a filtration of $F(V)$ and hence a filtration of $T(\mathfrak{s}F^{\geq 1}(V))$. With respect to this, $kD + Dk - \mathrm{id}$ is strictly filtration decreasing in $\ker \rho_A$, as follows from the explicit formula for $k$.

Since all filtrations are increasing we may conclude that for $u \in \ker \rho_A$, $(kD + Dk - \mathrm{id})^{n(u)}(u) = 0$. This implies that $H(\ker \rho_A, D) = 0$ and hence that $H(\rho_A)$ is an isomorphism.

(i) (b) $H(\rho_M)$ *is an isomorphism.* If $\rho$ is the projection of Proposition 2.4 then $\rho_M = \mathrm{id} \otimes \rho : W \otimes F(V) \otimes T(\mathfrak{s}F^{\geq 1}(V)) \to W$. Let $h$ be the operator defined in the proof of 2.4, so that $hD + Dh = \mathrm{id}$ in $\ker \rho$. Thus if $d = d_W$ in $W \otimes 1$ then $(\mathrm{id} \otimes h)D + D(\mathrm{id} \otimes h) = \mathrm{id}$ in $\ker \rho_M$ and in general $[(\mathrm{id} \otimes h)D +$

$D(\mathrm{id}\otimes h)-\mathrm{id}]$ is strictly filtration decreasing. Hence, as above, $H(\ker\rho_M)=0$ and $H(\rho_M)$ is an isomorphism.

(ii) The proof is exactly the same as in (i) except that now $k$ is defined by

$$k\colon\begin{cases} s^{-1}[v]\otimes s^{-1}[v_1|\cdots|v_l]\otimes\cdots\mapsto s^{-1}[v|v_1|v_2|\cdots|v_l]\otimes\cdots\,, & \text{and}\\ s^{-1}[v_1|\cdots|v_i]\otimes\cdots\mapsto 0\,, \qquad i\geq 2\,; \end{cases}$$

and we use the canonical filtration of $T(V)$ and $W\otimes T(V)$ instead of that induced by the filtration $V(i)$ of $V$ above. In the case of $\Omega T(V)$ we identify it (removing suspensions and shifting degrees) as a direct sum of modules each of which is canonically isomorphic with some $\bigotimes^l V$, and a module of this form is declared to have filtration degree $l$. The definition in $\Omega(N;C)$ is analogous. Q.E.D.

We will need the following application. Suppose

$$\alpha\colon C\to C' \quad\text{and}\quad \beta\colon N\to N',$$

are respectively a morphism of supplemented DGC's and a morphism (with respect to $\alpha$) of DGC comodules. Suppose that $C$, $C'$ and $N$, $N'$ have the form

$$C=T(V),\qquad C'=T(V'),\qquad N=W\otimes T(V),\qquad N'=W'\otimes T(V').$$

Then (cf. Example 2.2) $\alpha$ and $\beta$ restrict to maps

$$\alpha_V\colon V\to V' \quad\text{and}\quad \beta_W\colon W\to W'.$$

**Proposition 2.10.** *With the notation above, assume that $V$, $V'$, $W$, $W'$ are $R$ projective and that $V=V_{\geq 2}$, $V'=V'_{\geq 2}$, $W=W_{\geq 0}$, and $W'=W'_{\geq 0}$. Then the following conditions are equivalent*:

(i) *$\alpha$ and $\beta$ are quisms.*
(ii) *$\Omega\alpha$ and $\Omega(\beta;\alpha)$ are quisms.*
(iii) *$\alpha_V$ and $\beta_W$ are quisms.*
(iv) *In the canonical spectral sequences for $C$, $C'$, $N$, and $N'$, $E^1(\alpha)$ and $E^1(\beta)$ are isomorphisms.*

*Proof.* By Remark 2.3, (i) $\Rightarrow$ (ii). Proposition 2.8(ii) shows that (ii) $\Leftrightarrow$ (iii). The description in Example 2.2 identifies $E^0(\alpha)=T(\alpha_V)$ and $E^0(\beta)=\beta_W\otimes T(\alpha_V)$. Since the tensor product of quisms between $R$ projectives is a quism, (iii) $\Rightarrow$ (iv). Finally, because the canonical filtrations are increasing, the canonical spectral sequences converge, so (iv) $\Rightarrow$ (i). Q.E.D.

We complete this section by recalling the adjoint equivalences of Moore. First, recall that a supplemented graded coalgebra $C$ is *locally conilpotent* if $C=\bigcup_k\mathscr{F}_k(C)$; i.e., if for each $x\in\overline{C}$ and some $k$, $\overline{\Delta}^{(k)}x=0$. Tensor coalgebras, as well as coalgebras satisfying $\overline{C}_{\leq 0}=0$ or $\overline{C}_{\geq 0}=0$ are locally conilpotent.

