

A SHARP INEQUALITY FOR MARTINGALE TRANSFORMS AND THE UNCONDITIONAL BASIS CONSTANT OF A MONOTONE BASIS IN $L^p(0, 1)$

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ABSTRACT. Let $1 < p < \infty$. Let $d = (d_1, d_2, \dots)$ be a real-valued martingale difference sequence, $\theta = (\theta_1, \theta_2, \dots)$ is a predictable sequence taking values in $[0, 1]$. We show that the best constant of the inequality,

$$\left\| \sum_{k=1}^n \theta_k d_k \right\|_p \leq c_p \left\| \sum_{k=1}^n d_k \right\|_p, \quad n \geq 1,$$

satisfies

$$c_p = \frac{p}{2} + \frac{1}{2} \log \left(\frac{1+\gamma}{2} \right) + \frac{\alpha_2}{p} + \dots,$$

where $\gamma = e^{-2}$ and $\alpha_2 = [\frac{1}{2} \log \frac{1+\gamma}{2}]^2 + \frac{1}{2} \log \frac{1+\gamma}{2} - 2(\frac{\gamma}{1+\gamma})^2$. The best constant equals the unconditional basis constant of a monotone basis of $L^p(0, 1)$.

1. INTRODUCTION

More than fifty years ago, Paley [10] proved the following inequality for the Walsh system of functions ψ_n on the Lebesgue unit interval. If $1 < p < \infty$, there is a positive real number c_p with the property that if $b_1, b_2, \dots$ are real numbers and

$$e_n = \sum_{2^n \leq m < 2^{n+1}} b_m \psi_m,$$

then

$$(1.1) \quad c_p^{-1} \left\| \sum_{k=1}^n e_k \right\|_p \leq \left\| \sum_{k=1}^n \varepsilon_k e_k \right\|_p \leq c_p \left\| \sum_{k=1}^n e_k \right\|_p,$$

for all signs $\varepsilon_k \in \{1, -1\}$ and all positive integer n . Notice that the left-hand side of this inequality follows at once from the right-hand side.

It was then observed by Marcinkiewicz [8] that Paley's inequality can be given an equivalent formulation in terms of the Haar system of functions h_n :

$$(1.2) \quad c_p^{-1} \left\| \sum_{k=1}^n a_k h_k \right\|_p \leq \left\| \sum_{k=1}^n \varepsilon_k a_k h_k \right\|_p \leq c_p \left\| \sum_{k=1}^n a_k h_k \right\|_p, \quad 1 < p < \infty.$$

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Here $a_1, a_2, \dots$ are real numbers and the constant c_p is the same as in Paley's inequality.

Throughout this work, we adopt the following convention. The constant c_p may change from one use to the next; however, if it is necessary to be more specific, the best constant in an inequality, say (1.1), is denoted by $c_p(1.1)$. With this notation, Marcinkiewicz's result can be stated as follows:

$$c_p(1.1) = c_p(1.2).$$

In 1966, Burkholder [1] extended the result of Paley and Marcinkiewicz to martingales:

$$(1.3) \quad \left\| \sum_{k=1}^n v_k d_k \right\|_p \leq c_p \left\| \sum_{k=1}^n d_k \right\|_p, \quad 1 < p < \infty,$$

where $v = (v_1, v_2, \dots)$ is a predictable sequence uniformly bounded in absolute value by 1 and $d = (d_1, d_2, \dots)$ is a martingale difference sequence. Here the constant c_p is independent of both v and d .

An important special case of (1.3) is

$$(1.4) \quad \left\| \sum_{k=1}^n \varepsilon_k d_k \right\|_p \leq c_p \left\| \sum_{k=1}^n d_k \right\|_p, \quad 1 < p < \infty,$$

where, again, $\varepsilon_k \in \{1, -1\}$. Clearly, $c_p(1.4) \leq c_p(1.3)$. In 1981, Burkholder [2] showed that $c_p(1.3) \leq c_p(1.4)$, so equality holds. The Haar system $h = (h_1, h_2, \dots)$ is a martingale difference sequence, as is $d = (a_1 h_1, a_2 h_2, a_3 h_3, \dots)$ for real numbers a_k . Therefore $c_p(1.2) \leq c_p(1.4)$. Maurey [9] proved the reverse inequality. Therefore

$$c_p(1.1) = c_p(1.2) = c_p(1.3) = c_p(1.4).$$

In 1984, Burkholder [3] derived the value of this best constant. It is

$$p^* - 1$$

where p^* is the maximum of p and its conjugate $q = p/(p-1)$. The proof rests on solving a system of nonlinear partial differential equations and inequalities. See his paper [4] for a shorter proof.

Inequality (1.3) carries over to stochastic integrals with no change in the value of the best constant (see [3]). It has applications not only in probability theory but also in Fourier analysis and the theory of singular integrals. It carries over to B -valued martingales for a large class of Banach spaces B where the constant depends both on p and B . A geometrical characterization of this class is given in [2]. For a discussion of some of this, see [5].

The main contribution of this paper is a set of equations (see Theorem 3.3 and (3.11), (3.12) in §3) that determine the best constant in the inequality,

$$(1.5) \quad \left\| \sum_{k=1}^n \theta_k d_k \right\|_p \leq c_p \left\| \sum_{k=1}^n d_k \right\|_p.$$

Here, as before, $1 < p < \infty$ and $d = (d_1, d_2, \dots)$ is a martingale difference sequence, but $\theta = (\theta_1, \theta_2, \dots)$ is a predictable sequence taking values 0 or 1.

The gambling interpretation of (1.5) is obvious: as long as the gambler cannot look into the future and the game is fair in the sense that $d = (d_1, d_2, \dots)$ forms a martingale difference sequence then his fortune $\sum_{k=1}^n \theta_k d_k$ is controlled by the fortune $\sum_{k=1}^n d_k$ that would have been achieved without skipping bets. This holds for any p in $(1, \infty)$, but does not hold for $p = 1$ or $p = \infty$ in general. Inequality (1.5) can be extended to θ_k taking values in $[0, 1]$ with the same optimal constant $c_p(1.5)$.

Using a discretization argument (see §16 of [3]), we can extend the inequality (1.5) to stochastic integrals with $c_p(1.5)$ as the best constant.

The inequality (1.5) has another important connection, a connection with the unconditional basis constant. Let $1 < p < \infty$ and $e = (e_1, e_2, \dots)$ be a basis of real $L^p(0, 1)$. The unconditional basis constant, denoted by $K_p(e)$, is the extended real number (see [7], for example)

$$\sup \left\{ \left\| \sum_{k=1}^n \theta_k a_k e_k \right\|_p : \text{where } \left\| \sum_{k=1}^n a_k e_k \right\|_p = 1, \text{ for } n \geq 1, \right. \\ \left. a_1, \dots, a_n \text{ are real numbers and } \theta_k \in \{0, 1\} \right\}.$$

Clearly, the unconditional basis constant of the Haar system satisfies

$$K_p(h) \leq c_p(1.5)$$

and by the method of Maurey [9] the reverse inequality is true. Therefore, we have the following theorem.

Theorem A. *If $K_p(h)$ is the unconditional basis constant of the Haar system, then*

$$K_p(h) = c_p(1.5), \quad 1 < p < \infty.$$

Let $(\Omega, \mathcal{A}, \mu)$ be a positive measure space and $(P_1, P_2, \dots)$ be a non-decreasing sequence of contractive projections in $L^p(\Omega, \mathcal{A}, \mu)$: for every $n, m \geq 1$,

$$P_m P_n = P_n P_m = P_{\min\{m, n\}} \quad \text{and} \quad \|P_n\| \leq 1.$$

Theorem B. *Let $P = (P_1, P_2, \dots)$ be any nondecreasing sequence of contractive projections in $L^p(\Omega, \mathcal{A}, \mu)$ and let $P_0 = 0$. If $f \in L^p(\Omega, \mathcal{A}, \mu)$, then*

$$\left\| \sum_{k=1}^n a_k (P_k - P_{k-1}) f \right\|_p \leq c_p(1.5) \|f\|_p, \quad 1 < p < \infty,$$

for all integers $n \geq 1$ and all numbers $a_k \in [0, 1]$. And this inequality is sharp.

A basis (for definition, see [7]) $e = (e_1, e_2, \dots)$ in a real Banach space B with norm $\|\cdot\|_B$ is said to be monotone if

$$\left\| \sum_{k=1}^n a_k e_k \right\|_B \leq \left\| \sum_{k=1}^{n+1} a_k e_k \right\|_B,$$

for every $n \geq 1$ and all $a_k \in \mathbf{R}$. If B does have a monotone basis and $x = \sum_{k=1}^{\infty} a_k e_k$, then let $P_0 x = 0$ and $P_n x = \sum_{k=1}^n a_k e_k$ for all $n \geq 1$. Then it is easy to verify that $P = (P_0, P_1, \dots)$ is a nondecreasing sequence of contractive projections so we have the following conclusion.

Corollary C. *The unconditional basis constant of a monotone basis of $L^p(0, 1)$ is $c_p(1.5)$, i.e.*

$$K_p(e) = c_p(1.5).$$

See [3] for the proof of Theorem B. These theorems illustrate the interest in knowing the value of $c_p(1.5)$.

