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GROWTH RATES, $\mathbb{Z}_p$-HOMOLOGY, AND VOLUMES OF HYPERBOLIC 3-MANIFOLDS

PETER B. SHALEN AND PHILIP WAGREICH

Dedicated to Professor Heisuke Hironaka on his sixtieth birthday

ABSTRACT. It is shown that if M is a closed orientable irreducible 3-manifold and n is a nonnegative integer, and if $H_1(M, \mathbb{Z}_p)$ has rank $\geq n+2$ for some prime p, then every n-generator subgroup of $\pi_1(M)$ has infinite index in $\pi_1(M)$, and is in fact contained in infinitely many finite-index subgroups of $\pi_1(M)$. This result is used to estimate the growth rates of the fundamental group of a 3-manifold in terms of the rank of the $\mathbb{Z}_p$-homology. In particular it is used to show that the fundamental group of any closed hyperbolic 3-manifold has uniformly exponential growth, in the sense that there is a lower bound for the exponential growth rate that depends only on the manifold and not on the choice of a finite generating set. The result also gives volume estimates for hyperbolic 3-manifolds with enough $\mathbb{Z}_p$-homology, and a sufficient condition for an irreducible 3-manifold to be almost sufficiently large.

This paper addresses several different problems in the geometric and topological theory of 3-manifolds. In particular, we obtain new results on the problem of estimating the growth rate of the fundamental group of a 3-manifold (§4); on the problem of estimating numerical invariants of a hyperbolic 3-manifold, such as its volume or its maximal injectivity radius (§5); and on certain older problems in the topological theory, such as when a 3-manifold M is almost sufficiently large, and when $\pi_1(M)$ contains a free subgroup of rank 2 (§2). All these problems will be seen to be related to a circle of intriguing questions about 3-manifold groups which are treated in §1.

In the following discussion we shall let M denote a closed, orientable 3-manifold with a smooth or piecewise-linear structure. (The results of the paper can be adapted to nonclosed 3-manifolds, but we have concentrated on the closed case as it is the most interesting.) We shall assume that M is *irreducible*: this means that M is connected and that every (smooth or PL) 2-sphere in M bounds a 3-ball. (According to the prime decomposition theorem for 3-manifolds [Mi1], this is not a serious restriction. Furthermore, any hyperbolic 3-manifold is irreducible since it is covered by $\mathbb{R}^3$: see [Mi1, proof of Theorem 2].)

In §1 we exploit the idea that the presence of "enough" first homology in the 3-manifold M can be helpful in studying $\pi_1(M)$. For rational homology

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this idea was extensively developed by F. Waldhausen; for homology modulo a prime p, it seems to have been first successfully used by V. G. Turaev. Turaev showed [Tur, Remark 1.II, p. 359] that if $\operatorname{rank} H_1(M; \mathbb{Z}_p) \geq 3$, then $\pi_1(M)$ contains infinitely many subgroups of finite index. (In this paper $\mathbb{Z}_p$ will always mean $\mathbb{Z}/p\mathbb{Z}$.)

In §1 we give a refined version of [Tur, Remark 1.II]; this is our Proposition 1.1. It asserts that if $H_1(M; \mathbb{Z}_p)$ has rank at least $n+2$ for a given integer $n \geq 1$, then every n-generator subgroup of $\pi_1(M)$ is contained in infinitely many different finite-index subgroups. Our proof of 1.1 is similar in spirit to Turaev's arguments. The statement of 1.1 is also very similar in flavor to a result of Lubotzky's [L, Theorem 5.4]; the two results are compared in the discussion in 1.6 below. Much of the present paper is inspired by Lubotzky's work.

In the remainder of §1, we combine 1.1 with results from [JaS], [Tuc], and [BaS] to obtain criteria for subgroups of $\pi_1(M)$ to be free. A group G is said to be *k-free*, where k is a nonnegative integer, if every subgroup of G which can be generated by k elements is a free group (of some rank $\leq k$). In 1.9 we show that if $\pi_1(M)$ has no free abelian subgroup of rank 2, and if either $\operatorname{rank} H_1(M; \mathbb{Q}) \geq 3$ or $\operatorname{rank} H_1(M; \mathbb{Z}_p) \geq 4$ for some prime p, then $\pi_1(M)$ is 2-free.

More generally according to Proposition 1.8, if $\pi_1(M)$ contains no isomorphic copy of a genus-g surface group for any positive integer $g < k$, and if either $\operatorname{rank} H_1(M; \mathbb{Q}) \geq k+1$ or $\operatorname{rank} H_1(M; \mathbb{Z}_p) \geq k+2$ for some prime p, then $\pi_1(M)$ is k-free.[1] The latter result is used, in 1.13, to prove that if G is the fundamental group of any closed hyperbolic 3-manifold, then for any k, the group G is *virtually k-free* in the sense that it has a subgroup of finite index which is k-free. It would be interesting to know which groups besides hyperbolic 3-manifold groups have the property of being virtually k-free for every k.

Some of the results of §1 have recently been improved on by Mess [Mes2] and [Parry P]. Their proofs involve quoting the results proved in §1.

In §2 we apply the results of §1 to some classical problems in 3-dimensional topology. Generally speaking, the strongest known results in 3-manifold theory apply to the case where the manifold M is "sufficiently large." This means that M contains an embedded closed, connected, orientable surface F of positive genus which is *incompressible* in the sense that the inclusion homomorphism maps $\pi_1(F)$ injectively to $\pi_1(M)$. It is a consequence of the loop theorem of Papakyriakopoulos that if $H_1(M; \mathbb{Q}) \neq 0$ then M is sufficiently large (see [He, Lemma 6.6].) In §2 we show that certain results that are known in the sufficiently large case are true under the assumption that $H_1(M; \mathbb{Z}_p)$ has big enough rank for some prime p.

