

A SIMPLIFIED TRACE FORMULA FOR HECKE OPERATORS FOR $\Gamma_0(N)$

SHEPLEY L. ROSS II

ABSTRACT. Let N and n be relatively prime positive integers, let χ be a Dirichlet character modulo N , and let k be a positive integer. Denote by $S_k(N, \chi)$ the space of cusp forms on $\Gamma_0(N)$ of weight k and character χ , a space denoted simply $S_k(N)$ when χ is the trivial character. Beginning with Hijikata's formula for the trace of T_n acting on $S_k(N, \chi)$, we develop a formula which essentially reduces the computation of this trace to looking up values in a table. From this formula we develop very simple formulas for (1) the dimension of $S_k(N, \chi)$ and (2) the trace of T_n acting on $S_k(N)$.

PRELIMINARIES

For each positive integer N , let

$$\Gamma_0(N) = \left\{ \gamma = \begin{pmatrix} a & b \\ cN & d \end{pmatrix} \middle| a, b, c, d \in \mathbf{Z}, \det(\gamma) = 1 \right\};$$

$\Gamma_0(N)$ is a congruence subgroup of $SL_2(\mathbf{Z})$. Let χ be a (Dirichlet) character mod N . Suppose $N = \prod_{l|N} l^{\nu_l}$, where each l is a prime and $\nu_l = \text{ord}_l(N)$. Then χ can be written as a product $\chi = \prod_{l|N} \chi_l$ of characters, where for each prime $l|N$, χ_l is a character mod l^{ν_l} . The *exponential conductor* $e = e(\chi_l)$ is the smallest value e such that χ_l is a character mod l^e ; note that $e = e(\chi_l) \leq \nu_l$. If χ is a character and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, with $a, b, c, d \in \mathbf{Z}$, then by $\chi(\gamma)$ we mean $\chi(a)$. In this paper we will use “ $|$ ” and “ $\nmid$ ” for “divides” and “does not divide,” respectively.

Fix a positive integer k . For any complex-valued function f and matrix $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in \mathbf{R}$ and $\det(\gamma) > 0$ define

$$f| \gamma = (\det(\gamma))^{k/2} (c\tau + d)^{-k} f\left(\frac{a\tau + b}{c\tau + d}\right)$$

(where we take the positive root if k is odd).

Let $\mathcal{H} = \{z \in \mathbf{C} \mid \text{Im}(z) > 0\}$ denote the complex upper half plane, and let f be any complex-valued function on $\mathcal{H}$. The *cusps* of $\Gamma_0(N)$ are the rational

Received by the editors April 30, 1989 and, in revised form, April 7, 1990. Presented to the joint meeting of the MAA-AMS at San Antonio, Texas, January 24, 1987.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 11F11; Secondary 11F25, 11F72, 11Y40.

Key words and phrases. Cusp forms, Hecke operators, modular forms, trace formula.

numbers, along with the point $i\infty$ at infinity. We say that f is a *cuspidal form* on $\Gamma_0(N)$ of weight k and character χ if f satisfies

- (i) f is holomorphic on $\mathcal{H}$,
- (ii) f is 0 at each cusp,
- (iii) $f \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \chi(a)^{-1} f$ for each $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$.

See [Sha, A-L, or Li] for details. The space of cuspforms on $\Gamma_0(N)$ of weight k and character χ is denoted by $S_k(N, \chi)$, or by $S_k(N)$ if χ is the trivial character.

For each n with $(n, N) = 1$, let T_n be the standard Hecke operator whose action on $S_k(N, \chi)$ is defined by

$$(1) \quad f \mid T_n = n^{k-1} \sum_{a, d} \sum_{b=0}^{d-1} \chi(a) f\left(\frac{a\tau + b}{d}\right) d^{-k},$$

where the first sum is over all pairs of integers a, d satisfying $a > 0$, $ad = n$, and $(a, N) = 1$. Note this is the same as the T'_n used by Shimura (see 3.5.7 of [Sha]), and therefore by Hijikata, Pizer, and Shemanske in [H-P-S₁, H-P-S₂]. If $n = p$, where p is a prime not dividing N and χ is the trivial character, then our T_p is the same as the T_p operator of Atkin and Lehner in [A-L]. (Note however that our weight k is twice the weight k of Atkin and Lehner.)

THE SIMPLIFIED FORMULA; APPLICATIONS

We begin by stating the version of the trace formula for the operator T_n acting on $S_k(N, \chi)$ as given in Theorem 2.2 of [H-P-S₁] and also in Theorem 2.2 of [H-P-S₂]. Denote this trace by $\text{tr}_{N, \chi, k} T_n$.

Theorem 1 (Hijikata-Pizer-Shemanske). *Let k be an integer, $k \geq 2$. Let χ be a character mod N and assume $(-1)^k \chi(-1) = 1$. Write $\chi = \prod_{l|N} \chi_l$, where for each prime l dividing N , χ_l is a character mod l^ν , where $\nu = \text{ord}_l(N)$. Then for $(n, N) = 1$ we have*

$$\begin{aligned} \text{tr}_{N, \chi, k} T_n = & - \sum_s a(s) \sum_f b(s, f) \prod_{l|N} c'_\chi(s, f, l) \\ & + \delta(\chi) \deg(T_n) + \delta(\sqrt{n}) \frac{k-1}{12} N \prod_{l|N} \left(1 + \frac{1}{l}\right) \\ & - \delta(\sqrt{n}) \frac{\sqrt{n}}{2} \prod_{l|N} \text{par}(l), \end{aligned}$$

where

$$\begin{aligned} \delta(\chi) &= \begin{cases} 1 & \text{if } k = 2 \text{ and } \chi \text{ is trivial,} \\ 0 & \text{otherwise,} \end{cases} \\ \delta(\sqrt{n}) &= \begin{cases} n^{k/2-1} \chi(\sqrt{n}) & \text{if } n \text{ is a perfect square,} \\ 0 & \text{otherwise,} \end{cases} \\ \text{par}(l) &= \begin{cases} 2l^{\nu-e} & \text{if } e \geq \mu + 1, \\ l^\mu + l^{\mu-1} & \text{if } e \leq \mu \text{ and } \nu \text{ is even,} \\ 2l^\mu & \text{if } e \leq \mu \text{ and } \nu \text{ is odd.} \end{cases} \end{aligned}$$

Here for a fixed prime $l \mid N$, $\nu = \text{ord}_l(N)$, $\mu = [\frac{\nu}{2}]$, and $e = e(\chi_l)$.

The meanings of s , $a(s)$, $b(s, f)$, and $c'_\chi(s, f, l)$ are given as follows:

Let s run over all integers such that $s^2 - 4n$ is a positive square or any negative integer. Hence for some positive integer t and squarefree negative integer m , $s^2 - 4n$ has one of the following forms which we classify into the cases (h) or (e) as follows:

$$s^2 - 4n = \begin{cases} t^2, & \text{(h)} \\ t^2 m, & 0 > m \equiv 1 \pmod{4}, \quad \text{(e)} \\ t^2 4m, & 0 > m \equiv 2, 3 \pmod{4}. \quad \text{(e)} \end{cases}$$

Let $\Phi(X) = \Phi_s(X) = X^2 - sX + n$ and let x and y be the roots in $\mathbf{C}$ of $\Phi(X) = 0$. Corresponding to the classification of s , put

$$a(s) = \begin{cases} (\min\{|x|, |y|\})^{k-1} |x - y|^{-1} \text{sgn}(x)^k, & \text{(h)} \\ \frac{1}{2}(x^{k-1} - y^{k-1})/(x - y). & \text{(e)} \end{cases}$$

For each fixed s , let f run over all positive divisors of t and let

$$b(s, f) = \begin{cases} \frac{1}{2} \phi((s^2 - 4n)^{1/2}/f), & \text{(h)} \\ h((s^2 - 4n)/f^2)/\omega((s^2 - 4n)/f^2), & \text{(e)} \end{cases}$$

where ϕ is Euler's function and $h(d)$ (respectively $\omega(d)$) denotes the class number of locally principal ideals (resp. $\frac{1}{2}$ the cardinality of the unit group) of the order of $\mathbf{Q}(\sqrt{d})$ with discriminant d .

Fix a pair (s, f) and let l be a prime divisor of N ; let $\nu = \text{ord}_l(N)$ and $\rho = \text{ord}_l(f)$. Put

$$\begin{aligned} \tilde{A} &= \{x \in \mathbf{Z} \mid \Phi(x) \equiv 0 \pmod{l^{\nu+2\rho}}, 2x \equiv s \pmod{l^\rho}\}, \\ \tilde{B} &= \{x \in \tilde{A} \mid \Phi(x) \equiv 0 \pmod{l^{\nu+2\rho+1}}\}. \end{aligned}$$

Let $A_\rho = A(s, f, l)$ (resp. $B_\rho = B(s, f, l)$) be a complete set of representatives of $\tilde{A}$ (resp. $\tilde{B}$) mod $l^{\nu+\rho}$, and let $B'_\rho = B'(s, f, l) = \{s - z \mid z \in B_\rho\}$. Then

$$c'_\chi(s, f, l) = \begin{cases} \sum_x \chi_l(x) & \text{if } (s^2 - 4n)/f^2 \not\equiv 0 \pmod{l}, \\ \sum_x \chi_l(x) + \sum_y \chi_l(y) & \text{if } (s^2 - 4n)/f^2 \equiv 0 \pmod{l}, \end{cases}$$

where x (resp. y) runs over all elements of A_ρ (resp. B'_ρ). This ends the statement of the theorem.

Proof. See [Hij, H-P-S₁, H-P-S₂].

We introduce a classification of prime numbers to be used throughout this paper. Fix integers n and s , with $n \geq 1$, such that $s^2 - 4n$ is a positive square or any negative integer and write $s^2 - 4n$ as one of t^2 , $t^2 m$, or $t^2 4m$ as in the statement of Theorem 1. Let l be any prime that divides either N or t , and classify l into one of six cases, A , B , C , D , E , or F , depending on how l

divides $s^2 - 4n$ and whether or not l is odd, as follows:

$$l \text{ is case } \begin{cases} A & \text{if } s^2 - 4n = l^{2a}d^2, l \text{ is odd, and } d \text{ is a unit of } \mathbf{Z}_l, \\ B & \text{if } s^2 - 4n = l^{2a}u, l \text{ is odd, and } u \text{ is a nonsquare unit of } \mathbf{Z}_l, \\ C & \text{if } s^2 - 4n = l^{2a+1}u, l \text{ is odd, and } u \text{ is a unit of } \mathbf{Z}_l, \text{ or} \\ & s^2 - 4n = l^{2a}4w, l = 2, w \in \mathbf{Z}_2, w \equiv 2 \pmod{4}, \\ D & \text{if } s^2 - 4n = 2^{2a}d^2, l = 2, \text{ and } d \text{ is a unit of } \mathbf{Z}_2, \\ E & \text{if } s^2 - 4n = 2^{2a}u, l = 2, u \in \mathbf{Z}_2, \text{ and } u \equiv 5 \pmod{8}, \\ F & \text{if } s^2 - 4n = 2^{2a}4w, l = 2, w \in \mathbf{Z}_2, \text{ and } w \equiv 3 \pmod{4}. \end{cases}$$

We will sometimes denote the case into which l falls by $\text{Case}(l)$. Note that $a = \text{ord}_l(t)$, where a is the ' a ' which appears in the expression for $s^2 - 4n$, in whatever case l is actually classified; we will sometimes write $a_l(s^2 - 4n)$ to mean this a .

Also let us introduce a convention to be adhered to throughout this paper:

Convention A. Let l be a prime and let n be any integer. We agree that any expression of the form l^n or $l^n - 1$ is taken to be 0 if $n < 0$.

We are ready to state the new version of the trace formula.

Theorem 2. Let k be an integer, $k \geq 2$. Let χ be a character mod N and assume $(-1)^k \chi(-1) = 1$. Write $\chi = \prod_{l|N} \chi_l$, where for each prime l dividing N , χ_l is a character mod l^ν , where $\nu = \text{ord}_l(N)$. Then for $(n, N) = 1$ we have

$$\begin{aligned} \text{tr}_{N, \chi, k} T_n = & - \sum_s \left(a(s)b(s) \prod_{l|t, l \nmid N} \gamma(s, l) \prod_{l|N} c(s, l) \right) \\ & + \delta(\chi) \deg(T_n) + \delta(\sqrt{n}) \frac{k-1}{12} N \prod_{l|N} \left(1 + \frac{1}{l} \right) \\ & - \delta(\sqrt{n}) \frac{\sqrt{n}}{2} \prod_{l|N} \text{par}(l), \end{aligned}$$

where $\delta(\chi)$, $\delta(\sqrt{n})$, and $\text{par}(l)$ are exactly the same as in Theorem 1.