In §4 (see Theorem 4.3), it is shown that

$$K_p(e) = c_p(1.5) \sim \frac{p}{2} + \frac{1}{2} \log \left(\frac{1 + e^{-2}}{2} \right).$$

2. ZIGZAG MARTINGALES

Let $f = (f_1, f_2, \dots)$ be a real-valued martingale on a probability space $(\Omega, \mathcal{A}, P)$ and $d = (d_1, d_2, \dots)$ be its difference sequence. Suppose $f_1 = x - y$, where $x, y \in \mathbf{R}$ and $\theta = (1, \theta_2, \theta_3, \dots)$ is a sequence of real numbers taking values in $\{0, 1\}$. A sequence $g = (g_1, g_2, \dots)$ is the transform of f by $\theta = (1, \theta_2, \theta_3, \dots)$ if, for every $n \geq 1$, $g_n = \sum_{k=1}^n \theta_k d_k$, where $\theta_1 = 1$. Indeed, g is also a real-valued martingale.

Let $(X_1, Y_1) = (x, y)$ and for all $n \geq 2$,

$$(2.1) \quad X_n = x + \sum_{k=2}^n \theta_k d_k,$$

$$(2.2) \quad Y_n = y + \sum_{k=2}^n (\theta_k - 1) d_k.$$

Then $Z = (Z_1, Z_2, \dots)$, where $Z_n = (X_n, Y_n)$, is an $\mathbf{R}^2$ -valued martingale starting at (x, y) . Since $\theta_k \in \{0, 1\}$, it is obvious that for each $n \geq 2$ either $X_n - X_{n-1} \equiv 0$ or $Y_n - Y_{n-1} \equiv 0$. In other words, if Z moves at all at the n th step ($n \geq 2$), it moves either horizontally or vertically, which way depending on n only. In the terminology of [3], Z is a zigzag martingale. Furthermore, we can recover f_n and g_n by

$$(2.3) \quad f_n = X_n - Y_n,$$

$$(2.4) \quad g_n + y = X_n.$$

3. SHARP INEQUALITIES

By a standard duality argument, it can be proved that

$$(3.1) \quad c_p(1.5) = c_q(1.5), \quad 1 < p, q < \infty, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

Therefore, we will determine $c_p(1.5)$, $2 < p < \infty$.

Let p_0 be the unique solution to the equation

$$(3.2) \quad p - 2 = \left[\frac{(p-1)(p-2)}{-p^2 + 5p - 5} \right]^{p-1}, \quad 2 < p < 3,$$

$$(3.3) \quad p_0 \simeq 2.5455458.$$

Indeed, putting $x = p - 2$ into (3.2) and simplifying, we obtain an equivalent equation

$$(3.4) \quad (x + 1) \log(-x^2 + x + 1) = x \log x + (x + 1) \log(x + 1), \quad 0 < x < 1.$$

The left-hand side is concave in x and the right-hand side is convex in x . The existence and uniqueness of solution to (3.4) can then be deduced readily by considering the behavior of the two sides near the endpoints of $(0, 1)$. Indeed, we have

$$(3.5) \quad (p - 2)[-p^2 + 5p - 5]^{p-1} \geq [(p - 1)(p - 2)]^{p-1}, \quad 2 < p \leq p_0,$$

and,

$$(3.6) \quad (p - 2)[-p^2 + 5p - 5]^{p-1} < [(p - 1)(p - 2)]^{p-1}, \quad p_0 < p \leq 3.$$

For $t \in [-1, 1]$, we define

$$(3.7) \quad E(t) = \operatorname{sgn}(t)|t|^{p-1} - (p - 1)t + p - 2,$$

$$(3.8) \quad A(t) = (p - 1)(1 - t)^2 - [(p - 2) - pt]E(t),$$

$$(3.9) \quad D(t) = (p - 1)(1 - t)^2 + tE(t), \quad \text{and}$$

$$(3.10) \quad B(t) = (p - 1)(1 - t)^2 E(t) - tA(t),$$

or equivalently,

$$(3.10') \quad B(t) = [(p - 1) - pt]E(t) - tD(t).$$

For $p_0 < p < \infty$, let $I_p = (0 \vee (p - 3)/(p - 1), (p - 2)/p)$, and for $2 < p \leq p_0$, $I_p = (-(3 - p)/2p, 0]$.

Lemma 3.1. *For $2 < p < \infty$, there exists a unique solution, $t_p \in I_p$, to the equation*

$$(3.11) \quad [p - 2 - (p - 1)t][A(t)]^{p-1} = [B(t)]^{p-1}.$$

For $2 < p < \infty$, let

$$(3.12) \quad k_p = \frac{A(t_p)}{(p - 1)(1 - t_p)^2} \left\{ \frac{D(t_p)}{A(t_p)} \right\}^p, \quad \text{and}$$

$$(3.13) \quad v(x, y) = |x|^p - k_p |x - y|^p, \quad (x, y) \in \mathbf{R}^2.$$

Lemma 3.2. *For $2 < p < \infty$, there exists a biconcave function $u : \mathbf{R}^2 \rightarrow \mathbf{R}$ such that $u(0, 0) = 0$, $u(x, y) \geq v(x, y)$ for all $(x, y) \in \mathbf{R}^2$, and u satisfies the following bounds:*

$$(3.14) \quad |u(x, y)| \leq c_p(|x|^p + |y|^p),$$

$$(3.15) \quad |u_x(x, y)| \leq c_p(|x|^{p-1} + |y|^{p-1}), \quad \text{and},$$

$$(3.16) \quad |u_y(x, y)| \leq c_p(|x|^{p-1} + |y|^{p-1}).$$

Lemmas 3.1 and 3.2 will be proved in §5.

The following theorem is the main result of this paper.

Theorem 3.3. For $2 < p < \infty$,

$$c_p(1.5) = (k_p)^{1/p}.$$

Proof. The proof is based on the idea of Burkholder in [4]. The proof consists of two parts. Part I makes use of Lemma 3.2 to show that $c_p^p(1.5) \leq k_p$ and Part II shows by an example that $k_p \leq c_p^p(1.5)$.

Part I. Let $d = (d_1, d_2, \dots)$ be a martingale difference sequence of an L^p -bounded martingale f and let g be the martingale transform of f by a predictable sequence $\theta = (\theta_1, \theta_2, \dots)$, where $\theta_k \in [0, 1]$. By a reduction argument (see §2 in [3]), we may assume $d_1 \equiv 0$, $\theta_1 \equiv 1$ and that $\theta = (1, \theta_2, \dots)$ is a sequence of real numbers taking values in $\{0, 1\}$. Construct the zigzag martingale $Z = (Z_1, Z_2, \dots)$, $Z_n = (X_n, Y_n)$ by (2.1) and (2.2) where $x = y = 0$. By Lemma 3.2, there exists a biconcave function $u(x, y)$ that majorizes $v(x, y)$, therefore

$$v(X_n, Y_n) = |X_n|^p - k_p |X_n - Y_n|^p \leq u(X_n, Y_n).$$

By (2.3) and (2.4), this implies that

$$|g_n|^p - k_p |f_n|^p \leq u(X_n, Y_n).$$

Using the bound (3.14), we see that $u(X_n, Y_n)$ is integrable. Taking expectation, we have that

$$(3.17) \quad \|g_n\|_p^p - k_p \|f_n\|_p^p \leq Eu(X_n, Y_n).$$

We also observe that

$$(3.18) \quad \begin{aligned} u(X_n, Y_n) &\leq u(X_{n-1}, Y_{n-1}) + \theta_n u_x(X_{n-1}, Y_{n-1}) d_n \\ &\quad + (\theta_n - 1) u_y(X_{n-1}, Y_{n-1}) d_n, \end{aligned}$$

since u is biconcave and $\theta_n = 0$ or $1 - \theta_n = 0$.

Now d_n is L^p -integrable and by the bounds (3.15) and (3.16), it follows that $u_x(X_{n-1}, Y_{n-1})$ and $u_y(X_{n-1}, Y_{n-1})$ are L^q -integrable. Indeed

$$\begin{aligned} \|u_x(X_{n-1}, Y_{n-1})\|_q^q &\leq c_p E(|X_{n-1}|^{p-1} + |Y_{n-1}|^{p-1})^q \\ &\leq c_p (E|X_{n-1}|^{(p-1)q} + E|Y_{n-1}|^{(p-1)q}) \\ &= c_p (\|X_{n-1}\|_p^p + \|Y_{n-1}\|_p^p) \\ &= c_p \left(\left\| \sum_{k=1}^n \theta_k d_k \right\|_p^p + \left\| \sum_{k=1}^n (1 - \theta_k) d_k \right\|_p^p \right) \\ &\leq c_p \sum_{k=1}^n \|d_k\|_p^p < \infty. \end{aligned}$$

Similarly $\|u_y(X_{n-1}, Y_{n-1})\|_q < \infty$. By Hölder's inequality, $u_x(X_{n-1}, Y_{n-1})d_n$ and $u_y(X_{n-1}, Y_{n-1})d_n$ are integrable. Therefore

$$Eu_x(X_{n-1}, Y_{n-1})d_n = E(E(d_n | \mathcal{A}_{n-1})u_x(X_{n-1}, Y_{n-1})) = 0$$

and, similarly, $Eu_y(X_{n-1}, Y_{n-1})d_n = 0$.