The usefulness of this notion lies in the fact that if $C=\overline{C}\oplus R$ is locally conilpotent then any $R$ module map, $\phi\colon\overline{C}\to U$, of degree zero lifts to a unique coalgebra morphism $\Phi\colon C\to T(U)$; $\Phi$ is given explicitly by $\Phi\colon x\mapsto T(\phi)\sum_{k=0}^{\infty}\overline{\Delta}^{(k)}x$, $x\in\overline{C}$. Since tensor coalgebras are locally conilpotent, the bar and cobar construction can be regarded as a pair of adjoint functors

$$\mathrm{DGA}\underset{\Omega}{\overset{B}{\leftrightarrows}}\mathscr{N}\text{-DGC},$$

where DGA is the category of supplemented DGA's and $\mathscr{N}$-DGC is the category of locally conilpotent supplemented DGC's.

Suppose now we fix an adjoint pair of morphisms

$$f: \Omega C \to A \quad \text{and} \quad g: C \to BA.$$

These convert $A$ modules to $\Omega C$ modules and $C$ comodules to $BA$ comodules, and we have

**Proposition 2.11.** *The functors* $\Omega(-; C)$ *and* $B(-; A)$ *are adjoint under a natural bijection*:

$$\mathrm{Hom}_{\Omega C}(\Omega(N; C); M) = \mathrm{Hom}_{BA}(N; B(M; A)).$$

*Proof.* Restriction on the left and projection on the right define bijections of both sides with $\mathrm{Hom}_R(N; M)$. Q.E.D.

These adjoint relations lead to natural morphisms

$$\Omega BA \xrightarrow{\tau_A} A \quad \text{and} \quad \Omega(B(M; A); BA) \xrightarrow{\tau_M} M, \tag{2.12}$$

of algebras (resp., modules) if $M$ is an $A$ module, and to natural morphisms

$$B\Omega C \xleftarrow{\sigma_C} C \quad \text{and} \quad B(\Omega(N; C); \Omega C) \xleftarrow{\sigma_N} N, \tag{2.13}$$

of coalgebras (resp. comodules) if $N$ is a $C$ comodule.

**Proposition 2.14** [10], [8, II, §4, 5]. *Suppose* $M$ *is a* DGA *module over the supplemented* DGA, $A$ *and* $N$ *is a* DGC *comodule over the locally conilpotent supplemented* DGC, $C$. *Then the maps* (2.12) *and* (2.13) *commute with the differentials and induce isomorphisms of homology.*

*Proof.* Since $C$ is locally conilpotent, its canonical filtration exhibits $\Omega C$ as locally conilpotent, cf. Definition 2.5. Similarly $\Omega(N; C)$ is locally conilpotent. Thus we can apply Proposition 2.8 to obtain quisms

$$\rho_C: B\Omega C \to C \quad \text{and} \quad \rho_N: B(\Omega(N; C); \Omega C) \to N.$$

They satisfy $\rho\sigma = \mathrm{id}$ and so $\sigma_C$ and $\sigma_N$ are quisms.

Similarly, $\tau_A$ and $\tau_M$ are quisms. Q.E.D.

## 3. The cobar equivalence

In this section we prove Theorems I and II. We do so in seven steps, as follows: First we define $C_*(X)$. Then we recall some path space terminology and define $C_*^{\omega}(Y)$. Next following Adams [1], we make some universal constructions in the cube and simplex. We use these to define natural morphisms $\psi_X$ and $\psi_\omega$ and, finally, show that $H(\psi_X)$ and $H(\psi_\omega)$ are isomorphisms. The proof of this last assertion is a little longer, but more elementary, than that given in Adams [1]. Now we proceed to the details.

*Step* 1. *Definition of* $C_*(X)$. Recall that $CS_*(-)$ denotes the DGC of ordinary singular chains, with Alexander-Whitney diagonal. Let $CS_*^1(X)$ be the sub DGC spanned by the singular simplices $\sigma: \Delta^n \to X$ that map the 1-skeleton $\Delta_1^n$, to the basepoint $x_0$.

Let $\alpha: \Delta^1 \to x_0$, $\beta: \Delta^2 \to x_0$ be the constant simplices. Thus $\alpha$ is primitive and $d\beta = \alpha$. Moreover $\beta$ represents a primitive cycle $\overline{\beta}$ in $CS_*^1(X)/R\alpha$. We let $\gamma$ be a symbol of degree 3, $R\gamma$ the free $R$ module on $\gamma$ and we define a DGC:

$$C_*(X) = [CS^1_*(X)/R\alpha] \oplus R\gamma,$$

by setting $\gamma$ primitive and $d\gamma = \overline{\beta}$. Notice that $C_*$ is a functor from simply-connected pointed spaces to simply-connected DGC's, and that

$$CS_*(X) \leftarrow CS^1_*(X) \rightarrow C_*(X),$$

are natural DGC morphisms inducing isomorphisms in homology.

