NESTS OF SUBSPACES IN BANACH SPACE
AND THEIR ORDER TYPES

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ABSTRACT. This paper addresses some questions which arise naturally in the theory of nests of subspaces in Banach space. The order topology on the index set of a nest is discussed, as well as the method of spatial indexing by a vector; sufficient geometric conditions for the existence of such a vector are found. It is then shown that a continuous nest exists in any Banach space.

   Applications and examples follow; in particular, an extension of the Volterra nest in $L^\infty[0, 1]$ to a continuous one, a continuous nest in a Banach space having no two elements isomorphic to one another, and a characterization of separable $\mathscr{L}_p$-spaces in terms of nests.

INTRODUCTION

We begin with some basic definitions and examples; further background may be found in [L1, L2 and ALWW]; in particular, [L2] contains an excellent bibliography of the subject.

A *nest* **N** is a family of closed subspaces of a Banach space X totally ordered by inclusion. **N** is *complete* if it contains (0), X and the join (closed linear span) and meet (intersection) of any subfamily.

We will consider only complete nests. An *index set* for a given nest is any ordered set which is order isomorphic to the nest. The order properties of a complete nest are characterized by the order topology of its index set, in the following sense: the meet of any collection of subspaces is the subspace indexed by the order infimum of the indices, and similarly for joins and order suprema.

A nest is said to have a *gap* between two elements if the index set is disconnected between the two indices; a nest is continuous if it contains no gaps (i.e. its index set is order connected). The index set of a complete nest is always order compact; one may ask of any compact ordered set if it is the index of a nest in some particular space.

A subnest **M** of a nest **N** is said to be *order dense* if its completion is **N**, where the *completion* of **M** is the smallest complete nest containing **M**. The completion is formed by including meets and joins of all subcollections of nest

elements; clearly $\mathbf{M}$ is order dense in $\mathbf{N}$ iff the index set of $\mathbf{M}$ is dense in the index set of $\mathbf{N}$.

It can be shown that any nest in a separable space can have at most countably many gaps [ALWW]; and also that any such nest contains a countable order dense subset. One can conclude from this that the order type (the index set with order topology) of any nest in separable Banach space is order isomorphic to a compact subset of the reals (§4). In §4 we also show that given any such compact set and any separable Banach space, there exists a nest in the space with that order type.

Next we consider some important motivating examples.

Example (the discrete nest). Let $(e_i)_{i=1}^n$ be a basis for $H_n = n$-dimensional Hilbert space $(n \in \mathbf{Z}^+)$. Define: $N_k = \text{span}\{e_1, \ldots, e_k\}$ $\forall k \leq n$. Then

$$\mathbf{N} = \{(0), N_1, N_2, \ldots, N_n = H_n\}$$

is a complete nest.

Example (the Volterra nest). $X = L_p([0, 1], \mu)$ where μ is Lebesgue measure (or any finite nonatomic Borel measure). For $0 \leq t \leq 1$ let $N_t = M_{\chi[0, t]}X$ (the set of all L_p functions μ-essentially supported on $[0, t]$) and $\mathbf{N} = \{N_t | 0 \leq t \leq 1\}$. It is routine to verify that for $1 \leq p < \infty$ this nest is complete. Of some interest is the fact that if $p = \infty$ the nest is not complete; for if we let $t = \frac{1}{2}$ (say) and $\mathbf{F} = \{N_s | s < \frac{1}{2}\}$, then

$$L = \bigvee \{N | N \in \mathbf{F}\}$$

is not in $\mathbf{N}$. It can be verified that the addition of these sets will complete the L_∞ nest. The order type of the Volterra nest for $p < \infty$ is $[0, 1]$; for $p = \infty$ it is $[0, 1] \times \{0, 1\}$ in the lexicographic order.

The Volterra nest in $L_p([0, 1], \mu)$ for $p < \infty$ is a continuous nest. In §3 we will show that a continuous nest may also exist in a sequence space (we will eventually build such a nest in any Banach space).

Example (the Cantor nest). Let H be a separable Hilbert space. Let $(e_r)_{r \in \mathbf{Q}}$ be an orthonormal basis for H. $\forall t \in \mathbf{R}$ define $N_t = \overline{\text{span}}\{e_r | r \in \mathbf{Q}, r < t\}$ and for all $t \in \mathbf{Q}$ define $M_t = \overline{\text{span}}\{e_r | r \in \mathbf{Q}, r \leq t\}$. Then let

$$\mathbf{N} = \{N_t | t \in \mathbf{R}\} \cup \{M_t | t \in \mathbf{Q}\} \cup \{(0)\} \cup \{H\}.$$

It is easy to verify that $\mathbf{N}$ is complete. This nest is order isomorphic to the Cantor ternary set.

We now consider the idea of an *indexing vector* for a nest.

Let $\mathbf{N}$ be a complete nest in a Banach space X. For each $x \in X$, the function $\varphi : \mathbf{N} \to \mathbf{R}$ by $\varphi(N) = \text{dist}(x, N)$ has the following properties.

(i) it is continuous from the order topology on $\mathbf{N}$ to the usual topology on $\mathbf{R}$.

(ii) its range is compact (in fact, $\text{ran}(\varphi) \subset [0, \|x\|]$).

(iii) if the map is injective, it is a homeomorphism onto its range.

We can ask the question: If $\mathbf{N}$ is complete and order separable (i.e. has a countable order dense subset) does there exist a vector $x \in X$ making the above function injective? We call such a vector an *indexing vector*.

The vector $f \equiv 1$ is an indexing vector for the Volterra nest, $p < \infty$. In separable Hilbert space, every nest has an indexing vector; in fact, D. Larson has pointed out to us that unitary equivalence of complete nests in separable Hilbert space can be characterized using these vectors. This result provides a proof of the Kadison-Singer theorem on similarity (see [KS]) that does not use the spectral theorem.

Larson has asked which Banach spaces will always contain an indexing vector for an arbitrary nest; we will begin by giving some sufficient conditions for this to be true.

INDEXING VECTORS FOR NESTS

The main goal of this section is to give sufficient conditions on a Banach space so that every nest in it will have an indexing vector.

Definition. Let X be a Banach space. We say that a nest **N** in X has *unique best approximations* if for every $N \in \mathbf{N}$ and $x \in X$, there exists a unique $n_x \in \mathbf{N}$ such that $\mathrm{dist}(x, N) = \|x - n_x\|$. That is, the distance from a subspace of the nest to an outside vector is attained by a unique element of the subspace. If this is true for every possible nest in X we simply say that X has unique best approximations.

Theorem 2.1. *Let X be a separable Banach space. If **N** is a complete nest in X with unique best approximations, then there exists an indexing vector for the nest **N**.*

We have as an immediate consequence that any complete nest in a separable Banach space which is reflexive and strictly convex (in particular, if X is uniformly convex) will have an indexing vector; for reflexivity guarantees the existence of a best approximating vector for any subspace, and the strict convexity ensures its uniqueness.

