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ISOPARAMETRIC SUBMANIFOLDS OF HYPERBOLIC SPACES

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ABSTRACT. In this paper we prove a decomposition theorem for isoparametric submanifolds of hyperbolic spaces. And as a consequence we obtain all polar actions on hyperbolic spaces. We also prove that any isoparametric submanifold of infinite dimensional hyperbolic space is either totally geodesic, or finite dimensional.

0. INTRODUCTION

In the late 1930s Élie Cartan defined the notion of isoparametric hypersurface of a space form, and he proved that an isoparametric hypersurface of Euclidean space is a totally umbilic hypersurface; an isoparametric hypersurface of a hyperbolic space is either a totally umbilic hypersurface, or the standard product $S^k \times H^{n-k}$ in H^{n+1}. But for the sphere case, isoparametric hypersurfaces turn out to be very complicated [Cal–4]. In the last ten years many people carried forward this research, see [Ab, FKM, Mü, OT], but the complete classification is still not known. Recently, the general theory of higher codimensional isoparametric submanifolds of Euclidean space and Hilbert space has been studied in [Ha, CW1–3, Te1–4, HPT, PT2].

Let $R^{n+k,1}$ be the Lorentz space with the nondegenerate symmetric bilinear form $\langle x, y\rangle = \sum_{i=1}^{n+k} x_i y_i - x_{n+k+1}y_{n+k+1}$, and $H^{n+k} = \{x \in R^{n+k,1} \mid \langle x, x\rangle = -1, x_{n+k+1} > 0\}$, the standard isometric embedding of hyperbolic space with sectional curvature -1 into $R^{n+k,1}$. It is well known that any totally umbilic complete submanifold of H^{n+k} is $L(V, u) = H^{n+k} \cap (V+u)$, where V is a linear subspace of $R^{n+k,1}$ and $u \in R^{n+k,1}$. In fact, if $V \subset R^{n+k,1}$ is a Euclidean subspace, i.e., $\langle\ ,\ \rangle|V$ is positive definite, then $L(V, u)$ is a sphere with sectional curvature $-1/(\langle u, u\rangle + 1)$, where $u \perp V$; if $V \subset R^{n+k,1}$ is a Lorentz subspace, i.e., $\langle\ ,\ \rangle|V$ is a nondegenerate symmetric bilinear form with index 1, then $L(V, u)$ is a hyperbolic space with sectional curvature $-1/(\langle u, u\rangle + 1)$, where $u \perp V$; if $V \subset R^{n+k,1}$ is degenerate, i.e., $\langle\ ,\ \rangle|V$ is a degenerate symmetric bilinear form, then $L(V, u)$ is flat and isometric to a Euclidean space. We will call $L(V, u)$ *spherical* if $V \subset R^{n+k,1}$ is a Euclidean subspace, *hyperbolic* if $V \subset R^{n+k,1}$ is a Lorentz subspace, and *flat* if $V \subset R^{n+k,1}$ is degenerate.

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A submanifold M^n of H^{n+k} is called *isoparametric* if it has a globally flat normal bundle and the principal curvatures along any parallel normal vector field are constant. In this paper we study isoparametric submanifolds of hyperbolic space and show that any isoparametric submanifold M^n of H^{n+k} is either an isoparametric submanifold of a totally umbilic hypersurface of H^{n+k}, or a standard product of an isoparametric submanifold of a spherical umbilic submanifold and a hyperbolic umbilic submanifold in H^{n+k}. As a consequence we are able to classify all polar actions of connected, closed Lie subgroups of $O(n, 1)$ on H^n.

Let V be a separable Hilbert space. Let $\widetilde{V} = R \oplus V$ with the inner product

$$\langle (s, x), (t, y) \rangle = -st + \langle x, y \rangle \quad \text{where } s, t \in R, \ x, y \in V.$$

Then $\widetilde{V}$ is a Lorentz Hilbert space. Let $H(\widetilde{V}) = \{(s, x) \in \widetilde{V} \mid -s^2 + \langle x, x \rangle = -1, s > 0\}$. It is well known that $H(\widetilde{V})$ is a Riemannian Hilbert hyperbolic space with constant sectional curvature -1. In this paper we prove that if M is a submanifold of a Hilbert hyperbolic space $H(\widetilde{V})$ with finite codimension, and M satisfies: (1) $\nu(M)$, the normal bundle of M in $H(\widetilde{V})$, is globally flat, (2) for any parallel normal vector field v, the shape operator $A_{v(x)}$ is orthogonally equivalent to the shape operator $A_{v(y)}$ for all $x, y \in M$, (3) the shape operator A_v is a compact operator for any $v \in \nu(M)_x$, then M is either of finite dimension, or a totally geodesic submanifold of $H(\widetilde{V})$, i.e., $M = H(\widetilde{V}) \cap V_1$, where V_1 is a Lorentz Hilbert subspace of $\widetilde{V}$.

This paper is organized as follows: in §1 we prove the basic properties of isoparametric submanifolds of hyperbolic space, and the decomposition theorem is given in §2. In §3 we obtain all polar actions of connected closed Lie subgroups of $O(n, 1)$ on H^n, and in §4, we study the infinite dimensional case.

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1. Basic properties

Suppose a submanifold M^n of H^{n+k} is isoparametric, i.e., the normal bundle of M in H^{n+k} is globally flat and the principal curvatures along any parallel normal vector field are constant. Let $\nu(M)$ be the normal bundle of M in $R^{n+k,1}$. Then $\nu(M)$ is globally flat, hence $\{A_v \mid v \in \nu(M)_q\}$ is a family of commuting selfadjoint operators on TM_q for any $q \in M$. So, there exists a common eigendecomposition $TM_q = \bigoplus_{i=1}^p E_i(q)$, at q. Since the principal curvatures along any parallel normal vector field are constant, the E_i, $i = 1, 2, \ldots, p$ are smooth distributions and there exist $v_1, v_2, \ldots, v_p$, parallel normal vector fields, such that for any normal vector $v \in \nu(M)_q$, we have

$$A_v | E_i = \langle v, v_i \rangle \mathrm{id}_{E_i}.$$

The E_i are called the *curvature distributions* of M, and the v_i are called the *curvature normals* of M in $R^{n+k,1}$.

1.1. We will make the following standing assumptions:

(1) M has p curvature distributions, $E_1, E_2, \ldots, E_p$, and $\mathrm{rank}(E_i) = m_i$.

(2) Let $\{e_j \mid 1 \le j \le n\}$ be a local orthonormal tangent frame for M, such that E_i is spanned by $\{e_j \mid \mu_{i-1} < j \le \mu_i\}$, where $\mu_i = \sum_{s=1}^{i} m_s$, $\mu_0 = 0$.

(3) Let $\{e_\alpha \mid n+1 \le \alpha \le n+k+1\}$ be a local orthonormal parallel normal frame for M in $R^{n+k,1}$, where $e_{n+k+1}(x) = x$ for $x \in M$.

Remark. (a) Since $M^n \subset H^{n+k}$, we have $A_q = -I$ on T_qM. Hence there is a smooth vector field u_i on M, such that $v_i(q) = u_i(q) + q$.

(b) Since the orthonormal frame field $\{e_\alpha\}$ for M is parallel, we have

$$\omega_{\alpha\beta} = 0 \quad \text{for all } \alpha, \beta, \text{ and}$$
$$\omega_{\alpha j} = -\langle v_i, e_\alpha\rangle\omega_j \quad \text{where } \mu_{i-1} < j \le \mu_i .$$

1.2 **Definition.** A submanifold M of H^{n+k} is called full if M is not included in any totally umbilic hypersurface of H^{n+k}.

1.3 **Proposition.** *Let $L(V, u) \subset H^{n+k}$ be a totally umbilic hypersurface and suppose $M \subset L(V, u) \subset H^{n+k}$. Then M is isoparametric in $L(V, u)$ iff M is isoparametric in H^{n+k}.*

Proof. If M is isoparametric in H^{n+k}, then it is obvious that M is isoparametric in $L(V, u)$.