For example, Evans and Moser ([EM]; see also [Wa]) show that when M is sufficiently large, $\pi_1(M)$ usually contains a free subgroup of rank 2; they classify all the exceptions, which in particular turn out to have fundamental groups generated by at most three elements. In 2.9 we prove, without assuming M sufficiently large, that if $\operatorname{rank} H_1(M; \mathbb{Z}_p) \geq 4$ for some prime p then $\pi_1(M)$ always contains a free subgroup of rank 2.

[1]This is a much stronger result than the one announced in [BaS], where $\operatorname{rank} H_1(M; \mathbb{Z}/p\mathbb{Z}) \geq k+2$ was replaced by a considerably more restrictive condition. A suggestion of Marc Culler's was crucial in arriving at the present version of Proposition 1.8.

One of the fundamental conjectures in 3-manifold theory is that whenever $\pi_1(M)$ is infinite, M is *almost sufficiently large* in the sense that it has a sufficiently large finite-sheeted covering. (According to [MeeSY], any finite-sheeted covering of the irreducible 3-manifold M is itself irreducible.) An obvious necessary condition for M to be almost sufficiently large is that $\pi_1(M)$ contain a surface group, i.e. a subgroup isomorphic to the fundamental group of some closed, connected, orientable surface S_g of genus $g > 0$. It is natural to ask whether this condition is also sufficient. In 2.5 we show that if $\pi_1(M)$ contains an isomorphic copy of $\pi_1(S_g)$ for some $g > 0$, and if $H_1(M; \mathbb{Z}_p)$ has rank at least $2g+2$, then some finite-sheeted cover of M contains an incompressible surface.

The proof of Proposition 2.4 depends on combining Proposition 1.1 with a theorem of Jaco's which has not been previously published; we give a proof of Jaco's theorem in 2.1–2.3.

Another basic result in the sufficiently large case is Waldhausen's torus theorem, which asserts (in the refined version proved in [Jo and JaS]) that if $\pi_1(M)$ contains a free abelian subgroup of rank 2 then either M contains an incompressible torus or M is a Seifert fibered space. (For an account of the theory of Seifert fibered spaces, see [He, Chapter 12].) In [Tur], Turaev points out that the results of his paper, combined with those of [Sh], imply that the torus theorem remains true if the assumption that M is sufficiently large is replaced by the assumption that $\operatorname{rank} H_1(M; \mathbb{Z}_p) \geq 3$. In §2 we give a self-contained proof of this case of the torus theorem (Corollary 2.8), including a self-contained proof of the relevant result from [Sh]; the main reason for doing this is to point out explicitly the connection between the results of [Sh] and Jaco's Corollary 2.3.

Turaev's Corollary 2.8 can also be deduced from a theorem proved by G. Mess in [Mes1], using entirely different methods. Furthermore, A. Casson and D. Gabai have recently announced proofs that the torus theorem remains true for an *arbitrary* closed, orientable, irreducible 3-manifold. We emphasize that the arguments of §2 are much more elementary than those used by Mess, Casson and Gabai.

Our proof of Theorem 2.9, on the existence of free subgroups, combines Corollary 2.8 with the results from §1.

Recent work by Cannon and others has stimulated interest in growth functions of finitely generated groups, and in particular of fundamental groups of hyperbolic manifolds. Let G be a group and let S be a finite generating set. For any integer $n \geq 0$, let b_n denote the number of elements of G that can be expressed as words of length $\leq n$ in the generating set S. The formal power series $g(x) = \sum b_n x^n$ turns out to reflect interesting properties of the group G; for example, Cannon has shown [C] that if G is the fundamental group of a closed hyperbolic n-manifold then $g(x)$ is a rational function.

We say that G has *exponential growth* if there is a constant $B > 1$ such that $b_n > B^n$ for all sufficiently large n. The supremum of all such B will be called the *growth rate* of G with respect to S and denoted $\omega(S)$. Equivalently, ω is the reciprocal of the radius of convergence of $g(x)$.

The condition that G have exponential growth, i.e. that $\omega(S)$ be > 1, is independent of the generating set S (see [Mi2]); but the value of ω depends on S. We define the *minimal growth rate* of G to be $\omega(G) = \inf \omega(S)$, where the infimum is taken over all finite generating sets S. If $\omega(G) > 1$ we shall say that G has *uniformly exponential growth.* It is an intriguing question whether every finitely generated group G of exponential growth is in fact of uniformly exponential growth. The question seems particularly interesting for the case of fundamental groups of closed hyperbolic n-manifolds, which are known always to have exponential growth.

Section 3 contains a brief introduction to the theory of growth of groups and proofs of some general results that are used elsewhere in the paper. Some of these results may be of independent interest, particularly Proposition 3.1, which describes the growth of a free product, and Corollary 3.6, which asserts that virtually 2-free groups that are not cyclic-by-finite have uniformly exponential growth.

In §4 we show that the fundamental group of a hyperbolic 3-manifold M always has uniformly exponential growth. In fact, this follows easily from the results of §1 and the elementary Corollary 3.6. Again using the results of §1, we obtain explicit lower bounds for the minimal growth rate of $\pi_1(M)$ in the case where M has enough homology: for example, if $\operatorname{rank} H_1(M;\mathbb{Q}) \geq 3$, or if $\operatorname{rank} H_1(M;\mathbb{Z}_p) \geq 4$ for some prime p, then $\omega(\pi_1(M)) \geq 3$. Likewise, we obtain explicit estimates if $\operatorname{rank} H_1(M;\mathbb{Q})$ is 1 or 2. However, we have not been able to determine whether there is a uniform lower bound for $\omega(\pi_1(M))$ if M ranges over all hyperbolic 3-manifolds.