The meanings of s , $a(s)$, $b(s)$, t , $\gamma(s, l)$, and $c(s, l)$ are as follows:

Let s , $a(s)$, and t be exactly as in Theorem 1. Now fix s and write $s^2 - 4n$ as one of t^2 , t^2m , t^24m as in Theorem 1. Let

$$b(s) = \begin{cases} \frac{1}{2} & \text{if } s^2 - 4n = t^2, \\ h(m)/\omega(m) & \text{if } s^2 - 4n = t^2m \text{ or } t^24m, m < 0, \end{cases}$$

where $h(m)$ is the class number of $\mathbf{Q}(\sqrt{m})$ and $\omega(m)$ is one-half the cardinality of the unit group of $\mathbf{Q}(\sqrt{m})$.

Keeping s fixed, now fix a prime l with $l|t$, $l \nmid N$, and, according to the classification of l , let $a = a_l(s^2 - 4n)$ be the ' a ' which appears in the expression for $s^2 - 4n$ and define

$$\gamma(s, l) = \begin{cases} l^a & \text{if } l \text{ is a case A or D prime,} \\ (l^a(l+1) - 2)/(l-1) & \text{if } l \text{ is a case B or E prime,} \\ (l^{a+1} - 1)/(l-1) & \text{if } l \text{ is a case C or F prime.} \end{cases}$$

Case(l)	ν	k_1	k_2	k_3	g	k_4	ε	k_5	k_6
A	even	1	0	$l-1$	1	$l+1$	0	$d(\mu+1, e)d(\mu+1, a)$	$d(e, a)d(\mu, a)$
	odd					2		0	0
B	even	0	0	$l+1$	1	$l+1$	0	$d(\mu+1, e)d(\mu+1, a)$	$d(e, a)d(\mu, a)$
	odd					2		0	0
C	even	0	0	1	1	$l+1$	1	$d(\mu+1, e)d(\mu, a)$	0
	odd					$2l$	0	0	$d(e, a+1)d(\mu, a)$
D	even	1	$d(e, a) \min_0(a-\mu+1, 2)$	1	0	$l+1$	-1	$d(\mu+1, e)d(\mu+2, a)$	$d(e, a-1)d(\mu, a-1)$
	odd		$2d(e, a)d(\mu+1, a)$			2		0	0
E	even	0	$d(e, a) \min_0(a-\mu+1, 2)$	$l+1$	0	$l+1$	-1	$d(\mu+1, e)d(\mu+2, a)$	$d(e, a-1)d(\mu, a-1)$
	odd		$2d(e, a)d(\mu+1, a)$			2		0	0
F	even	0	$d(e, a+1)d(\mu, a)$	1	0	$l+1$	0	$d(\mu+1, e)d(\mu+1, a)$	$d(e, a)d(\mu, a)$
	odd					2		0	0

TABLE 1

Keeping s fixed, now fix a prime $l \mid N$. Let $\nu = \text{ord}_l(N)$, and write $\nu = 2\mu + \delta$, where $\delta = 0$ or 1 . Let $e = e(\chi_l)$. Classify l into one of the six cases $A, \dots, F$, and, according to the classification, let $a = a_l(s^2 - 4n)$; if l is a case A or D prime let d be the 'd' which appears in $s^2 - 4n$, otherwise let $d = 1$. Let $\chi_l^* = \chi_l(\frac{s+l^a d}{2}) + \chi_l(\frac{s-l^a d}{2})$; let $\hat{\chi}_l = \chi_l(\frac{s+l^{a+f} d}{2})$, where $f = 1$ if $\text{Case}(l) = F$, and 0 otherwise. Then $c(s, l)$ is given by the expression

$$(2) \quad \chi_l^* k_1 l^{\min(2a, \nu-1, a+\nu-e)} + \hat{\chi}_l k_2 k_3 l^{\nu-1} \\ + \chi_l(\frac{s}{2}) k_3 l^{\nu-g} (k_4 (l^{a-\max(\mu, e-\delta)+\varepsilon} - 1)/(l-1) + k_5 l^{a-e+\varepsilon} + k_6),$$

where $k_1, \dots, k_6, g$, and ε are determined from Table 1, by knowing the parity of ν and the classification of the prime l . In the table, define $d(x, y) = 1$ if $x \leq y$ and 0 otherwise, for any integers x and y . Also, let $\min_0(x, y) = \max(0, \min(x, y))$. Note also that Convention A must be followed when evaluating (2).

We remark that we have used Theorem 2 to write a Turbo Pascal program which finds $\text{tr}_{N, \chi, k} T_n$ for small values of k, N , and n , and real characters χ .

The proof of Theorem 2 consists of transforming the first line of the formula given in Theorem 1 into the first line of that given in Theorem 2. We need two lemmas from [H-P-S₁] or [H-P-S₂]. The first of these is

Lemma 3. *Let the notation be as Theorem 1. In particular, write $s^2 - 4n = t^2, t^2 m$, or $t^2 4m$ as illustrated there. Let l be any prime dividing N or t and put $t = l^a t_0$ where $(l, t_0) = 1$. Let $f \mid t$ and put $f = l^p f_0$ with $(l, f_0) = 1$. Then $b(s, f) = \alpha(s, \rho, l) \cdot b(s, l^a f_0)$ where*

$$\alpha(s, \rho, l) = \begin{cases} l^{a-\rho} - l^{a-\rho-1} & \text{if } l \text{ is a case } A \text{ or } D \text{ prime,} \\ l^{a-\rho} + l^{a-\rho-1} & \text{if } l \text{ is a case } B \text{ or } E \text{ prime,} \\ l^{a-\rho} & \text{if } l \text{ is a case } C \text{ or } F \text{ prime.} \end{cases}$$

Note our Convention A in effect here; if $\rho = a$ we take $l^{a-\rho-1} = 0$.

Proof. See Lemma 2.4 of either [H-P-S₁] or [H-P-S₂].

Fix s as in Theorem 1 and write $s^2 - 4n = t^2, t^2 m$, or $t^2 4m$ as illustrated there. Let $l \mid N$ and define $c''_{\chi}(s, \rho, l) = c'_{\chi}(s, l^{\rho}, l)$ for $\rho = 0, \dots, \text{ord}_l(t)$. Now let $f \mid t$, and note that $c'_{\chi}(s, f, l)$ depends only on $\text{ord}_l(f)$ once s and l are fixed. Write $f = l^{\rho} f_0$ where $\rho = \text{ord}_l(f)$; then

$$c'_{\chi}(s, f, l) = c'_{\chi}(s, l^{\rho} f_0, l) = c'_{\chi}(s, l^{\rho}, l) = c''_{\chi}(s, \rho, l).$$

Now, if there are $v > 0$ distinct primes l satisfying $l \mid t$ and $l \nmid N$, let $\{l_i\}$, $i = 1, \dots, v$, be a list of them; if there are no such primes, then for convenience set $v = 1$ and define $l_1 = 1$ and $\alpha(s, 0, 1) = 1$. Next, if $N \neq 1$ let $l_{v+1}, \dots, l_{v+w}$ be a list of the w distinct primes dividing N , while if $N = 1$ then for convenience set $w = 1$ and define $l_{v+1} = 1$, $\alpha(s, 0, 1) = 1$ and $c''_{\chi}(s, 0, 1) = 1$. We can write $t = \prod_{i=1}^{v+w} l_i^{a_i}$ where $a_i = \text{ord}_{l_i}(t)$ if l_i is a (bona-fide) prime and $a_i = 0$ if $l_i = 1$. Let f be any divisor of t ; then we

can write f uniquely as $f = \prod_{i=1}^{v+w} l_i^{\rho_i}$, where $0 \leq \rho_i \leq a_i$. We have

$$(3) \quad \sum_{f|t} b(s, f) \prod_{l|N} c'_\chi(s, f, l) \\ = \sum_{\rho_1=0}^{a_1} \cdots \sum_{\rho_{v+w}=0}^{a_{v+w}} \left(b \left(s, \prod_{i=1}^{v+w} l_i^{\rho_i} \right) \prod_{i=v+1}^{v+w} c''_\chi(s, \rho_i, l_i) \right).$$

The statement of Lemma 3 in our current notation is that, for fixed $\rho_1, \dots, \rho_{v+w}$, with $0 \leq \rho_i \leq a_i$ for each i , $i = 1, \dots, v+w$, and for some particular $i = i_0$ with $1 \leq i_0 \leq v+w$, we have

$$b \left(s, \prod_{i=1}^{v+w} l_i^{\rho_i} \right) = \alpha(s, \rho_{i_0}, l_{i_0}) b \left(s, l_{i_0}^{a_{i_0}} \cdot \prod_{\substack{i \neq i_0 \\ i=1}}^{v+w} l_i^{\rho_i} \right)$$

(and this is clearly also true if $l_{i_0} = 1$). Repeated applications of this equality transform (3) into

$$\sum_{\rho_1=0}^{a_1} \cdots \sum_{\rho_{v+w}=0}^{a_{v+w}} \left(\alpha(s, \rho_1, l_1) \cdots \alpha(s, \rho_{v+w}, l_{v+w}) \right. \\ \left. \cdot b \left(s, \underbrace{l_1^{a_1} \cdots l_{v+w}^{a_{v+w}}}_{=t} \prod_{i=v+1}^{v+w} c''_\chi(s, \rho_i, l_i) \right) \right).$$

Noting that $b(s, t) = b(s)$, this last expression equals

$$b(s) \sum_{\rho_1=0}^{a_1} \cdots \sum_{\rho_{v+w}=0}^{a_{v+w}} \left(\prod_{i=1}^{v+w} \alpha(s, \rho_i, l_i) \cdot \prod_{i=v+1}^{v+w} c''_\chi(s, \rho_i, l_i) \right).$$

Again for convenience let $c''_\chi(s, \rho_i, l_i) = 1$ for $i = 1, \dots, v$; the above becomes

$$b(s) \sum_{\rho_1=0}^{a_1} \cdots \sum_{\rho_{v+w}=0}^{a_{v+w}} \left(\prod_{i=1}^{v+w} \alpha(s, \rho_i, l_i) c''_\chi(s, \rho_i, l_i) \right) \\ = b(s) \prod_{i=1}^{v+w} \left(\sum_{\rho=0}^{a_i} \alpha(s, \rho, l_i) c''_\chi(s, \rho, l_i) \right) \\ = b(s) \prod_{i=1}^v \left(\sum_{\rho=0}^{a_i} \alpha(s, \rho, l_i) \right) \cdot \prod_{i=v+1}^{v+w} \left(\sum_{\rho=0}^{a_i} \alpha(s, \rho, l_i) c''_\chi(s, \rho, l_i) \right).$$

All that is needed to prove Theorem 2 is to show two things. First, that $\sum_{\rho=0}^{a_i} \alpha(s, \rho, l_i) = \gamma(s, l_i)$ for each i , $i = 1, \dots, v$, in the case there are $v > 0$ primes l satisfying $l | t$, $l \nmid N$; if there are no such primes then we arranged things so that $\prod_{i=1}^v (\sum_{\rho=0}^{a_i} \alpha(s, \rho, l_i)) = \alpha(s, 0, 1) = 1$ which agrees with the “empty” product $\prod_{l|t, l \nmid N} \alpha(l)$. Second, we need to show that

$$\sum_{\rho=0}^{a_i} \alpha(s, \rho, l_i) c''_\chi(s, \rho, l_i) = c(s, l_i)$$

for each i , $i = v + 1, \dots, v + w$, in the case that there are $w > 0$ primes l dividing N ; if $N = 1$ we arranged for $\prod_{i=v+1}^{v+w} \sum_{\rho=0}^{a_i} \alpha(s, \rho, l_i) c''_{\chi}(s, \rho, l_i) = \alpha(s, 0, 1) c''_{\chi}(s, 0, 1) = 1 \cdot 1 = 1$ which agrees with the “empty” product $\prod_{l|N} c(s, l)$.

Suppose then that l is a prime, $l | t$, $l \nmid N$. Then $l = l_i$ for some i with $1 \leq i \leq v$. Let $a = \text{ord}_l(t)$; we show $\sum_{\rho=0}^a \alpha(s, \rho, l) = \gamma(s, l)$.