Hence, by (3.18) we get $Eu(X_n, Y_n) \leq Eu(X_{n-1}, Y_{n-1})$. Working backward, we have

$$Eu(X_n, Y_n) \leq Eu(X_1, Y_1) = u(0, 0) = 0.$$

Combining this and (3.17), we have

$$\|g_n\|_p^p \leq k_p \|f_n\|_p^p.$$

Therefore $c_p^p(1.5) \leq k_p$.

Part II. We will exhibit an example here to show that $c_p^p(1.5) \geq k_p$. We need the following definitions. Let t_p be as in Lemma 3.2,

$$(3.19) \quad \alpha = [p - 2 - (p - 1)t_p] \frac{E(t_p)}{D(t_p)},$$

$$(3.20) \quad \beta = \frac{t_p}{[p - 1 - pt_p]} \frac{D(t_p)}{E(t_p)},$$

$$(3.21) \quad \lambda = [p - 1 - pt_p] \frac{E(t_p)}{D(t_p)},$$

$$(3.22) \quad \omega = 2 \frac{[p - 1 - pt_p]}{(p - 1)(1 - t_p)^2} E(t_p).$$

Note that $\beta > 0$ when $p_0 < p < \infty$; $\beta \leq 0$ when $2 < p \leq p_0$.

We need the following technical lemma which can be verified by straightforward computation. Its proof is given in §5.

Lemma 3.4. For $2 < p < \infty$,

- (i) $\frac{\lambda}{\lambda - \alpha} + \frac{|\beta\lambda|^p}{1 - \beta\lambda} = k_p \left[\frac{(1 - \alpha)^p \lambda}{\lambda - \alpha} + \frac{(1 - \beta)^p \lambda^p}{1 - \beta\lambda} \right],$
- (ii) $\frac{\lambda}{\lambda - \alpha} + \frac{1}{1 - \beta\lambda} = p,$
- (iii) $0 < \alpha < \lambda < \beta^{-1}$, for $p_0 < p < \infty$, and
- (iii') $0 < \alpha < \lambda$, $\beta \leq 0$, for $2 < p \leq p_0$.

Returning to the proof of Theorem 3.3, we proceed as follows. For $x > 0$, there exists a unique $\eta \in (\alpha, \lambda)$ such that

$$(3.23) \quad x^p = \frac{\lambda}{\lambda - \eta} + \frac{1}{1 - \beta\lambda} - p.$$

We have, by (ii) in Lemma 3.4, that η converges to α as x converges to 0.

Case (1). $p_0 < p < \infty$. Fix $x > 0$ and choose $\delta \in (0, x)$, actually we will eventually let $\delta \rightarrow 0$. For all $k \geq 1$, define

$$\begin{aligned} \xi_k &= 1 - \frac{\lambda\delta}{(\lambda - \eta)(x + k\delta) + \eta\delta}, \\ \gamma_k &= 1 - \frac{\delta}{(1 - \beta\lambda)(x + k\delta)}, \\ \pi_0 &= 1, \quad \text{and} \quad \pi_k = \prod_{j=1}^k \gamma_j \xi_j. \end{aligned}$$

Note that $\beta > 0$.

When there is no risk of ambiguity, we use $[a, b)$ to denote either the interval $\{x : a \leq x < b\}$ or the indicator function of the set $[a, b)$.

On the probability space $([0, 1), \mathscr{A}, m)$, where $\mathscr{A}$ is the σ -field of all Borel measurable sets in $[0, 1)$ and m is the Lebesgue measure on $[0, 1)$, we

define a sequence of functions on $[0, 1)$, $d = (d_1, d_2, \dots)$ as follows:

$$\begin{aligned} d_1 &\equiv (1 - \lambda)x, \quad \text{and for all } k \geq 1, \\ d_{2k} &= -\lambda\delta[0, \pi_{k-1}\xi_k] + (\lambda - \eta)[x + (k - 1)\delta][\pi_{k-1}\xi_k, \pi_{k-1}), \\ d_{2k+1} &= \delta[0, \pi_k] - [(1 - \beta\lambda)(x + k\delta) - \delta][\pi_k, \pi_{k-1}\xi_k]. \end{aligned}$$

It is not difficult to see that $d = (d_1, d_2, \dots)$ forms a martingale difference sequence. Let $\theta_{2k} = 0$ and $\theta_{2k-1} = 1$ for all $k \geq 1$. Let f be the martingale with the martingale difference sequence $d = (d_1, d_2, \dots)$ and g the martingale transform of f by $\theta = (\theta_1, \theta_2, \dots)$. Therefore, we have

$$\begin{aligned} \|f\|_p &= \lim_{n \rightarrow \infty} \|f_{2n+1}\|_p^p \\ &= \lim_{n \rightarrow \infty} \|X_{2n+1} - Y_{2n+1}\|_p^p \\ &= \lim_{n \rightarrow \infty} \left\{ (1 - \lambda)^p (x + n\delta)^p \pi_n + \sum_{k=1}^n (1 - \eta)^p [x + (k - 1)\delta]^p \pi_{k-1} (1 - \xi_k) \right. \\ &\quad \left. + \sum_{k=1}^n (1 - \beta)^p \lambda^p (x + k\delta)^p \pi_{k-1} \xi_k (1 - \gamma_k) \right\}, \end{aligned}$$

and similarly,

$$\begin{aligned} \|g + \lambda x\|_p^p &= \lim_{n \rightarrow \infty} \|X_{2n+1}\|_p^p \\ &= \lim_{n \rightarrow \infty} \left\{ (x + n\delta)^p \pi_n + \sum_{k=1}^n [x + (k - 1)\delta]^p \pi_{k-1} (1 - \xi_k) \right. \\ &\quad \left. + \sum_{k=1}^n (\beta\lambda)^p (x + k\delta)^p \pi_{k-1} \xi_k (1 - \gamma_k) \right\}. \end{aligned}$$

We make use of the inequality $1 - w\delta \leq (1 - \delta)^w$ for all $w > 1$ and $0 < \delta < 1$. Since $\beta\lambda < 1$ by Lemma 3.4(iii), we introduce $s_1 = \lambda/(\lambda - \eta) > 1$ and $s_2 = 1/(1 - \beta\lambda) > 1$, then

$$\begin{aligned} \pi_k &= \left(\prod_{j=1}^k \gamma_j \right) \left(\prod_{j=1}^k \xi_j \right) \leq \prod_{j=1}^k \left(1 - \frac{\delta}{x + j\delta} \right)^{s_2} \times \prod_{j=1}^k \left(1 - \frac{\delta}{x + j\delta + \frac{\eta\delta}{\lambda - \eta}} \right)^{s_1} \\ &= \left(\frac{x}{x + k\delta} \right)^{s_1 + s_2} \left(1 + \frac{\eta\delta}{(\lambda - \eta)x} \right)^{s_1} \left(\frac{x + k\delta}{x + k\delta + \frac{\eta\delta}{\lambda - \eta}} \right)^{s_1} \\ &\leq H_1(\delta) \left(\frac{x}{x + k\delta} \right)^{s_1 + s_2}, \end{aligned}$$

where

$$H_1(\delta) = \left(1 + \frac{\eta\delta}{(\lambda - \eta)x} \right)^{s_1} \rightarrow 1 \quad \text{as } \delta \rightarrow 0.$$

Now

$$0 \leq (x + n\delta)^p \pi_n \leq H_1(\delta) \left(\frac{x}{x + n\delta} \right)^{s_1 + s_2} (x + n\delta)^p,$$

the right side goes to zero as n goes to infinity by (3.23). So

$$\begin{aligned}
& \limsup_{\delta \rightarrow 0} \|f\|_p^p \\
& \leq \limsup_{\delta \rightarrow 0} \left\{ \sum_{k=1}^{\infty} (1-\eta)^p [x + (k-1)\delta]^p \pi_{k-1} (1-\xi_k) \right. \\
& \quad \left. + \sum_{k=1}^{\infty} (1-\beta)^p \lambda^p (x + k\delta)^p \pi_{k-1} \xi_k (1-\gamma_k) \right\} \\
& = \limsup_{\delta \rightarrow 0} \left\{ (1-\eta)^p x^p (1-\xi_1) + \sum_{k=1}^{\infty} (1-\eta)^p (x + k\delta)^p \pi_k (1-\xi_{k+1}) \right. \\
& \quad \left. + \sum_{k=1}^{\infty} (1-\beta)^p \lambda^p (x + k\delta)^p \pi_k \frac{1-\gamma_k}{\gamma_k} \right\} \\
& \leq \limsup_{\delta \rightarrow 0} \left\{ \frac{\lambda(1-\eta)^p}{\lambda-\eta} + \frac{(1-\beta)^p \lambda^p}{1-\beta\lambda} \right\} \delta \sum_{k=1}^{\infty} (x + k\delta)^{p-1} \pi_k \\
& \leq \left\{ \frac{\lambda(1-\eta)^p}{\lambda-\eta} + \frac{(1-\beta)^p \lambda^p}{1-\beta\lambda} \right\} \limsup_{\delta \rightarrow 0} H_1(\delta) \delta \sum_{k=1}^{\infty} \frac{x^{s_1+s_2}}{(x + k\delta)^{s_1+s_2-p+1}} \\
& \leq \left\{ \frac{\lambda(1-\eta)^p}{\lambda-\eta} + \frac{(1-\beta)^p \lambda^p}{1-\beta\lambda} \right\} \limsup_{\delta \rightarrow 0} \delta \int_0^{\infty} \frac{x^{s_1+s_2}}{(x + t\delta)^{s_1+s_2-p+1}} dt \\
& = \left\{ \frac{\lambda(1-\eta)^p}{\lambda-\eta} + \frac{(1-\beta)^p \lambda^p}{1-\beta\lambda} \right\} \frac{x^p}{s_1+s_2-p} \\
& = \frac{\lambda(1-\eta)^p}{\lambda-\eta} + \frac{(1-\beta)^p \lambda^p}{1-\beta\lambda} \quad \text{by (3.23).}
\end{aligned}$$