*Step* 2. *Path spaces.* For any space $Z$, $LZ$ denotes the space of *Moore paths* in $Z$: an element of $LZ$ is thus a pair $(f\colon [0,\infty) \to Z;\ r \in [0,\infty))$ such that $f(t) = f(r)$, $t \geq r$; $r$ is called the *length of* the path $(f, r)$. We denote $f(r)$ by $f(\infty)$ and we may abuse notation by writing $f$ for $(f, r)$. If $(f, r)$, $(g, s) \in LZ$ and $f(\infty) = g(0)$ then the *composite* $(f, r)\cdot(g, s) = (f\cdot g, r+s)$ is defined by

$$(f\cdot g)(t) = \begin{cases} f(t), & t \leq r, \\ g(t-r), & t \geq r. \end{cases}$$

For any $z \in Z$, $L_z(Z)$ denotes the space of Moore paths *ending* at $z$ and $\Omega_z(Z)$ denotes the space of Moore loops at $z$. Thus composition makes $\Omega_z(Z)$ into a topological monoid whose identity is the constant path $(z, 0)$ of length zero. Moreover composition defines a right action of $\Omega_z(Z)$ on $L_z(Z)$. For pointed spaces, e.g. $(X, x_0)$, we write $\Omega X$ for $\Omega_{x_0}X$.

*Step* 3. *Definition of* $C^\omega_*(Y)$. Convert the continuous map $\omega\colon Y \to X$ into a fibration in the standard way; i.e. by replacing it by

$$\rho\colon Y \times_\omega LX \to X, \qquad \rho(y, f) = f(\infty);$$

here $Y \times_\omega LX \subset Y \times LX$ consists of the pairs $(y, f)$ such that $\omega(y) = f(0)$. The fibre, $F = Y \times_\omega L_{x_0}X$, of $\rho$ is the *homotopy fibre* of $\omega$ and the action of $\Omega X$ on $L_{x_0}X$ defines a natural right action $F \times \Omega X \to F$.

We now define $C^\omega_*(Y) \subset CS_*(Y \times_\omega LX)$ to be the sub DGC spanned by the singular simplices $(\tau, \sigma)\colon \Delta^n \to Y \times_\omega LX$ such that $\sigma(v)(\infty) = x_0$, $v \in \Delta^n_1$. Then $CS_*(\rho)$ restricts to a morphism $C^\omega_*(Y) \to CS^1_*(X)$ and this induces a morphism $C_*(\omega)\colon C^\omega_*(Y) \to C_*(X)$, which makes $C^\omega_*(Y)$ into a DGC comodule over $C_*(X)$. This morphism is functorial in $\omega$, and the natural DGC morphisms

$$CS_*(Y) \rightarrow CS_*(Y \times_\omega LX) \leftarrow C^\omega_*(Y),$$

all induce isomorphisms of homology.

*Step* 4. *Universal constructions.* Let $\Delta^n = \langle v_0, \dots, v_n\rangle$ be the standard $n$-simplex; as in [1], $L_{0,n}(\Delta^n) \subset L(\Delta^n)$ will denote the paths from $v_0$ to $v_n$; $L_n(\Delta^n) = L_{v_n}(\Delta^n)$ is the space of paths ending at $v_n$. The inclusions of the front face, back face, and face opposite $v_i$ will be denoted respectively by $f_i\colon \Delta^i \to \Delta^n$, $l_i\colon \Delta^{n-i} \to \Delta^n$, and $d_i\colon \Delta^{n-1} \to \Delta^n$.

Next, $\pi\colon \Delta^n \to \Delta^n/\Delta^n_1$ will denote the quotient map which identifies the 1-skeleton $\Delta^n_1$ to a single base point $*$, which we use to point the "1-*reduced*" *simplex*, $\Delta^n/\Delta^n_1$. In particular, $(*, 0)$ is the identity for $\Omega(\Delta^n/\Delta^n_1)$. The maps $f_i$, $l_i$, $d_i$ factor to give maps between the 1-reduced simplices and we denote these maps also by $f_i$, $l_i$, $d_i$.

Finally $I^n = I \times \cdots \times I$ denotes the $n$-cube, and for $\varepsilon = 0, 1$ and $1 \le i \le n$, $\lambda_i^\varepsilon : I^{n-1} \to I^n$ is the inclusion with $i$th coordinate $\varepsilon$.