If the closed, orientable, irreducible 3-manifold M is not assumed to be hyperbolic, it is not obvious whether $\pi_1(M)$ has exponential growth at all. Indeed, some restriction on M is obviously needed to ensure exponential growth, since there are examples in which $\pi_1(M)$ is finite or nilpotent. In [Tur], Turaev gives a lower bound for the growth under the assumption that $\operatorname{rank} H_1(M;\mathbb{Z}_p) \geq 3$ for some prime p and that $\pi_1(M)$ has no nilpotent subgroup of finite index; however, this lower bound is subexponential, being of the form $C^{n/\log n}$, where C is a constant > 1.

In 4.1 we point out that if $\operatorname{rank} H_1(M;\mathbb{Z}_p) \geq 4$ for some prime p, then $\pi_1(M)$ has exponential growth. In fact, this is an immediate consequence of Theorem 2.9, which guarantees the existence of free subgroups of rank 2.

In §5 we study the geometry of hyperbolic 3-manifolds. If the closed 3-manifold M has a hyperbolic metric (i.e. a Riemannian metric of constant curvature -1), there is a number ε such that any two smooth loops of length $< \varepsilon$ based at an arbitrary point of M define commuting elements of $\pi_1(M)$. We call the largest such ε the *Margulis number* of M and denote it $\mu(M)$. The Margulis number plays a key role in studying the geometry of M. For example, it is easy to show that M always contains an isometric copy of a ball B of radius $\varepsilon/2$ in hyperbolic 3-space. (This argument is reviewed in §5.) This gives an obvious estimate for the volume of M, namely that $\operatorname{vol}(M) \geq \operatorname{vol}(B)$. This estimate can be improved using the methods developed by Meyerhoff in [Mey2].

In [Mey1], Meyerhoff shows that $\mu(M) \geq .104$ for any closed hyperbolic 3-manifold M. In [Mey2] he uses this to obtain a lower bound of .00082 for the volume of M.

There is a general method, essentially due to Gromov [Gr], for estimating the Margulis number of a hyperbolic n-manifold in terms of the growth rates of 2-generator subgroups. We give an explicit statement and proof in 5.2. This implies in particular (see 5.3) that if the hyperbolic 3-manifold M has a 2-free fundamental group then[2] $\mu(M) \geq \frac{\log 3}{2}$. This gives $\operatorname{vol} M \geq 0.1124$. (Throughout this paper all logarithms are understood to be taken to the base e.)

In particular the lower bound $\frac{\log 3}{2}$ is valid if $\operatorname{rank} H_1(M; \mathbb{Q}) \geq 3$, or if $\operatorname{rank} H_1(M; \mathbb{Z}_p) \geq 4$ for some prime p (see 5.4).[3] Likewise, in 5.4 we get new estimates if $\operatorname{rank} H_1(M; \mathbb{Q})$ is 1 or 2.

We are indebted to Alex Lubotzky for his suggestion that his result cited above may have applications to 3-manifold theory; although this result is not used in the final versions of our proofs, it was crucial in developing the ideas that we use in this paper. We are grateful to Marc Culler for the computer calculations that yield the lower bounds for volumes given in 5.5 and, especially, for helping us find the right statement and proof of Proposition 1.1; the latter is a much stronger result than our original version (which was announced in [BaS]). Finally, we thank Cameron Gordon for telling us about Turaev's paper [Tur], and Craig Hodgson, Bob Brooks and Marc Culler for telling us about Gromov's book [Gr].

1. Free subgroups of 3-manifold groups

Let k be a nonnegative integer. A group G is *k-free* if every subset of G with cardinality $\leq k$ generates a free subgroup (of some $\operatorname{rank} \leq k$).

Our first result, Proposition 1.1, gives a sufficient condition for a finitely generated subgroup of a closed 3-manifold group to be of infinite index (and, what is stronger, to be contained in infinitely many different finite-index subgroups). We combine this with a result from [BaS] to obtain, in Proposition 1.8 and its Corollary 1.9, sufficient conditions for a closed 3-manifold group to be k-free. Some variants of the latter results, which will be used in §§4 and 5, are given in 1.11 and 1.12. Finally, we use 1.8, together with a result from [We], to show (Proposition 1.13) that if G is the fundamental group of any closed hyperbolic 3-manifold, then for every k there is a subgroup of finite index in G which is k-free.

Proposition 1.1. *Let p be a prime and let M be a closed 3-manifold. If p is odd assume that M is orientable. Let $n \geq 1$ be an integer, and let E be a subgroup generated by n elements of $\pi_1(M)$. If $\operatorname{rank} H_1(M, \mathbb{Z}_p) \geq n+2$, then E has infinite index in $\pi_1(M)$. In fact, E is contained in infinitely many distinct subgroups of finite index.*

The proof of 1.1 depends on several lemmas. In 1.2–1.5, we fix an arbitrary prime p. Whenever A is a subgroup of a group G, we let $G\#A$ denote the subgroup of G generated by all elements of the form $gag^{-1}a^{-1}b^p$ with $g \in G$ and $a, b \in A$. Thus if A is any normal subgroup of G, the group $G\#A$ is a normal subgroup of A, and $A/G\#A$ is an elementary abelian p-group. We shall denote by $(G_i)_{i \geq 0}$ the sequence of characteristic subgroups defined recursively by $G_0 = G$ and $G_{i+1} = G\#G_i$. This is the $\operatorname{mod} p$ *lower central series* of G.

[2]Recent work by Culler and Shalen [CS] gives strong evidence that it should be possible to replace the estimate $(\log 3)/2$ by $\log 3$.

[3]In the case where $\operatorname{rank} H_1(M; \mathbb{Q}) \geq 3$, Culler and Shalen [CS] have proved that $\mu(M) \geq \log 3$. (See the previous footnote and the remark following the statement of Proposition 5.4.) This gives a lower bound of about .92 for the volume.