Suppose M is isoparametric in $L(V, u)$. If $V = \{x \in R^{n+k,1} \mid \langle x, v\rangle = 0\}$, then $L(V, u) = \{x \in H^{n+k} \mid \langle x, v\rangle = a\}$, where $a = \langle u, v\rangle$. Suppose e_{n+k} is a normal vector field of $L(V, u)$ in H^{n+k}, and $X\colon H^{n+k} \to R^{n+k,1}$ is the standard isometric embedding. Then $v = -aX + be_{n+k}$, so $0 = -a\mathrm{d}X + b\mathrm{d}e_{n+k}$. For any tangent vector field Z on M, we have, $b\mathrm{d}e_{n+k}(Z) = aZ$. Since $b \ne 0$, e_{n+k} is a parallel normal vector field and $A_{e_{n+k}}$ has constant eigenvalues on M. Therefore the normal bundle of M in H^{n+k} is flat and the principal curvatures along any parallel normal vector field are constant, i.e., M is isoparametric in H^{n+k}. □

1.4 **Proposition.** *M is full in H^{n+k} iff $v_1, v_2, \dots, v_p$ span $\nu(M)$.*

Proof. If M is not full in H^{n+k} then M is included in a totally umbilic hypersurface of H^{n+k}, i.e., there exists $v \in R^{n+k,1}$, $v \ne 0$, such that $\langle v, x\rangle$ is constant on M. Hence v is a constant normal field on M, we have $A_v = 0$. On the other hand we know $A_v|E_i = \langle v, v_i\rangle \mathrm{id}_{E_i}$. Therefore $\langle v, v_i\rangle = 0$, $i = 1, 2, \dots, p$. Since $v \ne 0$, $v_1, v_2, \dots, v_p$ do not span $\nu(M)$.

If $v_1, v_2, \dots, v_p$ do not span $\nu(M)$, then there exists a normal field v on M, such that $\langle v, v_i\rangle = 0$, $i = 1, 2, \dots, p$, and we have $A_v|E_i = \langle v, v_i\rangle \mathrm{id}_{E_i} = 0$, $i = 1, 2, \dots, p$, $A_v = 0$; i.e., v is constant on M. Then

$$\mathrm{d}\langle v, x\rangle = \langle v, \mathrm{d}x\rangle = \sum_{i=1}^{n} \omega_i\langle v, e_i\rangle = 0,$$

hence $\langle v, x\rangle = c$ is constant, i.e., M is included in a totally umbilic hypersurface of H^{n+k}. □

1.5 **Lemma.** *If $\omega_{ij} = \sum_k \gamma_{ij}^k \omega_k$ then*

$$\gamma_{ij}^k(v_{i'} - v_{j'}) = \gamma_{ik}^j(v_{i'} - v_{k'}) = \gamma_{kj}^i(v_{k'} - v_{j'})$$

where $\mu_{i'-1} < i \le \mu_{i'}$, $\mu_{j'-1} < j \le \mu_{j'}$, $\mu_{k'-1} < k \le \mu_{k'}$, and $j \ne k$.

Proof. Suppose $\mu_{i'-1} < i \le \mu_{i'}$. Then we have $\omega_{\alpha i} = -\langle v_{i'}, e_\alpha\rangle\omega_i$ for all α, and because $\mathrm{d}\omega_{\alpha i} = \omega_{\alpha j} \wedge \omega_{ji}$, we have

$$\begin{aligned}
&\omega_{\alpha j} \wedge \omega_{ji} = -\langle v_{i'}, e_\alpha\rangle \,\mathrm{d}\omega_i = -\langle v_{i'}, e_\alpha\rangle\omega_{ij} \wedge \omega_j, \\
&\omega_{ij} \wedge \langle v_{j'}, e_\alpha\rangle\omega_j = \langle v_{i'}, e_\alpha\rangle\omega_{ij} \wedge \omega_j, \\
&\sum_k \gamma_{ij}^k\langle v_{j'}, e_\alpha\rangle\omega_k \wedge \omega_j = \sum_k \gamma_{ij}^k\langle v_{i'}, e_\alpha\rangle\omega_k \wedge \omega_j.
\end{aligned}$$

Comparing coefficients of $\omega_k \wedge \omega_j$ for $j \ne k$, if $\omega_{k'-1} < k \le \omega_{k'}$ we have

$$\begin{gathered}
\gamma_{ij}^k\langle v_{j'}, e_\alpha\rangle - \gamma_{ik}^j\langle v_{k'}, e_\alpha\rangle = \gamma_{ij}^k\langle v_{i'}, e_\alpha\rangle - \gamma_{ik}^j\langle v_{i'}, e_\alpha\rangle \quad \forall\alpha, \\
\gamma_{ij}^k\langle v_{i'} - v_{j'}, e_\alpha\rangle = \gamma_{ik}^j\langle v_{i'} - v_{k'}, e_\alpha\rangle \quad \forall\alpha,
\end{gathered}$$

hence, $\gamma_{ij}^k(v_{i'} - v_{j'}) = \gamma_{ik}^j(v_{i'} - v_{k'}) = \gamma_{kj}^i(v_{k'} - v_{j'})$. □

1.6 **Corollary.** *Suppose* $\mu_{i'-1} < i \le \mu_{i'}$, $\mu_{j'-1} < j \le \mu_{j'}$, *and* $i' \ne j'$. *Then*

(1) $\gamma_{ij}^k = 0$, *if* $\mu_{j'-1} < k \le \mu_{j'}$.

(2) *If* $\gamma_{ij}^k \ne 0$, *where* $\mu_{k'-1} < k \le \mu_{k'}$, *then*

$$\begin{aligned}
&v_{k'} = (1 - a_{i'j'}^{k'})v_{i'} + \alpha_{i'j'}^{k'}v_{j'}, \quad \textit{where } a_{i'j'}^{k'} \ne 0, 1, \\
&\gamma_{ij}^k = \gamma_{ik}^j a_{i'j'}^{k'} = \gamma_{kj}^i(1 - a_{i'j'}^{k'}).
\end{aligned}$$

1.7 **Proposition.** *Each curvature distribution* E_i *is integrable.*

Proof. For simplicity, we consider the case $i = 1$. Since E_1 is defined by the 1-form equations on M: $\omega_i = 0$, $m_1 < i \le n$, and

$$\begin{aligned}
\mathrm{d}\omega_i &= \sum_{j=1}^n \omega_{ij} \wedge \omega_j = \sum_{j \le m_1} \omega_{ij} \wedge \omega_j \\
&= \sum_{\substack{j \le m_1 \\ k \le m_1}} \gamma_{ij}^k\omega_k \wedge \omega_j = 0 \quad \text{by (1.6)}.
\end{aligned}$$

It follows that E_1 is integrable. □

1.8. Recall that the endpoint map $Y\colon \nu(M) \to R^{n+k,1}$ is defined by $Y(v) = x + v$ for $v \in \nu(M)_x$, and that a singular value of Y is called a *focal point* of M.

1.9 **Proposition.** *Let* M *be an isoparametric submanifold of* $H^{n+k} \subset R^{n+k,1}$ *and* Γ *its focal point set. For each* $q \in M$, *let* Γ_q *denote the intersection of* Γ *with the normal plane* $\nu(M)_q$ *to* M *at* q. *Then*

$$\Gamma = \bigcup_{q \in M} \Gamma_q \quad \textit{and} \quad \Gamma_q = \bigcup_{i=1}^p l_i(q),$$

where $l_i(q) = \{v \in \nu(M)_q \mid \langle v, v_i\rangle = 0\}$. *These* $l_i(q)$ *are called the focal hyperplanes associated to* E_i *at* q.

Proof. Let Y be the endpoint map,

$$Y\colon \nu(M) \to R^{n+k,1}, \qquad (x, v) \mapsto x + v,$$

so, we have

$$\begin{aligned} Y &= Y(x, z) = x + \sum_\alpha z_\alpha e_\alpha \\ \mathrm{d}Y &= \mathrm{d}x + \sum_\alpha \mathrm{d}z_\alpha e_\alpha + \sum_\alpha z_\alpha \mathrm{d}e_\alpha \\ &= I - A_{\sum_\alpha z_\alpha e_\alpha} + \sum_\alpha \mathrm{d}z_\alpha e_\alpha \\ &= \bigoplus_{i=1}^{p} \left(1 - \left\langle \sum_\alpha z_\alpha e_\alpha, v_i \right\rangle\right) \mathrm{id}_{E_i} + \sum_\alpha \mathrm{d}z_\alpha e_\alpha . \end{aligned}$$

Then

$$\Gamma = \bigcup_{q \in M} \Gamma_q$$

and

$$\begin{aligned} \Gamma_q &= \bigcup_{i=1}^{p} \{q + v \mid v \in \nu(M)_q, \langle v, v_i \rangle = 1\} \\ &= \bigcup_{i=1}^{p} \{v \mid v \in \nu(M)_q, \langle v, v_i \rangle = 0\} \\ &= \bigcup_{i=1}^{p} l_i(q), \end{aligned}$$

where $l_i(q) = \{v \in \nu(M)_q \mid \langle v, v_i \rangle = 0\}$. □

1.10 **Proposition.** *Let $X: M^n \subset H^{n+k} \subset R^{n+k,1}$ be an isoparametric submanifold, and v a parallel normal vector field on M.*

