

ON TWISTOR SPACES OF ANTI-SELF-DUAL HERMITIAN SURFACES

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ABSTRACT. We consider a complex surface M with anti-self-dual hermitian metric h and study the holomorphic properties of its twistor space Z . We show that the naturally defined divisor line bundle $[X]$ is isomorphic to the $-\frac{1}{2}$ power of the canonical bundle of Z , if and only if there is a Kähler metric of zero scalar curvature in the conformal class of h . This has strong consequences on the geometry of M , which were also found by C. Boyer [3] using completely different methods. We also prove the existence of a very close relation between holomorphic vector fields on M and Z in the case that M is compact and Kähler.

1. INTRODUCTION

The aim of this section is to give the basic definitions and results which will be used later.

In this work (M, h) will denote a complex surface M together with a hermitian metric h whose Weyl tensor W is anti-self-dual. We write $h = g - 2i\omega$ where g and ω are the associated Riemannian metric and fundamental 2-form, respectively. This is equivalent to having a Riemannian 4-dimensional manifold (M, g) with an integrable almost complex structure J satisfying $g(JX, JY) = g(X, Y)$ for all tangent vectors X, Y on M .

As the real dimension of M is four we have a famous splitting of the bundle of 2-forms: $\Lambda^2(M) = \Lambda_+^2(M) \oplus \Lambda_-^2(M)$ into the eigenspaces of the Hodge star operator $*$: $\Lambda^2(M) \rightarrow \Lambda^2(M)$, because $*^2 = 1$. Looking at the curvature operator $\mathcal{R}: \Lambda^2(M) \rightarrow \Lambda^2(M)$ we can then ask that its conformally invariant part $W = W_+ + W_-$ be anti-self-dual, that is $W_+ = 0$ with respect to the orientation of M given by J .

This conformal property of h has an important consequence [1] on the twistor space Z of M which we are briefly going to describe. The real 6-dimensional manifold Z can be defined to be the bundle of almost complex structures on M which are compatible with the metric g and the orientation given by J :

$$Z = \{I \in \mathcal{O}(TM) \mid I^2 = -\text{id}, I > 0\}.$$

We denote by $t: Z \rightarrow M$ the twistor fibration and notice that the fiber is $SO(4)/U(2) \cong S^2$. The important point is that Z has a natural almost complex

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structure J' given as follows: at each point $z \in Z$ we can use the Levi-Civita connection of M to split the tangent space to Z into vertical and horizontal components: $T_z Z = V_z \oplus H_z$. The vector space V_z is tangent to the fiber which is an oriented metric 2-sphere and then has a natural almost complex structure J_1 , namely rotation by $+90^\circ$. On the other hand, H_z can be identified with $T_{t(z)}M$ and given the tautological almost complex structure J_2 defined by z itself. Finally the almost complex structure of Z is $J' = J_1 \oplus J_2$. The important theorem [1] is that

(1) J' is integrable if and only if $W_+ = 0$ on W .

In this case the fibers of t become complex submanifolds of Z , isomorphic to $\mathbf{CP}_1$ and called twistor lines; the antipodal map on each line induces an anti-holomorphic involution σ of Z called the real structure.

The reason why the integrability of J' only depends on the conformal class of g , is because the whole construction is indeed conformally invariant [1].

In fact there is an important interplay, called the Penrose correspondence, between holomorphic properties of Z and conformal properties of M . An instance of this is the following: as the Lie algebra $\mathfrak{so}(4) = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$, the locally defined spin bundle of M splits into two complex subbundles, denoted by S_+ and S_- , which have rank 2 and satisfy $S_+ \oplus S_- = CT^*M$. By S_+^m and S_-^m we indicate their symmetric m th powers and notice that for m even these bundles are globally defined even when M is not spin.

One then considers covariant differentiation

$$\nabla: \Gamma(S_+^m) \rightarrow \Gamma(S_+^m \otimes CT^*M) = \Gamma(S_+^m \otimes S_+ \otimes S_-)$$

which, together with the orthogonal decomposition [1],

$$S_+^m \otimes S_+ \otimes S_- = (S_+^{m-1} \otimes S_-) \oplus (S_+^{m-1} \otimes S_-)$$

gives, by projection, the Dirac operator

$$D_m: \Gamma(S_+^m) \rightarrow \Gamma(S_+^{m-1} \otimes S_-)$$

and the twistor operator

$$\bar{D}_m: \Gamma(S_+^m) \rightarrow \Gamma(S_+^{m+1} \otimes S_-).$$

Finally recall that the canonical line bundle K of Z always admits a preferred square root $K^{1/2}$. And in fact a fourth root exactly when M is spin.