The main fact we will need is as follows:

Lemma 2.2. *Let X be a Banach space. If a nest **N** in X has unique best approximations, with $M \subsetneq N$ elements of **N**, and if $\mathrm{dist}(x, M) = \mathrm{dist}(x, N)$ and $y \in N \backslash M$ then*

$$\mathrm{dist}(x + y, M) > \mathrm{dist}(x + y, N).$$

Proof. By unique best approximations, there exists a unique $m_x \in M$ such that $\mathrm{dist}(x, N) = \mathrm{dist}(x, M) = \|x - m_x\|$. Now note that $\mathrm{dist}(x + y, N) = \mathrm{dist}(x, N)$. But there exists a unique $m_{x+y} \in M$ so that $\mathrm{dist}(x + y, M) = \|x + y - m_{x+y}\|$. If we had $\mathrm{dist}(x + y, M) = \mathrm{dist}(x + y, N)$, we could write

$$\mathrm{dist}(x, N) = \|x + y - m_{x+y}\|,$$

and since the best approximations are unique, we would have $y - m_{x+y} = -m_x$ or $y \in M$, a contradiction. $\square$

Proof of Theorem 2.1. We construct the indexing vector as follows. Let **S** be a countable order dense nest in **N**. Order (in any fashion) the countable collection of pairs from **S**:

$$\{(M_i, N_i)\}_{i=1}^{\infty} \quad \text{where } M_i \subsetneq N_i \subset X.$$

Take x_1 separating M_1 and N_1 by (say) $\varepsilon_1 > 0$ with $\|x_1\| = 1$ (by this we mean $\text{dist}(x_1, M_1) - \text{dist}(x_1, N_1) = \varepsilon_1$). If x_1 already separates M_2 from N_2 then choose $x_2 = 0$. Otherwise choose x_2 in $N_2 \backslash M_2$ with $\|x_2\| < \varepsilon_1/8$. Then let

$$\varepsilon_2 = \text{dist}(x_1 + x_2, M_2) - \text{dist}(x_1 + x_2, N_2).$$

By the main lemma we know that $\varepsilon_2 > 0$. Continuing in this way, we let $x_{n+1} = 0$ if M_{n+1} and N_{n+1} are already separated by $s_n = \sum_{i=1}^{n} x_i$. If not we choose x_{n+1} in $N_{n+1} \backslash M_{n+1}$ with

$$\|x_{n+1}\| < \frac{1}{2^{n+2}} \min\{\varepsilon_i\}_{i=1}^{n}$$

and let

$$\varepsilon_{n+1} = \text{dist}(s_{n+1}, M_{n+1}) - \text{dist}(s_{n+1}, N_{n+1}).$$

Again, by the lemma we have $\varepsilon_{n+1} > 0$. Define $x = \sum_{i=1}^{\infty} x_i$. The claim is that x indexes the nest. Note that it suffices to show that x indexes the order dense subnest $\mathbf{S}$.

So take $M, N \in \mathbf{S}$ with $M \subsetneq N$. Then $\exists k$ such that $M = M_k$ and $N = N_k$.

Let

$$s_k = \sum_{1}^{k} x_i \quad \text{and} \quad t_k = \sum_{k+1}^{\infty} x_i.$$

Then, using the triangle inequality and the estimates on the $\|x_i\|$'s we get

$$
\begin{aligned}
\text{dist}(x, &M_k) - \text{dist}(x, N_k) \\
&\geq \text{dist}(s_k, M_k) - \text{dist}(t_k, M_k) - [\text{dist}(s_k, N_k) + \text{dist}(t_k, N_k)] \\
&= \text{dist}(s_k, M_k) - \text{dist}(s_k, N_k) - \text{dist}(t_k, M_k) - \text{dist}(t_k, N_k) \\
&\geq \varepsilon_k - 2 \sum_{i=k+1}^{\infty} \|x_i\| > \varepsilon_k - 2 \sum_{i=k+1}^{\infty} \frac{\varepsilon_k}{2^{i+1}} \\
&= \varepsilon_k \left(1 - \frac{1}{2^k}\right) > 0.
\end{aligned}
$$

So x is an indexing vector for the nest $\mathbf{N}$ in X. $\square$

Using basically the same construction and the triangle inequality we can notice that the order type of a nest can be important in determining the existence of an indexing vector.

Corollary 2.3. *If a nest $\mathbf{N}$ in X has the order type of the extended positive integers, then the nest has an indexing vector.*

The condition of Theorem 2.1 cannot be dropped. We can construct counterexamples in the classical spaces not satisfying the unique best approximation property: $C[0, 1]$, c_0, $L_1[0, 1]$.

Example. Consider the space $C[0, 1]$. We define a nest indexed by the extended negative integers and zero as follows. Let $\{r_i\}_{i=1}^{\infty}$ be an ordering of the rationals in $(0, 1)$. Define

$$N_k = \{f \in C[0, 1] | f(r_i) = 0 \; \forall 0 \leq i \leq -k\} \quad \text{where } k \in \mathbf{Z}^-.$$

Of course $N_0 = C[0, 1]$ and $N_{-\infty} = (0)$. One can check that this nest is complete (and even countable) yet has no nonzero indexing vector; it is a simple consequence of the fact that the tails $\{r_i\}_{i=n+1}^{\infty}$ are dense in any subinterval of $(0, 1)$.

We can construct a somewhat similar example in the space $c_0(\mathbf{Q})$. Let

$$N_t = \overline{\mathrm{span}}\{e_{r_i} | r_i < t\}$$

and then form the completion by taking meets and joins. We then have a complete nest with no indexing vector (since for a given $\varepsilon > 0$ a c_0 vector can only have finitely many indices greater than ε).

A simple look at Lemma 2.2 could suggest a possible role for finite cotype in the existence of indexing vectors. The next example in $L_1[0, 1]$ due to W. B. Johnson says that this is not the case.

Proposition 2.4. *Let X be a Banach space. Suppose $\imath: l_1 \to X^*$ is an isometric embedding such that (defining $\imath e_i = x_i$) the sequence (x_i) either contains a weak* accumulation point of itself or converges weak* to zero. Then there is a nest in X with no indexing vector.*

Proof. Define

$$N_{1/k} = \{\overline{\mathrm{span}}\{x_i\}_{i=1}^{k}\}_{\perp} \quad \forall k = 1, 2, \ldots,$$

and then let $\mathbf{N}$ be the completion of the nest $\{N_{1/k} : k \geq 1\}$. Now if $\mathbf{N}$ were to have an indexing vector, say x, then $\mathrm{dist}(x, N_{1/k})$ would be strictly increasing. But

$$\begin{aligned}
\mathrm{dist}(x, N_{1/k}) &= \sup\{\langle x, x^* \rangle | x^* \in N_{1/k}^{\perp} \|x^*\| = 1\} \\
&= \sup\{\langle x, x^* \rangle | x^* \in \overline{\mathrm{span}}\{x_i\}_{i=1}^{k} \|x^*\| = 1\} \\
&= \sup\{\langle x, x^* \rangle | x^* \in \{x_i\}_{i=1}^{k}\}
\end{aligned}$$

(since $[x_i]_{i=1}^{\infty}$ is isometric to l_1 one may consider only the extreme points) which cannot be strictly increasing under the conditions of the hypothesis. □

If we consider the fact that the span of the Rademacher functions in $L^{\infty}[0, 1]$ is isometric to l_1 we may apply the above proposition to obtain a nest in $L^1[0, 1]$ with no indexing vector. It is interesting to note that the gaps in this nest are infinite dimensional; by a result proved later (Corollary 4.3) this means that it is contained in a continuous nest. We thus obtain a continuous nest in $L^1[0, 1]$ with no indexing vector.

We conclude this section with some remarks about nests in Banach lattices; for it may be that this is a more appropriate context in which to attempt to generalize some of the theory already developed in Hilbert space.

Consider the Volterra nest in L_1; note that while L_1 is neither strictly convex nor reflexive, this nest does have an indexing vector: $f \equiv 1$. This nest satisfies the unique best approximation property; thus the construction above will also apply. This situation can be profitably generalized: by putting some condition on the nests, we can get indexing vectors using different conditions on the Banach space. What we have is this

Proposition 2.5. *Let $\mathbf{N}$ be a nest of ideals in an order continuous Banach lattice X. If X has the property that for $x, y \in X$ $|x| \wedge |y| = 0$ $x, y \neq 0$ we always have $\|x + y\| > \|x\|$, then $\mathbf{N}$ has unique best approximations.*

As an immediate corollary we get an indexing vector for **N**.