We shall regard any elementary abelian p-group as a $\mathbb{Z}_p$ vector space; in particular such a group has a well-defined *rank*. Furthermore, all homology groups of groups and spaces will be understood to be taken with $\mathbb{Z}_p$ coefficients. If G is a group and $i \geq 0$ is an integer, we shall write $h_i(G)$ for the rank of $H_i(G)$.

1.2. It was shown by Stallings in [St2] that if N is a normal subgroup of a group G and $Q = G/N$, then there is an exact sequence

$$H_2(G) \to H_2(Q) \to N/(G\#N) \to H_1(G) \to H_1(Q) \to 0.$$

Lemma 1.3. *Let M be a closed 3-manifold. If p is odd suppose that M is orientable. Set $\Gamma = \pi_1(M)$ and $r = \operatorname{rank}(\Gamma/\Gamma_1)$. Then*

$$\operatorname{rank}\Gamma_1/\Gamma_2 \geq \tfrac{1}{2}r(r-1).$$

Proof. We apply 1.2 with $G = \Gamma$ and $N = \Gamma_1$, so that $Q = \Gamma/\Gamma_1$ is an elementary p-group of rank r, and $N/(G\#N) = \Gamma_1/\Gamma_2$. This gives an exact sequence

$$H_2(\Gamma) \to H_2(\Gamma/\Gamma_1) \to \Gamma_1/\Gamma_2 \to H_1(\Gamma) \to H_1(\Gamma/\Gamma_1) \to 0.$$

Hence we have

$$\operatorname{rank}(\Gamma_1/\Gamma_2) - h_2(\Gamma/\Gamma_1) + h_1(\Gamma/\Gamma_1) \geq h_1(\Gamma) - h_2(\Gamma).$$

Now for any space X there is a natural surjective homomorphism from $H_2(X; Z_p)$ to $H_2(\pi_1(X), Z_p)$. (Indeed, a $K(\pi_1(X), 1)$-space can be constructed from X by adding cells of dimension ≥ 3 to kill the higher homotopy groups.) Applying this to $X = M$ we conclude that $h_2(\Gamma) \leq \operatorname{rank} H_2(M; Z_p)$. Moreover, we have $h_1(\Gamma) = \operatorname{rank} H_1(M; Z_p) = \operatorname{rank} H_2(M; Z_p)$ by Poincaré duality. Thus $h_1(\Gamma) \geq h_2(\Gamma)$. We therefore conclude that

$$\operatorname{rank}(\Gamma_1/\Gamma_2) \geq h_2(\Gamma/\Gamma_1) - h_1(\Gamma/\Gamma_1).$$

Using the Künneth formula to compute $H_i(\Gamma/\Gamma_1) = H_i(\mathbb{Z}_p^{(r)})$, we find that

$$h_2(\Gamma/\Gamma_1) - h_1(\Gamma/\Gamma_1) = r(r+1)/2 - r = r(r-1)/2.$$

This gives the desired result. □

Lemma 1.4. *Let $n \geq 0$ be an integer. Let Γ be a group and let E be a subgroup generated by n elements of Γ. Then*

$$h_1(E\Gamma_1) \geq \operatorname{rank}(\Gamma_1/\Gamma_2) - \tfrac{1}{2}n(n-1).$$

Proof. We apply 1.2, taking $G = E\Gamma_1$ and $N = \Gamma_1$, so that $Q = E\Gamma_1/\Gamma_1$. This gives (ignoring the term $H_2(G)$ in 1.2) an exact sequence

$$H_2(E\Gamma_1/\Gamma_1) \to \Gamma_1/(E\Gamma_1\#\Gamma_1) \to H_1(E\Gamma_1) \to H_1(E\Gamma_1/\Gamma_1) \to 0.$$

Hence

$$h_1(E\Gamma_1) \geq \operatorname{rank}(\Gamma_1/(E\Gamma_1\#\Gamma_1)) + h_1(E\Gamma_1/\Gamma_1) - h_2(E\Gamma_1/\Gamma_1).$$

Since E is generated by n elements, $E\Gamma_1/\Gamma_1$ is an elementary abelian p-group of some rank $r \leq n$. Hence $h_1(E\Gamma_1/\Gamma_1) = r$; and the Künneth formula gives $\operatorname{rank} h_2(E\Gamma_1/\Gamma_1) = \frac{1}{2}r(r+1)$. On the other hand, since the subgroup Γ_2 of Γ contains $E\Gamma_1\#\Gamma_1$, we have

$$\operatorname{rank}(\Gamma_1/(E\Gamma_1\#\Gamma_1)) \geq \operatorname{rank}(\Gamma_1/\Gamma_2).$$

Hence

$$\begin{aligned} h_1(E\Gamma_1) &\geq \operatorname{rank}(\Gamma_1/\Gamma_2) + r - \tfrac{1}{2}r(r+1) \\ &= \operatorname{rank}(\Gamma_1/\Gamma_2) - \tfrac{1}{2}r(r-1) \geq \operatorname{rank}(\Gamma_1/\Gamma_2) - \tfrac{1}{2}n(n-1). \quad \square \end{aligned}$$

Lemma 1.5. *Let M be a closed 3-manifold. If p is odd assume that M is orientable. Let $n \geq 0$ be an integer, and let E be a subgroup generated by n elements of $\Gamma = \pi_1(M)$. Suppose that $\operatorname{rank} H_1(M) \geq n+2$. Then $D = E\Gamma_1$ is a proper subgroup of finite index in Γ, and $\operatorname{rank} H_1(D) \geq 2n+1$.*

Proof. Since Γ is finitely generated, Γ_1 has finite index in Γ, and hence so does D. Furthermore, we have $\operatorname{rank}\Gamma/\Gamma_1 = \operatorname{rank} H_1(M) \geq n+2$, and so the quotient homomorphism $\Gamma \to \Gamma/\Gamma_1$ cannot map the n-generator group E onto Γ/Γ_1; hence D is a proper subgroup of Γ.