Having said all this we can state a result of Hitchin [6] which says that global holomorphic sections of $K^{-m/4}$ exactly corresponds to solutions of the twistor equation:

$$(2) \quad H^0(Z, K^{-m/4}) \cong \text{Ker } \bar{D}_m \quad \text{for } m \geq 0.$$

We will be mainly interested in the case $m = 2$, in the case in fact $s_+^2 \cong \Lambda_+^2(M)$ and the Dirac operator D_2 is just exterior differentiation d restricted to self-dual 2-forms [6]. Now the fundamental 2-form ω of (M, h) is self-dual and it is very well known that ω is closed if and only if is parallel, h is called a Kähler metric in this case. In the same spirit we have the following result which will be needed later:

Lemma 1.1. *Let ω be the fundamental 2-form of a hermitian surface, then*

$$\overline{D}_2\omega = 0 \Leftrightarrow \nabla\omega = d\omega = 0.$$

Proof. In terms of spinor indices the formula relating covariant differentiation to the Dirac and twistor equations on $S_+^2 \cong \Lambda_+^2$ is

$$\nabla_{A'}^A \omega^{BC} = \nabla_{A'}^{(A} \omega^{BC)} + \frac{2}{3} \varepsilon^{A(B} \nabla_{A'D} \omega^{C)D}.$$

This says that for a self-dual 2-form ω ,

$$\overline{D}_2\omega = 0 \quad \text{if and only if} \quad \nabla\omega = d\omega.$$

But when ω is the fundamental 2-form of a hermitian metric one has [8, p. 148] that for all vector fields X, Y, Z , on M ,

$$(\nabla_X \omega)(Z, Y) = \frac{3}{2} d\omega(X, JY, JZ) - \frac{3}{2} d\omega(X, Y, Z)$$

therefore $\overline{D}_2\omega = 0$ if and only if

$$d\omega(X, Y, Z) = 3 d\omega(X, JY, JZ).$$

And using that $J^2 = -1$ we get $d\omega = 0$; then $\nabla\omega = 0$ also. $\square$

2. THE TWISTOR SPACE

Let $t: Z \rightarrow M$ denote the twistor fibration and suppose M is hermitian and anti-self-dual. Two things are clear from the definition of the almost complex structure of Z : first, t is never a holomorphic map; second, the complex structure J of M defines a cross section $J: M \rightarrow Z$, whose image we denote by Σ . By the integrability of J , Σ is indeed a complex hypersurface of Z biholomorphic to M [4]. Similarly $-J: M \rightarrow Z$ defines hypersurface $\overline{\Sigma}$. The “real structure” σ of Z switches the two hypersurfaces identifying one with the other in an antiholomorphic fashion. If X denotes the divisor $\Sigma + \overline{\Sigma}$ in Z , we can consider the holomorphic line bundle $[X]$; since $\sigma(X) = \sigma(\Sigma + \overline{\Sigma}) = \overline{\Sigma} + \Sigma = X$, $[X]$ is called a “real” bundle.

We then investigate the relation between the holomorphic line bundle $[X]$ and the complex structure of Z .

First, when M is compact, one has the following topological remark [13]:

$$(3) \quad c_1([X]) = c_1(K_Z^{-1/2})$$

where K_Z denotes the canonical bundle.

It is then natural to ask when is $[X]$ isomorphic to $K_Z^{-1/2}$.

Now if $H^1(Z, \mathcal{O}) = 0$, the Chern class map $c_1: H^1(Z, \mathcal{O}^*) \rightarrow H^2(Z, \mathbf{Z})$ is injective and the above implies $[X] \cong K_Z^{-1/2}$, however by the Ward correspondence [1, Theorem 5.2], $H^1(M, \mathbf{R})$ has to be zero in this case.

The general philosophy of the Twistor Program of R. Penrose is to relate the conformal geometry of M to the holomorphic properties of Z . In this context, whether M is compact or not, we have

Theorem 2.1.

$$[X] \cong K_Z^{-1/2}$$

if and only if h is conformal to a Kähler metric.