Similarly,

$$\limsup_{\delta \rightarrow 0} \|g + \lambda x\|_p^p \leq \frac{\lambda}{\lambda-\eta} + \frac{(\beta\lambda)^p}{1-\beta\lambda}.$$

We proceed to a lower estimate of π_k . Note that if $w > 1$ and $w_1 = w(x-\delta)/(x-w\delta)$, then

$$1 - ws \geq (1-s)^{w_1}, \quad s \in \left(0, \frac{\delta}{x}\right).$$

To see this, consider

$$\frac{\log(1-ws)}{\log(1-s)} = \frac{\log(1-ws) - \log 1}{\log(1-s) - \log 1},$$

by mean value theorem, there exists a $c \in (0, s)$,

$$= \frac{-w/(1-wc)}{-1/(1-c)} = \frac{w-1}{1-wc} + 1 \leq \frac{w-1}{1-w\delta/x} + 1 = w_1.$$

Therefore,

$$\log(1-ws) \geq w_1 \log(1-s),$$

i.e.,

$$1 - ws \geq (1-s)^{w_1}.$$

To obtain a lower estimate of π_k , let

$$\begin{aligned} r_1 &= \frac{x - \delta}{(1 - \beta\lambda)x - \delta}, \quad r_2 = \frac{\lambda(x - \delta)}{(\lambda - \eta)x - \lambda\delta}, \\ \pi_k &\geq \prod_{j=1}^k \left(1 - \frac{\delta}{x + j\delta}\right)^{r_1} \left(1 - \frac{\delta}{x + j\delta + \frac{\eta\delta}{\lambda - \eta}}\right)^{r_2} \\ &= \frac{x^{r_1+r_2}}{(x + k\delta)^{r_1+r_2}} \left(\frac{x + k\delta}{x + k\delta + \frac{\eta\delta}{\lambda - \eta}}\right)^{r_2} \left(\frac{x + \frac{\eta\delta}{\lambda - \eta}}{x}\right)^{r_2} \\ &\geq \frac{x^{r_1+r_2}}{(x + k\delta)^{r_1+r_2}} H_2(\delta), \end{aligned}$$

where

$$H_2(\delta) = \left[\frac{x + k\delta}{x + k\delta + \frac{\eta\delta}{\lambda - \eta}} \right]^{r_2} \rightarrow 1 \quad \text{as } \delta \rightarrow 0.$$

Furthermore, as $\delta \rightarrow 0$, $r_1 \rightarrow 1/(1 - \beta\lambda)$ and $r_2 \rightarrow \lambda/(\lambda - \eta)$. Therefore,

$$\begin{aligned} \liminf_{\delta \rightarrow 0} \|f\|_p^p &\geq \liminf_{\delta \rightarrow 0} \left\{ \frac{\lambda(1 - \eta)^p}{\lambda - \eta} + \frac{(1 - \beta)^p \lambda^p}{1 - \beta\lambda} \right\} \sum_{k=1}^{\infty} \frac{\delta x^{r_1+r_2} H_2(\delta)}{(x + k\delta)^{r_1+r_2-p+1}} \\ &= \liminf_{\delta \rightarrow 0} \left\{ \frac{\lambda(1 - \eta)^p}{\lambda - \eta} + \frac{(1 - \beta)^p \lambda^p}{1 - \beta\lambda} \right\} \int_0^{\infty} \frac{\delta x^{r_1+r_2}}{(x + t\delta)^{r_1+r_2-p+1}} dt \\ &= \frac{\lambda(1 - \eta)^p}{\lambda - \eta} + \frac{(1 - \beta)^p \lambda^p}{1 - \beta\lambda}, \end{aligned}$$

i.e.,

$$\lim_{\delta \rightarrow 0} \|f\|_p^p = \frac{\lambda(1 - \eta)^p}{\lambda - \eta} + \frac{(1 - \beta)^p \lambda^p}{1 - \beta\lambda}.$$

Similarly

$$\lim_{\delta \rightarrow 0} \|g + \lambda x\|_p^p = \frac{\lambda}{\lambda - \eta} + \frac{(\beta\lambda)^p}{1 - \beta\lambda},$$

so

$$\begin{aligned} \lim_{x \rightarrow 0} \lim_{\delta \rightarrow 0} \|f\|_p^p &= \frac{\lambda(1 - \alpha)^p}{\lambda - \alpha} + \frac{(1 - \beta)^p \lambda^p}{1 - \beta\lambda} \\ &= \frac{1}{k_p} \lim_{x \rightarrow 0} \lim_{\delta \rightarrow 0} \|g + \lambda x\|_p^p \end{aligned}$$

by Lemma 3.4(i) since $\beta > 0$ for $p_0 < p < \infty$

$$= \frac{1}{k_p} \lim_{x \rightarrow 0} \lim_{\delta \rightarrow 0} \|g\|_p^p,$$

i.e., $c_p^p(1.5) \geq k_p$.

Case (2). $2 < p \leq p_0$. Fix $x > 0$ and choose $\delta > 0$ such that $\delta \ll x$. For all $k \geq 1$, define ξ_k , γ_k , π_k , d_k and θ_k as in Case (1). Similar calculation as in Case (1) shows that $k_p \leq c_p^p(1.5)$. This completes the proof, apart from the lemmas, of Theorem 3.3. $\square$

Remark. The martingale f described in Part II of the proof of Theorem 3.3 is actually constructed from a zigzag martingale by (2.3). Here we provide a brief geometrical description of this zigzag martingale. Let $x > 0$, choose $\eta \in (\alpha, \lambda)$ by (3.23). The zigzag martingale first starts at $(x, \lambda x)$, it then moves vertically either down to the point $(x, \eta x)$ where it will stop or up to the point $(x, \lambda(x + \delta))$. From the point $(x, \lambda(x + \delta))$, the zigzag martingale moves horizontal either to the left to the point $(\beta\lambda(x + \delta), \lambda(x + \delta))$ where it will stop or to the right to the point $(x + \delta, \lambda(x + \delta))$. The pattern of movement is then repeated.

4. ASYMPTOTIC RESULTS AND REMARKS

We make use of a result of Burkholder in [3] and triangle inequality to obtain

Theorem 4.1. For $1 < p < \infty$,

$$\max \left\{ 1, \frac{p^*}{2} - 1 \right\} \leq c_p(1.5) \leq \frac{1}{2}p^*,$$

where $p^* = \max\{p, p/(p-1)\}$.

Proof. Let $d = (d_1, d_2, \dots)$ be a martingale difference sequence and $\theta_k \in \{0, 1\}$, $k \geq 1$. Let $\varepsilon_k = 2\theta_k - 1$. Then $\varepsilon_k \in \{1, -1\}$ and

$$\begin{aligned} \left\| \sum_{k=1}^n \theta_k d_k \right\| &= \frac{1}{2} \left\| \sum_{k=1}^n d_k + \sum_{k=1}^n \varepsilon_k d_k \right\|_p \\ &\leq \frac{1 + c_p(1.4)}{2} \left\| \sum_{k=1}^n d_k \right\|_p. \end{aligned}$$

Therefore, $c_p(1.5) \leq (1 + c_p(1.4))/2 = \frac{1}{2}p^*$.

Similarly, let $\varepsilon_k \in \{1, -1\}$ and define $\theta_k = (\varepsilon_k + 1)/2 \in \{0, 1\}$. Then

$$\begin{aligned} \left\| \sum_{k=1}^n \varepsilon_k d_k \right\|_p &= \left\| 2 \sum_{k=1}^n \theta_k d_k - \sum_{k=1}^n d_k \right\|_p \\ &\leq [2c_p(1.5) + 1] \left\| \sum_{k=1}^n d_k \right\|_p. \end{aligned}$$

Therefore, $c_p(1.4) - 1 \leq 2c_p(1.5)$. On the other hand, $c_p(1.5) \geq 1$, Theorem 4.1 follows immediately. $\square$

We assume that $t_p = \sum_{k=0}^{\infty} a_k p^{-k}$, $p_0 < p < \infty$. By (3.10) and (3.8), we rewrite (3.11) as

$$(4.1) \quad \{t_p + [p - 2 - (p-1)t_p]^{1/(p-1)}\}^{-1} = \frac{1}{E(t_p)} - \frac{p-2-pt_p}{(p-1)(1-t_p)^2}.$$

From Theorem 3.3, we have $a_0 = 1$, $a_1 = -2$. Substituting $t_p = \sum_{k=0}^{\infty} a_k p^{-k}$ into (4.1) and equating coefficients of p^{-k} , $k = 0, 1, 2, \dots$, we obtain

Theorem 4.2. Assuming $t_p = \sum_{k=0}^{\infty} a_k p^{-k}$, $p_0 < p < \infty$, then

$$t_p = 1 - \frac{2}{p} - \frac{2(1-\gamma)}{(1+\gamma)} \frac{1}{p^2} - \frac{4\gamma(3-\gamma)(1-\gamma)}{(1+\gamma)^3} \frac{1}{p^3} \\ + \frac{4\gamma(-9\gamma^4 + 56\gamma^3 - 186\gamma^2 + 120\gamma - 13)}{3(1+\gamma)^5} \frac{1}{p^4} + \cdots,$$

where $\gamma = e^{-2}$.