Suppose l is a case A or D prime. We have

$$\sum_{\rho=0}^a \alpha(s, \rho, l) = \sum_{\rho=0}^a (l^{a-\rho} - l^{a-\rho-1}) = l^a - l^{-1} = l^a = \gamma(s, l).$$

If $\text{Case}(l) = B$ or E ,

$$\begin{aligned} \sum_{\rho=0}^a \alpha(s, \rho, l) &= \sum_{\rho=0}^a l^{a-\rho} + \sum_{\rho=0}^a l^{a-\rho-1} \\ &= (l^{a+1} - 1)/(l - 1) + (l^a - 1)/(l - 1) = \gamma(s, l). \end{aligned}$$

If $\text{Case}(l) = C$ or F ,

$$\sum_{\rho=0}^a \alpha(s, \rho, l) = \sum_{\rho=0}^a l^{a-\rho} = (l^{a+1} - 1)/(l - 1) = \gamma(s, l).$$

So all that remains is to explicitly evaluate $\sum_{\rho=0}^{a_i} \alpha(s, \rho, l_i) c''_{\chi}(s, \rho, l_i)$ for each prime $l_i | N$, and show the result is $c(s, l_i)$. Fix $l | N$ and write $\alpha(\rho)$ for $\alpha(s, \rho, l)$ and $c(\rho)$ for $c''_{\chi}(s, \rho, l)$. Now, the task of evaluating $\sum_{\rho=0}^a \alpha(\rho) c(\rho)$ and showing that it equals $c(s, l)$ as given by Table 1 is a long straightforward one, but extremely tedious. We give some details concerning the explicit calculation/evaluation of $\sum_{\rho=0}^a \alpha(\rho) c(\rho)$ for $\text{Case}(l) = A$ and $\text{ord}_l(N)$ even, leaving all other calculations (i.e., those for $\text{Case}(l) = B, \dots, F$, $\text{ord}_l(N)$ even or odd, and $\text{Case}(l) = A$ with $\text{ord}_l(N)$ odd) to the reader. First, let us summarize all the calculations here:

$$\text{Explicit Value of } c(s, l) = \sum_{\rho=0}^a \alpha(\rho) c(\rho)$$

		$\sum_{\rho=0}^a \alpha(\rho) c(\rho)$
condition 1	condition 2	for $\text{Case}(l) = A$, $\nu = \text{ord}_l(N)$ even
$a \leq \mu - 1$	$e \leq \nu - a$	$\chi_l^* l^{2a}$
	$e > \nu - a$	$\chi_l^* l^{\nu+a-e}$
$a = \mu$	$e \leq a$	$\chi_l^* l^{\nu-1} + \chi_l(\frac{\nu}{2})(l-1)l^{\nu-1}$
	$e > a$	$\chi_l^* l^{\nu+a-e}$
$a > \mu$	$e \leq \mu$	$\chi_l^* l^{\nu-1} + \chi_l(\frac{\nu}{2})(l-1)l^{\nu-1}((l+1)(l^{a-\mu}-1)/(l-1)+1)$
	$\mu < e \leq a-1$	$\chi_l^* l^{\nu-1} + \chi_l(\frac{\nu}{2})(l-1)l^{\nu-1} \cdot ((l+1)(l^{a-e}-1)/(l-1)+l^{a-e}+1)$
	$e = a$	$\chi_l^* l^{\nu-1} + \chi_l(\frac{\nu}{2})(l-1)l^{\nu-1}(l^{a-e}+1)$
	$e > a$	$\chi_l^* l^{\nu+a-e}$
condition 1	condition 2	$\text{Case}(l) = A$, $\nu = \text{ord}_l(N)$ odd
$a \leq \mu$	$e \leq \nu - a$	$\chi_l^* l^{2a}$
	$e > \nu - a$	$\chi_l^* l^{\nu+a-e}$

$a > \mu$	$e \leq \mu + 1$	$\chi_l^* l^{\nu-1} + \chi_l(\frac{s}{2}) l^{\nu-1} 2(l^{a-\mu} - 1)$
	$\mu + 1 < e \leq a$	$\chi_l^* l^{\nu-1} + \chi_l(\frac{s}{2}) l^{\nu-1} 2(l^{a-e+1} - 1)$
	$e = a + 1$	$\chi_l^* l^{\nu-1} \quad (= \chi_l^* l^{\nu+a-e})$
	$e > a + 1$	$\chi_l^* l^{\nu+a-e}$
condition 1	condition 2	Case(l) = B , $\nu = \text{ord}_l(N)$ even
$a \leq \mu - 1$	(none)	0
$a = \mu$	$e \leq a$	$\chi_l(\frac{s}{2})(l+1)l^{\nu-1}$
	$e > a$	0
$a > \mu$	$e \leq \mu$	$\chi_l(\frac{s}{2})(l+1)l^{\nu-1}((l+1)(l^{a-\mu} - 1)/(l-1) + 1)$
	$\mu < e \leq a - 1$	$\chi_l(\frac{s}{2})(l+1)l^{\nu-1}((l+1)(l^{a-e} - 1)/(l-1) + l^{a-e} + 1)$
	$e = a$	$\chi_l(\frac{s}{2})(l+1)l^{\nu-1}(l^{a-e} + 1)$
	$e > a$	0
condition 1	condition 2	Case(l) = B , $\nu = \text{ord}_l(N)$ odd
$a \leq \mu$	(none)	0
$a > \mu$	$e \leq \mu + 1$	$\chi_l(\frac{s}{2})(l+1)l^{\nu-1} 2(l^{a-\mu} - 1)/(l-1)$
	$\mu + 1 < e \leq a$	$\chi_l(\frac{s}{2})(l+1)l^{\nu-1} 2(l^{a-e+1} - 1)/(l-1)$
	$e > a$	0
condition 1	condition 2	Case(l) = C , $\nu = \text{ord}_l(N)$ even
$a \leq \mu - 1$	(none)	0
$a \geq \mu$	$e \leq \mu$	$\chi_l(\frac{s}{2})l^{\nu-1}(l+1)(l^{a-\mu+1} - 1)/(l-1)$
	$\mu < e \leq a$	$\chi_l(\frac{s}{2})l^{\nu-1}((l+1)(l^{a-e+1} - 1)/(l-1) + l^{a-e+1})$
	$e = a + 1$	$\chi_l(\frac{s}{2})l^{\nu-1}l^{a-e+1}$
	$e > a + 1$	0
condition 1	condition 2	Case(l) = C , $\nu = \text{ord}_l(N)$ odd
$a \leq \mu - 1$	(none)	0
$a = \mu$	$e \leq a + 1$	$\chi_l(\frac{s}{2})l^{\nu-1}$
	$e > a + 1$	0
$a \geq \mu + 1$	$e \leq \mu + 1$	$\chi_l(\frac{s}{2})l^{\nu-1}(2l(l^{a-\mu} - 1)/(l-1) + 1)$
	$\mu + 1 < e \leq a$	$\chi_l(\frac{s}{2})l^{\nu-1}(2l(l^{a-e+1} - 1)/(l-1) + 1)$
	$e = a + 1$	$\chi_l(\frac{s}{2})l^{\nu-1}$
	$e > a + 1$	0
condition 1	condition 2	Case(l) = D , $\nu = \text{ord}_l(N)$ even
$a \leq \mu - 1$	$e \leq \nu - a$	$\chi_l^* l^{2a}$
	$e > \nu - a$	$\chi_l^* l^{\nu+a-e}$
$a = \mu$	$e \leq a$	$\chi_l^* l^{\nu-1} + \hat{\chi}_l l^{\nu-1}$
	$e > a$	$\chi_l^* l^{\nu+a-e}$
$a = \mu + 1$	$e \leq a - 1$	$\chi_l^* l^{\nu-1} + \hat{\chi}_l 2l^{\nu-1} + \chi_l(\frac{s}{2})l^{\nu}$
	$e = a$	$\chi_l^* l^{\nu-1} + \hat{\chi}_l 2l^{\nu-1}$
	$e > a$	$\chi_l^* l^{\nu+a-e}$
$a > \mu + 1$	$e \leq \mu$	$\chi_l^* l^{\nu-1} + 2\hat{\chi}_l l^{\nu-1} + \chi_l(\frac{s}{2})l^{\nu}((l+1)(l^{a-\mu-1} - 1) + 1)$
	$\mu < e \leq a - 2$	$\chi_l^* l^{\nu-1} + 2\hat{\chi}_l l^{\nu-1}$
		$+ \chi_l(\frac{s}{2})l^{\nu}((l+1)(l^{a-e-1} - 1) + l^{a-e-1} + 1)$
	$e = a - 1$	$\chi_l^* l^{\nu-1} + 2\hat{\chi}_l l^{\nu-1} + \chi_l(\frac{s}{2})l^{\nu}(l^{a-e-1} + 1)$
	$e = a$	$\chi_l^* l^{\nu-1} + 2\hat{\chi}_l l^{\nu-1}$
	$e > a$	$\chi_l^* l^{\nu+a-e}$
condition 1	condition 2	Case(l) = D , $\nu = \text{ord}_l(N)$ odd
$a \leq \mu$	$e \leq \nu - a$	$\chi_l^* l^{2a}$
	$e > \nu - a$	$\chi_l^* l^{\nu+a-e}$

$a = \mu + 1$	$e \leq a$	$\chi_l^* l^{\nu-1} + \hat{\chi}_l 2l^{\nu-1}$
	$e > a$	$\chi_l^* l^{\nu+a-e}$
$a > \mu + 1$	$e \leq \mu + 1$	$\chi_l^* l^{\nu-1} + \hat{\chi}_l 2l^{\nu-1} + \chi_l(\frac{s}{2})l^\nu 2(l^{a-\mu-1} - 1)$
	$\mu + 1 < e < a$	$\chi_l^* l^{\nu-1} + \hat{\chi}_l 2l^{\nu-1} + \chi_l(\frac{s}{2})l^\nu 2(l^{a-(e-1)-1} - 1)$
	$e = a$	$\chi_l^* l^{\nu-1} + \hat{\chi}_l 2l^{\nu-1}$
	$e > a$	$\chi_l^* l^{\nu+a-e}$
condition 1	condition 2	Case(l) = E , $\nu = \text{ord}_l(N)$ even
$a \leq \mu - 1$	(none)	0
$a = \mu$	$e \leq a$	$\hat{\chi}_l(l+1)l^{\nu-1}$
	$e > a$	0
$a = \mu + 1$	$e \leq a - 1$	$\hat{\chi}_l 2(l+1)l^{\nu-1} + \chi_l(\frac{s}{2})(l+1)l^\nu$
	$e = a$	$\hat{\chi}_l 2(l+1)l^{\nu-1}$
	$e > a$	0
$a > \mu + 1$	$e \leq \mu$	$\hat{\chi}_l 2(l+1)l^{\nu-1} + \chi_l(\frac{s}{2})(l+1)l^\nu((l+1)(l^{a-\mu-1} - 1) + 1)$
	$\mu < e \leq a - 2$	$\hat{\chi}_l 2(l+1)l^{\nu-1} + \chi_l(\frac{s}{2})(l+1)l^\nu$
		$\cdot ((l+1)(l^{a-e-1} - 1) + l^{a-e-1} + 1)$
	$e = a - 1$	$\hat{\chi}_l 2(l+1)l^{\nu-1} + \chi_l(\frac{s}{2})(l+1)l^\nu(l^{a-e-1} + 1)$
	$e = a$	$\hat{\chi}_l 2(l+1)l^{\nu-1}$
	$e > a$	0
condition 1	condition 2	Case(l) = E , $\nu = \text{ord}_l(N)$ odd
$a \leq \mu$	(none)	0
$a = \mu + 1$	$e \leq a$	$\hat{\chi}_l 2(l+1)l^{\nu-1}$
	$e > a$	0
$a > \mu + 1$	$e \leq \mu + 1$	$\hat{\chi}_l 2(l+1)l^{\nu-1} + \chi_l(\frac{s}{2})(l+1)l^\nu 2(l^{a-\mu-1} - 1)$
	$\mu + 1 < e < a$	$\hat{\chi}_l 2(l+1)l^{\nu-1} + \chi_l(\frac{s}{2})(l+1)l^\nu 2(l^{a-(e-1)-1} - 1)$
	$e = a$	$\hat{\chi}_l 2(l+1)l^{\nu-1}$
	$e > a$	0
condition 1	condition 2	Case(l) = F , $\nu = \text{ord}_l(N)$ even
$a \leq \mu - 1$	(none)	0
$a = \mu$	$e \leq a$	$\hat{\chi}_l l^{\nu-1} + \chi_l(\frac{s}{2})l^\nu$
	$e = a + 1$	$\hat{\chi}_l l^{\nu-1}$
	$e > a + 1$	0
$a \geq \mu + 1$	$e \leq \mu$	$\hat{\chi}_l l^{\nu-1} + \chi_l(\frac{s}{2})l^\nu((l+1)(l^{a-\mu} - 1) + 1)$
	$\mu < e \leq a - 1$	$\hat{\chi}_l l^{\nu-1} + \chi_l(\frac{s}{2})l^\nu((l+1)(l^{a-e} - 1) + l^{a-e} + 1)$
	$e = a$	$\hat{\chi}_l l^{\nu-1} + \chi_l(\frac{s}{2})l^\nu(l^{a-e} + 1)$
	$e = a + 1$	$\hat{\chi}_l l^{\nu-1}$
	$e > a + 1$	0
condition 1	condition 2	Case(l) = F , $\nu = \text{ord}_l(N)$ odd
$a \leq \mu - 1$	(none)	0
$a = \mu$	$e \leq a + 1$	$\hat{\chi}_l l^{\nu-1}$
	$e > a + 1$	0
$a \geq \mu + 1$	$e \leq \mu + 1$	$\hat{\chi}_l l^{\nu-1} + \chi_l(\frac{s}{2})l^\nu 2(l^{a-\mu} - 1)$
	$\mu + 1 < e \leq a$	$\hat{\chi}_l l^{\nu-1} + \chi_l(\frac{s}{2})l^\nu 2(l^{a-e+1} - 1)$
	$e = a + 1$	$\hat{\chi}_l l^{\nu-1}$
	$e > a + 1$	0

In all of what follows, by the “SUM”, we mean $\sum_{\rho=0}^a \alpha(\rho)c(\rho)$. The key to its explicit evaluation is

Lemma 4. *Let p be a prime and let ω be a character modulo some power of p . Let $e = e(\omega)$ be the exponential conductor of ω . If σ and b are nonnegative*

integers with $\sigma + b \geq e$ and $2b \geq e$ and if u is a unit mod p , then

$$\sum_{z \in \mathbf{Z}/p^e \mathbf{Z}} \omega(u + zp^b) = \begin{cases} p^\sigma \omega(u) & \text{if } e \leq b, \\ 0 & \text{if } e > b. \end{cases}$$

Proof. This is easy. See Lemma 2.1 of [H-P-S₁] or [H-P-S₂].