By Lemma 1.3 we have $\operatorname{rank}(\Gamma_1/\Gamma_2) \geq \frac{1}{2}(n+2)(n+1)$. By Lemma 1.4 we have

$$\begin{aligned} \operatorname{rank} H_1(D) &\geq \operatorname{rank}(\Gamma_1/\Gamma_2) - \tfrac{1}{2}n(n-1) \\ &\geq \tfrac{1}{2}(n+2)(n+1) - \tfrac{1}{2}n(n-1) = 2n+1. \quad \square \end{aligned}$$

Proof of Proposition 1.1. Let $\mathcal{S}$ denote the set of all subgroups Δ of $\Gamma = \pi_1(M)$ such that

(i) Δ has finite index in Γ.
(ii) $\operatorname{rank} H_1(\Delta) \geq n+2$, and
(iii) $\Delta \supset E$.

We shall prove the result by showing that the set $\mathcal{S}$ is infinite. Clearly $\Gamma \in \mathcal{S}$, so that $\mathcal{S} \neq \varnothing$. Hence it suffices to show that for any $\Delta \in \mathcal{S}$ there is a proper subgroup D of Δ such that $D \in \mathcal{S}$.

Let $\Delta \in \mathcal{S}$ be given. Since Δ has finite index in Γ, we may identify Δ with $\pi_1(\widetilde{M})$, where $\widetilde{M}$ is a finite-sheeted covering space of M; in particular, $\widetilde{M}$ is a closed 3-manifold, and is orientable if p is odd. Since $\Delta \in \mathcal{S}$ we have $\operatorname{rank} H_1(\widetilde{M}) = \operatorname{rank} H_1(\Delta) \geq n+2$, and Δ contains the n-generator subgroup E. Now set $D = E\Delta_1$. By Lemma 1.5, D is a proper subgroup of Δ. Again by Lemma 1.5, D has finite index in Δ and hence in Γ; and $\operatorname{rank} H_1(D) \geq 2n+1 \geq n+2$. Moreover, D clearly contains E. Hence $D \in \mathcal{S}$ as required. $\square$

1.6. An argument due to Lubotzky [L, proof of Theorem 5.4] shows that if $\operatorname{rank} H_1(M; \mathbb{Z}_p)$ is large enough in comparison to k, then the normal closure of any k-element subset of $\pi_1(M)$ has infinite index in $\pi_1(M)$. To obtain this conclusion, which is stronger than the conclusion of 1.1, one needs a stronger hypothesis: for example, when $k=2$, Proposition 1.1 requires $\operatorname{rank} H_1(M; \mathbb{Z}_p) \geq 4$, whereas the argument of [L] seems to require $\operatorname{rank} H_1(M; \mathbb{Z}_p)$ to be at least 10. Note that the proof of 1.1 is quite elementary, whereas the proof in [L] depends on the Golod-Shafarevich theorem and theory of p-adic analytic groups.

Setting $k=1$ in 1.1, we obtain the following result which sharpens a result of Turaev's [Tur, Remark 1.II, p. 359].

Corollary 1.7. *Let p be a prime and let M be a closed 3-manifold. If p is odd assume that M is orientable. Suppose that $\operatorname{rank} H_1(M, \mathbb{Z}_p) \geq 3$. Then any element of $\pi_1(M)$ lies in infinitely many distinct finite-index subgroups of $\pi_1(M)$.* $\square$

For any nonnegative integer g, let S_g denote the closed orientable surface of genus g.

Proposition 1.8. *Let M be an irreducible, closed orientable 3-manifold, and let k be a nonnegative integer. Suppose that $\pi_1(M)$ has no subgroup isomorphic to $\pi_1(S_g)$ for any g with $0 < g < k$. Suppose furthermore that either*

(a) $\operatorname{rank} H_1(M; \mathbb{Q}) \geq k + 1$, *or*
(b) $\operatorname{rank} H_1(M; \mathbb{Z}_p) \geq k + 2$ *for some prime p.*

Then $\pi_1(M)$ is k-free.

Proof. First note that every k-generator subgroup of $\pi_1(M)$ has infinite index; indeed, if (a) holds this is clear, and if (b) holds then it follows from Proposition 1.1. Now according to [BaS, Corollary A1], if G is the fundamental group of an irreducible, orientable 3-manifold, if k is a positive integer and if G has no subgroup isomorphic to $\pi_1(S_g)$ for any g with $0 < g < k$, then any infinite-index subgroup of G generated by at most k elements is free (of some rank $\leq k$). The assertion follows. □

Corollary 1.9. *Let M be a closed, orientable hyperbolic 3-manifold. If either*

(b) $\operatorname{rank} H_1(M; \mathbb{Q}) \geq 3$, *or*
(a) $\operatorname{rank} H_1(M; Z_p) \geq 4$ *for some prime p,*

then any two noncommuting elements of $\pi_1(M)$ generate a free subgroup of rank 2.

Proof. Since M is hyperbolic, its universal cover is homeomorphic to $\mathbb{R}^3$, and it follows from the proof of [Mi1, Theorem 2] that M is irreducible. It is shown in [F] that $\pi_1(M)$ contains no free abelian subgroup of rank 2; that is, $\pi_1(M)$ has no subgroup isomorphic to $\pi_1(S_1)$. Thus the hypotheses of 1.8 hold with $k = 2$. Hence $\pi_1(M)$ is 2-free. □

In later sections we shall need two variants of Corollary 1.9, Propositions 1.11 and 1.2 below. The proofs depend on the following elementary fact.