We substitute these known coefficients into (3.12), we have

Theorem 4.3. For $p_0 < p < \infty$,

$$c_p(1.5) = \frac{p}{2} + \frac{1}{2} \log \left(\frac{1+\gamma}{2} \right) + \frac{\alpha_2}{p} + \cdots,$$

where $\gamma = e^{-2}$ and

$$\alpha_2 = \left[\frac{1}{2} \log \left(\frac{1+\gamma}{2} \right) \right]^2 + \frac{1}{2} \log \left(\frac{1+\gamma}{2} \right) - 2 \left(\frac{\gamma}{1+\gamma} \right)^2.$$

Remark. That $a_2 = -2(1-\gamma)/(1+\gamma)$ can also be derived from Theorem 4.1 and the fact that $t_p = 1 - 2/p + a_2/p^2 + \cdots$. Indeed, Theorem 4.1 implies that $c_p(1.5) = p/2 + O(1)$. Substitute, $t_p = 1 - 2/p + a_2/p^2 + O(1/p^3)$ into (3.12) and by Theorem 3.3, we have $c_p(1.5) = p(1+\gamma)/(4 + (1+\gamma)a_2) + O(1)$. this implies $a_2 = -2(1-\gamma)/(1+\gamma)$.

5. PROOF OF LEMMAS

Proof of Lemma 3.1. We will give a sketch of the proof. For details, see [6].

Case 1. $p_0 < p < \infty$. Define $a(t) = A(t)/(1-t)^2$ and $b(t) = B(t)/(1-t)^2$.

Step 1. We shall show that $a(t)$ is increasing on I_p , $b(t)$ is decreasing on I_p and that they are both positive on I_p .

We note that $E(t)$ is positive and decreasing on $(0, 1)$. Convexity of $E(t)$ on $(0, 1)$ and $E(1) = 0$ imply that $E(t)/(1-t)$ is a positive decreasing function on $(0, 1)$. Note that

$$(5.1) \quad a(t) = p - 1 - \frac{[(p-2) - pt]}{(1-t)} \cdot \frac{E(t)}{1-t}, \quad \text{and}$$

$$(5.2) \quad b(t) = (p-1)E(t) - ta(t).$$

Now $a(t)$ is increasing on $(0, (p-2)/p)$ because $[(p-2) - pt]/(1-t)$ and $E(t)/(1-t)$ are both positive and decreasing on $(0, (p-2)/p)$. By direct verification, we have $a((p-2)/p) > 0$ and $b((p-2)/p) > 0$.

(i) When $p_0 < p \leq 3$, $a(0) \geq 0$. Therefore $a(t) > 0$ on $(0, (p-2)/p)$. It follows from (5.2) that $b(t)$ is decreasing on $(0, (p-2)/p)$ and hence positive on I_p .

(ii) When $3 < p < \infty$. There exists a unique $t_1 \in (0, (p-2)/p)$ such that $a(t_1) = 0$. Since $a((p-3)/(p-1)) > 0$, it implies that $a(t) > 0$ on I_p . It follows from (5.2) that $b(t)$ is decreasing on I_p and hence positive on I_p .

Step 2. Existence of a solution to (3.11). Define

$$(5.3) \quad \Delta(t) = (p-2) - (p-1)t - \left[\frac{B(t)}{A(t)} \right]^{p-1}, \quad t \in I_p.$$

(i) When $p_0 < p \leq 3$, we can verify that

$$\Delta(0) = p - 2 - \left[\frac{(p-2)(p-1)}{-p^2 + 5p - 5} \right]^{p-1} < 0.$$

(This explains equation (3.2).)

(ii) When $3 < p < \infty$, we have $A((p-3)/(p-1)) < B((p-3)/(p-1))$, therefore $\Delta((p-3)/(p-1)) < 0$. Now $\Delta(t)$ is continuous and $\Delta((p-2)/p) > 0$. Existence of a solution to (3.11) follows.

Step 3. Uniqueness of solution to (3.11). For $3 < p < \infty$, we first show that $a(t)$ is convex on $(0, 1)$. Differentiate (5.1) twice and let $M(t)$ be the numerator. When $4 < p < \infty$, $M''(t) < 0$ on $(0, \tilde{s})$, where $\tilde{s} = (p-4)(p-2)/p^2$; $M''(t) > 0$ on $(\tilde{s}, 1)$. Computing $M'(t)$ at 0 and 1, we see that $M'(t) < 0$ on $(0, 1)$. Therefore, $M(t) > M(1) = 0$. When $3 < p \leq 4$, $M''(t) > 0$ on $(0, 1)$, therefore $M'(t) < M'(1) = 0$ and this implies that $M(t) > 0$ on $(0, 1)$. Now we deduce that $[p-2-(p-1)t](a(t))^{p-1}$ is increasing on I_p . Its derivative equals $(a(t))^{p-2}\{[p-2-pt]a'(t) + ta'(t) - a(t)\}$ which is positive because $ta'(t) - a(t)$ is increasing, therefore $ta'(t) - a(t) > t_1a'(t_1) > 0$ for $t \in (t_1, (p-2)/p)$. Uniqueness of solution to (3.11) follows immediately since $(b(t))^{p-1}$ is decreasing. For $p_0 < p \leq 3$, we show that $b(t)/a(t)$ is convex on $(0, t_2)$. Uniqueness of solution to (3.11) follows because $[p-2-(p-1)t]$ is linear and $(b(t)/a(t))^{p-1}$ is convex. To show that $b(t)/a(t)$ is convex, we differentiate it twice and rewrite the numerator as

$$N(t) = a(t)[a(t)b''(t) - 2a'(t)b'(t)] + (-a''(t))a(t)b(t) + 2b(t)a'(t).$$

Since $a''(t) < 0$ in this case. To show that $N(t)$ is positive, it suffices to show that $\Delta_1(t) = a(t)b''(t) - 2a'(t)b'(t) \geq 0$. Recall $b(t) = (p-1)E(t) - ta(t)$ from (5.2), $\Delta_1(t) = (p-1)a(t)E''(t) + ta(t)[-a''(t)] + 2(p-1)a'(t)(-E'(t)) + t(a'(t))^2$. Each of these four terms is nonnegative on I_p . This completes the proof of Lemma 3.1 for Case 1.

Case 2. $2 < p \leq p_0$. We will be brief in the proof. We are able to show that $A(t)$ and $B(t)$ are positive on I_p , and by (3.5), $\Delta(0) > 0$ (see (5.3)). It can be shown that $\Delta(-(3-p)/2p) < 0$, existence of a solution follows. Lastly we can show that $[p-2-(p-1)t](B(t)/A(t))^{p-1}$ is decreasing and uniqueness is proved.

This completes the proof of Lemma 3.1. $\square$

Proof of Lemma 3.4. We note that $(p-1-pt_p)$, $[p-2-(p-1)t_p]$, $D(t_p)$ and $E(t_p)$ are positive, so $\alpha > 0$. Also, $\lambda - \alpha = (1-t_p)E(t_p)/D(t_p) > 0$ and $1 - \beta\lambda = 1 - t_p > 0$. When $p_0 < p < \infty$, $t_p > 0$ so $\beta > 0$. Since $\beta\lambda = t_p < 1$, this implies $\lambda < \beta^{-1}$. When $2 < p \leq p_0$, $t_p \leq 0$ so $\beta \leq 0$. This completes the proof of (iii) and (iii').

It is easy to show (ii) by (3.19) to (3.21).

From (3.7) and (3.9), we see readily that the left-hand side of (i) equals $D(t_p)/(1-t_p)$. By definition of $A(t)$ in (3.8), we have

$$\frac{\lambda(1-\alpha)^p}{\lambda-\alpha} = \frac{[p-1-pt_p]}{(1-t_p)} \left[\frac{A(t_p)}{D(t_p)} \right]^p.$$

From (3.10'),

$$\frac{(1-\beta)^p \lambda^p}{\lambda - \alpha} = \frac{1}{(1-t_p)} \cdot \left[\frac{B(t_p)}{D(t_p)} \right]^p.$$

Since t_p is a solution to (3.11), the right-hand side of (i) can be simplified as

$$\frac{(p-1-pt_p)A(t_p) + [p-2-p-1]t_p B(t_p)}{(p-1)(1-t_p)^3}$$

which can be further simplified as $D(t_p)/(1-t_p)$ by (3.10'), (3.8) and (3.9). This completes the proof of (i) and hence Lemma 3.4. $\square$

Let

$$(5.4) \quad A = 1 - k_p(1-\alpha)^p = v(x, \alpha x)|x|^{-p},$$

$$(5.5) \quad B = |\beta|^p - k_p(1-\beta)^p = v(\beta y, y)|y|^{-p}.$$

To prove Lemma 3.2, we need the following identities and inequalities which are grouped under the following lemma.

