

INVARIANT AFFINE CONNECTIONS ON LIE GROUPS

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ABSTRACT. The space of bi-invariant affine connections is determined for arbitrary compact Lie groups. In particular, there is a surprising new family of such connections on $SU(n)$.

INTRODUCTION

Invariant connections on homogeneous spaces have been well understood for years but much of the interest ceased after consideration of the canonical connection in general and the Levi-Civita connection in the case of the frame bundle [5]. The main objective of this paper is to give a complete description of the bi-invariant affine connections on compact Lie groups, i.e., those connections in the frame bundle which are invariant under both left and right translations. A sequel to this paper will extend these results to arbitrary compact irreducible Riemannian symmetric spaces [8].

The first five sections of this paper describe invariant connections in general and affine connections in particular from the point of view of homogeneous principal bundles. These are essentially those principal bundles over homogeneous spaces which admit a lifting of the group action on the base manifold to give a fiber transitive action on the total space. This categorical viewpoint is quite useful for dealing with different descriptions of isomorphic objects. It helps separate the ideas of homogeneity from actual constructions involving Lie groups. The main result in these sections is Theorem 5.1 which gives several equivalent descriptions of the set of invariant affine connections on a homogeneous space.

Sections 6 through 10 deal with the determination of all bi-invariant affine connections on compact Lie groups. Theorem 8.1 gives a complete classification of such connections in the compact simple case. This theorem was essentially the result of a case by case computation using representation theory. The surprising result is that while in most cases the only such connections are from a well-known one-dimensional family (containing the canonical, $(+)$ -, and $(-)$ -connections), in the case of $SU(n)$ there is a second family of such connections. These results are then extended in Theorem 9.1, where the dimension of the space of bi-invariant affine connections is determined for arbitrary compact Lie groups. Finally, §10 includes a more detailed description of these connections

Received by the editors July 21, 1987 and, in revised form, June 25, 1990.

1991 *Mathematics Subject Classification*. Primary 53C05; Secondary 53C30, 22E15.

Key words and phrases. Affine, connection, invariant, homogeneous principal bundle, Lie group.

in the cases of $SU(n)$ and $U(n)$. The six-dimensional space of such connections on $U(n)$ is also quite interesting. In both cases, the extra connections are related to the bilinear operation $\theta(x, y) = i \cdot (x \cdot y + y \cdot x)$ on the Lie algebra $\mathfrak{u}(n)$.

1. HOMOGENEOUS PRINCIPAL BUNDLES

A *homogeneous principal bundle* is a triple $(G, P(M, U, \pi), \alpha)$, where G is a Lie group, $\pi: P \rightarrow M$ is a principal U -bundle over a homogeneous space M , and α is a *homogeneous G -action* on P , i.e., there must be smooth group actions $\alpha_P: G \times P \rightarrow P$ and $\alpha_M: G \times M \rightarrow M$, written multiplicatively, with α_M transitive and such that the compatibility condition $\pi(g \cdot p) = g \cdot \pi(p)$ holds. In addition, the left G -action and the right U -action on P must commute, i.e., $(g \cdot p) \cdot u = g \cdot (p \cdot u)$.

The homogeneous principal bundles form a category, denoted HPB, by letting a morphism $\phi: (G_1, P_1(M_1, U_1, \pi_1), \alpha_1) \rightarrow (G_2, P_2(M_2, U_2, \pi_2), \alpha_2)$ consist of differentiable maps $\phi_P: P_1 \rightarrow P_2$ and $\phi_M: M_1 \rightarrow M_2$ and Lie group homomorphisms $\phi_G: G_1 \rightarrow G_2$ and $\phi_U: U_1 \rightarrow U_2$ satisfying the following conditions:

- (i) $\pi_2 \circ \phi_P = \phi_M \circ \pi_1$,
- (ii) $\phi_P(p \cdot u) = \phi_P(p) \cdot \phi_U(u)$,
- (iii) $\phi_P(g \cdot p) = \phi_G(g) \cdot \phi_P(p)$.

Note, it then follows that

- (iv) $\phi_M(g \cdot m) = \phi_G(g) \cdot \phi_M(m)$.

Let HPB_0 denote the category whose objects are homogeneous principal bundles with base points— $(G, P(M, U, \pi), \alpha, p_0)$, where p_0 is some point in the total space P —and in which morphisms must preserve base points. Let HS and HS_0 be the related categories of homogeneous spaces—with objects (G, M, α) and (G, M, α, m_0) and in which morphisms $\phi = (\phi_G, \phi_M)$ must satisfy condition (iv).

As with homogeneous spaces where the choice of base point gives a correspondence to quotient manifolds, it is the category HPB_0 which is more closely related to standard constructions involving Lie groups. Given $(G, P(M, U, \pi), \alpha, p_0) \in \text{HPB}_0$, let H be the isotropy subgroup at $m_0 = \pi(p_0)$. Define a homomorphism $\lambda: H \rightarrow U$ by letting $h \cdot p_0 = p_0 \cdot \lambda(h)$ and let P_λ be the principal bundle (cf. [5, 6, 7])

$$P_\lambda = G \times_H U = G \times U / (gh, u) \sim (g, \lambda(h)u).$$

This principal U -bundle over G/H has a standard homogeneous G -action defined by $g_1 \cdot [g, u] = [g_1 \cdot g, u]$ and the original bundle is isomorphic in HPB_0 to the bundle $(G, P_\lambda(G/H, U, \pi), \alpha, [e, e])$. So up to isomorphism in HPB_0 , it is the “structure homomorphism” λ which really determines the homogeneous principal bundle.