Let A_ρ and B'_ρ be as in Theorem 1. Specific sets of representatives for A_ρ and B'_ρ are calculated by the authors of [H-P-S₁, H-P-S₂] in their Lemma 2.5 by “easy but tedious calculations”; we copy them here for reference as

Lemma 5. Let $A(s, f, l) = A_\rho$ and $B'(s, f, l) = B'_\rho$ be the sets appearing in Theorem 1. For fixed N , n , s , and l , A_ρ and B'_ρ depend only on $\rho = \text{ord}_l(f)$. Let $\nu = \text{ord}_l(N)$ and set $\nu = 2\mu$ or $\nu = 2\mu + 1$. Classify l as case A , B , etc., setting a and d according to how l is classified. Then the sets A_ρ and B'_ρ are as follows:

Case(l)	ν	condition	
A	odd	$a - \rho \leq \mu$	$A_\rho = \left\{ \frac{s \pm l^a d}{2} + z l^{2\mu+2\rho-a+1} \mid z \in \mathbf{Z}/l^{a-\rho} \mathbf{Z} \right\}$ $B'_\rho = \left\{ \frac{s \pm l^a d}{2} + z l^{2\mu+2\rho-a+2} \mid z \in \mathbf{Z}/l^{a-\rho-1} \mathbf{Z} \right\}^\dagger$
		$a - \rho \geq \mu + 1$	$A_\rho = B'_\rho = \left\{ \frac{s}{2} + z l^{\mu+\rho+1} \mid z \in \mathbf{Z}/l^\mu \mathbf{Z} \right\}$
	even	$a - \rho \leq \mu - 1$	$A_\rho = \left\{ \frac{s \pm l^a d}{2} + z l^{2\mu+2\rho-a} \mid z \in \mathbf{Z}/l^{a-\rho} \mathbf{Z} \right\}$ $B'_\rho = \left\{ \frac{s \pm l^a d}{2} + z l^{2\mu+2\rho-a+1} \mid z \in \mathbf{Z}/l^{a-\rho-1} \mathbf{Z} \right\}^\dagger$
		$a - \rho = \mu$	$A_\rho = \left\{ \frac{s}{2} + z l^a \mid z \in \mathbf{Z}/l^\mu \mathbf{Z} \right\}$ $B'_\rho = \left\{ \frac{s \pm l^a d}{2} + z l^{a+1} \mid z \in \mathbf{Z}/l^{\mu-1} \mathbf{Z} \right\}$
		$a - \rho \geq \mu + 1$	$A_\rho = \left\{ \frac{s}{2} + z l^{\mu+\rho} \mid z \in \mathbf{Z}/l^\mu \mathbf{Z} \right\}$ $B'_\rho = \left\{ \frac{s}{2} + z l^{\mu+\rho+1} \mid z \in \mathbf{Z}/l^{\mu-1} \mathbf{Z} \right\}$
	B	odd	$a - \rho \leq \mu$
		$a - \rho \geq \mu + 1$	$A_\rho = B'_\rho = \left\{ \frac{s}{2} + z l^{\mu+\rho+1} \mid z \in \mathbf{Z}/l^\mu \mathbf{Z} \right\}$
even		$a - \rho \leq \mu - 1$	$A_\rho = B'_\rho = \emptyset$
		$a - \rho = \mu$	$A_\rho = \left\{ \frac{s}{2} + z l^a \mid z \in \mathbf{Z}/l^\mu \mathbf{Z} \right\}$ $B'_\rho = \emptyset$
		$a - \rho \geq \mu + 1$	$A_\rho = \left\{ \frac{s}{2} + z l^{\mu+\rho} \mid z \in \mathbf{Z}/l^\mu \mathbf{Z} \right\}$ $B'_\rho = \left\{ \frac{s}{2} + z l^{\mu+\rho+1} \mid z \in \mathbf{Z}/l^{\mu-1} \mathbf{Z} \right\}$
C	odd	$a - \rho \leq \mu - 1$	$A_\rho = B'_\rho = \emptyset$
		$a - \rho = \mu$	$A_\rho = \left\{ \frac{s}{2} + z l^{a+1} \mid z \in \mathbf{Z}/l^\mu \mathbf{Z} \right\}$ $B'_\rho = \emptyset$
		$a - \rho \geq \mu + 1$	$A_\rho = B'_\rho = \left\{ \frac{s}{2} + z l^{\mu+\rho+1} \mid z \in \mathbf{Z}/l^\mu \mathbf{Z} \right\}$
	even	$a - \rho \leq \mu - 1$	$A_\rho = B'_\rho = \emptyset$
		$a - \rho \geq \mu$	$A_\rho = \left\{ \frac{s}{2} + z l^{\mu+\rho} \mid z \in \mathbf{Z}/l^\mu \mathbf{Z} \right\}$ $B'_\rho = \left\{ \frac{s}{2} + z l^{\mu+\rho+1} \mid z \in \mathbf{Z}/l^{\mu-1} \mathbf{Z} \right\}$
D	odd	$a - \rho \leq \mu$	A_ρ and B'_ρ are the same as for a case A prime
		$a - \rho = \mu + 1$	$A_\rho = B'_\rho = \left\{ \frac{s \pm l^a d}{2} + z l^a \mid z \in \mathbf{Z}/l^\mu \mathbf{Z} \right\}$
		$a - \rho \geq \mu + 2$	A_ρ and B'_ρ are the same as for a case A prime

E	even	$a - \rho \leq \mu - 1$	A_ρ and B'_ρ are the same as for a case A prime
		$a - \rho = \mu$	$A_\rho = \left\{ \frac{s+l^a d}{2} + z l^a \mid z \in \mathbf{Z}/l^\mu \mathbf{Z} \right\}$ $B'_\rho = \left\{ \frac{s \pm l^a d}{2} + z l^{a+1} \mid z \in \mathbf{Z}/l^{\mu-1} \mathbf{Z} \right\}$
		$a - \rho = \mu + 1$	$A_\rho = \left\{ \frac{s}{2} + z l^{a-1} \mid z \in \mathbf{Z}/l^\mu \mathbf{Z} \right\}$ $B'_\rho = \left\{ \frac{s+l^a d}{2} + z l^a \mid z \in \mathbf{Z}/l^{\mu-1} \mathbf{Z} \right\}$
		$a - \rho \geq \mu + 2$	A_ρ and B'_ρ are the same as for a case A prime
	odd	$a - \rho \leq \mu$	$A_\rho = B'_\rho = \emptyset$
		$a - \rho \geq \mu + 1$	A_ρ and B'_ρ are the same as for a case D prime, with d set to 1
	even	$a - \rho \leq \mu - 1$	$A_\rho = B'_\rho = \emptyset$
		$a - \rho = \mu$	$A_\rho = \left\{ \frac{s+l^a}{2} + z l^a \mid z \in \mathbf{Z}/l^\mu \mathbf{Z} \right\}$ $B'_\rho = \emptyset$
		$a - \rho \geq \mu + 1$	A_ρ and B'_ρ are the same as for a case D prime, with d set to 1
	odd	$a - \rho \leq \mu - 1$	$A_\rho = B'_\rho = \emptyset$
		$a - \rho = \mu$	$A_\rho = \left\{ \frac{s+l^{a+1}}{2} + z l^{a+1} \mid z \in \mathbf{Z}/l^\mu \mathbf{Z} \right\}$ $B'_\rho = \emptyset$
		$a - \rho \geq \mu + 1$	A_ρ and B'_ρ are the same as for a case A prime
F	even	$a - \rho \leq \mu - 1$	$A_\rho = B'_\rho = \emptyset$
		$a - \rho = \mu$	$A_\rho = \left\{ \frac{s}{2} + z l^a \mid z \in \mathbf{Z}/l^\mu \mathbf{Z} \right\}$ $B'_\rho = \left\{ \frac{s+l^{a+1}}{2} + z l^{a+1} \mid z \in \mathbf{Z}/l^{\mu-1} \mathbf{Z} \right\}$
	odd	$a - \rho \geq \mu + 1$	A_ρ and B'_ρ are the same as for a case A prime

†If $\rho = a$ then set $B'_\rho = \emptyset$.

We can now make one more observation: With s fixed and $l \mid N$, let $f \mid t$ and $\rho = \text{ord}_l(f)$ for some $\rho \geq 0$. Refer to the definition of $c'_\chi(s, f, l)$ as in Theorem 1. We can actually write

$$c''_\chi(s, \rho, l) = c'_\chi(s, f, l) \quad \text{as} \quad \sum_{x \in A_\rho} \chi_l(x) + \sum_{y \in B'_\rho} \chi_l(y).$$

For suppose $(s^2 - 4n)/(l^\rho)^2 \not\equiv 0 \pmod{l}$. Write $s^2 - 4n$ as t^2 , $t^2 m$, or $t^2 4m$ as in Theorem 1 and write $t = l^a t_0$, where $(l, t_0) = 1$; recall this $a = a_l(s^2 - 4n)$. Clearly if $(s^2 - 4n)/(l^\rho)^2 \not\equiv 0 \pmod{l}$ then $t^2/l^{2\rho} = l^{2a-2\rho} t_0^2 \not\equiv 0 \pmod{l}$, which implies $\rho = a$. However, in every case in which $\rho = a$, Lemma 5 shows that the set B'_ρ is empty; consequently $\sum_{y \in B'_\rho} \chi_l(y) = 0$. Let us write " A_ρ sum" and " B'_ρ sum" for $\sum_{x \in A_\rho} \chi_l(x)$ and $\sum_{y \in B'_\rho} \chi_l(y)$, respectively, so that $c''_\chi(s, \rho, l) = A_\rho \text{sum} + B'_\rho \text{sum}$. Finally we are ready to begin evaluating $\sum_{\rho=0}^a \alpha(\rho) c(\rho)$. Suppose $\text{Case}(l) = A$ and $\nu = \text{ord}_l(N)$ is even; write $\nu = 2\mu$. Set $a = a_l(s^2 - 4n)$ and $e = e(\chi_l)$. If $a - \rho \leq \mu - 1$ then by applying Lemma 4 twice on each of the sets A_ρ and B'_ρ as given in Lemma 5, we have

$$A_\rho \text{sum} = \begin{cases} l^{a-\rho} \chi_l^* & \text{if } e \leq 2\mu + 2\rho - a, \\ 0 & \text{if } e > 2\mu + 2\rho - a \end{cases}$$

and

$$B'_\rho \text{sum} = \begin{cases} l^{a-\rho-1} \chi_l^* & \text{if } e \leq 2\mu + 2\rho - a + 1, \\ 0 & \text{if } e > 2\mu + 2\rho - a + 1. \end{cases}$$

Notice that if $\rho = a$ then $B'_\rho = \emptyset$; in this case the above formula gives $l^{-1} \chi_l^*$ which is 0 by our convention. Adding $A_\rho \text{sum}$ and $B'_\rho \text{sum}$, we have for $a - \rho \leq \mu - 1$

$$(4) \quad c(\rho) = \begin{cases} (l^{a-\rho} + l^{a-\rho-1}) \chi_l^* & \text{if } e \leq 2\mu + 2\rho - a, \\ l^{a-\rho-1} \chi_l^* & \text{if } e = 2\mu + 2\rho - a + 1, \\ 0 & \text{if } e > 2\mu + 2\rho - a + 1. \end{cases}$$