Proof. We start by assuming that h is a Kähler metric and prove that $[X] \cong K_Z^{-1/2}$, in two steps. We first define a holomorphic section $\tilde{\omega} \in H^0(Z, K^{-1/2})$ by using the Kähler form ω of M ; then we show that $X = \{\tilde{\omega} = 0\}$. In the course of this proof we will often use the following (see [1]): $Z = \mathbf{P}(S_+)$; the symplectic form of S_+ defines a linear isomorphism $\varepsilon: S_+ \rightarrow S_+^*$ and the hermitian form an antilinear isomorphism $h: S_+ \rightarrow \bar{S}_+$ so that if $\eta \in S_+$, $\bar{\eta}$ will denote its image and we will write $\eta \otimes \bar{\eta} \in S_+^2$.

Step 1. Recall that $\bigwedge_+^2(M) = S_+^2$, then the Kähler form ω of M is a section of S_+^2 . Now according to [6, §2] any section $\psi \in S_+^2$ tautologically defines a complex valued function on $S_+ \setminus 0$ which is a homogeneous polynomial of degree 2 on each fiber; this in turns gives a section $\tilde{\psi} \in \Gamma(Z, \mathcal{O}(2)) = \Gamma(Z, K^{-1/2})$. And furthermore $\tilde{\psi}$ is a holomorphic section, i.e. $\tilde{\psi} \in H^0(Z, \mathcal{O}(K^{-1/2}))$, if and only if ψ satisfies the twistor equation $\bar{D}_2\psi = 0$. It is clear from the definition of the operators D_m and $\bar{D}_m$ that in general every parallel section of S_+^m is a solution to both the Dirac and twistor equations (in fact, by the Weitzenböck formulas, these are the only solutions when M is compact and $R = 0$). Therefore since ω is parallel, $\tilde{\omega} \in H^0(Z, K^{-1/2})$ is holomorphic.

Step 2. Since M is hermitian we have two sections ϕ and $\bar{\phi}: M \rightarrow Z$ representing the almost complex structures J and $\bar{J}$. Let $\omega \in \bigwedge_+^2(M) = S_+^2$ be the Kähler form. According to [1, §1], at each point $p \in M$, $\omega = \phi \otimes \bar{\phi}$ where $\phi \in S_+$ and $\bar{\phi} \in S_+$ represents ϕ and $\bar{\phi}$ respectively. Now let $\alpha \in Z = \mathbf{P}(S_+)$ be a twistor at p . By using the isomorphism $\varepsilon: S_+ \rightarrow S_+^*$ it makes sense to solve the equation $\phi(\alpha) = 0$. Since ε is given by the symplectic form and S_+ has complex dimension 2, the only solution is $\alpha = \phi$. Similarly for $\bar{\phi}$ and we have shown that $\tilde{\omega}(\alpha) = 0$ if and only if $\alpha = \phi$ or $\alpha = \bar{\phi}$ that is $X = \{\tilde{\omega} = 0\}$. This proves one direction of the statement.

To complete the proof we assume now that $[X] \cong K_Z^{-1/2}$, and show that there is a Kähler metric in the conformal class of h . By hypothesis we have a holomorphic section $\tilde{\rho}$ of $K_Z^{-1/2}$ vanishing exactly on X . Furthermore since $H^0(Z, K_Z^{-1/2})$ has a “real” structure, we can choose $\tilde{\rho}$ to be invariant under the anti-holomorphic involution σ of Z . The corresponding self-dual 2-form ρ is then real and satisfies the twistor equation: $\bar{D}_2\rho = 0$ [6]. By Lemma 1.1 is then enough to prove that ρ is the fundamental 2-form of a hermitian metric in the conformal class of h .

Now if ω is the fundamental 2-form of h , we have already shown that $\tilde{\omega}$ also vanishes exactly on X , but is not necessarily holomorphic. However, on each twistor line $\mathbf{P}(S_{+x})$, $\tilde{\rho}$ and $\tilde{\omega}$ are homogeneous polynomial of degree two vanishing on the same two antipodal points and therefore they differ by a nonzero multiplicative constant $f(x)$ which is real. It follows that $\rho = f\omega$ for a never-zero real function f on M . Assume that M is connected, this means that either ρ or $-\rho$ is the fundamental 2-form of the hermitian metric $|f|h$. $\square$

Remark 2.2. It was proved for example in [9] that a metric is anti-self-dual and Kähler if and only if it is Kähler of zero scalar curvature. So that the hermitian

anti-self-dual surfaces for which $[X] \cong K_Z^{-1/2}$ are precisely the Kähler surfaces of zero scalar curvature. The problem of their classification was posed in [16].