2. THE FRAME BUNDLE AS A HOMOGENEOUS PRINCIPAL BUNDLE

The frame bundle of a homogeneous space (G, M, α, m_0) can be viewed as a homogeneous principal bundle in a variety of ways. Clearly, it is the differential of left translation (TL_g) which provides the left G -action on the total space. However, the structure group can change, depending on what is meant

by a “frame”, and in addition, different base points give different structure homomorphisms.

A frame p at a point $m \in M$ can be viewed as a real linear isomorphism $p: V \rightarrow T_m M$, where V is some “model” vector space. The set of such frames becomes a homogeneous principal $\text{Aut}(V)$ -bundle by letting

$$g \cdot p \cdot u = TL_g \circ p \circ u.$$

Clearly this action is compatible with the given action on the homogeneous space M . A base point is then some fixed isomorphism $A: V \rightarrow T_{m_0} M$. If W is some other model and if $B: W \rightarrow T_{m_0} M$ is a base point in the corresponding homogeneous principal bundle, then the two bundles are naturally isomorphic in HPB_0 . Indeed, the mappings $\phi_G(g) = g$, $\phi_M(m) = m$, $\phi_P(p) = p \circ A^{-1} \circ B$, and $\phi_U(u) = B^{-1} \circ A \circ u \circ A^{-1} \circ B$ give an explicit HPB_0 -isomorphism between them.

There are three models which arise naturally in this context. The model $V = \mathbb{R}^n$ (where n is the dimension of M) corresponds to the ordered basis viewpoint of frames (by looking at the image under p of the standard basis of $\mathbb{R}^n$). In this case, a base point is the choice of a distinguished basis $(E_1, \dots, E_n)$ for $T_{m_0} M$. The resulting structure homomorphism $\lambda: H \rightarrow GL(n, \mathbb{R})$ is the isotropy representation

$$TL_h(E_i) = E_j \cdot (\lambda(h))_i^j.$$

A second model is given by $V = T_{m_0} M$. In this case the natural base point is the identity isomorphism $V \rightarrow T_{m_0} M$. The resulting structure homomorphism is the differential of left translation $TL: H \rightarrow \text{Aut}(T_{m_0} M)$. Finally, if the homogeneous space is *reductive*, i.e., if the Lie algebra $\mathfrak{g}$ can be expressed as a direct sum $\mathfrak{h} \oplus \mathfrak{m}$ with $\text{Ad}(H)\mathfrak{m} \subseteq \mathfrak{m}$, then the subspace $\mathfrak{m}$ can serve as the model. In this case the differential at the identity of the mapping $g \mapsto g \cdot m_0$ restricts to give an isomorphism $\mathfrak{m} \subseteq \mathfrak{g} = T_e G \rightarrow T_{m_0} M$. This is the natural choice of base point for this version of the frame bundle. The corresponding structure homomorphism is then given by $\text{Ad}: H \rightarrow \text{Aut}(\mathfrak{m})$.

3. INVARIANT CONNECTIONS AND WANG'S THEOREM

By definition, a *connection* A in a principal bundle $P(M, U, \pi)$ is a smoothly varying choice of horizontal subspaces $\mathcal{H}_p^A \subseteq T_p P$. These subspaces must be complementary to the vertical subspaces $\mathcal{V}_p = \text{kernel}(T\pi)$ and they must be invariant under the right action of U on P , i.e. $\mathcal{H}_{p \cdot u}^A = TR_u \mathcal{H}_p^A$. Homogeneous principal bundles can have *homogeneous* or *G-invariant* connections in which the horizontal subspaces are also left G -invariant, i.e. $\mathcal{H}_{g \cdot p}^A = TL_g \mathcal{H}_p^A$. The set of such G -invariant connections will be denoted $\mathcal{A}_G(P)$.

Wang's theorem [11] describes the set of G -invariant connections in the bundle P_λ . Let $\mathfrak{g}$, $\mathfrak{h}$, and $\mathfrak{u}$ be the Lie algebras of G , H , and U respectively and let $T\lambda: \mathfrak{h} \rightarrow \mathfrak{u}$ be the differential of λ . Then the invariant connections in P_λ are in one-to-one correspondence with linear mappings $\xi: \mathfrak{g} \rightarrow \mathfrak{u}$ satisfying the following two conditions:

- (i) $\xi(x) = T\lambda(x) \forall x \in \mathfrak{h}$,
- (ii) $\xi(\text{Ad}_h x) = \text{Ad}_{\lambda(h)} \xi(x) \forall x \in \mathfrak{g}, h \in H$.

In this description, the set of all vectors of the form $x \oplus -\xi(x) \in \mathfrak{g} \oplus \mathfrak{u}$ gives the horizontal subspace at (e, e) of the connection pulled back to the trivial bundle

$G \times U \rightarrow G$. Equivalently, if $\rho: G \times U \rightarrow P_\lambda$ is the mapping $(g, u) \mapsto [g, u]$ then $y = -\xi(x)$ if and only if $T\rho(x \oplus y) \in \mathcal{H}_{[e, e]}^\xi$. Note that this description of invariant connections in terms of linear maps $\xi: \mathfrak{g} \rightarrow \mathfrak{u}$ remains valid for arbitrary bundles in HPB_0 —recall that the choice of base point defines the subgroup H and the homomorphism $\lambda: H \rightarrow U$ canonically.