If $a - \rho = \mu$, then by Lemmas 4 and 5 we have

$$A_\rho \text{sum} = \begin{cases} l^\mu \chi_l(\frac{s}{2}) & \text{if } e \leq a, \\ 0 & \text{if } e > a \end{cases}$$

and

$$B'_\rho \text{sum} = \begin{cases} l^{\mu-1} \chi_l^* & \text{if } e \leq a + 1, \\ 0 & \text{if } e > a + 1. \end{cases}$$

Adding $A_\rho \text{sum}$ and $B'_\rho \text{sum}$, we have for $a - \rho = \mu$

$$(5) \quad c(\rho) = \begin{cases} l^{\mu-1} \chi_l^* + l^\mu \chi_l(\frac{s}{2}) & \text{if } e \leq a, \\ l^{\mu-1} \chi_l^* & \text{if } e = a + 1, \\ 0 & \text{if } e > a + 1. \end{cases}$$

Now suppose $a - \rho \geq \mu + 1$. By Lemmas 4 and 5 we have

$$A_\rho \text{sum} = \begin{cases} l^\mu \chi_l(\frac{s}{2}) & \text{if } e \leq \mu + \rho, \\ 0 & \text{if } e > \mu + \rho \end{cases}$$

and

$$B'_\rho \text{sum} = \begin{cases} l^{\mu-1} \chi_l(\frac{s}{2}) & \text{if } e \leq \mu + \rho + 1, \\ 0 & \text{if } e > \mu + \rho + 1. \end{cases}$$

Adding $A_\rho \text{sum}$ and $B'_\rho \text{sum}$ we have for $a - \rho \geq \mu + 1$

$$(6) \quad c(\rho) = \begin{cases} (l^\mu + l^{\mu-1}) \chi_l(\frac{s}{2}) & \text{if } e \leq \mu + \rho, \\ l^{\mu-1} \chi_l(\frac{s}{2}) & \text{if } e = \mu + \rho + 1, \\ 0 & \text{if } e > \mu + \rho + 1. \end{cases}$$

Now we are ready to calculate the $\text{SUM} = \sum_{\rho=0}^a \alpha(\rho) c(\rho)$ under the various possibilities.

Suppose $a \leq \mu - 1$. Then for $\rho = 0, \dots, a$ we have $a - \rho \leq \mu - 1$ and therefore $c(\rho)$ is given by (4).

Suppose $e \leq 2\mu - a (= \nu - a)$. Then for $\rho = 0, \dots, a$ we have $e \leq 2\mu + 2\rho - a$; by (4) we have

$$\begin{aligned} \sum_{\rho=0}^a \alpha(\rho) c(\rho) &= \sum_{\rho=0}^a (l^{a-\rho} - l^{a-\rho-1}) (l^{a-\rho} + l^{a-\rho-1}) \chi_l^* \\ &= \chi_l^* \sum_{\rho=0}^a (l^{2a-2\rho} - l^{2a-2\rho-2}) = \chi_l^* (l^{2a} - l^{-2}) = \chi_l^* l^{2a}. \end{aligned}$$

Suppose $e > 2\mu - a$. Then either $e = 2\mu + 2\rho_1 - a$ for some $\rho_1 > 0$, or $e = 2\mu + 2\rho_1 - a + 1$ for some $\rho_1 \geq 0$; we show the SUM is $\chi_l^* l^{\nu+a-e}$ in either case. Suppose then that $e = 2\mu + 2\rho_1 - a$ for some $\rho_1 > 0$; note $\rho_1 \leq a$. If ρ satisfies $\rho_1 > \rho$ then $2\rho_1 > 2\rho + 1$ implies $e = 2\mu + 2\rho_1 - a > 2\mu + 2\rho - a + 1$; if $\rho_1 \leq \rho$ then $e = 2\mu + 2\rho_1 - a \leq 2\mu + 2\rho - a$, so that by (4)

$$c(\rho) = \begin{cases} (l^{a-\rho} + l^{a-\rho-1})\chi_l^* & \text{if } \rho \geq \rho_1, \\ 0 & \text{if } \rho < \rho_1, \end{cases}$$

and the SUM becomes

$$\begin{aligned} & \sum_{\rho=0}^{\rho_1-1} \alpha(\rho)c(\rho) + \sum_{\rho=\rho_1}^a \alpha(\rho)c(\rho) \\ &= 0 + \sum_{\rho=\rho_1}^a (l^{a-\rho} - l^{a-\rho-1})(l^{a-\rho} + l^{a-\rho-1})\chi_l^* \\ &= \chi_l^* \sum_{\rho=\rho_1}^a (l^{2a-2\rho} - l^{2a-2\rho-2}) = \chi_l^* l^{2a-2\rho_1} = \chi_l^* l^{\nu+a-e}. \end{aligned}$$

Suppose $e = 2\mu + 2\rho_1 - a + 1$ for some $\rho_1 \geq 0$; note $\rho_1 + 1 \leq a$. If ρ satisfies $\rho_1 > \rho$ then $e = 2\mu + 2\rho_1 - a + 1 > 2\mu + 2\rho - a + 1$; if $\rho_1 < \rho$ then $e < 2\mu + 2\rho - a + 1$, i.e., $e \leq 2\mu + 2\rho - a$. We have

$$c(\rho) = \begin{cases} (l^{a-\rho} + l^{a-\rho-1})\chi_l^* & \text{if } \rho > \rho_1, \\ l^{a-\rho_1-1}\chi_l^* & \text{if } \rho = \rho_1, \\ 0 & \text{if } \rho < \rho_1, \end{cases}$$

so that the SUM equals

$$\begin{aligned} & \sum_{\rho=0}^{\rho_1-1} \alpha(\rho)0 + \alpha(\rho_1)l^{a-\rho_1-1}\chi_l^* + \sum_{\rho=\rho_1+1}^a \alpha(\rho)(l^{a-\rho} + l^{a-\rho-1})\chi_l^* \\ &= 0 + (l^{a-\rho_1} - l^{a-\rho_1-1})l^{a-\rho_1-1}\chi_l^* \\ & \quad + \sum_{\rho=\rho_1+1}^a (l^{2a-2\rho} - l^{2a-2\rho-2})\chi_l^* \\ &= \chi_l^* (l^{2a-2\rho_1-1} - l^{2a-2\rho_1-2} + l^{2a-2\rho_1-2}) \\ &= \chi_l^* l^{2a-2\rho_1-1} = \chi_l^* l^{\nu+a-e}. \end{aligned}$$

Suppose $a \geq \mu$; then we can write $a - \rho_0 = \mu$ for some $\rho_0 \geq 0$. Write the SUM as

$$(7) \quad \sum_{\rho=0}^{\rho_0-1} \alpha(\rho)c(\rho) + \alpha(\rho_0)c(\rho_0) + \sum_{\rho=\rho_0+1}^a \alpha(\rho)c(\rho),$$

where we take $\sum_{\rho=0}^{\rho_0-1} \alpha(\rho)c(\rho) = 0$ in the case $\rho_0 = 0$. Suppose $\rho_0 > 0$ and consider $\sum_{\rho=0}^{\rho_0-1} \alpha(\rho)c(\rho)$. If $\rho \leq \rho_0 - 1$ then $a - \rho \geq a - (\rho_0 - 1) = a - \rho_0 + 1 = \mu + 1$ so that $c(\rho)$ is determined using (6).

Suppose $e \leq \mu$; then $e \leq \mu + \rho$ for $\rho = 0, \dots, \rho_0 - 1$. From (6),

$$\begin{aligned} \sum_{\rho=0}^{\rho_0-1} \alpha(\rho) c(\rho) &= \sum_{\rho=0}^{\rho_0-1} (l^{a-\rho} - l^{a-\rho-1}) (l^\mu + l^{\mu-1}) \chi_l\left(\frac{s}{2}\right) \\ &= \chi_l\left(\frac{s}{2}\right) (l^\mu + l^{\mu-1}) \sum_{\rho=0}^{\rho_0-1} (l^{a-\rho} - l^{a-\rho-1}) \\ &= \chi_l\left(\frac{s}{2}\right) l^{\mu-1} (l+1) (l^a - l^{a-\rho_0}) = \chi_l\left(\frac{s}{2}\right) l^{\mu-1} (l+1) l^\mu (l^{a-\mu} - 1) \\ &= \chi_l\left(\frac{s}{2}\right) l^{\nu-1} (l+1) (l^{a-\mu} - 1). \end{aligned}$$

Suppose $e > \mu$; then for some $\rho_1 \geq 0$ we have $e = \mu + \rho_1 + 1$. Suppose $\rho \leq \rho_0 - 1$. Then $e = \mu + \rho_1 + 1 \leq \mu + \rho$ iff $\rho_1 + 1 \leq \rho$; by (6) we have

$$(8) \quad c(\rho) = \begin{cases} (l^\mu + l^{\mu-1}) \chi_l\left(\frac{s}{2}\right) & \text{if } \rho \geq \rho_1 + 1, \\ l^{\mu-1} \chi_l\left(\frac{s}{2}\right) & \text{if } \rho = \rho_1, \\ 0 & \text{if } \rho < \rho_1. \end{cases}$$

Now, $\rho_1 + 1 \leq \rho_0 - 1$ iff $\mu + \rho_1 + 1 \leq \mu + \rho_0 - 1$ iff $e \leq a - 1$. Suppose this is the case. We have

$$\begin{aligned} \sum_{\rho=0}^{\rho_0-1} \alpha(\rho) c(\rho) &= \sum_{\rho=0}^{\rho_0-1} \alpha(\rho) \cdot 0 + (l^{a-\rho_1} - l^{a-\rho_1-1}) l^{\mu-1} \chi_l\left(\frac{s}{2}\right) \\ &\quad + \sum_{\rho=\rho_1+1}^{\rho_0-1} (l^{a-\rho} - l^{a-\rho-1}) (l^\mu + l^{\mu-1}) \chi_l\left(\frac{s}{2}\right) \\ &= \chi_l\left(\frac{s}{2}\right) ((l^{a-e+\mu+1} - l^{a-e+\mu}) l^{\mu-1} \\ &\quad + (l^\mu + l^{\mu-1}) (l^{a-\rho_1-1} - l^{a-\rho_0})) \\ &= \chi_l\left(\frac{s}{2}\right) (l^{a-e} l^\mu (l-1) l^{\mu-1} + l^{\mu-1} (l+1) (l^{a-e+\mu} - l^\mu)) \\ &= \chi_l\left(\frac{s}{2}\right) (l^{a-e} l^{\nu-1} (l-1) + l^{\nu-1} (l+1) (l^{a-e} - 1)) \\ &= \chi_l\left(\frac{s}{2}\right) (l-1) l^{\nu-1} ((l+1) (l^{a-e} - 1) / (l-1) + l^{a-e}). \end{aligned}$$

Next, $\rho_1 = \rho_0 - 1$ iff $\mu + \rho_1 + 1 = \mu + \rho_0$ iff $e = a$. If this is the case, we have

$$\begin{aligned} \sum_{\rho=0}^{\rho_0-1} \alpha(\rho) c(\rho) &= \sum_{\rho=0}^{\rho_1-1} \alpha(\rho) \cdot 0 + (l^{a-\rho_1} - l^{a-\rho_1-1}) l^{\mu-1} \chi_l\left(\frac{s}{2}\right) \\ &= \chi_l\left(\frac{s}{2}\right) (l-1) l^{\nu-1} l^{a-e}. \end{aligned}$$

Finally, if $e > a$ then $\mu + \rho_1 + 1 > \mu + \rho_0$ so $\rho_1 > \rho_0 - 1$. Thus, $\rho \leq \rho_0 - 1$ implies $\rho \leq \rho_1 - 1$ so that by (8)

$$\sum_{\rho=0}^{\rho_0-1} \alpha(\rho) c(\rho) = \sum_{\rho=0}^{\rho_0-1} \alpha(\rho) \cdot 0 = 0.$$

Now consider $\alpha(\rho_0) c(\rho_0)$. We have $\alpha(\rho_0) = l^{a-\rho_0} - l^{a-\rho_0-1} = l^\mu - l^{\mu-1} =$

$l^{\mu-1}(l-1)$, so that by (5) directly

$$\begin{aligned} \text{if } e \leq a, \quad \alpha(\rho_0)c(\rho_0) &= (l^\mu - l^{\mu-1})l^{\mu-1}\chi_l^* + l^{\mu-1}(l-1)l^\mu\chi_l(\tfrac{s}{2}) \\ &= (l^{\nu-1} - l^{\nu-2})\chi_l^* + l^{\nu-1}(l-1)\chi_l(\tfrac{s}{2}); \\ \text{if } e = a+1, \quad \alpha(\rho_0)c(\rho_0) &= (l^\mu - l^{\mu-1})l^{\mu-1}\chi_l^* = (l^{\nu-1} - l^{\nu-2})\chi_l^*; \\ \text{if } e > a+1, \quad \alpha(\rho_0)c(\rho_0) &= 0. \end{aligned}$$

Consider next $\sum_{\rho=\rho_0+1}^a \alpha(\rho)c(\rho)$. Note that $\rho_0+1 \leq a$, for otherwise $1 > a - \rho_0 = \mu > 0$, a contradiction. If $\rho \geq \rho_0+1$ then $a - \rho \leq a - \rho_0 - 1 = \mu - 1$; hence we use (4) to find $c(\rho)$.