We claim now that there are four families of maps:

(a) $\theta_n : I^{n-1} \to L_{0,n}(\Delta^n)$, $n \ge 1$; (b) $\Phi_n : I^{n-1} \times I \to \Omega(\Delta^n/\Delta_1^n)$, $n \ge 1$;

(c) $\eta_n : I^n \to L_n(\Delta^n)$, $n \ge 0$; (d) $H_n : I^n \times I \to L_*(\Delta^n/\Delta_1^n)$, $n \ge 0$;

such that:

(a) (i) $\theta_n \circ \lambda_i^0 = [L(f_i)\theta_i] \cdot [L(l_i)\theta_{n-i}]$, $n \ge 2$.

(ii) $\theta_n \circ \lambda_i^1 = L(d_i)\theta_{n-1}$, $n \ge 2$.

(b) (i) $\Phi_n(-, 0) = L(\pi)\theta_n$, $n \ge 1$.

(ii) $\Phi_1(pt, 1) = (*, 0)$.

(iii) $\Phi_n(-, t) \circ \lambda_i^0 = [\Omega(f_i)\Phi_i(-, t)] \cdot [\Omega(l_{n-i})\Phi_{n-i}(-, t)]$, $n \ge 2$.

(iv) $\Phi_n(-, t) \circ \lambda_i^1 = \Omega(d_i)\Phi_{n-1}(-, .t)$, $n \ge 2$.

(c) (i) $\eta_n \circ \lambda_i^0 = [L(f_{i-1})\eta_{i-1}] \cdot [L(l_{i-1})\theta_{n-i+1}]$, $n \ge 1$.

(ii) $\eta_n \circ \lambda_i^1 = L(d_{i-1})\eta_{n-1}$, $n \ge 1$.

(d) (i) $H_n(-, 0) = L(\pi)\eta_n$, $n \ge 0$.

(ii) $H_n(-, t)(0) = \pi\eta_n(-)(0)$, $n \ge 0$.

(iii) $H_n(-, t) \circ \lambda_i^0 = [L(f_{i-1})H_{i-1}(-, t)] \cdot [\Omega(l_{i-1})\Phi_{n-i+1}(-, t)]$, $n \ge 1$.

(iv) $H_n(-, t) \circ \lambda_i^1 = L(d_{i-1})H_{n-1}(-, t)$, $n \ge 1$.

In fact, in each case the inductive procedure of Adams applies. Suppose the map is defined in dimensions $< n$. Then in dimension $n$ the conditions above define it coherently in a subcomplex. In each case, either this subcomplex is a retract of the domain or else the target is contractible, and so an extension to the full domain always exists.

Finally, using these maps we define:

$$\begin{aligned} &\psi_n : I^{n-1} \to \Omega(\Delta^n/\Delta_1^n), && \psi_n(-) = \Phi_n(-, 1); && n \ge 1. \\ &\xi_n : I^n \to \Delta^n, && \xi_n(-) = \eta(-)(0); && n \ge 0. \\ &\mu_n : I^n \to L_*(\Delta^n/\Delta_1^n), && \mu_n(-) = H_n(-, 1); && n \ge 0. \end{aligned}$$

The properties of $\psi_n$, $\xi_n$ and $\mu_n$ can be read off from the equations above.

*Step* 5. *Definition of* $\psi_X$ *and* $\psi_\omega$. Consider the space $\{pt\}$ and the unique 2-simplex $\lambda : \Delta^2 \to pt$. It is immediate that $\Omega(\lambda) \circ \psi_1 : I^1 \to \Omega(pt)$ is a cycle and hence a boundary in $CU_*(\Omega(pt))$. We fix $v \in CU_2(\Omega(pt))$ so that $\Omega(\lambda) \circ \psi_1 = dv$.

Now observe (cf. §2) that $\Omega C_*(X)$ is the free associative $R$ algebra on the set of singular simplices $\sigma : \Delta^n/\Delta_1^n \to (X, x_0)$, $n \ge 2$, together with $\gamma$, the degrees having been decreased by one. Thus we can define an algebra homomorphism

$$\psi_X : \Omega C_*(X) \to CU_*(\Omega X),$$

by

$$\psi_X : \sigma \mapsto \Omega(\sigma) \circ \psi_n \quad \text{and} \quad \psi_X : \gamma \mapsto CU_*(\Omega(x_0))(v).$$

That $\psi_X$ commutes with the differentials is a straightforward translation of the formulae for $\Phi_n \circ \lambda_i^\varepsilon$ that follow from those given above for $\Psi_n \circ (\lambda_i^\varepsilon \times 1)$; this is the same as in [1]. The naturality of $\psi_X$ is obvious.