4. PULLBACK AND ASSOCIATION IN THE CATEGORY HPB

Two standard constructions involving principal bundles (pullback and association) carry over to the category of homogeneous principal bundles. If $(G_2, P_2(M_2, U_2, \pi_2), \alpha_2) \in \text{HPB}$ and if $\phi: (G_1, M_1, \alpha_1) \rightarrow (G_2, M_2, \alpha_2)$ is an HS-morphism then the *pullback bundle*

$$\phi^*(P_2) = \{(m, p) \in M_1 \times P_2 \mid \phi_M(m) = \pi_2(p)\}$$

has a homogeneous G_1 -action given by $g \cdot (m, p) = (g \cdot m, \phi_G(g) \cdot p)$. In addition, ϕ extends to give an HPB-morphism $\phi^*(P_2) \rightarrow P_2$ by letting $\phi_P(m, p) = p$ and $\phi_U(u) = u$. More generally, if

$$\phi: (G_1, P_1(M_1, U_1, \pi_1), \alpha_1) \rightarrow (G_2, P_2(M_2, U_2, \pi_2), \alpha_2)$$

is an HPB-morphism with ϕ_U an isomorphism then P_1 and $\phi^*(P_2)$ are isomorphic in HPB. In addition, the following proposition shows that invariant connections can be pulled back under such mappings.

Proposition 4.1. *If*

$$\phi: (G_1, P_1(M_1, U_1, \pi_1), \alpha_1, p_1) \rightarrow (G_2, P_2(M_2, U_2, \pi_2), \alpha_2, p_2)$$

is an HPB-morphism with ϕ_U an isomorphism then there is an induced mapping $\phi^: \mathcal{A}_{G_2}(P_2) \rightarrow \mathcal{A}_{G_1}(P_1)$ defined by*

$$\mathcal{H}_p^{\phi^*(A)} = \{x \in T_p P_1 \mid T\phi_P(x) \in \mathcal{H}_{\phi_P(p)}^A\}.$$

The mapping ϕ^ is injective if the tangent mapping $T\phi_G: \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ of Lie algebras is surjective. If the morphism ϕ is HPB_0 then ϕ^* is also given by*

$$\phi^*(\xi) = T\phi_U^{-1} \circ \xi \circ T\phi_G: \mathfrak{g}_1 \rightarrow \mathfrak{u}_1.$$

Proof. The fact that this gives a well-defined connection in P_1 follows from Proposition II.6.2 in [5]. What needs to be proven is that the induced connection is G_1 -invariant and for that it suffices to show $TL_g \mathcal{H}_p^{\phi^*(A)} \subseteq \mathcal{H}_{g \cdot p}^{\phi^*(A)}$. Let $x \in \mathcal{H}_p^{\phi^*(A)}$. Then by property (iii) for HPB-morphisms, it follows that

$$T\phi_P TL_g(x) = TL_{\phi_G(g)} T\phi_P(x) \in TL_{\phi_G(g)} \mathcal{H}_{\phi_P(p)}^A = \mathcal{H}_{\phi_P(g \cdot p)}^A$$

and thus the connection $\phi^*(A) \in \mathcal{A}_{G_1}(P_1)$. If $T\phi_G$ is surjective then the mapping $T\phi_P: T_p P_1 \rightarrow T_{\phi_P(p)} P_2$ is surjective for all $p \in P_1$. This is clear because ϕ_U is an isomorphism and because the combination of the left G -action and the right U -action on a homogeneous principal bundle is transitive. So the horizontal subspaces of the original connection A can be recovered from the horizontal subspaces of the induced connection $\phi^*(A)$ and the mapping ϕ^* is injective. As for the description of ϕ^* in terms of the linear maps $\xi: \mathfrak{g} \rightarrow \mathfrak{u}$, let $\rho_i: G_i \times U_i \rightarrow P_i$ for $i = 1, 2$ be defined by $\rho_i(g, u) = g \cdot p_i \cdot u$ and let $\hat{\phi}(g, u) = (\phi_G(g), \phi_U(u)): G_1 \times U_1 \rightarrow G_2 \times U_2$.

Now $y = -(\phi^*(\xi))(x)$ if and only if $T\rho_1(x \oplus y) \in \mathcal{H}_{p_1}^{\phi^*(\xi)}$, i.e., if and only if $T\phi_P T\rho_1(x \oplus y) \in \mathcal{H}_{p_2}^{\xi}$. Since $\phi_P \circ \rho_1 = \rho_2 \circ \tilde{\phi}$, this is true if and only if $T\tilde{\phi}(x \oplus y) = z \oplus -\xi(z)$ for some $z \in \mathfrak{g}_2$. This implies $-\xi(T\phi_G(x)) = T\phi_U(y)$ or equivalently $\phi^*(\xi) = T\phi_U^{-1} \circ \xi \circ T\phi_G$. $\square$

The process of forming associated principal bundles also carries over to the homogeneous category. If $(G_1, P_1(M_1, U_1, \pi_1), \alpha_1) \in \text{HPB}$ and if $\mu: U_1 \rightarrow U_2$ is a Lie group homomorphism then the associated principal U_2 -bundle

$$P_1 \times_{U_1} U_2 = P_1 \times U_2 / (p \cdot u_1, u_2) \sim (p, \mu(u_1) \cdot u_2)$$

has a homogeneous G_1 -action given by $g \cdot [p, u] = [g \cdot p, u]$. In addition, the mappings $\phi_P(p) = [p, e]$, $\phi_M(m) = m$, $\phi_G(g) = g$, and $\phi_U(u) = \mu(u)$ define an HPB-morphism $P_1 \rightarrow P_1 \times_{U_1} U_2$. More generally, if $\phi: (G_1, P_1(M_1, U_1, \pi_1), \alpha_1) \rightarrow (G_2, P_2(M_2, U_2, \pi_2), \alpha_2)$ is an HPB-morphism for which the induced map (ϕ_G, ϕ_M) on the base manifolds is an HS-isomorphism then the bundle P_2 is isomorphic in HPB to the associated bundle defined by the homomorphism $\mu = \phi_U: U_1 \rightarrow U_2$. In addition, the following proposition shows that invariant connections move forward under such mappings.