Proof. Ideals in order continuous Banach lattices are projection bands (this is due to Ando; see [LT, 1.a.11]). Let M be an ideal of X and P_M its projection band. Since for every $x \in X$, $x - P_M(x)$ is disjoint from M, the hypothesis of the proposition gives that $P_M(x)$ is the unique best approximation of x. $\square$

In particular, if X has a *lower p-estimate*, i.e., if the x_i's are disjoint then

$$\left\| \sum_{i=1}^{n} x_i \right\| \geq K \left(\sum_{i=1}^{n} \|x_i\|^p \right)^{1/p},$$

with $K = 1$, then we get indexing vectors for nests of projection bands as above.

If we ask whether we can obtain an indexing vector by an equivalent renorming of the space, then we can do more: Any lattice in which c_0 cannot be finitely lattice represented has a lower p-estimate for some p with some constant; it can be renormed to have constant 1 [LT, 1.f.12ff]. In fact, c_0 itself can be renormed (by $1 + \varepsilon$) to have the property required in the proposition:

$$\||(a_n)_{n=1}^{\infty}\|| = \|(a_n)_{n=1}^{\infty}\|_{c_0} + \varepsilon \sum_{n=1}^{\infty} \frac{|a_n|}{2^n}.$$

This says that we can find an indexing vector for the nest in the second example above under an arbitrarily small renorming. This is in contrast to the example above in $C[0, 1]$ (which is not order continuous).

It may thus be that the appropriate interesting nests to study are nests of projection bands in Banach lattices, or perhaps nests with uniformly bounded commuting projections (see, for example [W]). One certainly needs some condition on the nest; for, as we show next, every Banach space has a continuous nest.

A CONTINUOUS NEST IN BANACH SPACES WITH BASES

The aim of this section is to construct a continuous nest in any Banach space with a basis. We assume that $(e_n)_{i=1}^{\infty}$ is a normalized basis of X with constant 1, i.e.

$$\left\| \sum_{1}^{n} a_i e_i \right\| \leq \left\| \sum_{1}^{n+1} a_i e_i \right\| \quad \forall (a_i)_1^{n+1} \text{ and } n \in \mathbf{Z}^+.$$

We denote by e_i^* the biorthogonal functionals, $e_i^*(e_j) = \delta_{ij}$.

The typical continuous nest in $L^2[0, 1]$ is the Volterra nest (in fact, in some sense, this is the only continuous nest in $L^2[0, 1]$). Notice that for this nest, the map

$$[0, 1] \to L^2[0, 1] :: t \to \chi_{[0, t]}$$

is continuous and that

$$N_t = \overline{\mathrm{span}}\{\chi_{[0, s]} | 0 \leq s \leq t\}.$$

This is the kind of structure we wish to build in X; i.e., we would like to define

a continuous function $y : [0, 1] \to X$ with the properties:

$$y(0) = 0,$$
$$N_t = \overline{\text{span}}\{y_s | 0 \leq s \leq t\},$$
$$t < s \Rightarrow d(y_s, N_t) > 0,$$
$$\mathbf{N} = \{N_t | 0 \leq t \leq 1\} \text{ is a continuous nest}.$$

We look at what must happen if such a function exists: given y as above, define

$$g_n : [0, 1] \to \mathbf{R} :: t \to g_n(t) = \langle y_t, e_n^* \rangle.$$

It is clear that these g_n are continuous and that

$$y_t = \sum_{n=1}^{\infty} g_n(t)e_n.$$

Since we will want to have $N_1 = X$, we need the y_t to span X; we thus need:

$$x^* \in X^* \quad \text{and} \quad \langle y_t, x^* \rangle = 0 \quad \forall t \Rightarrow x^* = 0$$

or, in terms of the functions g_n,

$$\sum_{n=1}^{\infty} \langle e_n, x^* \rangle g_n(t) = 0 \quad \Rightarrow \quad \forall n \quad \langle e_n, x^* \rangle = 0.$$

The natural candidates for functions having this property are biorthogonal systems and basic sequences in $C[0, 1]$. We will take for our functions the classical Schauder basic sequence:

$$g_{2^k+j}(t) = \begin{cases} t - \frac{(j-1)}{2} & \text{for } t \in \left[\frac{2j-2}{2^{k+1}}, \frac{2j-1}{2^{k+1}}\right], \\ -t + \frac{j}{2^k} & \text{for } t \in \left[\frac{2j-1}{2^{k+1}}, \frac{j}{2^k}\right], \\ 0 & \text{otherwise}, \end{cases}$$

for all $k \in \mathbf{Z}^+$ and $1 \leq j \leq 2^k$ and with $g_1(t) = t$.

We then define the vectors which will form the path y: for $t \in [0, 1]$

$$y_t = g_1(t)e_1 + \sum_{k=0}^{\infty} \sum_{j=1}^{2^k} g_{2^k+j}(t)e_{2^k+j}.$$

These vectors are well defined since the second sum contains only one nonzero element for each value of k and $\|g_{2^k+j}\|_\infty \leq 2^{-(k+1)}$. If t is dyadic, y_t is a finite sum.

The continuity of the function $y(t) = y_t$ follows easily from the continuity of the functions g_n and the fact that

$$\left\| \sum_{k=N}^{\infty} \sum_{j=1}^{2^k} g_{2^k+j} \right\|_\infty \leq 2^{-N}.$$

We define the nest $\mathbf{N}$ as follows: for $t \in [0, 1]$

$$N_t = \overline{\text{span}}\{y_s | 0 \leq s \leq t, s \text{ dyadic}\}.$$

Proposition 3.1. $\bigvee_{r<t} N_r = N_t \quad \forall 0 < t \leq 1$.

Proof. If t is not dyadic, the proposition is trivially true. If it is, then it is enough to prove that $y_t \in \bigvee_{r<t} N_r$. Since $\lim_{r\uparrow t} \|y_t - y_r\| = 0$ and $\bigvee_{r<t} N_r$ is closed, this follows. $\square$

Remark. We only used the fact that the function taking t to y_t is continuous. The main difficulty will be to establish continuity of the nest in the other direction.

Theorem 3.2. $\bigwedge_{r>t} N_r = N_t \quad \forall 0 \leq t < 1$.

The key to the proof of Theorem 3.2 is to characterize the subspace N_t when t is dyadic. In preparation for doing this we will develop some notation and prove some elementary facts about the nest **N**.

It is convenient to associate with every integer $2^k + j$ the dyadic number at which the function g_{2^k+j} peaks, $t(2^k + j) = (2j - 1)/2^{k+1}$ (in the sequel we will always assume when we write this that $1 \leq j \leq 2^k$) and vice versa; i.e., if $t = (2j - 1)/2^{k+1}$ for some j, k then $n(t) = 2^k + j$. We define the "path" to the vector y_t as the set of integers where $g_n(t)$ is nonzero:

$$\text{path } y_t = \{n \,|\, g_n(t) \neq 0\}.$$

It is easy to see from the definition that

$$\text{path } y_t = \{1\} \cup \left\{ 2^k + j \,\left|\, \frac{j-1}{2^k} < t < \frac{j}{2^k}, \ 1 \leq j \leq 2^k \text{ and } k \geq 0 \right.\right\}$$

and in fact

(1) $$t \in \text{supp } g_n \Leftrightarrow n \in \text{path } y_t.$$

If t is dyadic, i.e. if $t = t(2^q + p)$, then

(2) $$\text{path } y_t = \{1, 2, n_1, \ldots, n_q\}$$

where $n_k = 2^k + j_k$ and $(j_k - 1)/2^k < t < j_k/2^k$ with $1 \leq k \leq q$.