The above theorem also gives a “twistor proof” of a result of C. Boyer:

Corollary 2.3 [3]. *Let (m, h) be a compact anti-self-dual surface then:*

If $b_1(M)$ is even,

h is globally conformal to a Kähler metric of zero scalar curvature.

If $b_1(M)$ is odd,

h is locally conformal to a Kähler metric of zero scalar curvature.

Proof. By (3), $[X] = K_Z^{-1/2}F$ where F is a holomorphic line bundle of zero Chern class on Z . By the Ward correspondence then, $F = t^*E$ where E is a hermitian line bundle over M , with anti-self-dual connection and zero Chern class. In particular the curvature of the connection is harmonic and therefore zero, by Hodge theory. Now we consider the following commutative diagram:

$$\begin{array}{ccc} \tilde{Z} & \xrightarrow{q} & Z \\ i \downarrow & & \downarrow t \\ \tilde{M} & \xrightarrow{p} & M \end{array}$$

where $p: \tilde{M} \rightarrow M$ and $q: \tilde{Z} \rightarrow Z$ are universal coverings, and $\tilde{i}: \tilde{Z} \rightarrow \tilde{M}$ is the twistor fibration. The pulled-back connection on the line bundle p^*E over $\tilde{M}$ is trivial, because it is flat and $\tilde{M}$ is simply connected. As a consequence $\tilde{F} := \tilde{i}(p^*E) = q^*(t^*E) = q^*F$ is trivial [1]. And therefore $[\tilde{X}] = K_{\tilde{Z}}^{-1/2}$ on $\tilde{Z}$, where $\tilde{X}$ is the universal covering of X . Then $\tilde{M}$ is globally conformally Kähler by 2.1.

It follows that M is locally conformally Kähler (l.c.k. in the notation of Vaisman). But any compact l.c.k. surface is globally conformally Kähler exactly when $b_1(M)$ is even [3], because the Hodge decomposition $H^1(M, \mathbb{C}) = H^{1,0}(M) \oplus H^{0,1}(M)$ holds in this case. $\square$

3. COMPACT KÄHLER SURFACES

From now on we will assume that M is compact and $b_1(M)$ is even. All known examples of anti-self-dual compact complex surfaces of this type are the following:

- Flat tori and $K3$ surfaces with a Yau metric. These are the hyperkähler surfaces and are the universal coverings of:
- The other Ricci-flat Kähler surfaces, i.e. the hyperelliptic and the Enriques surfaces.
- $S_g \times \mathbb{CP}_1$, where S_g is a compact Riemann surface of genus $g \geq 2$ with a metric of constant scalar curvature -1 , and $\mathbb{CP}_1$ is the Riemann sphere with constant curvature $+1$, or, more generally, ruled surfaces which are flat S^2 -bundle over S_g , $g \geq 2$.
- Recently LeBrun has constructed zero scalar curvature Kähler metrics on ruled surfaces blown up at two or more points [10].

The reason why these are hermitian anti-self-dual manifolds is that they have a Kähler metric of zero scalar curvature.

Notice that the complex projective plane $\mathbf{CP}_2$ with its standard orientation and metric is self-dual and Kähler, while the same manifold with orientation reversed, $\overline{\mathbf{CP}}_2$, does not even admit an almost complex structure; otherwise c_1^2 would be equal to $2\chi + 3\tau = 3$, which implies that the first Chern class c_1 cannot be represented by an integral 2-form.

The above theorem also gives precise informations on the normal bundle of X in Z , denoted by $\nu_{X/Z}$.

Corollary 3.1. *When M is compact and $b_1(M)$ is even, the normal bundle of X in Z is isomorphic to the anticanonical bundle: $\nu_{X/Z} \cong K_X^{-1}$, similarly $\nu_{\Sigma/Z} \cong K_{\Sigma}^{-1}$ and $\nu_{\overline{\Sigma}/Z} \cong K_{\overline{\Sigma}}^{-1}$.*

Proof [13]. The adjunction formulas [5] state that $\nu_{X/Z} \cong [X]_X$ and $K_X \cong (K_Z \otimes [X])|_X$ therefore $\nu_{X/Z} \cong K_{Z|X}^{-1/2}$ and $K_X \cong (K_Z \otimes K_Z^{-1/2})|_X \cong K_{Z|X}^{1/2}$ as wanted. The rest clearly follows from $X = \Sigma \amalg \overline{\Sigma}$ $\square$