Proposition 4.2. *If*

$$\phi: (G_1, P_1(M_1, U_1, \pi_1), \alpha_1, p_1) \rightarrow (G_2, P_2(M_2, U_2, \pi_2), \alpha_2, p_2)$$

is an HPB-morphism for which the induced map (ϕ_G, ϕ_M) of base manifolds is an HS-isomorphism then there is an induced map $\phi_: \mathcal{A}_{G_1}(P_1) \rightarrow \mathcal{A}_{G_2}(P_2)$ defined by*

$$\mathcal{H}_{\phi_P(p) \cdot u}^{\phi_*(A)} = TR_u T\phi_P \mathcal{H}_p^A.$$

The mapping ϕ_ is injective if the tangent mapping $T\phi_U: \mathfrak{u}_1 \rightarrow \mathfrak{u}_2$ of Lie algebras is injective. If the morphism ϕ is HPB₀ then ϕ_* is also given by*

$$\phi_*(\xi) = T\phi_U \circ \xi \circ T\phi_G^{-1}: \mathfrak{g}_2 \rightarrow \mathfrak{u}_2.$$

Proof. The fact that this gives a well-defined connection in P_2 follows from Proposition II.6.1 in [5]. What needs to be proven is that the induced connection is G_2 -invariant. Property (iii) for HPB-morphisms implies $L_{\phi_G(g)} \circ R_u \circ \phi_P = R_u \circ \phi_P \circ L_g$. This gives

$$TL_{\phi_G(g)} \mathcal{H}_{\phi_P(p) \cdot u}^{\phi_*(A)} = TR_u T\phi_P TL_g \mathcal{H}_p^A = \mathcal{H}_{\phi_P(g \cdot p) \cdot u}^{\phi_*(A)} = \mathcal{H}_{\phi_G(g) \cdot \phi_P(p) \cdot u}^{\phi_*(A)}.$$

Since the ϕ_G in an HS-isomorphism (ϕ_G, ϕ_M) must be a Lie group isomorphism, it follows that the connection $\phi_*(A) \in \mathcal{A}_{G_2}(P_2)$. If $T\phi_U$ is injective then $TR_u \circ T\phi_P: T_p P_1 \rightarrow T_{\phi_P(p) \cdot u} P_2$ is injective for all $p \in P_1$ and $u \in U_2$ and thus ϕ_* is injective. This uses the assumption that (ϕ_G, ϕ_M) is an HS-isomorphism so that tangent vectors corresponding to the action of G_1 are not annihilated by $T\phi_P$. As for the description of ϕ_* in terms of the linear maps $\xi: \mathfrak{g} \rightarrow \mathfrak{u}$, let ρ_1 and ρ_2 be as in the proof of Proposition 4.1. Then

$$\mathcal{H}_{p_2}^{\phi_*(\xi)} = T\phi_P T\rho_1 \{x \oplus -\xi(x)\} = T\rho_2 \{T\phi_G(x) \oplus -T\phi_U(\xi(x))\}.$$

Since $T\phi_G$ is an isomorphism, this implies $\phi_*(\xi) = T\phi_U \circ \xi \circ T\phi_G^{-1}$. $\square$

5. INVARIANT AFFINE CONNECTIONS

An invariant affine connection on a homogeneous space (G, M, α) is a connection in the frame bundle which is also G -invariant. By applying the results

in §3 to the different models of the frame bundle in §2, several descriptions of the space of invariant affine connections can be obtained. Specifically, there is the following (cf. [10]).

Theorem 5.1. *Let (G, M, α, m_0) be a reductive homogeneous space with reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, where $\text{Ad}(H)\mathfrak{m} \subseteq \mathfrak{m}$. Then the set $\mathcal{A}_G(\text{Fr}(M))$ of invariant affine connections is given by any of the following six equivalent descriptions:*

- (i) $\{\xi : \mathfrak{g} \rightarrow \mathfrak{gl}(n, \mathbb{R}) \mid \xi \text{ linear}, \xi|_{\mathfrak{h}} = T\lambda, \xi(\text{Ad}_h x) = \lambda(h)\xi(x)\lambda(h)^{-1}\},$
- (ii) $\{\xi : \mathfrak{g} \rightarrow \text{End}(\mathfrak{m}) \mid \xi \text{ linear}, \xi|_{\mathfrak{h}} = \text{ad}, \xi(\text{Ad}_h x) = \text{Ad}_h \circ \xi(x) \circ \text{Ad}_{h^{-1}}\},$
- (iii) $\{\xi : \mathfrak{m} \rightarrow \text{End}(\mathfrak{m}) \mid \xi \text{ linear}, \xi(\text{Ad}_h x) = \text{Ad}_h \circ \xi(x) \circ \text{Ad}_{h^{-1}}\},$
- (iv) $\{\eta : \mathfrak{m} \times \mathfrak{m} \rightarrow \mathfrak{m} \mid \eta \text{ bilinear}, \eta(\text{Ad}_h x, \text{Ad}_h y) = \text{Ad}_h \eta(x, y)\},$
- (v) $\{\eta : T_{m_0} \times T_{m_0} \rightarrow T_{m_0} \mid \eta \text{ bilinear}, \eta(TL_h x, TL_h y) = TL_h \eta(x, y)\},$
- (vi) $\text{Hom}_H^{\mathbb{R}}(\mathfrak{m} \otimes \mathfrak{m}, \mathfrak{m}).$