Then $X + v: M \to R^{n+k,1}$ is an immersion if and only if $\langle v, v_i \rangle \neq 1$ for $i = 1, 2, \dots, p$. Moreover, if $X + v$ is an immersion, then

(i) *The parallel set $M_v = \{x + v \mid x \in M\}$ is an isoparametric submanifold of $R^{n+k,1}$, i.e., M_v is a space-like submanifold of $R^{n+k,1}$, the normal bundle of M_v in $R^{n+k,1}$ is globally flat, and the principal curvatures along any parallel normal field are constant.*

(ii) *Let $q^* = q + v(q)$, then*

$$\begin{gathered} TM_q = T(M_v)_{q^*}, \quad \nu(M_q) = \nu(M_v)_{q^*}, \\ q + \nu(M)_q = q^* + \nu(M_v)_{q^*}. \end{gathered}$$

(iii) *If $\{e_\alpha\}$ is a local parallel normal frame on M, then $\{\overline{e}_\alpha\}$ is a local parallel normal frame on M_v, where $\overline{e}_\alpha(q^*) = e_\alpha(q)$.*

(iv) *$E_i^*(q^*) = E_i(q)$ are the curvature distributions of M_v, and the corresponding curvature normals are given by*

$$v_i^*(q^*) = \frac{v_i(q)}{1 - \langle v, v_i \rangle}.$$

(v) *The focal hyperplane $l_i^*(q^*)$ of M_v associated to E_i^* is the same as the focal hyperplane $l_i(q)$ of M associated to E_i.*

Proof. The same proof as for Euclidean space works here. For details see [PT2, Te1]. □

1.11 **Theorem.** *Let M be a complete isoparametric submanifold of $H^{n+k} \subset R^{n+k,1}$, E_i the curvature distributions, v_i the curvature normals, and $l_i(q)$ the focal hyperplanes associated to E_i at $q \in M$. Let $S_i(q)$ denote the leaf of E_i through q.*

(1) *If $\langle v_i, v_i\rangle > 0$, then*
  (i) *$E_i(x) \oplus \nu(M)_x$ is a fixed m_i+k+1 Lorentz subspace η_i in $R^{n+k,1}$ for all $x \in S_i(q)$.*
  (ii) *$E_i(x) \oplus Rv_i(x)$ is a fixed m_i+1 Euclidean subspace ξ_i in $R^{n+k,1}$ for all $x \in S_i(q)$.*
  (iii) *$l_i(x) = l_i(q)$ for all $x \in S_i(q)$.*
  (iv) *$x + v_i(x)/\langle v_i, v_i\rangle = c_i$ is a constant for all $x \in S_i(q)$.*
  (v) *$S_i(q)$ is the standard sphere of $\xi_i + c_i$, with radius $1/|v_i|$ and centered at c_i.*

(2) *If $\langle v_i, v_i\rangle < 0$, then*
  (i) *$E_i(x) \oplus \nu(M)_x$ is a fixed m_i+k+1 Lorentz subspace η_i in $R^{n+k,1}$ for all $x \in S_i(q)$.*
  (ii) *$E_i(x) \oplus Rv_i(x)$ is a fixed m_i+1 Lorentz subspace ξ_i in $R^{n+k,1}$ for all $x \in S_i(q)$.*
  (iii) *$l_i(x) = l_i(q)$ for all $x \in S_i(q)$.*
  (iv) *$x + v_i(x)/\langle v_i, v_i\rangle = c_i$ is a constant for all $x \in S_i(q)$.*
  (v) *$S_i(q)$ is the standard hyperbolic space of $\xi_i + c_i$, with radius $1/|v_i|$ and centered at c_i, i.e.,*

$$S_i(q) = \{x \in c_i + \xi_i \mid \langle x - c_i, x - c_i\rangle = 1/\langle v_i, v_i\rangle\}.$$

(3) *If $\langle v_i, v_i\rangle = 0$, then*
  (i) *$v_i(x) = c_i$ is a constant on $S_i(q)$.*
  (ii) *$E_i(x) \oplus Rv_i(x) + Rx$ is a fixed m_i+2 Lorentz subspace ζ_i for $x \in S_i(q)$.*
  (iii) *$S_i(q)$ is a flat umbilic hypersurface of the hyperbolic space $H^{n+k} \cap \zeta_i$.*

Proof. The proof for (1) and (2) is the same as for Euclidean space. For details see [PT2, Te1].

Proof of (3). For simplicity, we assume $i = 1$.

(i) Since $\nabla_{e_j} v_1 = A_{v_1} e_j = \langle v_1, v_1\rangle = 0$, if $e_j \in E_1$, so $v_1(x) = c_1$ is constant on $S_1(q)$.
(ii) Let $v_1(x) = u_1(x) + x$. Then, we have

$$E_1(x) \oplus Rv_1 + Rx = E_1(x) \oplus Ru_1 \oplus Rx,$$

and

$$\begin{aligned}
\mathrm{d}(e_1 \wedge \cdots \wedge e_{m_1} \wedge u_1 \wedge x) &= \sum_j e_1 \wedge \cdots \wedge \mathrm{d}e_j \wedge \cdots \wedge e_{m_1} \wedge u_1 \wedge x \\
&= \sum_j e_1 \wedge \cdots \wedge \left(\sum_k \omega_{jk} e_k + \sum_\alpha \omega_{j\alpha} e_\alpha \right) \wedge \cdots \wedge e_{m_1} \wedge u_1 \wedge x \\
&= \sum_{j\alpha} e_1 \wedge \cdots \wedge \omega_{j\alpha} e_\alpha \wedge \cdots \wedge u_1 \wedge x \\
&= \sum_j e_1 \wedge \cdots \wedge \sum_{\alpha \le n+k} \omega_{j\alpha} e_\alpha \wedge \cdots \wedge u_1 \wedge x \\
&= \sum_j \omega_j e_1 \wedge \cdots \wedge \sum_{\alpha \le n+k} \langle v_1 , e_\alpha \rangle e_\alpha \wedge \cdots \wedge u_1 \wedge x \\
&= 0 .
\end{aligned}$$

So $\zeta_1 = E_1(x) \oplus Rv_1 + Rx$ is a fixed $m_1 + 2$ Lorentz subspace for all $x \in S_1(q)$.

(iii) For any $x \in S_1(q)$, we have

$$\langle x , x \rangle = -1 , \qquad \langle x , c_1 \rangle = -1 \text{ and } x \in \zeta_1 ,$$

so $S_1(q)$ is a flat umbilic hypersurface in the hyperbolic space of $H^{n+k} \cap \zeta_1$. □

1.12 **Definition.** If $v \in R^{n+k,1}$, $\langle v , v \rangle \ne 0$, then $R_v(x) = x - 2\langle x , v \rangle v / \langle v , v \rangle \in O(n+k , 1)$ is called the linear reflection along v.

1.13. Suppose $M^n \subset H^{n+k}$ is an isoparametric submanifold with curvature normals $v_1 , v_2 , \ldots , v_p$, where $\langle v_i , v_i \rangle \le 0$ if $i = 1 , 2 , \ldots , l$ and where $\langle v_i , v_i \rangle > 0$ if $i = l+1 , \ldots , p$.

Some notations:

(1) R_i^q denotes the linear reflection of $\nu(M)_q$ along $v_i(q)$, $i = l+1 , \ldots , p$.
(2) Let φ_i be the diffeomorphism of M defined by $\varphi_i(q) =$ the antipodal point of q in the leaf sphere $S_i(q)$ of E_i for $i = l+1 , \ldots , p$.
(3) S_p denotes the group of permutations of $\{1 , 2 , \ldots , p\}$.

It follows from (1) of (1.11) that

$$\varphi_i = X + 2 \frac{v_i}{\langle v_i , v_i \rangle} , \qquad i = l+1 , \ldots , p ,$$

and $\varphi_i(q) = R_i^q(q)$. Since φ_i is a diffeomorphism, it follows from (1.10) that

1.14 **Proposition.** *If $i \ge l+1$, then $2(v_i / \langle v_i , v_i \rangle , v_j) \ne 1$ for all j, i.e.,*

$$1 - 2\langle v_i , v_j \rangle / \langle v_i , v_i \rangle \ne 0 \quad \textit{for all } j .$$

1.15 **Theorem.** *There exist permutations $\sigma_{l+1} , \ldots , \sigma_p \in S_p$ such that*

(1) $$E_j(\varphi_i(q)) = E_{\sigma_i(j)}(q) , \quad \textit{i.e.,} \quad \varphi_i^*(E_j) = E_{\sigma_i(j)} .$$

In particular, $m_j = m_{\sigma_i(j)}$.