Suppose $e \leq 2\mu + 2(\rho_0+1) - a (= a+2)$; then for $\rho = \rho_0+1, \dots, a$ we have $e \leq 2\mu + 2\rho - a$, so that

$$\begin{aligned} \sum_{\rho=\rho_0+1}^a \alpha(\rho)c(\rho) &= \sum_{\rho=\rho_0+1}^a (l^{a-\rho} - l^{a-\rho-1})(l^{a-\rho} + l^{a-\rho-1})\chi_l^* \\ &= \chi_l^* \sum_{\rho=\rho_0+1}^a (l^{2a-2\rho} - l^{2a-2\rho-2}) = \chi_l^* l^{2a-2\rho_0-2} = \chi_l^* l^{\nu-2}. \end{aligned}$$

Suppose $e > 2\mu + 2(\rho_0+1) - a (= a+2)$. Then either $e = 2\mu + 2\rho_1 - a$ for some $\rho_1 > \rho_0+1$ (note $\rho_1 \leq a$ or a contradiction arises) or $e = 2\mu + 2\rho_1 - a + 1$ for some $\rho_1 \geq \rho_0+1$. In either case, $\sum_{\rho=\rho_0+1}^a \alpha(\rho)c(\rho) = \chi_l^* l^{\nu+a-e}$; the work done to show this is virtually the same as that which showed $\sum_{\rho=0}^a \alpha(\rho)c(\rho) = \chi_l^* l^{\nu+a-e}$ under the conditions $a \leq \mu - 1$ and $e > 2\mu - a$, except that $\sum_{\rho=0}^{\rho_1} \alpha(\rho)c(\rho)$ must be replaced with $\sum_{\rho=\rho_0+1}^{\rho_1-1} \alpha(\rho)c(\rho)$; this has no effect on the outcome, as $c(\rho) = 0$ for each ρ in either of these two sums.

Now then, to write explicit formulas for (7) let us first add $\alpha(\rho_0)c(\rho_0)$ to $\sum_{\rho=\rho_0+1}^a \alpha(\rho)c(\rho)$ and simplify. We have

$$\begin{aligned} \text{if } e \leq a: \quad \chi_l^* (l^{\nu-1} - l^{\nu-2}) + \chi_l(\tfrac{s}{2})l^{\nu-1}(l-1) + \chi_l^* l^{\nu-2} \\ = \chi_l^* l^{\nu-1} + \chi_l(\tfrac{s}{2})l^{\nu-1}(l-1); \\ \text{if } e = a+1: \quad \chi_l^* (l^{\nu-1} - l^{\nu-2}) + \chi_l^* l^{\nu-2} = \chi_l^* l^{\nu-1} = \chi_l^* l^{\nu+a-e}; \\ \text{if } e = a+2: \quad 0 + \chi_l^* l^{\nu-2} = \chi_l^* l^{\nu+a-e}; \\ \text{if } e > a+2: \quad \chi_l^* l^{\nu+a-e}. \end{aligned}$$

Note two things: First, we can combine the last three lines above into

$$\text{if } e \geq a+1: \quad \chi_l^* l^{\nu+a+e}.$$

Second, in (7) we take $\sum_{\rho=0}^{\rho_0-1} \alpha(\rho)c(\rho) = 0$ in case $\rho_0 = 0$, and this is the case iff $a = \mu$. Therefore, $\sum_{\rho=0}^a \alpha(\rho)c(\rho)$ is given by the above results in the case $a = \mu$. If $\rho_0 > 0$ (i.e., if $a > \mu$) then we add $\sum_{\rho=0}^{\rho_0-1} \alpha(\rho)c(\rho)$ to the above

results and simplify to find $\sum_{\rho=0}^a \alpha(\rho)c(\rho)$. We have:

if $e \leq \mu$:

$$\begin{aligned} & \chi_l(\tfrac{s}{2})l^{\nu-1}(l+1)(l^{a-\mu}-1) + \chi_l^* l^{\nu-1} + \chi_l(\tfrac{s}{2})l^{\nu-1}(l-1) \\ &= \chi_l^* l^{\nu-1} + \chi_l(\tfrac{s}{2})(l-1)l^{\nu-1}((l+1)(l^{a-\mu}-1)/(l-1) + 1); \end{aligned}$$

if $\mu < e \leq a-1$:

$$\begin{aligned} & \chi_l(\tfrac{s}{2})(l-1)l^{\nu-1}((l+1)(l^{a-e}-1)/(l-1) + l^{a-e}) \\ &+ \chi_l^* l^{\nu-1} + \chi_l(\tfrac{s}{2})l^{\nu-1}(l-1) \\ &= \chi_l^* l^{\nu-1} + \chi_l(\tfrac{s}{2})(l-1)l^{\nu-1}((l+1)(l^{a-e}-1)/(l-1) + l^{a-e} + 1); \end{aligned}$$

if $e = a$:

$$\begin{aligned} & \chi_l(\tfrac{s}{2})(l-1)l^{\nu-1}l^{a-e} + \chi_l^* l^{\nu-1} + \chi_l(\tfrac{s}{2})l^{\nu-1}(l-1) \\ &= \chi_l^* l^{\nu-1} + \chi_l(\tfrac{s}{2})(l-1)l^{\nu-1}(l^{a-e} + 1); \end{aligned}$$

if $e \geq a+1$:

$$0 + \chi_l^* l^{\nu+a-e} = \chi_l^* l^{\nu+a-e}.$$

Refer now to Table 1 for $c(s, l)$. If $\text{Case}(l) = A$ and $\nu = \text{ord}_l(N) = 2\mu$ is even, we have $c(s, l)$ is equal to

$$\begin{aligned} & \chi_l^* l^{\min(2a, \nu-1, \nu+a-e)} \\ &+ \chi_l(\tfrac{s}{2})(l-1)l^{\nu-1}\{(l+1)(l^{a-\max(\mu, e)}-1)/(l-1) + k_5 l^{a-e} + k_6\}, \end{aligned}$$

where $k_5 = d(\mu+1, e)d(\mu+1, a)$ and $k_6 = d(e, a)d(\mu, a)$. We show that $c(s, l) = \sum_{\rho=0}^a \alpha(\rho)c(\rho)$ for $\text{Case}(l) = A$ and even $\nu = \text{ord}_l(N) = 2\mu$.

Suppose $a \leq \mu-1$. We have $2a \leq 2\mu-2 = \nu-2 < \nu-1$ so that $\min(2a, \nu-1, \nu+a-e) = \min(2a, \nu+a-e)$. Now, $e \leq \nu-a$ iff $2a \leq \nu+a-e$, so that $c(s, l)$ gives the χ_l^* -term of the SUM properly. Next, $a - \max(\mu, e) \leq a - \mu \leq \mu-1 - \mu = -1$; by our convention, then, $l^{a-\max(\mu, e)} - 1 = 0$. Also, $a \leq \mu-1 < \mu < \mu+1$ so that $d(\mu, a) = 0$ and $d(\mu+1, a) = 0$ so that $k_6 = k_5 = 0$. Therefore, since each term in the $\{\}$'s is 0, the $\chi_l(\tfrac{s}{2})$ -term is 0.

Suppose $a = \mu$. Then $\min(2a, \nu-1, \nu+a-e) = \min(\nu-1, \nu+a-e)$, and moreover, $e > a$ iff $\nu+a-e = \min(\nu-1, \nu+a-e)$ so that $c(s, l)$ gives the χ_l^* -term properly. If $e \leq a$, we have $a - \max(\mu, e) = a - \mu = 0$ so that $l^{a-\max(\mu, e)} - 1 = 0$. Clearly $k_5 = 0$ and k_6 is 1 so the $\chi_l(\tfrac{s}{2})$ -term is given by $c(s, l)$ to be $\chi_l(\tfrac{s}{2})(l-1)l^{\nu-1}$. If $e > a$, we have $a - \max(\mu, e) \leq a - e < 0$, so that by convention, $l^{a-\max(\mu, e)} - 1 = 0$; $k_5 = k_6 = 0$ so that each term in the $\{\}$'s is 0, and so no $\chi_l(\tfrac{s}{2})$ -term appears.

Suppose $a > \mu$. Then $\min(2a, \nu-1, \nu+a-e) = \min(\nu-1, \nu+a-e)$; again $e > a$ iff $\nu+a-e = \min(\nu-1, \nu+a-e)$ so that $c(s, l)$ correctly gives the χ_l^* -term. Now consider the $\chi_l(\tfrac{s}{2})$ -term. If $e \leq \mu$ we have $l^{a-\max(\mu, e)} = l^{a-\mu}$; $d(\mu+1, e) = 0$ so that $k_5 = 0$, while $k_6 = 1$. The terms in $\{\}$'s become $(l+1)(l^{a-\mu}-1)/(l-1) + 1$. If $\mu < e \leq a-1$ we have $l^{a-\max(\mu, e)} = l^{a-e}$ while $k_5 = k_6 = 1$ and the terms in $\{\}$'s become $(l+1)(l^{a-e}-1)/(l-1) + l^{a-e} + 1$. If $e = a$ we get $l^{a-\max(\mu, e)} - 1 = l^{a-e} - 1 = 0$; $k_5 = k_6 = 1$ so the terms in $\{\}$'s become $l^{a-e} + 1$. Lastly, if $e > a$, we have $a - \max(\mu, e) = a - e < 0$ so that $l^{a-\max(\mu, e)} - 1 = 0$; also $k_5 l^{a-e} = 0$, and $k_6 = 0$ so that each term in the $\{\}$'s and therefore the entire $\chi_l(\tfrac{s}{2})$ -term is 0, and we have shown $c(s, l)$ gives the $\chi_l(\tfrac{s}{2})$ -term correctly in each case.

This concludes the proof that $\sum_{\rho=0}^a \alpha(\rho)c(\rho) = c(s, l)$ for $\text{Case}(l) = A$ and even $\nu = \text{ord}_l(N)$; we leave verification in the other cases to the reader.

Example 6. Let $N = 3$, $k = 7$, and $n = 7$, and suppose that $\chi = \left(\frac{*}{3}\right)$ is the Legendre symbol. We show how easy Theorem 2 makes the computation of $\text{tr}_{N, \chi, k} T_n$. First note that since $k > 2$, the $\delta(\chi)$ -term of the trace formula is 0, while both $\delta(\sqrt{n})$ -terms are 0 because n is not a perfect square. All we need to do is evaluate $\sum_s$. Now, $s^2 - 4n < 0$ for $0 \leq \pm s \leq 5$; $s^2 - 4n$ is a perfect square for $\pm s = 8$.

Suppose $s = 1$. We have $s^2 - 4n = -27$, so that $t = 3$ and $m = -3$. The contribution to $\sum_s$ for $s = 1$ is then $a(1)b(1)c(1, 3)$. To find $c(1, 3)$, we note that $\nu = \text{ord}_3(N) = 1$, so $\mu = 0$ and $\delta = 1$, that $e = e_3(\chi) = 1$, and we determine that 3 is a case C prime, with $a = a_3(s^2 - 4n) = 1$. By Table 1, $k_1 = k_2 = k_5 = \varepsilon = 0$, $k_3 = g = 1$, $k_4 = 6$, and $k_6 = d(1, 2)d(0, 1) = 1$; also $a - \max(\mu, e - \delta) + \varepsilon = 1$. By (2), $c(1, 3) = \chi_3(1/2)3^0(6(3-1)/2+1) = -7$. We compute $a(1) = 60$, while $b(1) = h(-3)/\omega(-3) = 1/3$. Therefore, the contribution to $\sum_s$ for $s = 1$ is $60 \cdot (1/3) \cdot -7 = -140$.

Similarly, when $s = 2$, the contribution to $\sum_s$ is $a(2)b(2)c(2, 3) = 51 \cdot (2/1) \cdot 1 = 102$; if $s = 4$ the contribution to $\sum_s$ is $a(4)b(4)\gamma(4, 2)c(4, 3) = -90 \cdot (1/3) \cdot 4 \cdot -1 = 120$; for $s = 5$ the contribution is $a(5)b(5)c(5, 3) = 180 \cdot (1/3) \cdot 1 = 60$; for each of these values of s , 3 is a case C prime in the evaluation of $c(s, 3)$.