To define $\psi_\omega$ we recall (cf. §2) that $\Omega(C_*^\omega(Y); C_*(X)) = C_*^\omega(Y)\otimes\Omega C_*(X)$ is a free $\Omega C_*(X)$ module on the basis of singular simplices $(\tau, \sigma)\colon \Delta^n \to Y\times_\omega L_{x_0}X$ such that $\sigma_\infty = \sigma(-)(\infty)\colon \Delta_1^n \to x_0$. Thus we can define

$$\psi_\omega\colon \Omega(C_*^\omega(Y); C_*(X)) \to CU_*(F)$$

by requiring

$$\psi_\omega(\tau, \sigma) = (\tau\circ\xi_n, (\sigma\circ\xi_n)\cdot(L(\sigma_\infty)\circ\mu_n)).$$

Again $\psi_\omega$ is clearly natural and a straightforward verification from the formulae of Step 4 shows that it commutes with the differentials.

*Step* 6. *$H(\psi_X)$ is an isomorphism.* We first reduce to the case that $X$ is a $CW$ complex with a single 0-cell and no 1-cell; it suffices to apply the naturality of $\psi_X$ and the fact that if $f$ is a homology equivalence between simply-connected pointed spaces then so is $\Omega f$, while $\Omega(C_*(f))$ is a quism by Remark 2.3. A direct limit argument reduces us further to the case $X$ is finite. Now we may consider a filtration $X = Y(1)\supset\cdots\supset Y(N) = \{x_0\}$ in which $Y(i)$ is obtained from $Y(i+1)$ by the addition of a single cell. Let $\omega(i)\colon Y(i)\to X$ be the inclusion.

In particular, $R\to C_*^{\omega(N)}(Y(N))$ is a DGC quism, and so, by Remark 2.3, it identifies $H(\psi_X)$ with $H(\psi_{\omega(N)})$. On the other hand, the standard homotopy equivalence $X\to X\times_{\mathrm{id}}LX$ gives a quism $C_*(X)\to C_*^{\omega(1)}(Y(1))$. Thus combining Remark 2.3 and Proposition 2.4 we find that $H_*(C_*^{\omega(1)}(Y(1))\otimes\Omega C_*(X)) = R$. Since this is also true for the homology of the homotopy fibre of $\omega(1)$ we have, trivially, that $H(\psi_{\omega(1)})$ is an isomorphism.

To simplify notation denote $C_*^{\omega(n)}(Y(n))$ by $C_*(n)$, $\overline{C}_*(n)=C_*(n)/C_*(n+1)$ and let $F(n)$ be the homotopy fibre of $\omega(n)$. Then we have the row exact commutative diagram of DGA modules over $\Omega(C_*(X))$:

(3.1)

$$\begin{array}{ccccccccc} 0 & \to & C_*(n+1)\otimes\Omega(C_*(X)) & \to & C_*(n)\otimes\Omega(C_*(X)) & \to & \overline{C}_*(n)\otimes\Omega(C_*(X)) & \to & 0 \\ & & \downarrow{\scriptstyle\psi_{\omega(n+1)}} & & \downarrow{\scriptstyle\psi_{\omega(n)}} & & \downarrow{\scriptstyle\overline{\psi}} & & \\ 0 & \to & CU_*(F(n+1)) & \to & CU_*(F(n)) & \to & CU_*(F(n), F(n+1)) & \to & 0. \end{array}$$

Now $H(\overline{C}_*(n)) = H(Y(n), Y(n+1)) = R\cdot c$, where $c$ represents the cell added to $Y(n+1)$ to give $Y(n)$. This implies that

$$H(C_*(n)/H(\overline{C}_*(n))\otimes\Omega(C_*(X)))$$

is the free $H(\Omega(C_*(X)))$ module on $c$. On the other hand, the diagram of fibrations

$$\begin{array}{ccccc} \Omega X & \longrightarrow & F(n+1) & \longrightarrow & Y(n+1) \\ \| & & \downarrow & & \downarrow \\ \Omega X & \longrightarrow & F(n) & \longrightarrow & Y(n), \end{array}$$

identifies $H(F(n), F(n+1))$ as the free $H(\Omega X)$ module on $c$. And it is easy to see that $H(\overline{\psi})\colon c\mapsto c$, so that $H(\overline{\psi})$ is identified with $H(\psi_X)$ with a shift of degrees of at least 2.

Call a map of graded modules $k$ *regular* if it is an isomorphism in degrees $\le k$ and surjective in degree $k+1$. Suppose we know $H_*(\psi_{\omega(n)})$ is $k$ regular for all $n$. Then $H_*(\psi_X) = H_*(\psi_{\omega(N)})$ is $k$ regular, and so by what we have just seen, $H_*(\overline{\psi})$ in (3.1) is $(k+2)$ regular. Now the long exact homology sequence for (3.1) shows that if $H_*(\psi_{\omega(n)})$ is $(k+1)$ regular so is $H_*(\psi_{\omega(n+1)})$. Since $H_*(\psi_{\omega(1)})$ is, a priori, an isomorphism, it follows by induction on $n$ that $H_*(\psi_{\omega(n)})$ is $(k+1)$ regular for all $n$.