Lemma 5.1. *For $2 < p < \infty$, we have*

$$(i) \quad 1 - \alpha = \frac{A(t_p)}{D(t_p)} > 0,$$

$$(ii) \quad A = \frac{[(p-2)-pt_p]}{(p-1)(1-t_p)^2} E(t_p) > 0,$$

$$(iii) \quad 1 - k_p(1-\alpha)^{p-2} < 0,$$

$$(iv) \quad \omega - A = \frac{pE(t_p)}{(p-1)(1-t_p)} > 0,$$

$$(v) \quad \frac{\omega - A}{\lambda - \alpha} = pk_p(1-\alpha)^{p-1},$$

$$(vi) \quad pA - \frac{\alpha(\omega - A)}{\lambda - \alpha} = p[1 - k_p(1-\alpha)^{p-1}],$$

$$(vii) \quad pA + (p-2)(\omega - A) - \frac{2\alpha(\omega - A)}{\lambda - \alpha} = 0,$$

$$(viii) \quad 1 - \beta = \frac{B(t_p)}{[(p-1)-pt_p]E(t_p)} > 0,$$

$$(ix) \quad \omega - B\lambda^p = \frac{p[(p-1)-pt_p]}{(p-1)(1-t_p)} E(t_p) > 0,$$

$$(x) \quad p[\operatorname{sgn}(\beta)|\beta|^{p-1} + k_p(1-\beta)^{p-1}] = \frac{\lambda}{1-\beta\lambda} \left[\frac{\omega}{\lambda^p} - B \right],$$

$$(xi) \quad -pk_p(1-\beta)^{p-1} = pB - \frac{\beta\lambda}{1-\beta\lambda} \left[\frac{\omega}{\lambda^p} - B \right],$$

$$(xii) \quad pB + (p-2) \left[\frac{\omega}{\lambda^p} - B \right] - \frac{2\beta\lambda}{1-\beta\lambda} \left[\frac{\omega}{\lambda^p} - B \right] = 0,$$

$$(xiii) \quad \frac{\omega - B\lambda^p}{1-\beta\lambda} = pA - \frac{\alpha(\omega - A)}{\lambda - \alpha} + (p-1)(\omega - A),$$

$$(xiv) \quad pB\lambda^{p-1} + (p-1) \left[\frac{\omega}{\lambda^p} - B \right] \lambda^{p-1} - \frac{\beta\lambda}{1-\beta\lambda} \left[\frac{\omega}{\lambda^p} - B \right] \lambda^{p-1} = \frac{\omega - A}{\lambda - \alpha},$$

$$(xv) \quad |\beta|^{p-2} - k_p(1-\beta)^{p-2} < 0.$$

Proof of Lemma 5.1. From (3.19), (3.8) and (3.9) we get (i). By (i), (3.12) and (3.8), we get (ii). Using (i), (3.12) and (3.9), we obtain

$$1 - k_p(1 - \alpha)^{p-1} = 1 - \frac{D(t_p)}{(p-1)(1-t_p)^2} = \frac{-t_p E(t_p)}{(p-1)(1-t_p)^2} < 0.$$

Therefore, $1 < k_p(1 - \alpha)^{p-1} < k_p(1 - \alpha)^{p-2}$, so (iii) is proved. From (ii) and (3.22), we prove (iv). From (iv), (3.19) and (3.21), both sides of (v) equal $D(t_p)/(p-1)(1-t_p)^2$, so (v) follows.

Now,

$$\begin{aligned} pA - \frac{\alpha(\omega - A)}{\lambda - \alpha} &= pA - p\alpha k_p(1 - \alpha)^{p-1} \quad \text{by (v)} \\ &= p[1 - k_p(1 - \alpha)^p] - p\alpha k_p(1 - \alpha)^{p-1} \quad \text{by (5.4)} \\ &= p[1 - k_p(1 - \alpha)^{p-1}], \end{aligned}$$

which is (vi). From (ii), (iv) and (v), we can prove (vii). By (3.20) and (3.10'), we prove (viii).

To show (ix). From (viii), we obtain

$$\begin{aligned} \omega - B\lambda^p &= \frac{2[p-1-pt_p]}{(p-1)(1-t_p)^2} E(t_p) - |t_p|^p + \frac{[B(t_p)]^p}{(p-1)(1-t_p)^2[A(t_p)]^{p-1}} \\ &= \frac{2[p-1-pt_p]}{(p-1)(1-t_p)^2} E(t_p) + [p-1-pt_p] - D(t_p) \\ &\quad + \frac{[p-2-(p-1)t_p]\{[p-1-pt_p]E(t_p) - t_p D(t_p)\}}{(p-1)(1-t_p)^2} \\ &\hspace{15em} (3.9)-(3.11) \\ &= \frac{[p-1-pt_p][p-(p-1)t_p]E(t_p)}{(p-1)(1-t_p)^2} - \frac{[p-1-pt_p]}{(p-1)(1-t_p)^2} D(t_p) \\ &\quad + [p-1-pt_p] \\ &= \frac{[p-1-pt_p][p-(p-1)t_p]E(t_p)}{(p-1)(1-t_p)^2} - \frac{[p-1-pt_p]t_p E(t_p)}{(p-1)(1-t_p)^2} \\ &= \frac{p[(p-1)-pt_p]E(t_p)}{(p-1)(1-t_p)} > 0. \end{aligned}$$

To show (x). It is enough to show that

$$\frac{\omega - B\lambda^p}{1 - \beta\lambda} = p[\operatorname{sgn}(\beta)|\beta|^{p-1} + k_p(1 - \beta)^{p-1}]\lambda^{p-1}.$$

Now,

$$\begin{aligned} \text{R. H. S.} &= p \left\{ \operatorname{sgn}(t_p)|t_p|^{p-1} + \frac{D(t_p)[(p-2)-(p-1)t_p]}{(p-1)(1-t_p)^2} \right\} \\ &= p \left\{ E(t_p) - [(p-2)-(p-1)t_p] \right. \\ &\quad \left. + \frac{D(t_p)[(p-2)-(p-1)t_p]}{(p-1)(1-t_p)^2} \right\} \\ &= p \frac{[(p-1)-pt_p]E(t_p)}{(p-1)(1-t_p)^2} = \text{L. H. S.}, \quad \text{so (x) is verified.} \end{aligned}$$

Now,

$$\begin{aligned} B &= |\beta|^p - k_p(1 - \beta)^p \\ &= \beta \{ \operatorname{sgn}(\beta) |\beta|^{p-1} + k_p(1 - \beta)^{p-1} \} - k_p(1 - \beta)^{p-1}. \end{aligned}$$

Therefore, by (x), we have,

$$B = \frac{\beta\lambda}{p(1 - \beta\lambda)} \left[\frac{\omega}{\lambda^p} - B \right] - k_p(1 - \beta)^{p-1}.$$

Multiplying throughout by p and rearranging terms, we obtain (xi). Instead of verifying (xii), we will show that

$$pB\lambda^p = \left[\frac{2\beta\lambda}{1 - \beta\lambda} - (p - 2) \right] [\omega - B\lambda^p],$$

which is equivalent to (xii). Now,

$$\begin{aligned} B\lambda^p &= [B\lambda^p - \omega] + \omega \\ &= \frac{-p[(p - 1) - pt_p]E(t_p)}{(p - 1)(1 - t_p)} + \frac{2[(p - 1) - pt_p]E(t_p)}{(p - 1)(1 - t_p)^2} \quad \text{by (ix)} \\ &= \frac{p[(p - 1) - pt_p]E(t_p)}{(p - 1)(1 - t_p)} \left\{ \frac{2}{p(1 - t_p)} - 1 \right\} \\ &= \frac{p[(p - 1) - pt_p]E(t_p)}{(p - 1)(1 - t_p)} \left\{ \frac{2t_p}{p(1 - t_p)} - \left(1 - \frac{2}{p} \right) \right\}. \end{aligned}$$

Since $\beta\lambda = t_p$ and by (ix), so

$$B\lambda^p = \left\{ \frac{2\beta\lambda}{p(1 - \beta\lambda)} - \left(1 - \frac{2}{p} \right) \right\} [\omega - B\lambda^p],$$

this implies (xii). Rewriting (vii) and (xii), they become

$$\frac{\lambda A}{\lambda - \alpha} = \left[\frac{\lambda}{\lambda - \alpha} - \frac{p}{2} \right] \omega \quad \text{and} \quad \frac{B\lambda^p}{1 - \beta\lambda} = \left[\frac{1}{1 - \beta\lambda} - \frac{p}{2} \right] \omega.$$

Therefore,

$$\frac{\lambda A}{\lambda - \alpha} + \frac{B\lambda^p}{1 - p\lambda} = \left[\frac{\lambda}{\lambda - \alpha} + \frac{1}{1 - \beta\lambda} - p \right] \omega$$

which is equivalent to (xiii).

$$\begin{aligned} \text{L. H. S. of (xiv)} &= \frac{2\beta\lambda}{1 - \beta\lambda} \left[\frac{\omega}{\lambda^p} - B \right] \lambda^{p-1} - (p - 2) \left[\frac{\omega}{\lambda^p} - B \right] \lambda^{p-1} \\ &\quad + (p - 1) \left[\frac{\omega}{\lambda^p} - B \right] \lambda^{p-1} - \frac{\beta\lambda}{1 - \beta\lambda} \left[\frac{\omega}{\lambda^p} - B \right] \lambda^{p-1} \quad \text{by (xii)} \\ &= \left\{ 1 + \frac{\beta\lambda}{1 - \beta\lambda} \right\} \left[\frac{\omega}{\lambda^p} - B \right] \lambda^{p-1} \\ &= \frac{p}{(p - 1)} \frac{D(t_p)}{(1 - t_p)^2} \quad \text{by (ix) and (3.21)}. \end{aligned}$$

By (iv), (3.19) and (3.21), R.H.S. of (xiv) = $pD(t_p)/(p - 1)(1 - t_p)^2$, so we have proved (xiv).