(2) $$v_{\sigma_i(j)}(q) = \left(1 - 2 \frac{\langle v_i , v_{\sigma_i(j)} \rangle}{\langle v_i , v_i \rangle} \right) v_j(\varphi_i(q)) \quad \textit{and}$$

(3) $$R_i^q(v_j(q)) = \left(1 - 2 \frac{\langle v_i , v_{\sigma_i(j)} \rangle}{\langle v_i , v_i \rangle} \right)^{-1} v_{\sigma_i(j)}(q) ,$$

where $i \ge l+1$, $1 \le j \le p$.

Proof. The same proof as for Euclidean space with trivial modifications. □

1.16 **Corollary.** *The subgroup* W^q *of* $O(\nu(M))_q$ *generated by* $R^q_{v_i}$, $i = l+1, \ldots, p$, *is a finite group.*

2. The Decomposition Theorem

Suppose $M^n \subset H^{n+k} \subset R^{n+k,1}$ is a full isoparametric submanifold with curvature normals $v_1, v_2, \ldots, v_p$ such that

$$\langle v_i, v_i \rangle \le 0 \quad \text{for } i = 1, \ldots, l \quad \text{and} \quad \langle v_i, v_i \rangle > 0 \quad \text{for } i = l+1, \ldots, p.$$

2.1 **Lemma.** *Suppose* $\mu_{i'-1} < i \le \mu_{i'}$, $\mu_{j'-1} < j \le \mu_{j'}$, *and* $i' \neq j'$ *then*

$$\langle v_{i'}, v_{j'} \rangle = \sum_{\substack{k' \\ v_{k'} = (1-a^{k'}_{i'j'})v_{i'} + a^{k'}_{i'j'} v_{j'} \\ a^{k'}_{i'j'} \neq 0, 1}} \sum_{\mu_{k'-1} < k \le \mu_{k'}} (\gamma^k_{ij})^2 \frac{1}{a^{k'}_{i'j'}(a^{k'}_{i'j'} - 1)}.$$

Proof. By the Gauss equation

$$-\mathrm{d}\omega_{ij} + \omega_{ik} \wedge \omega_{kj} + \sum_{\alpha \le n+k} \omega_{i\alpha} \wedge \omega_{\alpha j} = -\omega_i \wedge \omega_j.$$

We have

$$-\sum_k \mathrm{d}(\gamma^k_{ij}\omega_k) + \sum_{ktm} \gamma^k_{it}\omega_k \wedge \gamma^m_{tj}\omega_m - \sum_{\alpha \le n+k} \langle v_{i'}, e_\alpha \rangle \langle v_{j'}, e_\alpha \rangle \omega_i \wedge \omega_j = -\omega_i \wedge \omega_j,$$

$$-\sum_k \mathrm{d}\gamma^k_{ij} \wedge \omega_k - \sum_{kt} \gamma^k_{ij}\omega_{kt} \wedge \omega_t + \sum_{ktm} \gamma^k_{it}\gamma^m_{tj}\omega_k \wedge \omega_m$$
$$= \left(\sum_{\alpha \le n+k} \langle v_{i'}, e_\alpha \rangle \langle v, e_\alpha \rangle - 1 \right) \omega_i \wedge \omega_j = \langle v_{i'}, v_{j'} \rangle \omega_i \wedge \omega_j.$$

Comparing the coefficients of $\omega_i \wedge \omega_j$, we have,

$$-\sum_k \gamma^k_{ij}\gamma^i_{kj} + \sum_k \gamma^k_{ij}\gamma^j_{ki} + \sum_k \gamma^i_{ik}\gamma^j_{kj} - \sum_k \gamma^j_{ik}\gamma^i_{kj} = \langle v_{i'}, v_{j'} \rangle,$$

$$\sum_{\substack{k \\ e_k \notin E_{i'}, E_{j'}}} (\gamma^k_{ij}\gamma^j_{ki} - \gamma^k_{ij}\gamma^i_{kj} - \gamma^j_{ik}\gamma^i_{kj}) = \langle v_{i'}, v_{j'} \rangle \quad \text{(by (1.6))}.$$

By (2) of (1.6), we have

$$\sum_{\substack{k' \\ v_{k'} = (1-a^{k'}_{i'j'})v_{i'} + a^{k'}_{i'j'} v_{j'} \\ a^{k'}_{i'j'} \neq 0, 1}} \sum_{\mu_{k'-1} < k \le \mu_{k'}} \left\{ -\frac{(\gamma^k_{ij})^2}{a^{k'}_{i'j'}} - \frac{(\gamma^k_{ij})^2}{1 - a^{k'}_{i'j'}} - \frac{(\gamma^k_{ij})^2}{a^{k'}_{i'j'}(1 - a^{k'}_{i'j'})} \right\}$$
$$= \langle v_{i'}, v_{j'} \rangle,$$

so

$$\sum_{\substack{v_{k'}=(1-a^{k'}_{i'j'})v_{i'}+a^{k'}_{i'j'}v_{j'} \\ a^{k'}_{i'j'}\neq 0,1}} \sum_{\mu_{k'-1}<k\le\mu_{k'}} (\gamma^k_{ij})^2 \frac{1}{a^{k'}_{i'j'}(a^{k'}_{i'j'}-1)} = \langle v_{i'}, v_{j'}\rangle. \quad \square$$

2.2 **Lemma.** *Suppose* $1 \le i, j \le l$. *Then,*

(1) $$\langle v_i, v_j\rangle < 0 \quad \text{if } i \neq j,$$

(2) $\langle v_i, v_j\rangle^2 \ge \langle v_i, v_i\rangle\langle v_j, v_j\rangle$ *and* $\langle v_i, v_j\rangle^2 = \langle v_i, v_i\rangle\langle v_j, v_j\rangle$ *iff* $i = j$.

Proof. (1) Suppose $q \in M$, then $v_i(q) = u_i + q$, where $\langle q, u_i\rangle = 0$, $i = 1, \dots, l$. So we have $\langle v_i, v_j\rangle = \langle u_i, u_j\rangle - 1$, $\forall i, j$. Since $\langle v_i, v_i\rangle \le 0$, $\langle u_i, u_i\rangle \le 1$, and (1) follows from the Schwarz inequality.

(2) If $\langle v_i, v_i\rangle = 0$, then by (1) we have $\langle v_i, v_j\rangle^2 > 0 = \langle v_i, v_i\rangle\langle v_j, v_j\rangle$ if $i \neq j$. If $\langle v_i, v_i\rangle < 0$, then, since

$$\left\langle v_j - \frac{\langle v_i, v_j\rangle}{\langle v_i, v_i\rangle} v_i, v_i \right\rangle = 0,$$

it follows that

$$\left\langle v_j - \frac{\langle v_i, v_j\rangle}{\langle v_i, v_i\rangle} v_i, v_j - \frac{\langle v_i, v_j\rangle}{\langle v_i, v_i\rangle} v_i \right\rangle \ge 0,$$

hence

$$\langle v_j, v_j\rangle - \frac{\langle v_i, v_j\rangle^2}{\langle v_i, v_i\rangle} = \left\langle v_j - \frac{\langle v_i, v_j\rangle}{\langle v_i, v_i\rangle} v_i, v_j - \frac{\langle v_i, v_j\rangle}{\langle v_i, v_i\rangle} v_i \right\rangle \ge 0.$$

Therefore we have

$$\langle v_i, v_j\rangle^2 \ge \langle v_i, v_i\rangle\langle v_j, v_j\rangle$$

and

$$\langle v_i, v_j\rangle^2 = \langle v_i, v_i\rangle\langle v_j, v_j\rangle \quad \text{iff } v_j = \frac{\langle v_i, v_j\rangle}{\langle v_i, v_i\rangle} v_i.$$

Since $\langle v_i(q), q\rangle = \langle v_j(q), q\rangle = -1$,

$$\langle v_i, v_j\rangle^2 = \langle v_i, v_i\rangle\langle v_j, v_j\rangle \quad \text{iff } i = j. \quad \square$$

2.3 **Theorem.** *Suppose* $v_1, \dots, v_p$ *are the curvature normals of* M, *where*

$$\langle v_i, v_i\rangle \le 0, \quad i = 1, \dots, l, \qquad \langle v_i, v_i\rangle > 0, \quad i = l+1, \dots, p,$$

then $l \le 1$.