Proof. Descriptions (i) and (ii) are immediate applications of Wang's theorem to the frame bundle—modeled on $\mathbb{R}^n$ in (i) and on $\mathfrak{m}$ in (ii). Note, in (i), that λ is the isotropy representation described in §2. The ξ in (iii) is just the restriction to $\mathfrak{m}$ of the ξ in (ii). Conversely, any $\xi : \mathfrak{m} \rightarrow \text{End}(\mathfrak{m})$ satisfying the conditions in (iii) can be extended to all of $\mathfrak{g}$ by linearity and by requiring $\xi|_{\mathfrak{h}} = \text{ad}$. The correspondence between (iii) and (iv) is given by $\xi(x)(y) = \eta(x, y)$. It is easy to see that the conditions in (iii) and (iv) are equivalent. Next, the equivalence between (iv) and (v) comes from the canonical isomorphism between $\mathfrak{m}$ and $T_{m_0}M$ given by the differential at the identity of the mapping $g \mapsto g \cdot m_0$. Finally, description (vi) comes from (iv) and the universal mapping property for tensor product. Any bilinear map $\eta : \mathfrak{m} \times \mathfrak{m} \rightarrow \mathfrak{m}$ defines a unique linear map $\tilde{\eta} : \mathfrak{m} \otimes \mathfrak{m} \rightarrow \mathfrak{m}$ satisfying $\tilde{\eta}(x \otimes y) = \eta(x, y) \quad \forall x, y \in \mathfrak{m}$. The condition that η define an invariant affine connection on G/H is equivalent to the requirement that $\tilde{\eta}$ intertwine the actions of H on $\mathfrak{m} \otimes \mathfrak{m}$ and $\mathfrak{m}$. $\square$

The frame bundle of a homogeneous space (G, M, α, m_0) is associated to the canonical bundle $(G, G(M, H, \pi), \alpha, e)$ by the HPB₀-morphism $\phi_P(g) = TL_g \circ T\pi : \mathfrak{m} \rightarrow T_{g \cdot m_0}M$, $\phi_M(m) = m$, $\phi_G(g) = g$, $\phi_U(h) = \text{Ad}_h$. By Wang's theorem invariant connections in the canonical bundle are given by linear mappings $\zeta : \mathfrak{m} \rightarrow \mathfrak{h}$ satisfying $\zeta(\text{Ad}_h x) = \text{Ad}_h \zeta(x)$. By Proposition 4.2, any such ζ gives an invariant affine connection $\xi = \phi_*(\zeta) = \text{ad} \circ \zeta$. Alternatively, this invariant affine connection is given by $\eta(x, y) = [\zeta(x), y]$. This mapping of connections is injective if the representation $\text{ad} : \mathfrak{h} \rightarrow \text{End}(\mathfrak{m})$ is faithful. In the reductive case, as is being assumed here, this will be true if the action of G on M is almost effective.

6. INVARIANT AFFINE CONNECTIONS ON LIE GROUPS

The group action of a Lie group G allows the group to be viewed as a homogeneous space in several ways. If the action under consideration is left translation, then the isotropy subgroup is trivial and the corresponding “left invariant” affine connections are given by arbitrary bilinear maps $\eta : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ —Theorem 5.1 gives no restriction in this case. A more interesting action is given by a combination of left and right translation. This amounts to viewing G as a homogeneous space for the group $G \times G$, the precise action being

$\alpha_M((g_1, g_2), g) = g_1 \cdot g \cdot g_2^{-1}$. The resulting invariant affine connections will then be “bi-invariant,” i.e., invariant under both the left and right actions of the Lie group G .

Theorem 6.1. *The space of bi-invariant affine connections on a Lie group G is given by*

$$\{\eta : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g} \mid \eta \text{ bilinear}, \eta(\text{Ad}_g x, \text{Ad}_g y) = \text{Ad}_g \eta(x, y) \forall g \in G\}$$

or, equivalently, by the space $\text{Hom}_G^{\mathbb{R}}(\mathfrak{g} \otimes \mathfrak{g}, \mathfrak{g})$.

Proof. The isotropy subgroup for the action of $G \times G$ is the diagonal subgroup $\Delta = \{(g, g)\}$. The Lie algebra of Δ is $\mathfrak{h} = \{(x, x) \in \mathfrak{g} \oplus \mathfrak{g}\}$ and the reductive complement $\mathfrak{m} = \{(x, -x) \in \mathfrak{g} \oplus \mathfrak{g}\}$. Thus $\mathfrak{h}$ and $\mathfrak{m}$ are both isomorphic to $\mathfrak{g}$ as $\Delta (= G)$ -modules. The theorem then follows immediately from Theorem 5.1. $\square$

When the Lie group G is compact and simple the canonical bundle $G \times G \rightarrow G$ has a one-dimensional family of invariant connections corresponding to the mappings $\zeta(x, -x) = (\mu x, \mu x) : \mathfrak{m} \rightarrow \mathfrak{h}$ for $\mu \in \mathbb{R}$. Cases $\mu = 0, 1$, and -1 correspond to the so-called canonical, $(+)$ -, and $(-)$ -connections on a Lie group. The corresponding invariant affine connections are then given by $\eta(x, y) = \mu \cdot [x, y] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$. In §8 it will be proven that these are the only invariant affine connections in all cases except for $SU(n)$ for $n \geq 3$.