Thus $H_*(\psi_{\omega(N)})$ is $k$ regular for all $k$ and $n$; i.e., they are isomorphisms and, in particular, $H_*(\psi_{\omega(N)}) = H_*(\psi_X)$ is an isomorphism.

*Remark.* The last part of the argument is an essential part of the proof of the Moore comparison theorem, which is used by Adams [1] to prove Step 6 for his map $\phi$.

*Step* 7. *$H_*(\psi_\omega)$ is an isomorphism.* As in Step 6 we may reduce to the case that $Y \subset X$ is a subcomplex of a finite $CW$ complex $X$, except that now $X$ may have 0-cells and 1-cells. Now we consider a filtration $X = Y(1) \supset Y(2) \supset \cdots \supset Y(N) = Y$ where $Y(i)$ is obtained from $Y(i+1)$ by adding a single cell.

Again we have the diagram (3.1) with $H(\overline{\psi})$ identified with $H(\psi_X)$ up to a shift of degrees; hence $H(\overline{\psi})$ is an isomorphism by Step 6. Again, $H(\psi_{\omega(1)})$ is an isomorphism and so it follows trivially from (3.1) by induction on $n$ that each $H(\psi_{\omega(n)})$ is an isomorphism.

In particular, $H(\psi_\omega) = H(\psi_{\omega(N)})$ is an isomorphism.

## 4. Free models

In this section $R$ is always a *principal ideal domain.* It will sometimes be convenient to work with the superscript degrees (in which case differentials have degree $+1$) and we connect with §§2, 3 via the convention $V^i = V_{-i}$. Recall also that $\mathrm{Hom}(-; R)$ is written $(-)^\vee$. Finally, a graded module $V = \{V^i\}$ has *finite type* if each $V^i$ is finitely generated.

Our goal is Theorem V of the introduction; we begin with some generalities on free models. These are only a little more subtle than the case where $R$ is a field, considered in [7].

**Definition 4.1.** A *simply-connected free model* of a DGA, $A$, is a DGA quism

$$(F(V), d) \xrightarrow{\simeq} A,$$

in which $V = \{V^i\}_{i \geq 2}$ is a free $R$ module.

**Proposition 4.2.** *A* DGA, *$A$, has a simply-connected free model if and only if $H^{<0}(A) = 0$, $H^0(A) = R$, $H^1(A) = 0$, and $H^2(A)$ is $R$ free.*

*If these conditions hold, and each $H^i(A)$ is a finitely generated R-module, then $F(V)$ can be chosen to be finite type.*

*Proof*. The necessity of the conditions is obvious. To show existence we assume $\rho\colon (F(W), d) \to A$ is constructed so that $W = \{W^i\}_{i \geq 2}$ is $R$ free and $H^i(\rho)$ is an isomorphism for $i \leq k$ and surjective for all $i$. (For $k = 2$ we can do this easily with $d = 0$.)

Since submodules of free $R$ modules are free, we can choose a submodule, $Z$, of $(\ker d)^{k+1}$, free on cycles $z_\alpha$, which maps onto $\ker H^{k+1}(\rho)$. Let $x_\beta = dy_\beta$ be a basis of $Z \cap \mathrm{Im}\, d$, and write $x_\beta = \sum_\alpha r_{\beta\alpha} z_\alpha$.

Now set $W' = W \oplus (\bigoplus_\alpha R \cdot u_\alpha) \oplus (\bigoplus_\beta R \cdot v_\beta)$ where $\deg u_\alpha = k$ and $\deg v_\beta = k-1$. (When $k = 2$, $d = 0$ and there are no $x_\beta$ or $v_\beta$.) Put $du_\alpha = z_\alpha$;

by hypothesis we may extend $\rho$ to the $u_\alpha$. Note that then $y_\beta - \sum_\alpha r_{\beta\alpha} u_\alpha$ is a cycle and, since $H(\rho)$ is surjective there are cycles $w_\beta \in F(W)^k$ such that $\rho(y_\beta - \sum r_{\beta\alpha} u_\alpha - w_\beta)$ is a boundary.

Put $dv_\beta = y_\beta - \sum_\alpha r_{\beta\alpha} u_\alpha - w_\beta$; we can then extend $\rho$ to the $v_\beta$. It remains to check that this extension is injective in homology in degrees $k-1$, $k$, $k+1$. Because the elements $\sum_\alpha r_{\beta\alpha} u_\alpha$ are linearly independent in $\bigoplus_\alpha RU_\alpha$, we have added no cycles in degree $k-1$. In degree $k$ we have added only $\bigoplus_\alpha Ru_\alpha$, and by construction the only new cycles are combinations of $y_\beta - \sum_\alpha r_{\beta\alpha} u_\alpha - w_\beta$, which are killed by the $v_\beta$.