Proof. Suppose $l > 1$. Then we can choose $v_{i'}, v_{j'}$ $i', j' \le l$ and $i' \neq j'$ such that

$$\langle v_{i'}, v_{i'}\rangle \ge \langle v_{k'}, v_{k'}\rangle \quad \text{for all } k' \le l,$$
$$\langle v_{i'}, v_{j'}\rangle \ge \langle v_{i'}, v_{k'}\rangle \quad \text{for all } k' \le l,\ k' \neq i'.$$

Claim. If $k' \le l$, $v_{k'} = (1 - a^{k'}_{i'j'})v_{i'} + a^{k'}_{i'j'}$ then, either $a^{k'}_{i'j'} \le 0$, or, $a^{k'}_{i'j'} \ge 1$. Otherwise, suppose $0 < a^{k'}_{i'j'} < 1$. Then, we have

$$\begin{aligned}\langle v_{i'}, v_{k'}\rangle &= (1 - a^{k'}_{i'j'})\langle v_{i'}, v_{i'}\rangle + a^{k'}_{i'j'}\langle v_{i'}, v_{j'}\rangle \\ &> (1 - a^{k'}_{i'j'})\langle v_{i'}, v_{j'}\rangle + a^{k'}_{i'j'}\langle v_{i'}, v_{j'}\rangle \quad \text{by (2.2)} \\ &= \langle v_{i'}, v_{j'}\rangle.\end{aligned}$$

Contradiction.

If $k' > l$ and $v_{k'} = (1 - a^{k'}_{i'j'})v_{i'} + a^{k'}_{i'j'}$, then either $a^{k'}_{i'j'} \leq 0$, or $a^{k'}_{i'j'} \geq 1$ because $\langle v_{k'}, v_{k'}\rangle > 0$.

By (2.1) we have

$$\langle v_{i'}, v_{j'}\rangle = \sum_{\substack{k' \\ v_{k'} = (1-a^{k'}_{i'j'})v_{i'} + a^{k'}_{i'j'}v_{j'} \\ a^{k'}_{i'j'} \neq 0, 1}} \sum_{\mu_{k'-1} < k \leq \mu_{k'}} (\gamma^k_{ij})^2 \frac{1}{a^{k'}_{i'j'}(a^{k'}_{i'j'} - 1)} \geq 0.$$

Contradiction. So, $l \leq 1$. □

2.4 **Theorem.** *Suppose M^n is a full isoparametric submanifold of $H^{n+k} \subset R^{n+k,1}$ with curvature normals $v_1, v_2, \ldots, v_p$, where $\langle v_i, v_i\rangle \leq 0$, $i = 1, \ldots, l$, and $\langle v_i, v_i\rangle > 0$, $i = l+1, \ldots, p$.*

Then $l = 1$, and $\langle v_1, v_i\rangle = 0$, $i > 1$.

Proof. Since M is full in H^{n+k}, $v_1(q), \ldots, v_p(q)$ span $\nu(M)_q$. Suppose there is no curvature normal with negative length. Then, by (2.3), there are two cases:

(i) $\langle v_1, v_1\rangle = 0$, and $\langle v_i, v_i\rangle > 0$, $i > 1$, or
(ii) $\langle v_i, v_i\rangle > 0$, $i = 1, 2, \ldots, p$.

Case (i). Since $R^q_{v_i}(v_1) = cv_{\sigma_i(1)}$, where $c \in R$, and $R^q_{v_i} \in O(\nu(M)_q)$, we have $v_{\sigma_i(1)} = v_1$. Hence

$$R^q_{v_i}(v_1) = cv_1, \qquad v_1 - 2\frac{\langle v_i, v_1\rangle}{\langle v_i, v_i\rangle} = cv_1.$$

This implies that $\langle v_i, v_1\rangle = 0$, $i > 1$, so, $v_1, \ldots, v_p$ do not span $\nu(M)_q$, a contradiction.

Case (ii). Let $u = \sum_{\tau \in W^q} \tau(q)$. Then $u \in \nu(M)_q$ is invariant under W^q. In particular $R^q_{v_i}(u) = u$ so $\langle v_i, u\rangle = 0$.

Claim. $\tau(q)$, $\forall \tau \in W^q$, are in the same component of the interior of the light cone, so $\langle u, u\rangle < 0$.

Proof of the claim. Since W^q is generated by $R^q_{v_i}$, so we only have to prove that for any v, $\langle v, v\rangle < 0$, then $R_{v_i}(v)$ and v are in the same component of the interior of the light cone. Suppose

$$v = v' + aq, \qquad a > 0,\ \langle q, v'\rangle = 0,$$
$$v_i = u_i + q, \qquad \langle q, u_i\rangle = 0,$$

then

$$\begin{aligned}\langle R_{v_i}(v), q\rangle &= -a - 2\frac{\langle v_i, v\rangle}{\langle v_i, v_i\rangle}\langle v_i, q\rangle = -a + 2\frac{\langle u_i, v'\rangle - a}{\langle u_i, u_i\rangle - 1} \\ &< -a + \frac{2a|u_i|}{\langle u_i, u_i\rangle - 1} \quad \text{since } |v'| < a \\ &< 0.\end{aligned}$$

So, $R_{v_i}(v)$, v are in the same component of the interior of the light cone.

Hence $\langle v_i, u\rangle = 0$, and $u \neq 0$. Contradiction.

Therefore M has one and only one curvature normal with negative length.

Since $R^q_{v_i}(v_1) = cv_{\sigma_i(1)}$, where $c \in R$, and $R^q_{v_i} \in O(\nu(M)_q)$, so we have $\sigma_i(1) = 1$ and $c = \pm 1$.

Since $R^q_{v_i}(v_1)$, v_1 are in the same component of the interior of the light cone, $c = 1$. i.e.,

$$R^q_{v_i}(v_1) = v_1 \quad \text{for } i = 2, \dots, p.$$

So, $\langle v_1, v_i \rangle = 0$ for $i = 2, \dots, p$. □

2.5 **Corollary.** *Suppose $M^n \subset H^{n+k}$ is a full isoparametric submanifold, $v_1, \dots, v_p$ are the curvature normals, where $\langle v_1, v_1 \rangle < 0$, $\langle v_i, v_i \rangle > 0$ if $i > 1$, E_i the corresponding curvature distributions.*

If $e_j \in E_1$, $e_k \in E_i$, $i > 1$, then $\omega_{jk} = 0$.

Proof. Let $\omega_{jk} = \sum_l \gamma^l_{jk} \omega_l$. By (1.6), we know that if $e_l \in E_1$, or E_i, then $\gamma^l_{jk} = 0$.

Suppose $\gamma^l_{jk} \neq 0$ and $e_l \in E_s$, $s \neq 1, i$, then

$$v_s = (1 - a^s_{1i}) v_1 + a^s_{1i} v_i \quad \text{by (1.6)}.$$

By (2.4) we have $\langle v_1, v_s \rangle = 0$, $\langle v_1, v_i \rangle = 0$. So, $a^s_{1i} = 1$, $v_s = v_i$, a contradiction. Hence we have $\gamma^l_{jk} = 0$, and therefore

$$\omega_{jk} = \sum_l \gamma^l_{jk} \omega_l = 0. \quad □$$

2.6 **Corollary.** *Suppose $M^n \subset H^{n+k} \subset R^{n+k,1}$ is a full isoparametric submanifold, $v_1, v_2, \dots, v_p$ curvature normals, such that $|v_1| < 0$, $|v_i| > 0$ for $i > 1$, $E_1, E_2, \dots, E_p$ are the corresponding curvature distributions and* $\dim E_1 = m$.

Then there exists a Lorentz subspace V^{m+1} of $R^{n+k,1}$, such that

$$M = H^m \left(\frac{1}{\langle v_1, v_1 \rangle} \right) \times M_1^{n-m}$$

where

$$H^m \left(\frac{1}{\langle v_1, v_1 \rangle} \right) = \left\{ x \in V^{m+1} \mid \langle x, x \rangle = \frac{1}{\langle v_1, v_1 \rangle} \right\}$$

and

$$M_1^{n-m} \subset S^{n+k-m-1} \left(-1 - \frac{1}{\langle v_1, v_1 \rangle} \right) \subset V^\perp$$

is an isoparametric submanifold.

Proof. Let

$$X^* = X + \frac{v_1}{\langle v_1, v_1 \rangle} : M \to R^{n+k,1}.$$

Then

$$dX^* = \bigoplus_{i=1}^p \left(1 - \frac{\langle v_1, v_i \rangle}{\langle v_1, v_1 \rangle} \right) \mathrm{id}_{E_i} = \bigoplus_{i>1} \mathrm{id}_{E_i}.$$

So, $X^*(M) = M_1$ is a submanifold, and $X^* \colon M \to M_1$ is a Riemannian submersion, E_1 is the vertical distribution, and $\bigoplus_{i>1} E_i$ is the horizontal distribution.

From (2.5), we have

$$\omega_{jk} = 0 \quad \text{if } e_j \in E_1, \ e_k \in \bigoplus_{i>1} E_i .$$

So, the O'Neil tensor A [O'N1] is zero, therefore the horizontal distribution $\bigoplus_{i>1} E_i$ is integrable. Let $N(q)$ denote the leaf $\bigoplus_{i>1} E_i$ through q, $q \in M$.