7. CASIMIR OPERATORS AND A COMPUTATIONAL LEMMA

Given a representation $\rho : G \rightarrow \text{Aut}(V)$, the *Casimir operator* of V is defined by $\Gamma(V) = \sum (T\rho(e_i))^2$, where $\{e_i\}$ is an orthonormal basis for the Lie algebra $\mathfrak{g}$. This depends on the choice of an inner product on $\mathfrak{g}$ but not on the choice of basis. If G is compact and simple, then a natural inner product on $\mathfrak{g}$ is provided by the negative of the Killing form $B_{\mathfrak{g}}(x, y) = \text{trace}(\text{ad}_x \text{ad}_y)$, but in general, any *ad*-invariant inner product ($\langle [x, y], z \rangle = \langle x, [y, z] \rangle$) can be used. If V is irreducible then $\Gamma(V)$ will be a negative or zero scalar. The following lemma will be needed for the computations in §8.

Lemma 7.1. *Suppose $\mathfrak{g}$ is simple and let $V \subseteq \wedge^2 \mathfrak{g}$ be the kernel of the natural map $\wedge^2 \mathfrak{g} \rightarrow \mathfrak{g}$ induced by the alternating bilinear map $(x, y) \mapsto [x, y] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$. Then V is a G -module and the Casimir operator of V is exactly twice that of the adjoint representation $\mathfrak{g}$.*

Proof. Since $\text{Ad}_g[x, y] = [\text{Ad}_g x, \text{Ad}_g y]$ the natural map $\wedge^2 \mathfrak{g} \rightarrow \mathfrak{g}$ is a G -module map and thus V is G -invariant. Let $\{e_i\}$ be an orthonormal basis for $\mathfrak{g}$ and let $[e_i, e_j] = C_{ij}^k e_k$. Because the basis is orthonormal relative to an *ad*-invariant inner product, the structure constants C_{ij}^k have a cyclic symmetry in i, j , and k along with the usual skew-symmetry. In addition, the Jacobi identity implies

$$C_{jm}^i C_{ni}^k = -C_{mn}^i C_{ji}^k - C_{nj}^i C_{mi}^k.$$

A general element of $\wedge^2 \mathfrak{g}$ is given by $a^{jk} e_j \wedge e_k$, where the summation goes over all j and k by assuming $a^{jk} = -a^{kj}$. Suppose $a^{jk} e_j \wedge e_k \in V$, i.e.,

$a^{jk}C_{jk}^i e_i = 0$. Then

$$\begin{aligned} \sum (\rho(e_i))^2 a^{jk} e_j \wedge e_k &= a^{jk} \rho(e_i) ([e_i, e_j] \wedge e_k + e_j \wedge [e_i, e_k]) \\ &= a^{jk} [e_i, [e_i, e_j]] \wedge e_k + a^{jk} e_j \wedge [e_i, [e_i, e_k]] + 2 \cdot a^{jk} [e_i, e_j] \wedge [e_i, e_k] \\ &= 2 \cdot \Gamma(\mathfrak{g}) \cdot (a^{jk} e_j \wedge e_k). \end{aligned}$$

This follows because

$$[e_i, [e_i, x]] = \Gamma(\mathfrak{g}) \cdot x \quad \text{for all } x \in \mathfrak{g}$$

and because

$$\begin{aligned} a^{jk} [e_i, e_j] \wedge [e_i, e_k] &= a^{jk} C_{ij}^m C_{ik}^n e_m \wedge e_n \\ &= -a^{jk} (C_{mn}^i C_{ji}^k + C_{nj}^i C_{mi}^k) e_m \wedge e_n \\ &= -a^{jk} C_{mj}^i C_{ni}^k e_n \wedge e_m \\ &= -a^{jk} C_{ij}^m C_{ik}^n e_m \wedge e_n = 0. \quad \square \end{aligned}$$

8. REPRESENTATION THEORY AND BI-INVARIANT AFFINE CONNECTIONS ON SIMPLE LIE GROUPS

The stage is now set for the main result of this paper.

Theorem 8.1. *If G is a compact simple Lie group then the space of bi-invariant affine connections on G is one-dimensional in all cases except for $SU(n)$ when $n \geq 3$. In the case of $SU(n)$ there is a two-dimensional family of bi-invariant affine connections. Note that $SO(6)$ behaves the same way as $SU(4)$.*

Proof. By Theorem 6.1, the space of bi-invariant affine connections consists of all intertwining maps $\mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$. Since $\mathfrak{g}$ is irreducible and of real type, Schur's lemma shows that the dimension of the space of such maps is equal to the multiplicity of $\mathfrak{g}$ in $\mathfrak{g} \otimes \mathfrak{g}$ (see [2]). This multiplicity can also be computed after complexifying all the representations—a step which makes the representation theory more convenient. A case by case computation was used to determine the actual multiplicities.