It remains to show (xv). It is enough to verify $k_p((1-\beta)/|\beta|)^{p-2} > 1$. Now, by (viii), (3.20), (3.12) and (3.11), we have

$$k_p \left(\frac{1-\beta}{|\beta|} \right)^{p-1} - 1 = \frac{[p-2-(p-1)t_p][D(t_p)]^2}{|t_p|^{p-2}(p-1)(1-t_p)^2 B(t_p)} - 1.$$

Therefore, it suffices to show that

$$(5.6) \quad [(p-2)-(p-1)t_p][D(t_p)]^2 - (p-1)(1-t_p)^2 |t_p|^{p-2} B(t_p) > 0.$$

Using (3.10'), we can rewrite the

$$\begin{aligned} \text{L. H. S.} &= \{[(p-2)-(p-1)t_p]D(t_p) + (p-1)(1-t_p)^2 \operatorname{sgn}(t_p)|t_p|^{p-1}\}D(t_p) \\ &\quad - (p-1)(1-t_p)^2 |t_p|^{p-2} [(p-1)-pt_p]E(t_p). \end{aligned}$$

We simplify the expression inside $\{ \}$:

$$\begin{aligned} &[(p-2)-(p-1)t_p]D(t_p) + (p-1)(1-t_p)^2 [E(t_p) - (p-2) + (p-1)t_p] \\ &= [(p-2)-(p-1)t_p]\{D(t_p) - (p-1)(1-t_p)^2\} + (p-1)(1-t_p)^2 E(t_p) \\ &= [(p-1)-pt_p]E(t_p) \quad \text{by (3.9)}. \end{aligned}$$

So,

$$\begin{aligned} \text{L. H. S.} &= [(p-1)-pt_p]E(t_p)D(t_p) - (p-1)(1-t_p)^2 |t_p|^{p-2} [(p-1)-pt_p]E(t_p) \\ &= [(p-1)-pt_p]E(t_p)\{D(t_p) - (p-1)(1-t_p)^2 |t_p|^{p-2}\} \\ &> [(p-1)-pt_p]E(t_p)(p-1)(1-t_p)^2 (1-|t_p|^{p-2}) \quad \text{by (3.9)} \\ &> 0. \end{aligned}$$

So (xv) is established and this concludes the proof of Lemma 5.1. $\square$

Proof of Lemma 3.2. Define

$$\begin{aligned} \Omega_1 &= \{(x, y) \in \mathbf{R}^2 : y > 0, y < \alpha x\}, \\ \Omega_2 &= \{(x, y) \in \mathbf{R}^2 : y > 0, \alpha x < y < \lambda x\}, \\ \Omega_3 &= \{(x, y) \in \mathbf{R}^2 : y > 0, \beta y < x < \lambda^{-1}y\}, \\ \Omega_4 &= \{(x, y) \in \mathbf{R}^2 : y > 0, x < \beta y\}. \end{aligned}$$

Let $u(x, y)$ be the continuous function from $\mathbf{R}^2$ to $\mathbf{R}$ satisfying $u(x, y) = u(-x, -y)$, and

$$u(x, y) = \begin{cases} v(x, y), & (x, y) \in \Omega_1 \cup \Omega_4, \\ Ax^p + \frac{y-\alpha x}{\lambda-\alpha}(\omega-A)x^{p-1}, & (x, y) \in \Omega_2, \\ By^p + \frac{\lambda(x-\beta y)}{1-\beta\lambda} \left[\frac{\omega}{\lambda^p} - B \right] y^{p-1}, & (x, y) \in \Omega_3. \end{cases}$$

The bounds on u , u_x and u_y , and $u(0, 0) = 0$ can be verified readily.

Step 1. To show that u is concave in y on $\bigcup_{i=1}^4 \Omega_i$. On $\Omega_1 \cup \Omega_4$, $u_{yy}(x, y) = -p(p-1)k_p|x-y|^{p-2} < 0$, therefore u is concave in y . On Ω_2 , $u_{yy} \equiv 0$, hence u is concave in y . On Ω_3 ,

$$\begin{aligned} u_{yy}(x, y) &= p(p-1)By^{p-2} - \frac{2(p-1)\beta\lambda}{1-\beta\lambda} \left[\frac{\omega}{\lambda^p} - B \right] y^{p-2} \\ &\quad + (p-1)(p-2) \frac{\lambda(x-\beta y)}{1-\beta\lambda} \left[\frac{\omega}{\lambda^p} - B \right] y^{p-3}. \end{aligned}$$

Now,

$$\begin{aligned} u_{yy} \left(\left(\frac{y}{\lambda} \right) -, y \right) &= (p-1) \left\{ pB + (p-2) \left[\frac{\omega}{\lambda^p} - B \right] - \frac{2\beta\lambda}{1-\beta\lambda} \left[\frac{\omega}{\lambda^p} - B \right] \right\} y^{p-2} \\ &= 0 \quad \text{by Lemma 5.1(xii).} \end{aligned}$$

So,

$$\begin{aligned} u_{yy}(x, y) &= u_{yy}(x, y) - u_{yy} \left(\left(\frac{y}{\lambda} \right) -, y \right) \\ &= \frac{(p-1)(p-2)(\lambda x - y)}{1-\beta\lambda} \left[\frac{\omega}{\lambda^p} - B \right] y^{p-3} \\ &< 0 \quad \text{by Lemma 5.1(ix) and that } y > \lambda x \text{ on } \Omega_3. \end{aligned}$$

Step 2. To show that u is concave in x on $\bigcup_{i=1}^4 \Omega_i$. On Ω_1 ,

$$\begin{aligned} u_{xx}(x, y) &= p(p-1)[x^{p-2} - k_p(x-y)^{p-2}] \\ &< p(p-1)[1 - k_p(1-\alpha)^{p-2}]x^{p-2} < 0 \end{aligned}$$

by Lemma 5.1(iii).

$$\begin{aligned} \text{On } \Omega_4, \quad u_{xx}(x, y) &= p(p-1)[|x|^{p-2} - k_p(y-x)^{p-2}] \\ &< p(p-1)[|\beta|^{p-2} - k_p(1-\beta)^{p-2}]y^{p-2} \\ &< 0 \quad \text{by Lemma 5.1(xv).} \end{aligned}$$

$$\begin{aligned} \text{On } \Omega_2, \quad u_{xx}(x, y) &= p(p-1)Ax^{p-2} - \frac{2(p-1)\alpha(\omega-A)}{\lambda-\alpha}x^{p-2} \\ &\quad + (p-1)(p-2)\frac{y-\alpha x}{\lambda-\alpha}(\omega-A)x^{p-3}. \end{aligned}$$

Since

$$u_{xx}(x, \lambda x-) = (p-1) \left\{ pA + (p-2)(\omega-A) - \frac{2\alpha(\omega-A)}{\lambda-\alpha} \right\} x^p = 0,$$

so

$$\begin{aligned} u_{xx}(x, y) &= u_{xx}(x, y) - u_{xx}(x, \lambda x-) \\ &= \frac{(p-1)(p-2)(\omega-A)}{\lambda-\alpha}(y-\lambda x)x^{p-3} < 0, \end{aligned}$$

by Lemma 5.1(iv) and the fact that $y < \lambda x$ on Ω_2 . The function, u , is affine in x on Ω_3 , so u is concave in x on $\bigcup_{i=1}^4 \Omega_i$.