Claim. $E_1(x) \oplus Rv_1(x)$ is a fixed $m+1$ plane in $R^{n+k,1}$ for x in $N(q)$.

Proof of the claim. If $e_j \in E_i$, $i > 1$, then $\nabla_{e_j} v_1 = -\langle v_1, v_i\rangle e_j = 0$. So, $e_1 \wedge \cdots \wedge e_m \wedge dv_1 = 0$.

Suppose $e_1, \dots, e_m \in E_1$, then

$$\begin{aligned}
&d(e_1 \wedge \cdots \wedge e_m \wedge v_1)\\
&= \sum_{j=1}^{m} e_1 \wedge \cdots \wedge de_j \wedge \cdots \wedge e_m \wedge v_1 + e_1 \wedge \cdots \wedge e_m \wedge dv_1\\
&= \sum_{j=1}^{m} e_1 \wedge \cdots \wedge \left(\sum_{k>m} \omega_{jk} e_k + \sum_{\alpha} \omega_{jk} e_\alpha \right) \wedge \cdots \wedge e_m \wedge v_1\\
&= \sum_{j=1}^{m} e_1 \wedge \cdots \wedge \left(\sum_{\alpha} \omega_{j\alpha} e_\alpha \right) \wedge \cdots \wedge e_m \wedge v_1 \quad \text{by (2.5)}\\
&= \sum_{j=1}^{m} e_1 \wedge \cdots \wedge \left(\sum_{\alpha \le n+k} \langle v_1, e_\alpha\rangle e_\alpha + e_{n+k+1} \right) \wedge \cdots \wedge e_m \wedge v_1\\
&= \sum_{j=1}^{m} e_1 \wedge \cdots \wedge v_1 \wedge e_{j+1} \wedge \cdots \wedge e_m \wedge v_1\\
&= 0.
\end{aligned}$$

So, $E_1(x) \oplus Rv_1(x)$ is constant for x in $N(q)$.

From (1.11) we know that $E_1(x) \oplus Rv_1(x)$ is also fixed on any fiber of X^*, so $V^{m+1} = E_1(x) \oplus Rv_1(x)$ is a fixed $m+1$ Lorentz subspace of $R^{n+k,1}$.

Since

$$\left\langle x + \frac{v_1(x)}{\langle v_1, v_1\rangle}, v_1(x) \right\rangle = 0, \qquad X^*(x) = x + \frac{v_1(x)}{\langle v_1, v_1\rangle} \in V^\perp$$

for any $x \in M$, i.e., $X^*(M) = M_1 \subset V^\perp$. Moreover, by direct computation we see that $X^*(M) = M_1$ is isoparametric in the sphere of $V^\perp$ with radius $(-1 - 1/\langle v_1, v_1\rangle)^{1/2}$, centered at the origin.

Suppose P is the orthogonal projection of $R^{n+k,1}$ onto V. Then

$$P|M \colon M \to H^m \left(\frac{1}{\langle v_1, v_1\rangle} \right), \qquad P|M(x) = -\frac{v_1(x)}{\langle v_1, v_1\rangle}$$

is surjective, and $X^* = 1 - P|M$. Therefore $M = H^m \times M_1$. $\square$

3. Polar actions on H^n

Suppose G is a connected, closed Lie subgroup of $O(n, 1)$, and the isometric action of G on H^n is polar (see [PT1]), i.e., H^n has a connected, closed submanifold which meets all orbits orthogonally. Any such submanifold is called a *section* of H^n. Suppose $q \in H^n$ is a regular point, $H(\Sigma_q)$ is the unique section through q. Since sections are totally geodesic submanifolds of H^n, therefore $H(\Sigma_q) = \Sigma_q \cap H^n$, where Σ_q is a $(m, 1)$ Lorentz subspace of $R^{n,1}$.

3.1 **Theorem.** *Suppose $q \in H^n$ is a regular point and $Gq \subset l$, where l is a hyperplane of $R^{n,1}$. Then*

$$G(H(\Sigma_q) \cap l) = l \cap H^n,$$

in particular $G(l \cap H^n) \subset l \cap H^n$. Moreover, the action of G on $l \cap H^n$ is polar.

Proof. Suppose v is a normal vector of l, where $l = \{x \in R^{n,1} \mid \langle x, v \rangle = c\}$. Then $v \in \nu_{gq}(Gq)$, the normal space of Gq in $R^{n,1}$ at gq, for all $g \in G$.

(1) If $x \in H(\Sigma_q) \cap l$, then $\langle \xi g x, v \rangle = 0$ for all $g \in G$, $\xi \in \mathbf{g}$, the Lie algebra of G, hence, $\langle gx, v \rangle$ is constant on G. Therefore, we have

$$\langle gx, v \rangle = \langle x, v \rangle = c \quad \forall g \in G,$$

i.e., $Gx \subset l$. So, $G(H(\Sigma_q) \cap l) \subset l \cap H^n$.

(2) Since $H^n = GH(\Sigma_q)$, for any $y \in H^n \cap l$ there exist $h \in G$, $x \in H(\Sigma_q)$, such that $y = hx$. Because $\langle gx, v \rangle$ is constant on G we have $\langle x, v \rangle = \langle hx, v \rangle = \langle y, v \rangle = c$, i.e., $x \in l$. So, $G(H(\Sigma_q) \cap l) \supset l \cap H^n$.

From (1) and (2) we have

$$G(H(\Sigma_q) \cap l) = l \cap H^n,$$

so G acts on $l \cap H^n$ and the action is polar. □

3.2. Suppose some principal orbit of G is not included in any hyperplane. Then all principal orbits of G in H^n are full isoparametric submanifolds of H^n.

3.3 **Theorem.** *Suppose Gq is a principal orbit of G, $q \in H^n$, v_1 is the curvature normal of Gq in $R^{n,1}$ such that $\langle v_1, v_1 \rangle < 0$, E_1 is the corresponding curvature distribution, $V = E_1(q) \oplus Rv_1(q)$, $\dim(E_1) = m$. Then*

(1) *V is G-invariant $(m, 1)$ subspace of $R^{n,1}$, and*

$$\rho\colon G \to SO(V), \qquad \rho^{\perp}\colon G \to SO(V^{\perp})$$

are polar representations.

(2) *There exists a polar representation $G_1 \subset SO(V^{\perp})$ such that the orbits of G in H^n coincide with the orbits*

$$\{SO(m, 1)u_1 \times G_1 u_2 \mid u_1 \in V,\ u_2 \in V^{\perp},\ \langle u_1, u_1 \rangle < -1, \text{ and } |u_1|^2 + |u_2|^2 = -1\}.$$

Proof. (1) From the proof of (2.6) we know,

$$V(gq) = E_1(gq) \oplus Rv_1(gq) = E_1(q) \oplus Rv_1(q) = V.$$

So $gV = V$ for all $g \in G$, i.e., V is invariant under G. Moreover ρ, $\rho^{\perp}$ are clearly polar.

(2) Let $q_1 = -v_1(q)/|v_1| \in H^n$. Then from (2.6) we know

$$Gq_1 = \{x \in V \mid \langle x, x \rangle = -1\},$$

so $V = T_{q_1}Gq_1 \oplus Rq_1$, hence the normal space of Gq_1 at q_1 in H^n is $V^{\perp}$.

Let G_1 be the identity component of G_{q_1}. Then $G_1 \subset O(V^\perp)$ is a slice representation of H^n, so it is a polar representation and $T_{q_1}H(\Sigma_q)$ is a section of G_1.

Let $q_2 = -v_1(q)/\langle v_1, v_1\rangle$, $q_3 = q + v_1(q)/\langle v_1, v_1\rangle$. Then $q = q_2 + q_3$, and $q_2 \in V$, $q_3 \in V^\perp$. Since q is a regular point of G, so q_3 is a regular point of $\rho^\perp : G \to SO(V^\perp)$.

Denote the identity component of G_{q_3} by K. Then K fixes every vector in $T_{q_1}H(\Sigma_q) \subset V^\perp$. Since

$$Gq = Gq_2 \times Gq_3,$$

so

$$G_1q_3 = Gq_3, \qquad Kq_2 = Gq_2.$$

Suppose $\mathbf{g}$, $\mathbf{g}_1$, $\mathbf{k}$ are the Lie algebras of G, G_1, K respectively. Then $\mathbf{g}_1q_3 = \mathbf{g}q_3$, $\mathbf{k}q_2 = \mathbf{g}q_2$.

Claim. $\mathbf{g} = \mathbf{k} + \mathbf{g}_1$ and $G = G_1K = KG_1$.