Lemma 1.10. *Let x and y be elements of the fundamental group of a closed, orientable hyperbolic 3-manifold. If x and yxy^{-1} commute, so do x and y.*

Proof. Since M is hyperbolic, we may regard G as a torsion-free subgroup of $\mathrm{PSL}_2(\mathbb{C})$. Since M is closed, G contains no parabolic elements; hence after a conjugation we may assume that x is represented by a diagonal matrix with distinct diagonal entries. The hypothesis that x and yxy^{-1} commute implies that yxy^{-1} is also represented by a diagonal matrix. Hence y is represented either by a diagonal matrix or a matrix with zeros on the diagonal. If y has zeros on the diagonal, then since it is unimodular it must have order 2 in $\mathrm{PSL}_2(\mathbb{C})$. This is impossible since G is torsion-free. Hence y must be diagonal, so that x and y commute. □

Proposition 1.11. *Let M be a closed, orientable hyperbolic 3-manifold such that $\operatorname{rank} H_1(M, \mathbb{Q}) \geq 2$. Then for any two noncommuting elements x, y of $\pi_1(M)$, the elements x and yxy^{-1} generate a rank-2 free subgroup of G.*

Proof. The subgroup Γ generated by x and yxy^{-1} has infinite index in G, since x and yxy^{-1} have the same image in $H_1(M, \mathbb{Q}) = (G/G') \otimes \mathbb{Q}$, where G' denotes the commutator subgroup of G. It therefore follows from [JaS, Theorem VI.4.1] or [BaS, Corollary A1] that Γ is a free group of some rank ≤ 2. But since x and y do not commute, it follows from Lemma 1.10 that x and yxy^{-1} do not commute. Hence Γ must have rank 2. □

Proposition 1.12. *Let M be a closed, orientable hyperbolic 3-manifold with $H_1(M, \mathbb{Q}) \neq 0$. Then for any two noncommuting elements x, y of $\pi_1(M)$, the elements $xyx^{-1}y^{-1}$ and $yx^{-1}y^{-1}x$ generate a rank-2 free subgroup of G.*

Proof. The subgroup Γ generated by $xyx^{-1}y^{-1}$ and $yx^{-1}y^{-1}x$ is contained in the commutator subgroup G' of G; since $H_1(M, \mathbb{Q}) \neq 0$, it follows that Γ has infinite index in G. It therefore follows from [JaS, Theorem VI.4.1] or [BaS, Corollary A1] that Γ is a free group of some rank ≤ 2. In order to complete the proof we must show that $xyx^{-1}y^{-1}$ and $yx^{-1}y^{-1}x$ do not commute. Suppose that they do. Applying Lemma 1.10 with $xyx^{-1}y^{-1}$ and x^{-1} playing the roles of x and y respectively, we conclude that $xyx^{-1}y^{-1}$ and x^{-1} commute. Hence x and $yx^{-1}y^{-1}$ commute. By Lemma 1.10 it therefore follows that x and y commute. This contradicts the hypothesis. □

A group is said to be *virtually k-free*, for a given integer $k \geq 0$, if it has a subgroup of finite index which is k-free.

Proposition 1.13. *Let M be a closed hyperbolic 3-manifold. Then for every nonnegative integer k the group $\pi_1(M)$ is virtually k-free.*

Proof. After possibly replacing M by a two-sheeted covering we may assume that M is orientable. It follows from [Th, Corollary 8.8.6(a)] that up to conjugacy, $\pi_1(M)$ has only finitely many subgroups isomorphic to $\pi_1(S_g)$ for any fixed integer $g > 0$. Thus there is a finite family of subgroups $L_1, \dots, L_n$, where n is a nonnegative integer, such that a subgroup of $\pi_1(M)$ is isomorphic to $\pi_1(S_g)$ for some g with $0 < g < k$ if and only if it is conjugate to one of the L_i. For $i = 1, \dots, n$, fix an element $l_i \neq 1$ in L_i.

Since M is hyperbolic, $\pi_1(M)$ is isomorphic to a subgroup of $\mathrm{PSL}_2(\mathbb{C})$. In particular $\pi_1(M)$ is a linear group, and is therefore residually finite by [Mal]. Thus for $i = 1, \dots, n$ there is a normal subgroup N_i of finite index in $\pi_1(M)$ such that $l_i \notin N_i$. Set $\Gamma = N_1 \cap \cdots \cap N_n$. Then Γ has no subgroup isomorphic to $\pi_1(S_g)$ for any g with $0 < g < k$.

Since M is closed and hyperbolic, $\pi_1(M)$ has no solvable subgroup of finite index. Hence Γ has no solvable subgroup of finite index. It follows from [We, Corollary 10.6 and Example 10.1, p. 139] that if Γ is any finitely generated linear group which has no solvable subgroup of finite index, then for every positive integer m, there is a subgroup Δ of finite index in Γ such that $\operatorname{rank} H_1(\Delta; \mathbb{Z}_2) \geq m$. Applying this with $m = k + 2$, we obtain a finite-index subgroup Δ of Γ which satisfies the hypotheses of Proposition 1.8 and is therefore k-free. □

Our original proof of Proposition 1.13 used [L, proof of Theorem 5.4] in place of 1.8; cf. our remark 1.6 above.

2. Topological applications

In this section we use the results of §1 to show that a number of long-standing questions in 3-manifold theory have affirmative answers for the case of a (closed, irreducible, orientable) 3-manifold M whose mod p first homology has large enough rank for some prime p. In 2.8 we prove a torus theorem for the case where $\operatorname{rank} H_1(M; \mathbb{Z}_p) \geq 3$; in 2.9 we show that $\pi_1(M)$ contains a rank-2 free subgroup whenever $\operatorname{rank} H_1(M; \mathbb{Z}_p) \geq 4$. In 2.5 we address the problem of whether a 3-manifold M, whose fundamental group containing a surface group $\pi_1(S_g)$, is almost sufficiently large; we obtain an affirmative answer when $\operatorname{rank} H_1(M; \mathbb{Z}_p)$ is large in comparison with g.