Now, let t be any number in $[0, 1]$, let $n_k \in \text{path } y_t$, and let $t(n_k) = (2j_k - 1)/2^{k+1}$ for $k = 1, 2, \ldots$. Then $\text{supp } g_{n_{k+1}} \subset \text{supp } g_{n_k}$. In fact,

$$t(n_k) < t \Rightarrow \text{supp } g_{n_{k+1}} = (t(n_k), t(n_k) + 2^{-(k+1)}),$$
$$t(n_k) > t \Rightarrow \text{supp } g_{n_{k+1}} = (t(n_k) - 2^{-(k+1)}, t(n_k)).$$

Notice that $\text{supp } g_{n_k} = (t(n_k) - 2^{-(k+1)}, t(n_k) + 2^{-(k+1)})$.

It will be useful to distinguish these two cases; we will say in the first case above that n_k is in the "left path" to y_t, and in the second that it is in the "right path" to y_t. The left path is an increasing sequence of integers whose associated dyadics form an increasing sequence approaching t. Notice that

$$\text{path } y_t = \text{right path } y_t \cup \text{left path } y_t \cup \{n \,|\, t(n) = t\}$$

the last set being empty when t is not dyadic. This decomposition of the path has the following properties.

Lemma 3.3. *Let* $t \in [0, 1]$ *and* $\text{path} \, y_t = \{1, 2, n_1, \ldots\}$. *For every* $k \geq 1$ *(1)–(3) are equivalent*:

(1) $n_k \in \text{left path} \, y_t$.

(2) $t(n_{k+1}) = t(n_k) + 2^{-(k+2)}$.

(3) $t(n_k) < t(n_j) \quad \forall j > k$ *(or equivalently* $\exists j > k$).

We have the analogous equivalences for the right path. We also have the above facts:

(4) $\text{path} \, y_{t(n_k)} = \{1, 2, n_1, \ldots, n_k\} \quad \forall k \geq 1$.

(5) *Whenever* n_k *and* n_{k+r} *are two "consecutive" elements of* $\text{left path} \, y_t$ *(i.e., if* $\{n_{k+1}, n_{k+2}, \ldots, n_{k+r-1}\} \subset \text{right path} \, y_t$) *then the left endpoint of* $\text{supp} \, g_{n_j}$ *is* $t(n_k)$ *for* $j = k + 1, \ldots, k + r$.

(6) g_{n_k} *is linear on* $\text{supp} \, g_{n_{k+1}}$.

Proof. All but (5) follow quickly from the definitions. For (5) use the formulas for support given above, noting that for elements of $\text{right path} \, y_t$, the left endpoint of the support of the Schauder function corresponds with that of the following element. $\square$

Let $t = t(2^q + p)$ be a fixed dyadic in $[0, 1]$. It follows from (1) that if $\text{supp} \, g_n \subset (t, 1]$ then $\langle y_s, e_n^* \rangle = 0$ for every $s \leq t$. Thus, $\langle x, e_n^* \rangle = 0$ (we will write $x_n = 0$). Precisely,

(3) $$\frac{j - 1}{2^k} \geq t \Rightarrow x_{2^k + j} = 0 \quad \forall x \in N_t.$$

Proposition 3.4. *For any dyadic* $t \in [0, 1]$ *we have the decomposition*

$$N_t \equiv \overline{\text{span}} \{e_{2^k + j} | j / 2^k \leq t\} \oplus F_t$$

where

$$F_t = \text{span}[\{y_{t(n)} | n \in \text{left path} \, y_t\} \cup \{y_t\}].$$

Remark. If $X = l_p$ for $1 \leq p < \infty$ and we take the usual basis, then $N_t \approx l_p$ for t dyadic. In §5 we will see that this is true for every t and use it to give a characterization of separable $\mathscr{L}_p$-spaces in terms of nests.

Remark. Once we have proved the continuity of the nest $\mathbf{N}$ it will follow that we have a similar formula for N_t for nondyadic t; F_t will be defined the same way, but will be infinite-dimensional and not necessarily complemented.

The proposition requires two technical lemmas.

Lemma 3.5. *Let* $t = t(2^q + p)$ *and* $P : X \to \text{span}\{e_n | n \in \text{path} \, y_t\}$ *be the coordinate projection. If* $s \leq t$ *then* $Py_s \in F_t$.

Proof. Notice that P is well defined because the path to y_t is a finite set of integers. By definition we have that $F_t = \text{span}\{y_{t_1}, y_{t_2}, \ldots, y_{t_m}\}$ for some $0 < t_1 < t_2 < \cdots < t_m = t$. Let $s < t$ and find i such that $t_{i-1} \leq s \leq t_i$. From Lemma 3.3(3) we have $n(t_{i-1}) < n(t_i)$. We claim that for $n \in \text{path} \, y_t$, g_n is affine on $[t_{i-1}, t_i]$. For $n > n(t_i)$, $g_n \equiv 0$ on $[t_{i-1}, t_i]$ (Lemma 3.3(2) and the definition of g_n). For $n < n(t_i)$ it will suffice (by 3.3(6)) to show that $(t_{i-1}, t_i) \subset \text{supp} \, g_{n(t_i)}$. But this is clear from 3.3(5). For $n = n(t_i)$ we only need note that there are no elements m of $\text{path} \, y_t$ in the interval (t_{i-1}, t_i). Since $g_{n(t_i)}$ "peaks" at t_i, it is affine in the interval as well. Since P involves only the g_n's in $\text{path} \, y_t$ it follows that

$$Py_s = \alpha y_{t_{i-1}} + (1 + \alpha) y_{t_i}$$

where $\alpha = (t_i - s)/(t_i - t_{i-1})$. $\square$

Lemma 3.6. *Let* $t = t(2^q + p)$. *If* $j/2^k \leq t$ *then* $e_{2^k + j} \in N_t$.

Proof. Let $s = t(2^k + j)$ with $j/2^k \leq t$. Consider path $y_s = \{1, 2, n_1, \ldots, n_k\}$. Let m be the first integer such that $\{1, 2, n_1, \ldots, n_m\} \subset$ path y_t and $n_{m+1} \notin$ path y_t. Since $s < t$ the only possibility is that $n_m \in$ left path y_t and $n_m \in$ right path y_s and hence $t(n_i) < t$ for $i = m+1, \ldots, k$. By Lemma 3.3(4) we know that path $y_{t(n_{m+1})} = \{1, 2, n_1, \ldots, n_{m+1}\}$. Hence for some $c \neq 0$,

$$y_{t(n_{m+1})} = P y_{t(n_{m+1})} + c e_{n_{m+1}}.$$

Lemma 3.5 says that $P y_{t(n_{m+1})} \in F_t$; we conclude that $e_{n_{m+1}} \in N_t$ because $y_{t(n_{m+1})} \in N_t$ by definition. If $m+1 \neq k$ we continue, writing for some $c_1, c_2 \neq 0$,

$$y_{t(n_{m+2})} = P y_{t(n_{m+2})} + c_1 e_{n_{m+1}} + c_2 e_{n_{m+2}}.$$

This gives $e_{n_{m+2}} \in N_t$; going on in this fashion gives the result, and Proposition 3.4 is now immediate. $\square$

Two more technical lemmas (to which we refer later as well) are needed.