Finally, the new elements of degree $k+1$ are in the modules $\bigoplus_\beta (W^2 \otimes Rv_\beta)$, and none of these are the projections mod $F(W)^{k+2}$ of cycles. It follows that the new $H(\rho)$ is injective in degrees $\leq k+1$, which completes the inductive step.

If the module $H(A)$ is of finite type then this procedure produces a free model of finite type, because $R$ is noetherian. Q.E.D.

The second fact that we need about free models is a certain nilpotence property. Let $(F(V), d)$ be a simply-connected free model and define $Z \subset V$ by

$$Z = d^{-1}(F^{\geq 2}(V)) \cap V;$$

here $F^k(V)$ denotes its $k$th tensor power, as usual. Because submodules of free modules are free we can write $V = Z \oplus C$, and both $Z$ and $C$ are free. A straightforward check (as in the proof of (4.2)) then shows that

$$d\colon Z^{k+1} \to F(V^{\leq k-1} \oplus Z^k) \quad \text{and} \quad d\colon C^k \to F(V^{\leq k-1} \oplus Z^{k+1}).$$

There follows the

**Lemma 4.3.** *If $(F(V), d)$ is a simply-connected free model then there is a direct sum decomposition $V = \bigoplus_{i\geq 1} V(i)$ such that*

$$d\colon V(i) \to F\left(\bigoplus_{j<i} V(j)\right).$$

With Lemma 4.3 we can deduce, as an immediate corollary, that simply connected free models have the usual "lifting property". The explicit statement is

**Proposition 4.4.** *In the diagram of* DGA *morphisms*

$$\begin{array}{ccc} & & A' \\ & & \eta \big\downarrow \simeq \\ (F(V), d) & \xrightarrow[\xi]{} & A, \end{array}$$

*assume that $\eta$ is a surjective quism and $(F(V), d)$ is a simply-connected free model. Then there is a morphism $\xi'\colon (F(V), d) \to A'$ such that $\eta\xi' = \xi$.*

Finally, we shall need the

**Proposition 4.5.** *Given a* DGA *quism* $\phi\colon A \xleftarrow{\simeq} A'$ *we can find* DGA *quisms*

$$A \underset{\psi}{\xleftarrow{\simeq}} \widehat{A} \underset{\chi}{\xrightarrow{\simeq}} A'$$

*such that* $\psi$ *is surjective and* $\chi$ *admits a right inverse* $\lambda\colon A' \longrightarrow \hat{A}$ *such that* $\psi\lambda = \phi$.

*Proof.* It suffices to choose a free graded $R$ module, $U$ which maps surjectively to $A$, to define an acyclic differential module $U \underset{d}{\xrightarrow{\cong}} dU$, and to put $\widehat{A} = A' \amalg F(U \oplus dU)$, where $\amalg$ denote free product. Q.E.D.

We next state the analogues of the observations above for DGA modules, leaving the verifications to the reader.

**Proposition 4.6.** *If* $(F(V), d) \xrightarrow{\simeq} A$ *is a simply-connected free model, and if* $M$ *is an* $A$ *module such that* $H^{<n}(M) = 0$ *and* $H^n(M)$ *is* $R$ *free, then there is a* DGA-*module quism of the form*

$$(F(V) \otimes W, d) \xrightarrow{\simeq} M,$$

*in which* $W = \{W^i\}_{i \geq n}$ *is* $R$ *free.*

*If* $H(M)$ *and* $F(V)$ *have finite type as* $R$ *modules, then* $W$ *can also be chosen of finite type.*

**Proposition 4.7.** *If* $(F(V), d)$ *is a simply-connected free model, and if* $(F(V) \otimes W, d)$ *is an* $(F(V), d)$ *module in which* $W = \{W^i\}_{i \geq n}$ *is* $R$*-free, then* $W$ *has the form* $W = \bigoplus_{j \geq 0} W(j)$ *with*

$$d\colon W(j) \longrightarrow F(V) \otimes \left( \bigoplus_{i<j} W(i) \right).$$

*Remark* 4.8. Proposition 4.7 exhibits $(F(V) \otimes W, d)$ as an $(F(V), d)$-semifree module in the terminology of [3] and [5].

**Proposition 4.9.** *In the situation of Proposition* 4.5, *suppose* $M \xleftarrow{\simeq} M'$ *is a quism, over* $\phi$, *of* DGA-*modules. There is then an* $\widehat{A}$ *module,* $\widehat{M}$, *and there are* DGA-*module quisms*

$$M \underset{\alpha}{\xleftarrow{\simeq}} \widehat{M} \underset{\beta}{\xrightarrow{\simeq}} M',$$

*over* $\xi$ *and* $\psi$ *such that* $\alpha$ *is surjective.*

We are now ready to prove Theorem V. Thus we fix a continuous map $\omega\colon Y \to X$ between spaces of finite $R$ type, and we fix a basepoint $x_0 \in X$, supposed simply connected. Let $F$ be the homotopy fibre of $\omega$.