The proofs of these results combine the results of §1 with a result due to Jaco; since Jaco's result has not been published, we shall state and prove it in 2.1–2.3 below.

In this section we shall work in the piecewise-linear category.

Let M be a closed, orientable, irreducible (PL) 3-manifold. Recall that M is said to be *sufficiently large* if it contains a closed, orientable (PL) surface F which has genus > 0 and is *incompressible* in the sense that the inclusion homomorphism $\pi_1(F) \to \pi_1(M)$ is injective. We shall say that M is *almost sufficiently large* if some finite-sheeted cover of M is sufficiently large. (According to [MeeSY], any covering space of M is irreducible.) One of the central conjectures in 3-manifold theory asserts that if $\pi_1(M)$ is infinite then M is almost sufficiently large. An obvious necessary condition for M to be sufficiently large is that $\pi_1(M)$ contain the fundamental group of the closed, orientable surface S_g of some genus > 0.

Jaco's theorem addresses the question of whether this necessary condition is also sufficient: that is, if $\pi_1(M)$ has a subgroup Γ isomorphic to $\pi_1(S_g)$ for some $g > 0$, is M almost sufficiently large? It provides an affirmative answer under the additional assumption that Γ is contained in infinitely many distinct subgroups of finite index in $\pi_1(M)$. Actually the argument does not require that Γ be a surface group, but only that it be freely indecomposable. A group is said to be *freely indecomposable* if it is neither trivial, nor infinite cyclic, nor isomorphic to the free product of two nontrivial groups.

Theorem 2.1 (Jaco). *Let M be an irreducible, closed, orientable 3-manifold. Suppose that $\pi_1(M)$ has a freely indecomposable subgroup Γ which is contained in infinitely many distinct subgroups of finite index in $\pi_1(M)$. Then M is almost sufficiently large.*

The proof of the theorem uses the following lemma. Recall that a path-connected subset Y of a path-connected space X is said to carry $\pi_1(X)$ if the inclusion homomorphism $\pi_1(Y) \to \pi_1(X)$ is surjective. (This condition is independent of the choice of base points.)

Lemma 2.2. *Let M be a closed, orientable, irreducible 3-manifold. Suppose that there is a compact, connected 3-manifold-with-boundary $N \subset M$ such that* (i) *N does not carry $\pi_1(M)$, and* (ii) *$\pi_1(N)$ has a freely indecomposable subgroup Γ such that $\pi_1(N) \to \pi_1(M)$ maps Γ injectively into $\pi_1(M)$. Then M is sufficiently large.*

Proof. For any closed, orientable 2-manifold F we shall denote by $k(N)$ the sum of the squares of the genera of the components of F.

Let $\mathcal{N}$ denote the set of all compact, connected 3-manifolds-with-boundary $N \subset M$ that satisfy conditions (i) and (ii) of the hypothesis of the lemma. Let $N_0 \in \mathcal{N}$ be chosen so that $k(\partial N_0) = \min_{N \in \mathcal{N}} k(\partial N)$. We shall prove the lemma by showing that (a) the inclusion homomorphism $\pi_1(F) \to \pi_1(M)$ is injective for every component F of ∂N_0, and (b) some component of ∂N_0 has positive genus.

To prove (a), suppose that J is a component of ∂N_0 for which $\pi_1(J) \to \pi_1(M)$ has nontrivial kernel. It follows from the loop theorem [He, Chapter 4] that there is a disk $D \subset M$ such that $D \cap J = \partial D$, and such that ∂D is homotopically nontrivial in J. Let A be an annulus neighborhood of ∂D in J, and let $E \subset M$ be a 3-ball such that $E \cap \partial N_0 = A \subset \partial E$. The surface $F = (\partial N_0 \cup \partial E) - (\operatorname{int} A)$ is obtained from ∂N_0 by replacing the homotopically nontrivial annulus A by two disks; it follows that $k(F) < k(\partial N_0)$. (Indeed, the component J of ∂N_0 is replaced in F either by a single component whose genus is one less than that of J, or by two components whose genera are strictly positive and add up to the genus of J.)

We have either $E \cap N_0 = A$ or $E \subset N_0$. If $E \cap N_0 = A$ then we set $N_1 = N_0 \cup E$. Note that $\partial N_1 = F$. We have $N_1 \in \mathscr{N}$. Indeed, N_1 satisfies (i) because N_0 carries $\pi_1(N_1)$; and if Γ is a freely indecomposable subgroup of $\pi_1(N_0)$ which is mapped injectively into $\pi_1(M)$, then the image of Γ in $\pi_1(N_1)$ is isomorphic to Γ and is again mapped injectively into $\pi_1(M)$. Now since $k(\partial N_1) = k(F) < k(\partial N_0)$, we have a contradiction to our choice of N_0.

Now suppose that $E \subset N_0$, and set $Q = \overline{N_0 - E}$. Note that $\partial Q = F$, and that Q has either one or two connected components. Consider the case where it has two components, say N_1 and N_2. Using van Kampen's theorem, we identify $\pi_1(N_0)$ with the free product $\pi_1(N_1) * \pi_1(N_2)$. Since $N_0 \in \mathscr{N}$, some indecomposable subgroup Γ of $\pi_1(N_0)$ is mapped injectively into $\pi_1(M)$. But it follows from the Kurosh subgroup theorem [K] that a freely indecomposable subgroup of a free product is contained in a conjugate of a factor. Thus we may assume that Γ is contained in a conjugate of $\pi_1(N_1)$. It follows that N_1 satisfies condition (ii). But N_1 certainly satisfies (i) since $N_1 \subset N_0$. Hence $N_1 \in \mathscr{N}$. On the other hand, since ∂N_1 is a union of components of $F = \partial Q$, we have $k(\partial N_1) \leq k(F) < k(\partial N_0)$, and again we have a contradiction to our choice of N_0.