Lemma 3.7. *Let* $t = t(2^q + p)$ *and* $s = t - 2^{-(q+2)}$ (*i.e.*, s *corresponds to the Schauder function on the next lower level whose support shares a left endpoint with* $g_{2^q + p}$). *If* $x \in N_t$ *then we can find* $w \in N_s$ *so that*

$$\|x - w\| \leq |x_{n(t)}| + \left\| \sum_{n=2^{q+1}+1}^{\infty} x_n e_n \right\|.$$

Proof. By Proposition 3.4 we can write

$$x = \sum_{j/2^k \leq t} x_{2^k + j} e_{2^k + j} + P_t x$$

where $P_t x \in F_t$. Let

$$w = \sum_{\substack{j/2^k \leq t \\ k \leq q}} x_{2^k + j} e_{2^k + j} + [P_t x + x_{n(s)} e_{n(s)}].$$

Now w clearly satisfies the norm condition. It is easy to compute that if $j/2^k \leq t = (2p - 1)/(2^q + 1)$ and $k \leq q$ then $j/2^k \leq s$.

By Lemma 3.8 below $P_t x + x_{n(s)} e_{n(s)} \in F_s$. So (by Proposition 3.4 again) we see that $w \in N_s$. $\square$

Lemma 3.8. *Let* $t = t(2^q + p)$, $s = t - 2^{-(q+2)}$ *and* $x \in \text{span}\{e_n | n \in \text{path } y_s\}$. *Then* $x \in F_s$ *if and only if* $P_t x \in F_t$ *and* $x_{n_t} = x_{n_s}$.

Proof. Now

$$x \in F_s \subset N_s \subset N_t = \text{span}\{y_r | r \leq t\}.$$

By Lemma 3.5, $P_t x \in F_t$. Since

$$x \in F_s = \text{span}[\{y_{t(n)} | n \in \text{left path } y_s\} \cup \{y_s\}]$$

and for $r \in [0, s]$ we have $g_{n(t)}(r) = g_{n(s)}(r)$ we obtain $x_{n(t)} = x_{n(s)}$. Hence

$$F_s \subset \{x \in \text{span}\{e_n | n \in \text{path } y_s\} | P_t x \in F_t \text{ and } x_{n_t} = x_{n_s}\} \equiv E.$$

Now notice that the left path y_s = left path y_t and thus $\dim F_s = \dim F_t$. Therefore, since $\operatorname{Ker} P_t \cap [e_n | n \in \operatorname{path} y_s] = [e_{n(s)}]$, we also get $\dim E = \dim F_t$ and so $E = F_s$. $\quad\square$

We are finally ready to prove Theorem 3.2 (that the nest is left continuous).

Proof of Theorem 3.2. (*Case* 1). Suppose t is not dyadic. Then both left path y_t and right path y_t are infinite sets. Let $x \in \bigwedge_{r>t} N_r$ and $\delta > 0$ be given. Take N so that

$$\left\| \sum_{i=n}^{\infty} x_i e_i \right\| < \delta \quad \forall n \geq N.$$

Choose $m > N$, $m = 2^k + j$ so that $m \in$ right path y_t but $n(t(m) - 2^{-(k+2)})$ (the next integer in path y_t) is in left path y_t. Let $s = t(m) - 2^{-(k+2)}$ and notice that $s < t < t(m)$ with $t(m)$ and s as in Lemma 3.7. Since $x \in N_{t(m)}$ we can find $w \in N_s \subset N_t$ satisfying

$$\|x - w\| \leq |x_m| + \left\| \sum_{i=n(s)}^{\infty} x_i e_i \right\| \leq 3\delta$$

since $n(s) > m > N$. But δ was arbitrary.

(*Case* 2). Suppose $t = t(2^q + p)$. Let $x \in \bigwedge_{r>t} N_r$. Since

$$X = \overline{[e_i]_{i=1}^{\infty}} = \overline{\operatorname{span}} \left[\left\{ e_{2^k+j} | \frac{j}{2^k} \leq t \right\} \cup \{ e_n | n \in \operatorname{path} y_t \} \cup \left\{ e_{2^k+j} | t \leq \frac{j-1}{2^k} \right\} \right]$$

and

$$N_t = \overline{\operatorname{span}} \{ e_{2^k+j} | j/2^k \leq t \} \oplus F_t$$

it suffices to show that $x_{2^k+j} = 0$ for $t \leq (j-1)/2^k$ and that $P_t x \in F_t$.

Let $s_i = t + 2^{-(q+1+i)}$ for $i = 1, 2, \ldots$. Then $s_i \downarrow t$. If $t < (j-1)/2^k$ we can find s_i such that $t < s_i < (j-1)/2^k$; since $x \in N_{s_i}$ we have by Proposition 3.4 (specifically comment (3) above it) that $x_{2^k+j} = 0$. Now if $t = (j-1)/2^k$ then $2^k + j = n(s_i)$ for some i. Applying Lemma 3.8 repeatedly to the pairs s_i, s_{i+1} we see that $x_{n(s_1)} = x_{n(s_2)} = \cdots$. Since (e_n) is a basis, these coordinates must be zero.

That $P_t x \in F_t$ now follows; for $x_{s_1} = 0$ implies

$$P_t x = P_{s_1} x \in F_{s_1} = \operatorname{span}[\{ y_{t(n)} | n \in \operatorname{left} \operatorname{path} y_{s_1} \} \cup \{ y_{s_1} \}].$$

But again left path y_{s_1} = left path y_t, so $P_t x \in F_t$. $\quad\square$

CONTINUOUS NESTS AND INDEX SETS IN SEPARABLE BANACH SPACE

In this section we will use the result of §3 to construct a continuous nest in any Banach space. We do this by "pasting together" nests formed in quotient spaces; the following lemma is essential.

Lemma 4.1. *Let* A *be a closed subspace of* X, $\mathbf{N}$ *a continuous nest in* A. *Suppose that* X/A *is infinite-dimensional, and* $\mathbf{M}$ *is a continuous nest in a*

closed subspace B of X/A. If π is the canonical projection onto X/A, then $\mathbf{N} \cup \pi^{-1}(\mathbf{M})$ is a continuous nest in $\pi^{-1}(B)$.

Proof. It is clear that we have continuity at every $N \in \mathbf{N}$ such that $N < A$, and continuity from the left at $N = A$. The continuity from the right at A and at every $\pi^{-1}(M)$, follows from the fact that $\pi^{-1}(0) = A$ and that

$$\pi^{-1}\left(\bigwedge_{\mathbf{M}_0} M \right) = \bigwedge_{\mathbf{M}_0} \pi^{-1}(M)$$

for any $\mathbf{M}_0 \subset \mathbf{M}$. To finish, we show continuity from the left at these places, i.e.,

$$\pi^{-1}(M_0) \subset \bigvee_{M < M_0} \pi^{-1}(M) \quad \forall M_0 \in \mathbf{M}$$

(the other inclusion is obvious). Let $x \in \pi^{-1}(M_0)$ and $\varepsilon > 0$. Since $\pi x \in M_0$ and $\mathbf{M}$ is continuous we can find $M < M_0$ and $\pi y \in M$ such that $\|\pi x - \pi y\|_{X/A} < \varepsilon$ which means that there exists $a \in A$ satisfying $\|x - y - a\| < \varepsilon$. Since $y + a \in \pi^{-1}(M)$ we are finished.   □

Theorem 4.2. *There is a continuous nest in every infinite-dimensional Banach space.*

Proof. Let X be a Banach space (not necessarily separable). Let $(x_\alpha)_{\alpha \in \Lambda}$ be dense in B_X. Well order Λ and construct a family of closed subspaces in the following way:

We know that any space contains a basic sequence starting with any fixed vector [LT]; choose a basic sequence starting with x_1,

$$\{x_1 = e_1^1, e_2^1, e_3^1, \dots \}.$$

Construct a continuous nest $\mathbf{N}_1$ in $X_1 = \overline{\operatorname{span}}(e_i^1)_{i=1}^\infty$ and assume that X/X_1 is not finite dimensional (if it were, we could extend the basic sequence to a basis for X and finish). Now let

$$\alpha_1 = \min\{\alpha | x_\alpha \notin X_1\}.$$

Choose a basic sequence $(e_i^2)_{i=1}^\infty$ of X/X_1 satisfying $e_1^2 = x_{\alpha_1} + X_1$ and construct another continuous nest $\mathbf{M}_2$ in $\overline{\operatorname{span}}(e_i^2)_{i=1}^\infty$. By the Lemma we know that $\mathbf{N}_2 = \mathbf{N}_1 \cup \pi^{-1}(\mathbf{M}_2)$ is a continuous nest in $X_2 = \pi^{-1}(\overline{\operatorname{span}}(e_i^2)_{i=1}^\infty)$. So we can choose a basic sequence $(e_i^3)_1^\infty$ of X/X_2 with $e_1^3 = x_{\alpha_2} + X_2$ where $\alpha_2 = \min\{\alpha | x_\alpha \notin X_2\}$ and proceed transfinitely as follows:

If α has an immediate predecessor, let $\mathbf{N}_\alpha = \mathbf{N}_{\alpha-1} \cup \pi^{-1}(\mathbf{M}_\alpha)$ as before. If α has no immediate predecessor, let

$$\mathbf{N}_\alpha = \bigcup_{\beta < \alpha} \mathbf{N}_\beta \cup \left\{ \bigvee_{\beta < \alpha} X_\beta \right\}$$

and of course

$$X_\alpha = \bigvee_{\beta < \alpha} X_\beta.$$

That this process ends is a consequence of the density of the $(x_\alpha)_{\alpha \in \Lambda}$. If the space is separable, we can index this nest by $[0, 1]$.   □

Now it is easy to see that every nest in a Banach space is contained in some maximal nest. Ringrose [R] proved that maximal nests are either continuous or all of the gaps are one dimensional. Larson has asked whether the Volterra nest in L_∞ is contained in a continuous nest. We obtain the positive answer to this question as a corollary to the above theorem.

Corollary 4.3. *A nest in a Banach space X is contained in a continuous nest if and only if the gaps in the nest are all infinite dimensional.*

Proof. It is clear that if there is a finite-dimensional gap we cannot extend the nest to a continuous one. Going the other way, let $\mathbf{N} = \{N_t | t \in T\}$ be a complete nest with infinite-dimensional gaps. We construct a continuous nest in each gap as follows: if t_0 is the immediate predecessor of t_1 and N_{t_0}/N_{t_1} is infinite dimensional, we construct a continuous nest there and lift it as above. Do this for every gap. □

If we focus on the index set of a nest it is natural to ask two questions: first, given a nest, what kind of index set can we have (i.e., what is its order type), and second, given an order type (say a compact subset of $\mathbf{R}$), is it always possible to construct a nest of that type in various spaces? A result in the direction of the first question is the following.

Proposition 4.4. *Let X be a separable Banach space. Let $\mathbf{N}$ be a complete nest in X. Then $\mathbf{N}$ is order isomorphic to a compact subset of $[0, 1]$.*

The proof is elementary; we include it for completeness.

Proof. We will actually define an order isomorphism between the nest and a compact subset of the extended reals. We recall two facts from the introduction. Since X is separable, $\mathbf{N}$ has a countable order dense subnest and at most countably many gaps. If there are only finitely many gaps, it is trivial to define an order isomorphism to $[0, 1]$. So let the set of gaps be $G = (M_i, N_i)_{i=1}^\infty$ (it may be that M_i coincides with N_j for some values i, j). We denote by $\mathbf{G}$ the union of the elements of pairs from G. We choose a countable order dense subnest $\mathbf{M}$ that contains all of the endpoints of the gaps (i.e. $\mathbf{G} \subset \mathbf{M}$); it may or may not contain other elements of the nest. We will define the order isomorphism on this subnest and then extend it. We intend first to define a map $\phi : A \subset \mathbf{R} \to \mathbf{N}$ associating the endpoints of the gaps with endpoints of intervals. Find $(a_i, b_i) \subset \mathbf{R}$ such that $M_i < M_j$ if and only if $a_i < b_i < a_j < b_j$ for all $i \in \mathbf{Z}^+$. Let

$$(c_i, d_i) = \left(a_i + \frac{b_i - a_i}{4}, b_i - \frac{b_i - a_i}{4} \right)$$

and let $A = \bigcup \{c_i, d_i | i \in \mathbf{Z}^+\}$. Define

$$\phi(c_i) = M_i \quad \text{and} \quad \phi(d_i) = N_i \quad \forall i \in \mathbf{Z}^+,$$

ϕ is an order homomorphism. The set A contains none of its accumulation points, so we can extend ϕ in a well-defined way, $\phi : \overline{A} \to \mathbf{G}^*$ where $\mathbf{G}^*$ is the completion of $\mathbf{G}$; when this is done in the obvious way (sups to joins, infs to meets) ϕ remains an order homomorphism (it may not be 1-1). By the construction it is easy to see that ϕ is at most 2-1 in some places; thus we can pick a nondecreasing function $f : \mathbf{R} \to \mathbf{R}$ which identifies the pairs

of points that ϕ does and the interval between the pair, but no others. Let $B = f(A)$. Then B is closed and we can define $g: B \to \mathbf{G}^*$ so that g is an order isomorphism and $f \circ g = \phi$. But now we are almost finished; for if $\mathbf{M} \subset \mathbf{G}^*$ then since $\mathbf{G}^*$ is complete, $\mathbf{G}^* = \mathbf{N}$. However, if $\mathbf{N} \backslash \mathbf{G}^*$ is not empty, it contains only elements at which the nest is continuous. These elements *must* fill in some gaps in $\mathbf{G}^*$ since $\mathbf{G}^*$ is order isomorphic to a closed subset B of $\mathbf{R}$; it is obvious that we can index these intervals continuously by the appropriate intervals in $\mathbf{R} \backslash B$ in an order-preserving fashion using the countable order-dense subset. By the previous comments, this completes the proof. $\square$

Observe that one can obtain in any Banach space a nest order isomorphic to any compact subset of the unit interval by simply taking a continuous nest in that space and deleting some appropriate elements, we thus have

Corollary 4.5. *Let X be a Banach space. For every compact subset K of $[0, 1]$ we can find a complete nest $\mathbf{N}$ order isomorphic to K.*

APPLICATIONS TO BANACH SPACE THEORY

This construction of a continuous nest in any space with a basis has some implications for the theory. In this section we will characterize separable $\mathscr{L}_p$-spaces and give examples of the variety of nests that can be constructed in Banach spaces. Our first task will be to prove the following fact about the continuous nest constructed in §3 (we prove it for every l_p, $1 \le p < \infty$):

Proposition 5.1. *Let $\mathbf{N}$ be the continuous nest constructed in §3 using the canonical basis of l_p. Then $\sup_{0 < t < 1} d(N_t, l_p) \le 20$.*

The distance here is the Banach-Mazur distance (or condition number):

$$d(X, Y) = \inf\{\|T\|^{-1}\|T\| \,|\, T : X \to Y \text{ is an isomorphism}\}.$$

In order to prove this proposition we will have to examine the properties of the construction made in §3 a bit more closely. For $t = t(2^q + p) = (2p - 1)/2^{q+1}$ we define the next "right" and "left" dyadics to be

$$r(t) = t + 2^{-(q+2)}, \qquad l(t) = t - 2^{-(q+2)}.$$

This is just another way of referring to the dyadics associated with integers in a given path; not that if we have $n_i \in \operatorname{path} y_t$ then

$$n_i \in \text{left path } y_t \quad \text{iff } t(n_{i+1}) = r(t(n_i)),$$
$$n_i \in \text{right path } y_t \quad \text{iff } t(n_{i+1}) = l(t(n_i)).$$

We will write $r^2(t)$ for $r(r(t))$.