Proof of the claim.

$$\mathbf{g}q = \mathbf{g}q_2 + \mathbf{g}q_3 = \mathbf{k}q_2 + \mathbf{g}_1q_3 = (\mathbf{k} + \mathbf{g}_1)q_2 + (\mathbf{k} + \mathbf{g}_1)q_3 = (\mathbf{k} + \mathbf{g}_1)q,$$

and if $Xq = 0$, $X \in \mathbf{g}$, then $Xq_2 = Xq_3 = 0$, so $X \in \mathbf{k} \cap \mathbf{g}_1$, hence $\mathbf{g} = \mathbf{k} + \mathbf{g}_1$ and $G = G_1K = KG_1$.

Now consider the $K \times G_1$ action on $V \oplus V^\perp$. For any $u_1 + u_2 \in H(\Sigma_q)$, where $u_1 \in Rv_1$, $u_2 \in T_{q_1}H(\Sigma_q)$,

$$G(u_1 + u_2) \subset Ku_1 \times G_1u_2,$$

and, because $\mathbf{g} = \mathbf{k} + \mathbf{g}_1$, by dimension and connectedness we have

$$G(u_1 + u_2) = Ku_1 \times G_1u_2.$$

By (2.6) we know

$$K\frac{v_1(q)}{\langle v_1, v_1\rangle} = \left\{x \in V \mid \langle x, x\rangle = \frac{1}{\langle v_1, v_1\rangle}\right\},$$

so, the orbits of G in H^n coincide with the orbits

$$\{SO(m, 1)u_1 \times G_1u_2 \mid u_1 \in V, u_2 \in V^\perp, \langle u_1, u_1\rangle < -1, \text{ and } |u_1|^2 + |u_2|^2 = -1\}. \quad \square$$

4. Isoparametric submanifold of Riemannian Hilbert hyperbolic space

Suppose V is a separable Hilbert space. Let $\widetilde{V} = R \oplus V$ with the inner product

$$\langle (s, x), (t, y)\rangle = -st + \langle x, y\rangle \quad \text{where } s, t \in R, \ x, y \in V.$$

Then $\widetilde{V}$ is a Lorentz Hilbert space. Let $H(\widetilde{V}) = \{(s, x) \in \widetilde{V} \mid -s^2 + \langle x, x\rangle = -1, s > 0\}$, then $H(\widetilde{V})$ is a Riemannian Hilbert hyperbolic space with constant sectional curvature -1.

4.1 **Definition.** A Hilbert submanifold M of $H(\tilde{V})$ is called isoparametric if M satisfies the following conditions:

(1) $\mathrm{Codim}(M) < \infty$.
(2) $\nu(M)$, the normal bundle of M in $H(\tilde{V})$, is globally flat.
(3) If v is a parallel normal vector field on M, then $A_{v(x)}$, $A_{v(y)}$ are orthogonally equivalent for all $x, y \in M$.
(4) A_v is compact for any $v \in \nu(M)_x$.

4.2 **Proposition.** *Suppose M is not a full isoparametric Hilbert submanifold of $H(\tilde{V})$, i.e., there is a nonzero vector v in $\tilde{V}$, such that $\langle x, v\rangle$ is a constant on M, then $\langle v, v\rangle > 0$, and $\langle x, v\rangle = 0$ on M.*

Proof. Suppose $X\colon M \subset H(\tilde{V}) \subset \tilde{V}$, then we have $\langle X, v\rangle = a$, $\langle \mathrm{d}X, v\rangle = 0$. So, there exists a normal vector field Z of M in $H(\tilde{V})$, such that $v = aX + bZ$, where $b \neq 0$. Hence we have

$$0 = a\mathrm{d}X + b\mathrm{d}Z, \qquad A_Z = \frac{a}{b}\mathrm{id}_{TM}.$$

Since A_Z is a compact operator, $a = 0$.

Therefore $\langle v, v\rangle > 0$ and $\langle x, v\rangle = 0$ for all $x \in M$.

4.3 **Theorem.** *If $M \subset H(\tilde{V})$ is a full isoparametric submanifold, then* $\dim(M) < \infty$.

In order to prove Theorem 4.3, we need the following lemmas, in which we will always assume that M is a full isoparametric in $H(\tilde{V})$ and $\nu(M)$ is the normal bundle of M in $H(\tilde{V})$.

4.4 **Lemma.** *There exist smooth distributions E_i, $i \in I$, and smooth parallel normal fields $v_i \in \nu(M)$, $i \in I$, such that*

$$TM = \bigoplus_{i \in I} E_i$$

and for any $v \in \nu(M)$,

$$A_v \mid E_i = \langle v, v_i\rangle \mathrm{id}_{E_i}.$$

Moreover, if we suppose $v_0 = 0$, then $\dim(E_i) < \infty$, *for* $i \neq 0$.

Proof. The same proof as for the finite dimension case works here, because A_v is compact and selfadjoint. □

E_i are called *curvature distributions*, v_i are called *curvature normals*. We will assume that $\{e_i \mid i \in N\}$ is an orthonormal frame for $H(\tilde{V})$, such that $\{e_i \mid i > k\}$ is an orthonormal tangent frame for M, where e_i is in some curvature distribution, and $\{e_\alpha \mid 1 \le \alpha \le k\}$ is an orthogonal normal frame of M in $H(\tilde{V})$.

4.5 **Lemma.** *M is full in $H(\tilde{V})$ iff v_i, $i \in I$, span $\nu(M)$.*

Proof. The same proof as for (1.4). □

4.6 **Lemma.** *For $\varepsilon > 0$, $I_\varepsilon = \{i \in I \mid |v_i|^2 > \varepsilon\}$ is finite.*

Proof. Since A_{e_α} is compact

$$\left\{ i \in I \mid |\langle v_i, e_\alpha\rangle| > \sqrt{\frac{\varepsilon}{k}} \right\} \quad \text{is finite},$$

hence

$$I_\varepsilon \subset \bigcup_{1\le\alpha\le k} \left\{ i \in I \mid |\langle v_i, e_\alpha\rangle| > \sqrt{\frac{\varepsilon}{k}} \right\} \quad \text{is finite.} \quad \square$$

4.7 **Lemma.** *Let* $\omega_{ij} = \sum_l \gamma_{ij}^l \omega_l$. *Then*

$$\gamma_{ij}^l (v_{i'} - v_{j'}) = \gamma_{il}^j (v_{i'} - v_{l'}) = \gamma_{lj}^i (v_{l'} - v_{j'}), \qquad j \ne l,$$

where $e_i \in E_{i'}$, $e_j \in E_{j'}$, $\varepsilon_l \in E_{l'}$.

Proof. The same proof as for (1.5). □

4.8 **Corollary.** *Suppose* $e_i \in E_{i'}$, $e_j \in E_{j'}$, $i' \ne j'$, *then*

(1) $\gamma_{ij}^l = 0$ *if* $e_l \in E_{j'}$.

(2) *If* $\gamma_{ij}^l \ne 0$, *where* $e_l \in E_{l'}$, $l' \ne j'$, *then*

$$v_{l'} = (1 - a_{i'j'}^{l'}) v_{i'} + a_{i'j'}^{l'} v_{j'}, \quad \text{where } a_{i'j'}^{l'} \ne 0, 1$$

and

$$\gamma_{ij}^l = \gamma_{il}^j a_{i'j'}^{l'} = \gamma_{lj}^i (1 - a_{i'j'}^{l'}).$$

4.9 **Lemma.** *Suppose* $v_i \ne v_j$, *then*

$$\langle v_i, v_j\rangle - 1 = \sum_{\substack{l \\ v_l = (1-a_{ij}^l)v_i + a_{ij}^l v_j \\ a_{ij}^l \ne 0, 1}} \sum_{\substack{t \\ e_t \in E_l}} (\gamma_{ij}^t)^2 \frac{1}{a_{ij}^l (a_{ij}^l - 1)}.$$

Proof. The same proof as for (2.1). □

4.10 **Lemma.** (1) M *has at most one curvature normal whose length* ≤ 1.

(2) *If* v_i *is a curvature normal of* M, *then* $v_i \ne 0$.

Proof. (1) Suppose there are at least two curvature normals whose length ≤ 1. Then, by (4.6), we can choose v_i, v_j, $i \ne j$, such that $|v_i| \le 1$, $|v_j| \le 1$, and if $|v_l| \le 1$, then $|v_i| \ge |v_l|$ and $\langle v_i, v_j\rangle \ge \langle v_i, v_l\rangle$, for $l \ne i$.