Now, if $s = 0$ or 3, then $c(s, 3) = 0$. Therefore, there is no need to evaluate $a(s)b(s)$, nor any products of the $\gamma(s, l)$ -terms; the contribution to $\sum_s$ for either of these values of s is simply 0.

If $s = 8$ then $s^2 - 4n = 36$, so that $t = 6$. The contribution to $\sum_s$ is then $a(8)b(8)\gamma(8, 2)c(8, 3)$. In finding $c(8, 3)$, we have $\nu = \text{ord}_3(N) = 1$, $\mu = 0$ and $\delta = 1$, $e = e_3(\chi) = 1$, and we determine that 3 is a case A prime, with $a = a_3(s^2 - 4n) = 1$. By Table 1, $k_2 = k_5 = k_6 = \varepsilon = 0$, $k_1 = g = 1$, and $k_3 = k_4 = 2$; also $\min(2a, \nu - 1, a + \nu - e) = 0$, $a - \max(\mu, e - \delta) + \varepsilon = 1$, $\chi_l^* = \chi_3(1) + \chi_3(7) = 2$, and $\chi_3(8/2) = 1$. By (2), $c(8, 3) = 2 \cdot 1 \cdot 1 + 0 + 1 \cdot 2 \cdot 1(2(3-1)/2 + 0 + 0) = 6$. We find $a(8) = 1/6$, while $b(8) = 1/2$. Also, $\gamma(8, 2) = 2$, as 2 is a case D prime and $a_2(s^2 - 4n) = 1$. Therefore, the contribution to $\sum_s$ for $s = 8$ is $(1/6)(1/2) \cdot 2 \cdot 6 = 1$.

By Proposition 7 below, the contributions to $\sum_s$ of s and $-s$ are equal, and so finally we obtain $\text{tr}_{N, \chi, k} T_n = -\sum_s = -2(-140 + 102 + 120 + 60 + 1) = -286$.

The following proposition states that for fixed s_0 the contributions of the terms corresponding to s_0 and $-s_0$ to the $\sum_s$ in the trace formula as given in Theorem 2 are the same. Therefore, the formula in Theorem 2 could be modified by taking the $\sum$ over all the *nonnegative* integers s satisfying $s^2 - 4n$ is a positive square or any negative integer, and replacing (say) $a(s)$ with $2a(s)$, except for $s = 0$.

Proposition 7. Let the notation be as in Theorem 2. Let $s \in \mathbb{Z}$ satisfy $s^2 - 4n$ is a positive square or any negative integer. Then

$$a(-s)b(-s) \prod_{l|t, l \nmid N} \gamma(-s, l) \prod_{l|N} c(-s, l) = a(s)b(s) \prod_{l|t, l \nmid N} \gamma(s, l) \prod_{l|N} c(s, l).$$

Proof. Fix s satisfying the hypothesis, and write $s^2 - 4n$ as t^2 , t^2m , or t^24m as in Theorem 1. Fix a prime l with $l \mid t$, $l \nmid N$. Note that $b(s)$ and $\gamma(s, l)$

depend only on $s^2 - 4n = (-s)^2 - 4n$ so that $b(s) = b(-s)$ and $\gamma(s, l) = \gamma(-s, l)$. Let x and y be the roots in $\mathbb{C}$ of $X^2 - sX + n$; then $-x$ and $-y$ are the roots of $X^2 - (-s)X + n$, and it follows that $a(-s) = (-1)^k a(s)$. Referring to its definition, note that $c(s, l)$ is of the form

$$C_1 \left(\chi_l \left(\frac{s + l^a d}{2} \right) + \chi_l \left(\frac{s - l^a d}{2} \right) \right) \\ + C_2 \left(\chi_l \left(\frac{s + l^{a+f} d}{2} \right) \right) + C_3 \chi_l \left(\frac{s}{2} \right),$$

where C_1 , C_2 , and C_3 are functions of l , $e = e(\chi_l)$, $\nu = \text{ord}_l(N)$, and $a = \text{ord}_l(t)$, where $s^2 - 4n = (-s)^2 - 4n = t^2$, $t^2 m$, or $t^2 4m$ as the case may be, so that C_1 , C_2 , and C_3 are independent of the sign of s . Therefore, for the same C_1 , C_2 , and C_3 , we have $c(-s, l)$ equals

$$C_1 \left(\chi_l \left(\frac{-s + l^a d}{2} \right) + \chi_l \left(\frac{-s - l^a d}{2} \right) \right) \\ + C_2 \left(\chi_l \left(\frac{-s + l^{a+f} d}{2} \right) \right) + C_3 \chi_l \left(\frac{-s}{2} \right).$$

First,

$$C_1 \left(\chi_l \left(\frac{-s + l^a d}{2} \right) + \chi_l \left(\frac{-s - l^a d}{2} \right) \right) \\ = \chi_l(-1) C_1 \left(\chi_l \left(\frac{s + l^a d}{2} \right) + \chi_l \left(\frac{s - l^a d}{2} \right) \right).$$

Next, it is clear that $C_3 \chi_l \left(\frac{-s}{2} \right) = \chi_l(-1) C_3 \chi_l \left(\frac{s}{2} \right)$. Furthermore, if $C_2 \neq 0$ then we must have $l = 2$ and $d(e, a + f) = 1$, that is, $e \leq a + f$. In this case, we have $0 \equiv 2^{a+f} d \equiv (s + 2^{a+f} d)/2 - (s - 2^{a+f} d)/2 \pmod{2^e}$, that is, $(s + 2^{a+f} d)/2 \equiv (s - 2^{a+f} d)/2 \pmod{2^e}$, so that

$$C_2 \chi_2 \left(\frac{-s + 2^{a+f} d}{2} \right) = C_2 \chi_2(-1) \chi_2 \left(\frac{s - 2^{a+f} d}{2} \right) \\ = \chi_2(-1) C_2 \chi_2 \left(\frac{s + 2^{a+f} d}{2} \right).$$

Therefore, $c(-s, l) = \chi_l(-1) c(s, l)$, and it follows that

$$\prod_{l|N} c(-s, l) = \chi(-1) \prod_{l|N} c(s, l).$$

Finally then,

$$a(-s) b(-s) \prod_{l|t, l \nmid N} \gamma(-s, l) \prod_{l|N} c(-s, l) \\ = (-1)^k \chi(-1) a(s) b(s) \prod_{l|t, l \nmid N} \gamma(s, l) \prod_{l|N} c(s, l).$$

This proves the result, because we assume (in both Theorems 1 and 2) that $(-1)^k \chi(-1) = 1$.

The following is easy to show using Theorem 2:

Corollary 8. *Let k , χ , and N be as in Theorems 1 and 2. The dimension of the space $S_k(N, \chi)$ is given by the formula*

$$\dim(S_k(N, \chi)) = -s_0 - s_1 + d + m - p,$$

where

$$s_0 = \begin{cases} 0 & \text{if any one of the following conditions is met: } k \text{ is} \\ & \text{odd; } 4 \mid N; \chi_l(-1) = -1 \text{ or } (\frac{-1}{l}) = -1 \text{ for some} \\ & \text{odd prime } l \mid N, \\ \frac{1}{4}(-1)^{k/2-1} \chi(r_0) 2^n & \text{otherwise, where } r_0 \in \mathbf{Z} \text{ satisfies } r_0^2 \equiv -1 \pmod{N} \\ & \text{and } n \text{ is the number of odd primes which divide } N, \end{cases}$$

$$s_1 = \begin{cases} 0 & \text{if any one of the following conditions is met: } k \equiv 1 \\ & \text{or } 4 \pmod{6}; 9 \mid N; 2 \mid N; \text{ or } (\frac{-3}{l}) = -1 \text{ for some} \\ & \text{odd prime } l \mid N, l \neq 3, \\ \frac{\alpha}{3} \chi(\frac{1}{2}) \prod_{l \mid N, l \neq 3} \beta_l & \text{otherwise, where } \alpha = 1 \text{ if } k \equiv 2 \text{ or } 3 \pmod{6} \text{ and } -1 \\ & \text{if } k \equiv 0 \text{ or } 5 \pmod{6}; \beta_l = \chi_l(1 + r_1) + \chi_l(1 - r_1) \\ & \text{where } r_1 \in \mathbf{Z} \text{ satisfies } r_1^2 \equiv -3 \pmod{N} \text{ if } (N, 3) = 1, \\ & \text{and } r_1^2 \equiv -3 \pmod{\frac{N}{3}} \text{ if } 3 \parallel N, \end{cases}$$

$$d = \begin{cases} 1 & \text{if } k = 2 \text{ and } \chi \text{ is trivial,} \\ 0 & \text{otherwise,} \end{cases}$$

$$m = \frac{k-1}{12} N \prod_{l \mid N} (1 + 1/l), \quad p = \frac{1}{2} \prod_{l \mid N} \text{par}(l),$$

where $\text{par}(l)$ is defined as in Theorem 1.

Proof. Since T_1 is the identity operator, the trace of T_1 acting on $S_k(N, \chi)$ gives the dimension of the space, so we need only evaluate Theorem 2 with n set to 1. Consider the sum over s in the first part of the trace formula as given in Theorem 2. Now, 0, 1, and -1 are the only values of s such that $s^2 - 4n$ is negative, and there are no integral values of s such that $s^2 - 4n$ is a positive square.

First, fix $s = 0$. We have $s^2 - 4n = -4 = t^2 4m$, where $t = 1$ and $m = -1 \equiv 3(4)$. Since i and $-i$ are the roots of $\Phi(X)$, we find $a(0) = (1/4)i^{k-2}(1 + (-1)^k)$. If k is odd, $a(0) = 0$; otherwise $a(0) = (1/2)(-1)^{k/2-1}$. The class number of $\mathbf{Q}\sqrt{-1}$ is 1, and one-half the cardinality of its unit group is 2 so that $b(0) = 1/2$. Since $t = 1$, $\prod_{l \mid t, l \nmid N} \gamma(0, l) = 1$. It remains to evaluate $\prod_{l \mid N} c(0, l)$. Let l be an odd prime dividing N , and set $\nu = \text{ord}_l(N)$; we have $s^2 - 4n = -4 = l^{2a} \cdot -4$, where $a = 0$. Suppose that $(\frac{-4}{l}) = 1$ so that $\text{Case}(l) = A$. Let $d_l \in \mathbf{Z}_l$ satisfy $d_l^2 = -1$, so that $(2d_l)^2 = -4$. Note that $2d_l$ is the 'd' which appears in the classification of l , so that $(s \pm l^a d)/2 = (0 \pm 1 \cdot 2d_l)/2 = \pm d_l$. Refer to Table 1 to find $c(0, l)$: We have $k_1 = 1$, $\min(2a, \nu - 1, a + \nu - e) = 0$, $k_2 = 0$, and $\chi_l(0/2) = 0$ so that $c(0, l) = \chi_l(d_l) + \chi_l(-d_l) = \chi_l(d_l)(1 + \chi_l(-1))$. If $\chi_l(-1) = -1$ then $c(0, l) = 0$ and hence the contribution of the $s = 0$ term to the trace is 0, while if $\chi_l(-1) = 1$ we have $c(0, l) = 2\chi_l(d_l)$. Now if $(\frac{-4}{l}) = -1$, so that $\text{Case}(l) = B$, then

referring to Table 1 for $c(0, l)$ we have $k_1 = k_2 = 0$ and $\chi_l(0/2) = 0$ so that $c(0, l) = 0$ and therefore the contribution of the $s = 0$ term to the trace is 0.

Keep $s = 0$, and suppose now that $l = 2$ and $l \mid N$. Then $\text{Case}(l) = F$, $a = 0$, $\chi_2(0/2) = 0$, and Table 1 for $c(0, l)$ gives $k_1 = 0$. Let $\nu = \text{ord}_2(N)$; write $\nu = 2\mu$ or $2\mu + 1$ as the case may be. If $4 \mid N$ then $\mu \geq 1$ so that $d(\mu, a) = d(\mu, 0) = 0$, thus $k_2 = 0$ and so $c(0, 2) = 0$. If $2 \parallel N$ then $\nu = 1$ and $\mu = 0$, and χ_2 is the trivial character, so that $e = e(\chi_2) = 0$. We have $k_2 = d(e, a + 1)d(\mu, a) = d(0, 1)d(0, 0) = 1$; in this case $c(0, 2) = k_2 k_3 \hat{\chi}_2 = 1 \cdot 1 \cdot \chi_2((0 + 2^{0+1} \cdot 1)/2) = \chi_2(1) = 1$.