Lemma 5.2. *Let $t = t(2^q + p)$ and $s = r^k(l(t))$. Then for*

$$x \in \operatorname{span}\{e_n | n \in \operatorname{path} y_s\}, \qquad x \in F_s,$$

if and only if $P_t x \in F_t$ and

$$(*) \qquad\qquad x_{n_q} = x_{n_{q+1}} + 2x_{n_{q+2}} + \cdots + 2^k x_{n_{q+k+1}}.$$

Proof. First notice that

$$\operatorname{path} y_t = \{1, 2, \dots, n_q\}, \quad \operatorname{path} y_{l(t)} = \{1, 2, \dots, n_q, n_{q+1}\},$$
$$\operatorname{path} y_s = \{1, 2, \dots, n_q, n_{q+1}, n_{q+2}, \dots, n_{q+k+1}\}.$$

We also know that $t(n_{q+i}) < t(n_q) = t$ for $i = 1, 2, \ldots, k+1$ (Lemma 3.3(3)). The proof will go by induction; first note that the case $k = 0$ is Lemma 3.8. We will assume that result true for $s = r^k(l(t))$ and prove that it holds for $r(s) = r^{k+1}(l(t))$.

First suppose that
$$x \in F_{r(s)} = F_s \oplus [y_{r(s)}].$$
So we can write $x = z + cy_{r(s)}$ with $z \in F_s$ and some $c \in R$. Now $P_t x \in F_t$ since $P_t z \in F_t$ (by induction) and $P_t y_{r(s)} \in F_t$ (by Lemma 3.5). Since z satisfies $(*)$ (again by induction) it of course satisfies

$(**)$ $\qquad\qquad x_{n_q} = x_{n_{q+1}} + 2x_{n_{q+2}} + \cdots + 2^k x_{n_{q+k+1}} + 2^{k+1} x_{n_{q+k+2}}$

and we can verify directly from the definitions that $y_{r(s)}$ does as well:
$$\langle r_{r(s)}, e_{n_q} \rangle = \langle y_s, e_{n_q} \rangle + 2^{-(q+k+2)},$$
$$\langle y_{r(s)}, e_{n_{q+i}} \rangle = \langle y_s, e_{n_{q+i}} \rangle - 2^{-(q+k+2)} \quad \forall i = 1, \ldots, k+1,$$
$$\langle y_{r(s)}, e_{n_{q+k+2}} \rangle = 2^{-(q+k+2)}.$$

Thus we have shown that
$$F_{r(s)} \subset \{x \,|\, x \in \operatorname{span}\{e_n | n \in \operatorname{path} y_{r(s)}\}, \, P_t x \in F_t \text{ and } (**) \text{ holds}\}.$$
But these subspaces both have the same dimension; namely, $\dim F_s + 1$. So they must be equal. $\square$

From the above lemma it is now obvious that for dyadic t, F_t is completely characterized by a set of conditions of the form $(*)$ above. If we take the elements of path y_t in order, we see that a new condition is added following each element of right path y_t (Lemma 3.8); following each element of the left path an existing condition is modified. The list of conditions will look like this
$$x_1 = x_2 + 2x_{n_1} + \cdots + 2^q x_{n_q}$$
$$x_q = x_{n_{q+1}} + 2x_{n_{q+2}} + \cdots + 2^k x_{n_{q+k}}$$
$$\vdots$$
where right path $y_t = \{1, n_q, n_{q+k}, \ldots\}$. The number of conditions is given exactly by the number of elements in right path y_t; this implies that
$$\dim F_t = \# \operatorname{left} \operatorname{path} y_t + 1.$$

The above characterization will allow us to exhibit a nice isomorphism between F_t and $l_p^{\dim F_t}$, proving Proposition 5.1. Suppose the first condition characterizing F_t is
$$x_1 = x_2 + 2x_{n_1} + \cdots + 2^q x_{n_q}.$$
We choose the following basis for F_t:
$$v_1 = e_1 + 2^{-(q-1)} e_{n_{q-1}}$$
$$v_2 = e_2 - 2^{-(q-1)} e_{n_{q-1}}$$
$$v_3 = e_{n_1} - 2^{-(q-2)} e_{n_{q-1}}$$
$$\vdots$$
$$v_q = e_{n_{q-2}} - 2^{-1} e_{n_{q-1}}$$
$$v_{q+1} = -2e_{n_{q-1}} + e_{n_q}.$$

If this is also the last condition defining F_t then we finish here. If not, there is a second condition

$$x_{n_q} = x_{n_{q+1}} + 2x_{n_{q+2}} + \cdots + 2^k x_{n_{q+k}}.$$

In that case, we continue. We used the element $e_{n_{q-1}}$ rather than e_{n_q} above for "cancellation" because $v_1, \ldots, v_q$ are not affected by the new condition; we redefine v_{q+1} and proceed as follows:

$$v_{q+1} = -2e_{n_{q-1}} + e_{n_q} + 2^{-(k-1)} e_{n_{q+k-1}}$$
$$v_{q+2} = e_{n_{q+1}} - 2^{-(k-1)} e_{n_{q+k-1}}$$
$$v_{q+3} = e_{n_{q+2}} - 2^{-(k-2)} e_{n_{q+k-1}}$$
$$\vdots$$
$$v_{q+k-1} = e_{n_{q+k-2}} - 2^{-1} e_{n_{q+k-1}}$$
$$v_{q+k} = -2e_{n_{q+k-1}} + e_{n_{q+k}}.$$

Again, if this is the last condition we will stop; if not, we will redefine the last basis element (in this case v_{q+k}) similarly and continue. Observe that the dimensionality conditions above are satisfied. Hence we may proceed in this manner until the set (v_i) spans F_t. It is clear that these vectors satisfy

$$(+) \qquad \frac{1}{4}\left(\sum_1^j |a_i|^p\right)^{1/p} \leq \left\|\sum_1^j a_i \frac{v_i}{\|v_i\|}\right\|_p \leq 5\left(\sum_1^j |a_i|^p\right)^{1/p}.$$

Now for nondyadic t we know that $N_t \approx l_p \oplus F$ with F infinite dimensional (this follows from the proof that the nest is continuous). But we still have a list of conditions characterizing F_t; now the list is infinite. Since these conditions are of the same form, we can define the sequence (v_i) inductively in exactly the same way. Then $\{v_i\}_{i=1}^\infty$ is a basis such that $F \approx l_p$ with constant ≤ 20. $\square$

Now 20 is definitely not the best constant here. We do not know if for every $\varepsilon > 0$ we can find a nest $\mathbf{N}$ in l_p so that $d(N, l_p) \leq 1 + \varepsilon$ for every $N \in \mathbf{N}$, but we know that we cannot find one with all of its elements isometric to l_p.