Step 3. To show that the first derivatives match up at the boundaries of the regions. At $y = \alpha x$, we have

$$\begin{aligned} u_x(x, \alpha x+) &= \left[pA - \frac{\alpha}{\lambda-\alpha}(\omega-A) \right] x^{p-1} \\ &= p[1 - k_p(1-\alpha)^{p-1}]x^{p-1} = u_x(x, \alpha x-) \end{aligned}$$

from Lemma 5.1(vi). Also, we have

$$\begin{aligned} u_y(x, \alpha x+) &= \frac{\omega-A}{\lambda-\alpha}x^{p-1} \\ &= pk_p(1-\alpha)^{p-1}x^{p-1} = u_y(x, \alpha x-) \quad \text{by Lemma 5.1(v).} \end{aligned}$$

At $y = \lambda x$,

$$\begin{aligned} u_x(x, \lambda x+) &= \frac{\omega - B\lambda^p}{1 - \beta\lambda} x^{p-1} \\ &= \left[pA - \frac{\alpha(\omega - A)}{\lambda - \alpha} + (p-1)(\omega - A) \right] x^{p-1} = u_x(x, \lambda x-) \end{aligned}$$

from Lemma 5.1(xiii); and

$$\begin{aligned} u_y(x, \lambda x+) &= \left\{ pB\lambda^{p-1} + (p-1) \left[\frac{\omega}{\lambda^p} - B \right] \lambda^{p-1} - \frac{\beta\lambda}{1 - \beta\lambda} \left[\frac{\omega}{\lambda^p} - B \right] \lambda^{p-1} \right\} x^{p-1} \\ &= \frac{\omega - A}{\lambda - \alpha} x^{p-1} = u_y(x, \lambda x-) \end{aligned}$$

from Lemma 5.1(xiv). At $x = \beta y$, we get

$$\begin{aligned} u_x(\beta y-, y) &= p[\operatorname{sgn}(\beta)|\beta|^{p-1} + k_p(1 - \beta)^{p-1}]y^{p-1} \\ &= \frac{\lambda}{1 - \beta\lambda} \left[\frac{\omega}{\lambda^p} - B \right] y^{p-1} = u_x(\beta y+, y) \end{aligned}$$

from Lemma 5.1(x); and

$$\begin{aligned} u_y(\beta y-, y) &= -pk_p(1 - \beta)^{p-1}y^{p-1} \\ &= \left\{ pB - \frac{\beta\lambda}{1 - \beta\lambda} \left[\frac{\omega}{\lambda^p} - B \right] \right\} y^{p-1} = u_y(\beta y+, y) \end{aligned}$$

from Lemma 5.1(xi).

Step 4. To show that $\lambda < 1$. Since

$$\begin{aligned} 1 - \lambda &= \{D(t_p) - [p - 1 - pt_p]E(t_p)\}/D(t_p) \\ &= (1 - t_p)^2 \left\{ a(t_p) - \frac{E(t_p)}{1 - t_p} \right\} / D(t_p), \end{aligned}$$

it is enough to show that $a(t_p) - E(t_p)/(1 - t_p) > 0$. From Step 1 of the proof of Lemma 3.1, we see that $a(t)$ and $E(t)/(1 - t)$ are increasing and decreasing functions respectively. Therefore, $\psi(t) = a(t) - E(t)/(1 - t)$ is an increasing function. For $p_0 \leq p \leq 3$, $\psi(0) = a(0) - E(0) = A(0) - E(0) = (3 - p)(p - 1) \geq 0$, therefore $\psi(t_p) > 0$; for $3 < p < \infty$,

$$\psi\left(\frac{p-3}{p-1}\right) = -(p-1) \left[\frac{p-3}{p-1} \right]^{p-1} < 0$$

and

$$\psi\left(\frac{p-2}{p}\right) = \frac{p}{2} \left[1 - \left(\frac{p-2}{p} \right)^{p-1} \right] > 0.$$

Therefore, there exists a unique $s_p \in I_p$ such that $\psi(s_p) = 0$ and $\psi(t) > 0$ on $(s_p, (p-2)/p)$. So, $\psi(t_p) > 0$ follows if we can prove that $s_p < t_p$. This can be reduced to proving

$$(5.7) \quad [p - 2 - (p-1)s_p] - (b(s_p)/a(s_p))^{p-1} < 0.$$

As $\psi(s_p) = 0$, it implies $(1 - s_p)a(s_p) = E(s_p)$. Using this and (5.2), we get $b(s_p)/a(s_p) = p - 1 - ps_p > 1$. Therefore,

$$\left(\frac{b(s_p)}{a(s_p)} \right)^{p-1} > \frac{b(s_p)}{a(s_p)} = p - 1 - ps_p > (p-2) - (p-1)s_p.$$

So, (5.7) follows and this finishes Step 4.

Step 5. To verify that $u(x, y) \geq v(x, y)$ on $\mathbf{R}^2$. By homogeneity and symmetry, it suffices to show that (i) $u(1, y) \geq v(1, y)$ for $y \in (\alpha, \lambda)$; and (ii) $u(x, 1) \geq v(x, 1)$ for $x \in (\beta, \lambda^{-1})$.

Case (i). We observe that the graph of $u(1, y)$ for $\alpha < y < \lambda$ is part of the line tangent to the graph of the concave function $v(1, y)$ at $y = \alpha$. Therefore, $u(1, y) > v(1, y)$ for $\alpha < y < \lambda$.

Case (ii). Observe that again the graph of $u(x, 1)$ is tangent to the graph of the function $v(x, 1)$ at the point $x = \beta$. Since $u(1, \lambda) > v(1, \lambda)$ from case (i), $u(\lambda^{-1}, 1) > v(\lambda^{-1}, 1)$ by homogeneity. Let m be the slope of the line defined by $u(x, 1)$ on (β, λ^{-1}) , i.e., $m = (\lambda/(1 - \beta\lambda))[\omega/\lambda^p - B]$. Consider the function $V(x) = v(x, 1)$. We have,

$$V'(x) = p\{\operatorname{sgn}(x)|x|^{p-1} - \operatorname{sgn}(x-1)|x-1|^{p-1}\}$$

and

$$V''(x) = p(p-1)\{|x|^{p-2} - k_p|x-1|^{p-2}\}.$$

There exist ξ_1, ξ_2 such that $0 < \xi_1 < 1 < \xi_2$ and $V''(\xi_i) = 0$, $i = 1, 2$. Furthermore, V is concave on each of the connected components of $\mathbf{R} \setminus [\xi_1, \xi_2]$; and V is convex on (ξ_1, ξ_2) . If $\xi_2 \geq \lambda^{-1}$. The affinity of $u(x, 1)$, the concave-convex situations as described above, the fact that $u(x, 1)$ is tangent to $V(x)$ at $x = \beta$ and $u(x, 1) \geq v(x, 1)$ at $x = \beta$ and $x = \lambda^{-1}$ imply that $u(x, 1) \geq v(x, 1)$ on (β, λ^{-1}) . If $\xi_2 < \lambda^{-1}$. Now $k_p(\xi_2 - 1)^{p-2} = \xi_2^{p-2}$ and it follows that $V'(\xi_2) = p\xi_2^{p-2} < p\lambda^{-(p-2)}$. Consider,

$$\begin{aligned} m - p\lambda^{-(p-2)} &= \frac{\lambda}{1 - \beta\lambda} \left[\frac{\omega}{\lambda^p} - B \right] - p\lambda^{-(p-2)} \\ &= \left\{ \frac{\omega - B\lambda^p}{1 - \beta\lambda} - p\lambda \right\} \lambda^{-(p-1)} \\ &= \frac{pt_p[p-1-pt_p][E(t_p)]^2}{(p-1)(1-t_p)^2D(t_p)} \lambda^{-(p-1)} > 0 \end{aligned}$$

from Lemma 5.1(ix), (3.21) and (3.9). Therefore, $m > p\lambda^{-(p-2)} > V'(\xi_2)$. Since $u(x, 1) \geq v(x, 1)$ at $x = \beta$, $x = \xi_2$ and $x = \lambda^{-1}$, it follows that $u(x, 1) \geq v(x, 1)$ on (β, λ^{-1}) . This completes the proof of Case (ii) and hence the proof of Lemma 3.2. $\square$

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REFERENCES

1. D. L. Burkholder, *Martingale transforms*, Ann. Math. Statist. **37** (1966), 1494–1504.
2. —, *A geometrical characterization of Banach spaces in which martingale difference sequences are unconditional*, Ann. Probab. **9** (1981), 997–1011.
3. —, *Boundary value problems and sharp inequalities for martingale transforms*, Ann. Probab. **12** (1984), 647–702.

4. —, *An elementary proof of an inequality of R. E. A. C. Paley*, Bull. London Math. Soc. **17** (1985), 474–478.
5. —, *Martingales and Fourier analysis in Banach spaces* (C.I.M.E. Lectures, Varenna, Italy, 1985), Lecture Notes in Math., vol. 1206, Springer-Verlag, Berlin and New York, 1986, pp. 61–108.
6. K. P. Choi, *Some sharp inequalities for martingale transforms*, Ph.D. dissertation, Univ. of Illinois at Urbana-Champaign, 1987.
7. R. C. James, *Bases in Banach spaces*, Amer. Math. Monthly **89** (1982), 625–640.
8. J. Marcinkiewicz, *Quelques théorèmes sur les séries orthogonales*, Ann. Soc. Polon. Math. **16** (1937), 84–96.
9. B. Maurey, *Système de Haar*, Séminaire Maurey-Schwartz (1974–1975), École Polytechnique, Paris.
10. R. E. A. C. Paley, *A remarkable series of orthogonal functions. I*, Proc. London Math. Soc. **34** (1932), 241–264.

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