Consider $CS^*(Y)$ as a $CS^*(X)$ module via $CS^*(\omega)$. Propositions 4.2 and 4.6 then give free models of finite type,

$$(F(V), d) \xrightarrow{\simeq} CS^*(X) \quad \text{and} \quad (F(V) \otimes W, d) \xrightarrow{\simeq} CS^*(Y),$$

with $F(V)$ simply connected and $W = \{W^i\}_{i \geq 0}$.

On the other hand, point $\Omega X$ by the identity, and let $CS^0_*(\Omega X)$ be the chain complex of singular simplices that map the 0-skeleton to the basepoint. Then $CS^0_*(\Omega X)$ is an $R$ free DGA which reduces to $R.1$ in degree zero. Since $CS^0_*(\Omega X) \to CS_*(\Omega X)$ is a quism, so is $B(CS^0_*(\Omega X)) \to B(CS_*(\Omega X))$, by Remark 2.3.

Consider this, together with the sequence of DGC quisms and DGC-comodule quisms provided by Theorem IV. Dualizing, we obtain a sequence of DGA quisms and of DGA-module quisms of the form

$$CS^*(X) \xrightarrow{\simeq} \xleftarrow{\simeq} \xrightarrow{\simeq} \cdots \xrightarrow{\simeq} B[CS^0_*(\Omega X)]^\vee,$$

and

$$CS^*(Y) \xrightarrow{\simeq} \xleftarrow{\simeq} \cdots \xrightarrow{\simeq} B[CS_*(F); CS^0_*(\Omega X)]^\vee.$$

Moreover, Propositions 4.5 and 4.9 allow us to modify this sequence (possibly increasing its length) so that the quisms which point to the left are surjective. And finally, Propositions 4.4 and 4.7 allow us to lift morphisms from $(F(V), d)$ and from $(F(V) \otimes W, d)$ through surjective quisms. Thus, at last, we obtain DGA and DGA-module quisms

$$(4.10) \qquad \begin{cases} (F(V), d) \xrightarrow{\simeq} B[CS^0_*(\Omega X)]^\vee & \text{and} \\ (F(V) \otimes W, d) \xrightarrow{\simeq} B[CS_*(F); CS^0_*(\Omega X)]^\vee. \end{cases}$$

Consider for a moment an $R$ module morphism of the form $M \to N^\vee$ where $M$ is finitely generated, free, and $N$ is free. Dualizing, we get $M^\vee \leftarrow N^{\vee\vee}$; composing with the natural map $N \to N^{\vee\vee}$ yields $N \to M^\vee$. Because $M$ and $N$ are free and $M$ is finitely generated, this map $N \to M^\vee$ dualizes to the original map $M \to N^\vee$.

Thus in the situation above, because $B[CS^0_*(\Omega X)]$ and $B[CS_*(F); CS^0_*(\Omega X)]$ are $R$ free we can identify the maps (4.10) as the duals of maps

$$(4.11) \qquad \begin{cases} B[CS^0_*(\Omega X)] \to (T(V^\vee), d^\vee), & \text{and} \\ B[CS_*(F); CS^0_*(\Omega X)] \to (W^\vee \otimes T(V^\vee), d^\vee). \end{cases}$$

Moreover these maps are respectively a DGC morphism and a DGC-comodule morphism, and since their duals are quisms they must be quisms as well.

Notice now that the morphisms (4.11) satisfy the hypotheses of Proposition 2.10: all the modules are $R$ free and concentrated in positive degrees, and the coalgebras are simply connected. Thus we obtain

**Theorem 4.12.** *The morphisms* (4.11) *induce*:

(i) *Isomorphisms* $sH_+(\Omega X) \xrightarrow{\cong} H_*(V^\vee)$ *and* $H_*(F) \xrightarrow{\cong} H(W^\vee)$, *and*

(ii) *Isomorphisms of the* $E^1$ *terms of the canonical spectral sequences.*

In the case of the bar constructions in (4.11) the canonical spectral sequences are just the classical Eilenberg-Moore spectral sequences, and there are inclusions

$$B(H_*(\Omega X)) \longrightarrow (E^1, d^1), \qquad B(H_*(F); H_*(\Omega X)) \longrightarrow (E^1, d^1),$$

which are isomorphisms when $H_*(\Omega X)$ has no $R$ torsion.

These remarks, together with Theorem 4.12, complete the proof of Theorem V.

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