Proposition 5.3. *There is no continuous nest in l_p, $1 \leq p \neq 2 < \infty$, having each of its elements isometrically isomorphic to l_p.*

Proof. Assume we can find a continuous nest $N = \{N_t : 0 \leq t \leq 1\}$ such that $N_t \equiv l_p$ for $0 < t \leq 1$. Then for every $t > 0$ we can find $\{x_n^t : n \geq 1\}$ normalized, disjointly supported such that their closed linear span is equal to N_t (by lattice properties of l_p). Notice that even in the case $p = 1$ the elements of this nest are weak* closed. For $j \in Z^+$ let

$$x(t, j) = \begin{cases} x_n^t, & \text{if } \langle e_j^*, x_n^t \rangle \neq 0 \quad \text{for some } n; \\ 0, & \text{if } \langle e_j^*, x_n^t \rangle = 0 \quad \text{for every } n. \end{cases}$$

It is clear that for every $j \in Z^+$ $x(0, j) = 0$ and $x(1, j) = e_j$; moreover, if $k \neq j$, either $x(t, j) = x(t, k)$ or they are disjointly supported. The main property needed is the fact that the map $t \mapsto |\langle e_j^*, x(t, j) \rangle|$ is nondecreasing and continuous. First observe that $|\langle e_j^*, x(t, j) \rangle| = \|e_{j|N_t}^*\|$. The continuity of

the map from below follows directly from the continuity of the nest. To see continuity from above, consider $|\langle e_j^*, x(t + 1/n, j)\rangle|$ for $n = 1, 2, \ldots$. Take a weak* accumulation point of the set $\{x(t + 1/n, j)\}_{n=1}^\infty$; since it is in N_t and the map is nondecreasing and left continuous, it must be right continuous. Let $0 < r < 1$ be such that $0 < |\langle e_j^*, x(r, j)\rangle| < 1$. Then we can find $k \neq j$ such that $x(r, j) = x(r, k)$. Let $s = \sup\{t : x(t, j) = x(t, k)\}$. Clearly $r \leq s < 1$ and $x(s, j) = x(s, k)$. Since for every $\varepsilon > 0$, $x(s + \varepsilon, j) \neq x(s + \varepsilon, k)$ then $\langle e_k^*, x(s + \varepsilon, j)\rangle = 0$ and $|\langle e_k^*, x(s + \varepsilon, j)\rangle| \geq |\langle e_j^*, x(s, j)\rangle| > 0$. As before let x be a weak*-limit of $\{x(s + 1/n, j) : n \geq 1\}$. We have that $x \in N_s$ (by continuity of the nest) and $\langle e_k^*, x\rangle = 0$ and $\langle e_j^*, x\rangle \neq 0$ contradicting $x(s, j) = x(s, k)$. □

We are ready to characterize the separable $\mathcal{L}_p$-spaces; first, some definitions:

Definition. Let X be a Banach space. Let E_1, E_2 be subspaces of X of the same finite dimension and $\varepsilon > 0$. We say that E_2 is ε-close to E_1 if there exists an invertible operator T from E_1 onto E_2 with

$$\|Tx - x\| \leq \varepsilon\|x\| \quad \forall x \in E_1.$$

Definition. We say that X is an $\mathcal{L}_{p,\lambda}$-space if given any finite-dimensional $F \subset X$ there exists $E \supset F$, $\dim E < \infty$ such that $d(E, l_p^{\dim E}) \leq \lambda$. We say X is an $\mathcal{L}_p$-space if X is an $\mathcal{L}_{p,\lambda}$-space for some λ.

Of course, $L^p(\mu)$ is a $\mathcal{L}_{p,\lambda}$-space for any $\lambda > 1$.

We are now ready to state the following proposition:

Proposition 5.4. *Let X be a separable Banach space. Then X is an $\mathcal{L}_p$-space if and only if there exists a continuous nest $\mathbf{N} = \{N_t | 0 \leq t \leq 1\}$ in X with $N_t \approx l_p$ for $0 < t < 1$ and $\sup_{0<t<1} d(N_t, l_p) < \infty$.*

The following elementary fact is needed: If Y, Z are Banach spaces with continuous nests $\mathbf{N}$ and $\mathbf{M}$ respectively, indexed by $[0, 1]$. Then $\mathbf{N} \oplus \mathbf{M}$ is a continuous nest indexed by $[0, 2]$ on $Y \oplus Z$; $\mathbf{N} \oplus \mathbf{M}$ is defined by N_t if $0 \leq t < 1$ and by $Y \oplus M_{t-1}$ if $1 \leq t \leq 2$.

Proof. ($\Rightarrow$) Let X be a separable $\mathcal{L}_p$-space. It follows from [LP] that $X \approx X \oplus l_p$; and from [JRZ] and [NW] that X has a basis (f_n) satisfying

$$\sup_n d([f_i]_{i=1}^n, l_p^n) = \lambda < \infty.$$

Since

$$l_p \equiv \left(\sum \bigoplus l_p\right)_p$$

we have that

$$X \approx \sum_i \bigoplus (f_i \oplus l_p)$$

in a natural way. Now each of these summands is isomorphic to l_p; inside each we can form a continuous nest indexed by $[0, 1]$ whose elements are all isomorphic to l_p. Applying the comment above and the estimate of Proposition 5.1 we obtain a continuous nest indexed by $[0, \infty]$ such that $d(N_t, l_p) < 20\lambda$ for $0 < t < \infty$. Because

$$\sum_{i=1}^n \bigoplus (f_i \oplus l_p) \equiv \overline{\mathrm{span}}(f_i)_{i=1}^n \oplus l_p \approx l_p$$

with constant λ; which upon re-indexing the nest, gives the proposition.

($\Leftarrow$) Let us suppose that we have a continuous nest in X such that

$$\mathbf{N} = \{N_t | 0 \leq t \leq 1\}, \quad N_t \approx l_p \quad \forall t \quad \text{and} \quad \sup_{0 \leq t \leq 1} d(N_t, l_p) = \lambda < \infty.$$

Let $E \subset X$ be a subspace, $\dim E = k$ and let $\varepsilon = 1/3k$ (for later use). Using the continuity of the nest at $t = 1$ we can find $E' \subset N_t$, $t < 1$, such that E' is ε-close to E (since we can do it for a single vector, we can do it for any finite number). Now $E' \subset N_t \approx l_p$ with constant λ. Since l_p is $\mathscr{L}_{p, 1+\delta}$ $\forall \delta > 0$, we can find a finite-dimensional subspace in l_p which is 2-close to l_p and contains the image of E' (under the isomorphism). Pull it back into N_t, calling it F'; we then have $F' \subset N_t$, $\dim F' = n < \infty$, with $E' \subset F'$ and $d(F', l_p^n) \leq 2\lambda$. By a standard perturbation argument (using Lemma 2.4 of [JRZ], which requires ε as above), we can obtain $F \subset X$ with $\dim F = \dim F'$, $E \subset F$ and $d(F, l_p^n) \leq 4\lambda$. Therefore X is an $\mathscr{L}_{p, 4\lambda}$-space. $\square$

The construction of a continuous nest in any space with a basis allows us to build some interesting nests. We finish with an example of a continuous nest with no two of its subspaces isomorphic.

Example. Let $X = (\sum_{p=3}^{\infty} \oplus l_p)_2$ with $(e_n^p)_{n=1}^{\infty}$ being the canonical basis for l_p. Construct the continuous nest as before, assigning basis elements to dyadics so as to satisfy the following:

Assign the basis elements for l_3 to dyadics in $[0, \frac{1}{2}]$.

Assign the basis elements for l_4 to dyadics in $[\frac{1}{2}, 1]$.

Assign the basis elements for l_5 to dyadics in $[0, \frac{1}{4}]$, ... etc. Then, given any two elements of the nest, there will be an l_p space contained in one which is not contained in the other.

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