The space $\mathfrak{g} \otimes \mathfrak{g}$ splits into a direct sum $S^2 \mathfrak{g} \oplus \wedge^2 \mathfrak{g}$ of symmetric and skew-symmetric parts. Table I gives the decomposition of these spaces into irreducible parts. In the table, representations are specified by maximal weights. These weights depend on an ordering of simple roots and the ordering which is used is that which appears in [4]. The results in the table were computed primarily by means of Young-tableaux [3] and dimension counting [1, 9]. Even in the case of the orthogonal groups, the tableau approach was reasonably simple because the adjoint representation is not spinor. The fact that the trivial representation occurs exactly once (and in $S^2 \mathfrak{g}$) was needed in the exceptional cases where dimension counting was used. Finally, Lemma 7.1 was particularly useful for determining the decomposition of $\wedge^2 \mathfrak{g}$. The space $\wedge^2 \mathfrak{g}$ had to include one copy of $\mathfrak{g}$ (because the natural map $\wedge^2 \mathfrak{g} \rightarrow \mathfrak{g}$ is surjective) and the remaining part, although possibly reducible, had to have a specific Casimir operator. These operators could be computed from the techniques in [6] and the tables in [9]. The copy of $\mathfrak{g}$ in $\wedge^2 \mathfrak{g}$ corresponds to the affine connections described after Theorem 6.1. The table shows that the only other bi-invariant affine connections occur on $SU(n)$ for $n \geq 3$. $\square$

TABLE I. Decompositions of $S^2\mathfrak{g}$ and $\wedge^2\mathfrak{g}$ into irreducible G -modules for the compact simple Lie groups. Representations specified by maximal weights

G	$\mathfrak{g}$	$S^2\mathfrak{g}$	$\wedge^2\mathfrak{g}$
A_1	$(2\lambda_1)$	$(4\lambda_1) \oplus (0)$	$(2\lambda_1)$
A_2	$(\lambda_1 + \lambda_2)$	$(2\lambda_1 + 2\lambda_2) \oplus (\lambda_1 + \lambda_2) \oplus (0)$	$(3\lambda_1) \oplus (3\lambda_2) \oplus (\lambda_1 + \lambda_2)$
A_3	$(\lambda_1 + \lambda_3)$	$(2\lambda_1 + 2\lambda_3) \oplus (2\lambda_2) \oplus (\lambda_1 + \lambda_3) \oplus (0)$	$(2\lambda_1 + \lambda_2) \oplus (\lambda_2 + 2\lambda_3) \oplus (\lambda_1 + \lambda_3)$
$A_n (n > 3)$	$(\lambda_1 + \lambda_n)$	$(2\lambda_1 + 2\lambda_n) \oplus (\lambda_2 + \lambda_{n-1}) \oplus (\lambda_1 + \lambda_n) \oplus (0)$	$(2\lambda_1 + \lambda_{n-1}) \oplus (\lambda_2 + 2\lambda_n) \oplus (\lambda_1 + \lambda_n)$
B_2	$(2\lambda_2)$	$(4\lambda_2) \oplus (2\lambda_1) \oplus (\lambda_1) \oplus (0)$	$(\lambda_1 + 2\lambda_2) \oplus (2\lambda_2)$
B_3	(λ_2)	$(2\lambda_2) \oplus (2\lambda_3) \oplus (2\lambda_1) \oplus (0)$	$(\lambda_1 + 2\lambda_3) \oplus (\lambda_2)$
B_4	(λ_2)	$(2\lambda_2) \oplus (2\lambda_4) \oplus (2\lambda_1) \oplus (0)$	$(\lambda_1 + \lambda_3) \oplus (\lambda_2)$
$B_n (n > 4)$	(λ_2)	$(2\lambda_2) \oplus (\lambda_4) \oplus (2\lambda_1) \oplus (0)$	$(\lambda_1 + \lambda_3) \oplus (\lambda_2)$
$C_n (n > 1)$	$(2\lambda_1)$	$(4\lambda_1) \oplus (2\lambda_2) \oplus (\lambda_2) \oplus (0)$	$(2\lambda_1 + \lambda_2) \oplus (2\lambda_1)$
D_3	$(\lambda_2 + \lambda_3)$	$(2\lambda_2 + 2\lambda_3) \oplus (2\lambda_1) \oplus (\lambda_2 + \lambda_3) \oplus (0)$	$(\lambda_1 + 2\lambda_2) \oplus (\lambda_1 + 2\lambda_3) \oplus (\lambda_2 + \lambda_3)$
D_4	(λ_2)	$(2\lambda_2) \oplus (2\lambda_1) \oplus (2\lambda_3) \oplus (2\lambda_4) \oplus (0)$	$(\lambda_1 + \lambda_3 + \lambda_4) \oplus (\lambda_2)$
D_5	(λ_2)	$(2\lambda_2) \oplus (\lambda_4 + \lambda_5) \oplus (2\lambda_1) \oplus (0)$	$(\lambda_1 + \lambda_3) \oplus (\lambda_2)$
$D_n (n > 5)$	(λ_2)	$(2\lambda_2) \oplus (\lambda_4) \oplus (2\lambda_1) \oplus (0)$	$(\lambda_1 + \lambda_3) \oplus (\lambda_2)$
E_6	(λ_2)	$(2\lambda_2) \oplus (\lambda_1 + \lambda_6) \oplus (0)$	$(\lambda_4) \oplus (\lambda_2)$
E_7	(λ_1)	$(2\lambda_1) \oplus (\lambda_6) \oplus (0)$	$(\lambda_3) \oplus (\lambda_1)$
E_8	(λ_8)	$(2\lambda_8) \oplus (\lambda_1) \oplus (0)$	$(\lambda_7) \oplus (\lambda_8)$
F_4	(λ_1)	$(2\lambda_1) \oplus (2\lambda_4) \oplus (0)$	$(\lambda_2) \oplus (\lambda_1)$
G_2	(λ_2)	$(2\lambda_2) \oplus (2\lambda_1) \oplus (0)$	$(3\lambda_1) \oplus (\lambda_2)$

9. BI-INVARIANT AFFINE CONNECTIONS ON COMPACT LIE GROUPS

The following theorem extends the results in Theorem 8.1 to arbitrary compact Lie groups.