Theorem 2.1 says that the line bundle $K^{-1/2}$ has global holomorphic sections and this easily implies that $K^{-m/2}$ has global holomorphic sections for each $m \geq 0$. In fact using the ideas of [13] one can show that these are the only line bundles, within their Chern class, to have global holomorphic sections:

Proposition 3.2. *If $c_1(L) = c_1(K^{-m/2})$ then $H^0(Z, L) \neq 0 \Leftrightarrow L \cong K^{-m/2}$ and $m \geq 0$.*

When one considers a compact twistor space $t: Z \rightarrow M$ as a complex manifold, there is a theorem of Hitchin [7] which states that Z is Kähler (in fact algebraic) if and only if M is either S^4 or $\mathbf{CP}_2$, with its standard conformal structure. It is then interesting to investigate “how far is a twistor space from being algebraic,” for example by looking at its algebraic dimension $a(Z)$. To this respect Poon has found some very interesting relations between $a(Z)$ and the geometry of M [12, 13].

The methods of [12] show that the algebraic dimension of a compact twistor space Z is achieved by the Kodaira dimension $k(Z, F)$ of some “real” holomorphic line bundle $F \rightarrow Z$, see [14] for definitions. Therefore

Remark 3.3.

$$a(Z) = k[Z, [X]] = k(Z, K^{-1/2})$$

when Z is the twistor space of a compact Kähler surface of zero scalar curvature.

The above discussion can then be used as in [11], to give a more direct proof of a theorem of Poon [13] which states that

$a(Z) \leq 1$ for the twistor space of a compact Kähler surface of zero scalar curvature. Furthermore equality holds precisely when M is Ricci-flat.

The situation is different when $b_1(M)$ is odd and in [11] we gave the first example of a twistor space with algebraic dimension equal to two. It is the twistor space of a Hopf surface.

4. HOLOMORPHIC VECTOR FIELDS

In this section we still assume that M is compact and Kähler; we show that there is a close relation between the Lie algebras of holomorphic vector fields of M and Z , which we denote by $H^0(M, \Theta)$ and $H^0(Z, \Theta)$. Again, as in Poon's theorem the results reflect whether or not M is Ricci-flat.

For a holomorphic vector bundle E over a compact complex manifold N , $h^0(N, E)$ will denote the complex dimension of $H^0(N, E)$. We will prove

Theorem 4.1. *If M is Ricci-flat*

$$H^0(Z, \Theta) \cong H^0(M, \Theta) \oplus H^0(M, \Theta)$$

which is also isomorphic to the complexification of the Lie algebra of real parallel vector fields on M ; so that

$$h^0(Z, \Theta) = b_1(M) = 2h^0(M, \Theta).$$

Theorem 4.2. *If M is not Ricci-flat*

$$H^0(Z, \Theta) \cong H^0(M, \Theta).$$

To explain this, recall that in the general case, by the Penrose correspondence, $H^0(Z, \Theta)$ is the complexification of the Lie algebra of conformal Killing vector fields on M . This in turn is closely related to $H^0(M, \Theta)$ when M is Kähler.

To prove the above theorems we will use the following [B]:

Theorem 4.3 (Bochner). *On a compact riemannian manifold (N, g) with $\text{Ric} \leq 0$, every Killing vector field is parallel.*

Similarly if g is Kähler, then every holomorphic vector field is parallel.

Theorem 4.4 (Lichnerowicz). *On a compact Kähler manifold of constant scalar curvature*

$$H^0(M, \Theta) \cong \mathfrak{a} \oplus \mathfrak{h}$$

where $\mathfrak{a}$ is the abelian Lie algebra of all parallel holomorphic vector fields and $\mathfrak{h}$ is the complexification of a Lie algebra consisting of Killing vector fields.

Another result of Lichnerowicz states that: on any compact Kähler manifold of dimension at least 2 a conformally Killing vector field is automatically Killing. In complex dimension 2 we also have an elementary proof of this fact:

Lemma 4.5. *If M is a compact Kähler surface every conformal vector field is real holomorphic and in fact Killing.*

Proof. Suppose $\mathcal{L}_V g = fg$ for some function f ; we start by showing that $\mathcal{L}_V \omega = 0$ where ω denotes the Kähler form. In fact let φ_t be the flow of V . For each t , φ_t is a conformal isometry homotopic to the identity. Since ω is a self-dual closed 2-form, it is also harmonic, and it is easy to check that the Hodge-star operator $*$: $\Lambda^n \rightarrow \Lambda^n$, on a manifold of real dimension $2n$, is invariant under a conformal rescaling of the metric; so that $\varphi_t^* \omega$ is again harmonic. But $[\varphi_t^* \omega] = [\omega] \in H_{dR}^2(M)$ and so by Hodge theory, $\varphi_t^* \omega = \omega$, i.e. $\mathcal{L}_V \omega = 0$.