Therefore the contribution of the $s = 0$ term to the trace is 0 unless k is even, $4 \nmid N$, $\chi_l(-1) = 1$ for all odd primes $l \mid N$, and $(\frac{-1}{l}) = 1$ for all odd $l \mid N$. Suppose that all these conditions are met. In particular, since $4 \nmid N$ and $(\frac{-1}{l}) = 1$ for all odd $l \mid N$, there is some $r \in \mathbf{Z}$ with $r^2 \equiv -1(N)$; note r is odd if $2 \mid N$ so that $\chi_2(r) = 1$. If l is an odd prime dividing N and d_l is a unit in $\mathbf{Z}_l$ satisfying $d_l^2 = -1$, then $\chi_l(d_l) = \chi_l(r)$, because $d_l \equiv \pm r \pmod{l^{\text{ord}_l(N)}}$ and $\chi_l(-1) = 1$. Finally then,

$$\prod_{l \mid N, l \text{ odd}} 2\chi_l(d_l) = \prod_{l \mid N, l \text{ odd}} 2\chi_l(r) = 2^n \chi(r),$$

where n is the number of odd primes dividing N .

Now fix $s = 1$. We have $s^2 - 4n = -3 = t^2 m$, where $t = 1$ and $m = -3$. The roots x and y of $\Phi(X) = X^2 - X + 1$ are $(1 \pm \sqrt{-3})/2$; deMoivre's formula gives $(x^{k-1} - y^{k-1})/(x - y) = 2i \sin((k-1)\pi/3)/(i\sqrt{3})$ so that

$$a(1) = \frac{1}{2} \cdot \begin{cases} 1 & \text{if } k \equiv 2, 3(6), \\ 0 & \text{if } k \equiv 1, 4(6), \\ -1 & \text{if } k \equiv 0, 5(6). \end{cases}$$

The class number of $\mathbf{Q}\sqrt{-3}$ is 1, and one-half the cardinality of the unit group is 3, so that $b(1) = 1/3$. Since $t = 1$, $\prod_{l \mid l, l \nmid N} \gamma(1, l) = 1$; it remains to evaluate $\prod_{l \mid N} c(1, l)$. Let $l \mid N$ be an odd prime, $l \neq 3$. Then $s^2 - 4n = -3 = l^{2a}(-3)$ with $a = 0$. Let $\nu = \text{ord}_l(N)$ and set $\nu = 2\mu$ or $2\mu + 1$ as appropriate. Suppose $(\frac{-3}{l}) = 1$, so that $\text{Case}(l) = A$. Let d_l be a unit in $\mathbf{Z}_l$ satisfying $d_l^2 = -3$. Referring to Table 1 for $c(1, l)$ we see that $k_2 = 0$, $k_1 = 1$, and $l^{\min(2a, \nu-1, \nu+a-e)} = l^0 = 1$. If ν is even, then $\mu > a$ so that $d(\mu + 1, a)$ and $d(\mu, a)$ are both 0, hence $k_5 = k_6 = 0$, while if ν is odd, k_5 and k_6 are 0 automatically. Since $a - \max(\mu, e - \delta) \leq 0$, the k_4 -term is 0, and hence for any ν , the contribution to $c(1, l)$ from the $\chi_l(\frac{e}{2})$ -term is 0, and so $c(1, l) = \chi_l((1 + d_l)/2) + \chi_l((1 - d_l)/2)$. If $(\frac{-3}{l}) = -1$ then $\text{Case}(l) = B$; here $k_1 = 0$ while the other k_i -terms are the same as for $\text{Case}(l) = A$. Thus if $(\frac{-3}{l}) = -1$, then $c(1, l) = 0$ and therefore the contribution of the $s = 1$ term to the trace is 0.

Suppose now that $l \mid N$ and $l = 3$; we have $s^2 - 4n = -3 = 3^{2a+1}(-1)$, where $a = 0$ and -1 is a unit in $\mathbf{Z}_3$, so that $\text{Case}(l) = C$. Let $\nu = \text{ord}_3(N)$ and set $\nu = 2\mu$ or $2\mu + 1$. Suppose first that ν is even; refer to Table 1 for $c(1, 3)$. We have $k_1 = k_2 = k_6 = 0$; also $d(\mu, a) = 0$ so $k_5 = 0$. Furthermore, $a - \max(\mu, e) + 1 \leq 0$ so the k_4 -term is 0. Therefore the $\chi_l(\frac{e}{2})$ -term is 0, so that if ν is even, $c(1, 3) = 0$. Now, if ν is odd, we have $k_1 = k_2 = k_5 = 0$,

and $a - \max(\mu, e - 1) \leq 0$ so the k_4 -term is 0. Consider k_6 . If $\mu \geq 1$ then $d(\mu, a) = 0$, so $k_6 = 0$, and therefore $c(1, 3) = 0$; combined with the fact that $c(1, 3) = 0$ if ν is even we have that the contribution of the $s = 1$ term to the trace is 0 if $3^2 \mid N$. Suppose $3 \parallel N$; then $\mu = 0$, and it follows that $k_6 = 1$. Therefore $c(1, 3) = \chi_3(\frac{1}{2}) \cdot 3^{1-1}(0 + 0 + 1) = \chi_3(\frac{1}{2})$; since χ_3 is now either the trivial character or $(\frac{\cdot}{3})$, we have $c(1, 3) = 1$ or -1 , respectively.

Suppose $l = 2$ and $2 \mid N$; then $s^2 - 4n = -3 = 2^{2a}(-3)$ with $a = 0$ and $-3 \equiv 5 \pmod{8}$ is a unit in $\mathbf{Z}_2$, so that $\text{Case}(l) = E$. Refer to Table 1 for $c(1, 2)$. We have $k_1 = 0$. Let $\nu = \text{ord}_2(N)$ and set $\nu = 2\mu$ or $2\mu + 1$. Suppose first that ν is even. Then $\mu > a$ so that $k_2 = k_5 = k_6 = 0$ and the k_4 -term is 0 and therefore $c(1, 2) = 0$. If ν is odd, then k_2 and the k_4 -term are again 0, while k_5 and k_6 are automatically 0, so that $c(1, 2) = 0$. Hence, if $2 \mid N$, the contribution of the $s = 1$ term to the trace is 0.

We have shown that $\prod_{l \mid N} c(1, l) = 0$ unless $9 \nmid N$, $2 \nmid N$, and $(\frac{-3}{7}) = 1$ for each odd prime $l \mid N$, $l \neq 3$. Suppose in fact that all these conditions are satisfied. It is then possible to find $r \in \mathbf{Z}$ such that $r^2 \equiv -3 \pmod{N}$ if $(3, N) = 1$, and $r^2 \equiv -3 \pmod{(\frac{N}{3})}$ if $3 \parallel N$, and satisfying the following: for each odd prime $l \mid N$, $l \neq 3$, we have $r \equiv \pm d_l \pmod{l^{\text{ord}_l(N)}}$, where $d_l \in \mathbf{Z}_l$ is a unit with $d_l^2 = -3$. For each such l we have $\chi_l(\frac{1}{2})(\chi_l(1 + r) + \chi_l(1 - r)) = \chi_l(\frac{1}{2})(\chi_l(1 \pm d_l) + \chi_l(1 \mp d_l)) = \chi_l((1 + d_l)/2) + \chi_l((1 - d_l)/2)$. Taking $\{\chi_3(\frac{1}{2})\}$ to mean 1 if $3 \nmid N$ and $\chi_3(\frac{1}{2})$ if $3 \parallel N$, we have $\prod_{l \mid N} c(1, l) = \{\chi_3(\frac{1}{2})\} \cdot \prod_{l \mid N, l \neq 3} (\chi_l((1 + d_l)/2) + \chi_l((1 - d_l)/2)) = \chi(\frac{1}{2}) \prod_{l \mid N, l \neq 3} (\chi_l(1 + r) + \chi_l(1 - r))$.

By Proposition 7, the contribution of the $s = -1$ term to the trace equals that of the $s = 1$ term. The remaining terms in the dimension formula come immediately from the corresponding terms in either Theorem 1 or 2.

Consider the trace formula as given in Theorem 2. If χ is the trivial character, we can make additional simplifications to the formula, the most important being that $c(s, l)$ can be given by a very simple table; this is the result of our next corollary.

Corollary 9. *Let k, χ, N , and n be as in Theorem 2, and suppose furthermore that χ is the trivial character. Then for $(n, N) = 1$ we have*

$$\begin{aligned} \text{tr}_{N, \chi, k} T_n = & - \sum_s \left(a(s)b(s) \prod_{l \mid t, l \nmid N} \gamma(s, l) \prod_{l \mid N} c_0(s, l) \right) \\ & + \left[\frac{2}{k} \right] \deg(T_n) + \delta_0(\sqrt{n}) \frac{k-1}{12} N \prod_{l \mid N} (1 + 1/l) \\ & - \delta_0(\sqrt{n}) \frac{\sqrt{n}}{2} \prod_{l \mid N} \text{par}_0(l), \end{aligned}$$

where $s, a(s), b(s), t$, and $\gamma(s, l)$ are exactly the same as in Theorem 2, and

$$\begin{aligned} \delta_0(\sqrt{n}) &= \begin{cases} n^{k/2-1} & \text{if } n \text{ is a perfect square,} \\ 0 & \text{otherwise,} \end{cases} \\ \text{par}_0(l) &= \begin{cases} l^\mu + l^{\mu-1} & \text{if } \nu = \text{ord}_l(N) = 2\mu, \\ 2l^\mu & \text{if } \nu = \text{ord}_l(N) = 2\mu + 1, \end{cases} \end{aligned}$$

and $c_0(s, l)$ is defined as follows. Fix s and a prime $l \mid N$. Let $\nu = \text{ord}_l(N)$, write $\nu = 2\mu + \delta$, where $\delta = 0$ or 1 . Classify l into one of the six cases $A, \dots, F$, and referring to how $\text{Case}(l)$ is determined, let $a = a_l(s^2 - 4n)$. Let $d(x, y) = 1$ if $x \leq y$ and 0 otherwise, for any $x, y \in \mathbf{Z}$. Then $c_0(s, l)$ is given by the expression

$$(9) \quad k_1 2^{l^{\min(2a, \nu-1)}} + k_3 l^{\nu-1} (c_4 (l^{a-\mu+\varepsilon} - 1) / (l - 1) + c_6),$$

where the values of k_1, k_3, c_4, c_6 , and ε are determined from Table 2.

TABLE 2

Case(l)	ν	k_1	k_3	c_4	ε	c_6
A or D	<i>even</i> <i>odd</i>	1	$l - 1$	$\frac{l+1}{2}$	0	$\frac{d(\mu, a)}{0}$
B or E	<i>even</i> <i>odd</i>	0	$l + 1$	$\frac{l+1}{2}$	0	$\frac{d(\mu, a)}{0}$
C or F	<i>even</i> <i>odd</i>	0	1	$\frac{l+1}{2l}$	1 0	0 $d(\mu, a)$

Remarks. Note that k_1 and k_3 are the same k_1 and k_3 as appear in Theorem 2, while c_4 and c_6 are similar to the k_4 and k_6 (respectively) of the same theorem. Also, one must heed Convention A in evaluating (9). In Theorems 1 and 2 we assume $(-1)^k \chi(-1) = 1$; the corollary's additional hypothesis that χ is trivial implies that k is even.

Proof. Let χ be the trivial character mod N , so that $\chi = \prod_{l \mid N} \chi_l$, where for each prime $l \mid N$, χ_l is the trivial character mod $l^{\text{ord}_l(N)}$; note $e = e(\chi_l) = 0$ for each prime $l \mid N$. Let the trace $\text{tr}_{N, \chi, k} T_n$ be as given by Theorem 2. Clearly the last three lines of the formula in the statement of Corollary 9 follow directly from the corresponding lines of Theorem 2. All one has to do is show how Table 1 “collapses” into Table 2 by showing that $c_0(s, l) = c(s, l)$ for each fixed s and fixed $l \mid N$, for any classification of the prime l , and any relationship between $a = a_l(s^2 - 4n)$ and μ where $\text{ord}_l(N) = 2\mu$ or $2\mu + 1$. We leave the details to the reader.

REFERENCES

- [A-L] A. O. L. Atkin and J. Lehner, *Hecke operators on $\Gamma_0(m)$* , Math. Ann. **185** (1970), 134–160.
- [Hij] H. Hijikata, *Explicit formula of the traces of the Hecke operators for $\Gamma_0(N)$* , J. Math. Soc. Japan **26** (1974), 56–82.
- [H-P-S₁] H. Hijikata, A. Pizer, and T. Shemanske, *The basis problem for modular forms on $\Gamma_0(N)$* , Mem. Amer. Math. Soc. No. 418, 1989.
- [H-P-S₂] ———, *Twists of newforms*, preprint, 1988.
- [Li] W. Li, *Newforms and functional equations*, Math. Ann. **212** (1975), 285–315.
- [Sha] G. Shimura, *Introduction to the arithmetic theory of automorphic functions*, Princeton Univ. Press, Princeton, N. J., 1971.