Theorem 9.1. *Suppose G is a compact Lie group and suppose the Lie algebra of G splits as $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_q$, where the center $\mathfrak{z}$ has dimension p and where the $\mathfrak{g}_i$ are the simple ideals in $\mathfrak{g}$. Let r be the number of $\mathfrak{su}(n)$'s ($n \geq 3$) in $\mathfrak{g}$. Then the dimension of the space of bi-invariant affine connections on G is given by*

$$\dim(\mathcal{A}_{G \times G} \text{Fr}(G)) = p^3 + 3 \cdot p \cdot q + q + r.$$

Proof. Irreducible representations of $\mathfrak{g}$ which are trivial on $\mathfrak{z}$ can be expressed as $(0, \rho_1, \dots, \rho_q)$. This represents the tensor product of the trivial representation of $\mathfrak{z}$ with representations ρ_i of $\mathfrak{g}_i$ for $i = 1, \dots, q$. In this notation

$$\mathfrak{g} = p \cdot (0, 0, \dots, 0) \oplus \bigoplus_{i=1}^q (0, 0, \dots, \mathfrak{g}_i, \dots, 0).$$

Computing $\mathfrak{g} \otimes \mathfrak{g}$ gives

$$\begin{aligned}
\mathfrak{g} \otimes \mathfrak{g} = & p^2 \cdot (0, 0, \dots, 0) \\
& \oplus \bigoplus_{i=1}^q 2 \cdot p \cdot (0, 0, \dots, \mathfrak{g}_i, \dots, 0) \\
& \oplus \bigoplus_{i=1}^q (0, 0, \dots, \mathfrak{g}_i \otimes \mathfrak{g}_i, \dots, 0) \\
& \oplus \bigoplus_{i < j} 2 \cdot (0, 0, \dots, \mathfrak{g}_i, \dots, \mathfrak{g}_j, \dots, 0).
\end{aligned}$$

The first line of this expression results in p^3 independent intertwining maps $\mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$. The second line gives $2 \cdot p \cdot q$ more. The third line gives $p \cdot q$ maps to the center part of $\mathfrak{g}$ and, by Theorem 8.1, $q + r$ more maps to the simple parts of $\mathfrak{g}$. Finally, there are no such maps between the representations in the fourth line and $\mathfrak{g}$. $\square$

10. AFFINE CONNECTIONS ON $SU(n)$ AND $U(n)$

Theorem 9.1 shows that the Lie group $U(n)$ has a six-dimensional family of bi-invariant affine connections. These connections are described more precisely in the following theorem.

Theorem 10.1. *In terms of bilinear maps $\eta: \mathfrak{u}(n) \times \mathfrak{u}(n) \rightarrow \mathfrak{u}(n)$, a basis for the six-dimensional space of bi-invariant affine connections on $U(n)$ ($n \geq 3$) is given by*

$$\begin{aligned}
\eta_1(x, y) &= x \cdot y - y \cdot x, & \eta_2(x, y) &= i \cdot (x \cdot y + y \cdot x), \\
\eta_3(x, y) &= i \cdot \text{tr}(x) \cdot y, & \eta_4(x, y) &= i \cdot \text{tr}(y) \cdot x, \\
\eta_5(x, y) &= \text{tr}(x \cdot y) \cdot iI, & \eta_6(x, y) &= \text{tr}(x) \cdot \text{tr}(y) \cdot iI.
\end{aligned}$$

Here, I is the $n \times n$ identity matrix.

Proof. First, it is straightforward to verify that the η_j are all bilinear and that they preserve $\mathfrak{u}(n)$. In addition, they all satisfy the condition $\eta(\text{Ad}_g x, \text{Ad}_g y) = \text{Ad}_g \eta(x, y)$. Since Theorem 9.1 shows that the space of bi-invariant affine connections is six-dimensional ($p = q = r = 1$), all that needs to be proven is that the η_j are linearly independent. Suppose $\sum a_j \eta_j(x, y) = 0 \quad \forall x, y \in \mathfrak{g}$. Let E_{jk} be the matrix with a 1 in the (j, k) -position and 0's elsewhere. Letting $x = i \cdot E_{11} - i \cdot E_{22}$ and $y = i \cdot E_{22} - i \cdot E_{33}$ gives $a_2 = a_5 = 0$. Next, by letting $x = i \cdot E_{11}$ and $y = i \cdot E_{22}$ it follows that $a_3 = a_4 = a_6 = 0$. Finally, the values $x = E_{12} - E_{21}$ and $y = i \cdot E_{12} + i \cdot E_{21}$ imply $a_1 = 0$. $\square$

Note that $U(2)$ only has a five-dimensional family of bi-invariant affine connections. The linear dependence of the η_j in the case $n = 2$ corresponds to the following identity in $\mathfrak{gl}(2, \mathbb{C})$:

$$(x \cdot y + y \cdot x) = \text{tr}(x) \cdot y + \text{tr}(y) \cdot x + \text{tr}(x \cdot y) \cdot I - \text{tr}(x) \cdot \text{tr}(y) \cdot I.$$

By Theorem 8.1 the Lie group $SU(n)$ has a two-dimensional family of bi-invariant affine connections. As with $U(n)$ it is simple to verify that these connections are given by

$$\eta(x, y) = \mu \cdot [x, y] + \nu \cdot \left(i \cdot (x \cdot y + y \cdot x) - \frac{2}{n} \cdot \text{tr}(x \cdot y) \cdot iI \right)$$

mapping $\mathfrak{su}(n) \times \mathfrak{su}(n) \rightarrow \mathfrak{su}(n)$ for arbitrary real numbers μ and ν .

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