Now the complex structure $J = g^{-1} \circ \omega$ as an endomorphism of the tangent bundle, therefore

$$\mathcal{L}_V J = (\mathcal{L}_V g^{-1}) \circ \omega + g^{-1} \circ (\mathcal{L}_V \omega) = f g^{-1} \circ \omega = f J,$$

on the other hand $J^2 = -\text{id}$ implies that

$$0 = \mathcal{L}_V(-\text{id}) = \mathcal{L}_V J^2 = J(\mathcal{L}_V J) + (\mathcal{L}_V J)J = -2f$$

i.e. $f = 0$, $\mathcal{L}_V g = 0$ and $\mathcal{L}_V J = 0$. $\square$

It is also straightforward to check that

Lemma 4.6. *On any Kähler manifold if V and JV are both Killing vector fields, they are also parallel with respect to the Levi-Civita connection.*

Proof of 4.1. By the Bochner theorem and 4.5 we have that $H^0(Z, \Theta)$ is the complexification of the Lie algebra of parallel vector fields. Now recall the Weitzenböch decomposition of 1-forms:

$$\Delta = dd^* + d^*d = \nabla^* \nabla + \text{Ric}$$

it says that on a Ricci-flat riemannian manifold a 1-form is harmonic if and only if is parallel with respect to the Levi-Civita connection. Using the metric to pass from 1-forms to vector fields we have:

$$h^0(Z, \Theta) = \dim_{\mathbb{R}}(\text{Lie algebra of parallel vector fields}) = b_1(M)$$

and we are left to prove that $2h^0(M, \Theta) = b_1(M)$. By the Bochner theorem every holomorphic vector field is parallel, so the dual $(0, 1)$ -form is parallel; since M is Kähler, $\Delta = 2\Box = \bar{\partial}\bar{\partial}^* = \bar{\partial}^*\bar{\partial}$ and a $(0, 1)$ -form is parallel if and only if is $\bar{\partial}$ -harmonic; we conclude that

$$h^0(M, \Theta) = h^0(M, \Omega^1) = \frac{1}{2}b_1(M). \quad \square$$

Proof of 4.2. Suppose M has no parallel vector fields, then by the Lichnerowicz theorem and 4.5, $H^0(M, \Theta)$ is the complexification of the Lie algebra of all conformal Killing vector fields on M and therefore isomorphic to $H^0(Z, \Theta)$, and we have proved the result. Then it is enough to show that M admits no parallel holomorphic vector fields.

To show this is true, we first reduce to the case of a minimal surface: suppose M is not minimal (i.e. it contains a holomorphically embedded, irreducible rational curve C with self-intersection $= -1$). Then if $\Theta_{M,C}$ denotes the sheaf of holomorphic vector fields on M which are tangent to C , along C , we have an exact sequence $0 \rightarrow \Theta_{M,C} \rightarrow \Theta_M \rightarrow \nu_{C/M} \rightarrow 0$. As $H^0(C, \nu_{C/M}) \cong H^0(\mathbb{CP}_1, \mathcal{O}(-1)) = 0$, it follows that every holomorphic tangent vector on M is tangent to C , along C . Since $C \cong \mathbb{CP}_1$, every holomorphic vector field vanishes somewhere. (In fact a direct image argument shows that it has to vanish identically, along C .)

However, if M is minimal and the total scalar curvature is nonnegative, Yau [15]¹ has shown that $M \cong \mathbb{CP}_2$ or else is a $\mathbb{CP}_1$ -bundle over a Riemann surface S_g . This says that $\chi(M) \neq 0$, and therefore M has no parallel vector fields unless it is a $\mathbb{CP}_1$ -bundle over a torus; in this case however $\chi(M) = \tau(M) = 0$.

¹Warning: Proposition 4 in [15] is false, counterexample: $\mathbb{CP}_1 \times S_g$.

On the other hand, by Chern-Weil theory [3], when M is anti-self-dual with zero scalar curvature, this implies that M is actually flat, which is absurd. $\square$

Notice that the result of 4.2 holds for any half-conformally flat compact Kähler surface with no parallel holomorphic vector field, e.g. $\mathbb{CP}_2$; or trivially, for any such surface of negative Ricci curvature.

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