By (4.9)

$$\langle v_i, v_j\rangle - 1 = \sum_{\substack{l \\ v_l = (1-a_{ij}^l)v_i + a_{ij}^l v_j \\ a_{ij}^l \ne 0, 1}} \sum_{\substack{t \\ e_t \in E_l}} (\gamma_{ij}^t)^2 \frac{1}{a_{ij}^l (a_{ij}^l - 1)}$$

and $\langle v_i, v_j\rangle < \langle v_i, v_i\rangle \le 1$, we have

$$\sum_{\substack{l \\ v_l = (1-a_{ij}^l)v_i + a_{ij}^l v_j \\ a_{ij}^l \ne 0, 1}} \sum_{\substack{t \\ e_t \in E_l}} (\gamma_{ij}^t)^2 \frac{1}{a_{ij}^l (a_{ij}^l - 1)} < 0.$$

On the other hand, if $|v_l| \le 1$, and $v_l = a_{ij}^l v_i + (1 - a_{ij}^l) v_j$, $0 < a_{ij}^l < 1$, then

$$\begin{aligned} \langle v_i, v_l\rangle &= a_{ij}^l \langle v_i, v_i\rangle + (1 - a_{ij}^l)\langle v_i, v_j\rangle \\ &> a_{ij}^l \langle v_i, v_j\rangle + (1 - a_{ij}^l)\langle v_i, v_j\rangle \\ &= \langle v_i, v_j\rangle, \end{aligned}$$

a contradiction. So, if $|v_l| \leq 1$, and $v_l = a^l_{ij}v_i + (1 - a^l_{ij})v_j$, then, either $a^l_{ij} < 0$, or $a^l_{ij} > 1$.

If $|v_l| > 1$, and $v_l = a^l_{ij}v_i + (1 - a^l_{ij})v_j$, then, either $a^l_{ij} < 0$, or $a^l_{ij} > 1$. So, we have

$$\sum_{\substack{l \\ v_l=(1-a^l_{ij})v_i+a^l_{ij}v_j \\ a^l_{ij}\neq 0,1}} \sum_{\substack{t \\ e_t\in E_l}} (\gamma^t_{ij})^2 \frac{1}{a^l_{ij}(a^l_{ij}-1)} \geq 0,$$

a contradiction. Thus M has at most one curvature normal whose length ≤ 1.

(2) Suppose $v_0 = 0$ is a curvature normal of M. By (1) and (4.6) we know that M has only finitely many curvature normals, and M has at least one nonzero curvature normal, since M is full. Hence we can choose v_j, such that $|v_j| \leq |v_l|$, for all $l \neq 0$. Then, by (4.9) we have

$$\begin{aligned}\langle v_0, v_j\rangle - 1 &= \sum_{\substack{l \\ v_l=(l-a^l_{ij})v_0+a^l_{ij}v_j \\ a^l_{ij}\neq 0,1}} \sum_{\substack{t \\ e_t\in E_l}} (\gamma^t_{ij})^2 \frac{1}{a^l_{ij}(a^l_{ij}-1)} - 1 \\ &= \sum_{\substack{l \\ v_l=a^l_{ij}v_j \\ a^l_{ij}\neq 0,1}} \sum_{\substack{t \\ e_t\in E_l}} (\gamma^t_{ij})^2 \frac{1}{a^l_{ij}(a^l_{ij}-1)}.\end{aligned}$$

On the other hand, if $0 < a^l_{ij} < 1$, and $v_l = a^l_{ij}v_j$, then $|v_l| = a^l_{ij}|v_j| < |v_j|$, a contradiction. So, if $v_l = a^l_{ij}v_j$, then either $a^l_{ij} < 0$, or $a^l_{ij} > 1$. Hence we have

$$\sum_{\substack{l \\ v_l=a^l_{ij}v_j \\ a^l_{ij}\neq 0,1}} \sum_{\substack{t \\ e_t\in E_l}} (\gamma^t_{ij})^2 \frac{1}{a^l_{ij}(a^l_{ij}-1)} \geq 0,$$

a contradiction. Therefore M has no zero curvature normal. □

4.11 *Proof of Theorem* 4.3. By (4.6) and (4.10), we know that M has only a finite number of curvature normals and none of them is zero. Therefore, by (4.4), M is of finite dimension. □

4.12 **Theorem.** *Suppose M is an isoparametric Hilbert submanifold of $H(\widetilde{V})$, then M is either of finite dimension, or a totally geodesic submanifold of $H(\widetilde{V})$, i.e., $M = H(\widetilde{V}) \cap V_1$, where V_1 is a Lorentz Hilbert subspace of $\widetilde{V}$.*

Proof. The direct corollary of (4.2) and (4.3). □

REFERENCES

[Ab] U. Abresch, *Isoparametric hypersurfaces with four or six distinct principal curvatures*, Math. Ann. **264** (1983), 283–302.

[Be] C. T. Benson and L. C. Grove, *Finite reflection groups*, Bogden & Quigley, 1971.

[Ca1] E. Cartan, *Familles de surface isoparametrique dans les space a courbure constante*, Ann. of Math. (2) **17** (1938), 177–191.

[Ca2] ——, *Sur des familles remarquables d'hypersurfaces isoparametriques dans les espaces spheriques*, Math. Z. **45** (1939), 335–367.

[Ca3] ——, *Sur quelques familles remarquables d'hypersurfaces*, C. R. Congres. Math. Liège, 1939, pp. 30–41.

[Ca4] ——, *Sur des familles d'hypersurfaces isoparametriques des espaces spheriques a* 5 *et a* 9 *dimension*, Univ. Nac. Tucuman Rev. Ser. A **1** (1940), 5–22.

[Co1] L. Conlon, *A class of variationally complete representations*, J. Differential Geom. **7** (1972), 149–160.

[Co2] ——, *Variational completeness and K-transversal domains*, J. Differential Geom. **7** (1972), 135–147.

[CR] T. E. Cecil and P. J. Ryan, *Tight and taut immersions of manifolds*, Research Notes in Math., vol. 107, Pitman, Boston, Mass., 1985.

[CW1] S. Carter and A. West, *Generalized Cartan polynomials*, J. London Math. Soc. (2) **32** (1984), 305–316.

[CW2] ——, *Isoparametric systems and transnormality*, Proc. London Math. Soc. (3) **51** (1985), 520–542.

[CW3] ——, *Isoparametric and totally focal submanifolds*, Preprint, University of Leeds, 1988.

[Da] J. Dadok, *Polar coordinates induced by actions of compact Lie groups*, Trans. Amer. Math. Soc. **288** (1985), 479–502.

[FKM] D. Ferus, H. Kacher and H. F. Münzner, *Cliffordalgebren and neue isoparametische hyperflächen*, Math. Z. **177** (1981), 479–502.

[Ha] C. E. Harle, *Isoparametric families of submanifolds*, Bol. Soc. Brasil. Mat. **13** (1982), 35–48.

[HL] W. Y. Hsiang and B. H. Lawson, Jr., *Minimal submanifolds of low cohomogeneity*, J. Differential Geom. **5** (1971), 1–38.

[HPT] W. Y. Hsiang, R. S. Palais and C. L. Terng, *The topology of isoparametric submanifolds*, J. Differential Geom. **27** (1988), 423–460.

[La] S. Lang, *Introduction to differentiable manifolds*, Interscience, New York, 1966.

[Mü] H. F. Münzner, *Isoparametrische hyperflächen in spharen*. I, II, Math. Ann. **251** (1980), 57–71; **256** (1981), 215–232.

[No] K. Nomizu, *Élie Cartan's work on isoparametric families of hypersurfaces*, Proc. Sympos. Pure Math., vol. 27, Amer. Math. Soc., Providence, R.I., 1975, pp. 191–200.

[O'N1] B. O'Neill, *The fundamental equations of a submersion*, Michigan Math. J. **13** (1966), 459–469.

[O'N2] ——, *Semi-Riemannian geometry with applications to relativity*, Academic Press, 1983.

[OT] H. Ozeki and M. Takeuchi, *On some types of isoparametric hypersurfaces in spheres*. I, II, Tohoku Math. J. **27** (1975), 515–559 and **28** (1976), 7–55.

[PT1] R. S. Palais and C. L. Terng, *A general canonical form theory*, Trans. Amer. Math. Soc. **300** (1987), 771–789.

[PT2] ——, *Critical point theory and submanifold geometry*, Lecture Notes in Math., vol. 1353, Springer-Verlag, 1988.

[Te1] C. L. Terng, *Isoparametric submanifolds and their Coxeter groups*, J. Differential Geom. **21** (1985), 79–107.

[Te2] ——, *Convexity theorem for isoparametric submanifolds*, Invent. Math. **85** (1986), 487–492.

[Te3] ——, *Submanifolds with flat normal bundle*, Math. Ann. **277** (1987), 95–111.

[Te4] ——, *Proper Fredholm submanifolds of Hilbert space*, J. Differential Geom. **29** (1989), 9–47.

[Th] G. Thorbergsson, *Isoparametric foliations and their buildings*, preprint.

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