In the case where Q is connected we set $N_1 = Q$ and use van Kampen's theorem to identify $\pi_1(N_0)$ with the free product $\pi_1(N_1) * \mathbb{Z}$; the same method again shows that $N_1 \in \mathscr{N}$ and $k(\partial N_1) < k(\partial N_0)$, giving a contradiction. This completes the proof of (a).

To prove (b), assume that the components $J_1, \ldots, J_n$ of ∂N_0 are all 2-spheres. Since M is irreducible, each J_i must bound a 3-ball $B_i \subset M$; and for each i with $1 \leq i \leq n$, we must have either $N_0 \subset B_i$ or $N_0 \cap B_i = J_i$. If some B_i contains N_0 then $\pi_1(N_0)$ maps trivially to $\pi_1(M)$; this contradicts condition (ii), since a freely indecomposable group is by definition nontrivial. On the other hand, if $N_0 \cap B_i = J_i$ for every i, we must have $M = N_0 \cup \bigcup_{i=1}^n B_i$; hence in this case N_0 carries $\pi_1(M)$, and we have a contradiction to condition (i). $\square$

Proof of Theorem 2.1. Let us fix a base point $y \in M$, and regard Γ as a subgroup of $\pi_1(M, y)$. By [Sc1], Γ is finitely presented; hence it is isomorphic to the fundamental group of a finite, connected simplicial 2-complex K. We fix a base point $x \in K$ and an isomorphism $j: \pi_1(K, x) \to \Gamma$. Since K is 2-dimensional, j is induced by a base point-preserving map $f: K \to M$. We may take f to be piecewise-linear; let us fix triangulations of K and M for which f is a simplicial map.

By hypothesis there is an infinite sequence $\Delta_0, \Delta_1, \ldots$ of distinct subgroups of $\pi_1(M, y)$ which all contain Γ. We may clearly choose the Δ_i so that $\Delta_0 = \pi_1(M, y)$ and $\Delta_{i-1} \supsetneq \Delta_i$ for every $i \geq 1$. Let $(\widetilde{M}_i, \tilde{y}_i)$ denote the based covering space of M determined by Δ_i. Each $\widetilde{M}_i$ is a closed 3-manifold and inherits a triangulation from M. For $i \geq 1$, since $\Delta_{i-1} \supsetneq \Delta_i$, we may regard $\widetilde{M}_i$ as a covering space of $\widetilde{M}_{i-1}$ in a natural way; we write $p_i \colon \widetilde{M}_i \to \widetilde{M}_{i-1}$ for the covering projection. Since all the Δ_i contain Γ, the map f has a unique base point-preserving lift $\tilde{f}_i : (K, x) \to (\widetilde{M}_i, \tilde{y}_i)$ for each i. We have $p_i \circ \tilde{f}_i = \tilde{f}_{i-1}$ whenever $i \geq 1$.

The p_i and the $\tilde{f}_i$ are simplicial maps. In particular, $K_i = \tilde{f}_i(K)$ is a subcomplex of $\widetilde{M}_i$ for each i.

We claim that for for some $n \geq 1$ the subcomplex K_n fails to carry $\pi_1(\widetilde{M}_n)$. The claim will be proved by assigning a "complexity" c_i to each simplicial map $\tilde{f}_i$ as in Stallings's proof [St1] of the loop theorem. We define c_i to be the number of ordered pairs (σ, σ') of simplices of K for which $f_i(\sigma) = f_i(\sigma')$. For $i \geq 1$, since $p_i \circ \tilde{f}_i = \tilde{f}_{i-1}$ where p_i is simplicial, it is clear that $c_i \leq c_{i-1}$ Since the c_i are positive integers we must have $c_n = c_{n+1}$ for some n. This means that p_{n+1} induces a bijection between the simplices of K_{n+1} and those of K_n. Thus p_{n+1} maps K_{n+1} homeomorphically onto K_n. If K_n were to carry $\pi_1(\widetilde{M}_n)$ it would follow that p_{n+1} induced a surjective homomorphism from $\pi_1(\widetilde{M}_{n+1})$ to $\pi_1(\widetilde{M}_n)$. This is impossible since $\Delta_{n+1} \neq \Delta_n$. This proves the claim.

We shall now apply Lemma 2.2 with $\widetilde{M}_n$ in place of M. Since M is irreducible, so is $\widetilde{M}_n$ [MeeSY]. Let N be a regular neighborhood of K_n in $\widetilde{M}_n$. By the claim just proved, N satisfies condition (i) of 2.2, i.e. it does not carry $\pi_1(\widetilde{M}_n)$. Condition (ii) also holds, with $(\tilde{f}_n)_\#(\pi_1(K, x))$ playing the role of Γ (since $\pi_1(K, x)$ is freely indecomposable and is mapped injectively by $f_\# : \pi_1(K, x) \to \pi_1(M, y))$. Thus Lemma 2.2 guarantees that $\widetilde{M}_n$ is sufficiently large; hence M is almost sufficiently large. □

Corollary 2.3 (Jaco). *Let M be an irreducible, closed, orientable 3-manifold. Suppose that $\pi_1(M)$ has a subgroup Γ which is isomorphic to $\pi_1(S_g)$ for some $g > 0$, and is contained in infinitely many distinct subgroups of finite index in $\pi_1(M)$. Then M is almost sufficiently large.* □

Combining Theorem 2.1 with Proposition 1.1, we obtain

Proposition 2.4. *Let M be an irreducible, closed, orientable 3-manifold. Let k be a positive integer, and let Γ be a freely indecomposable subgroup generated by k elements of $\pi_1(M)$. Suppose that for some prime p we have*

$$\operatorname{rank} H_1(M, \mathbb{Z}_p) \geq k + 2.$$

Then M is almost sufficiently large. □

Corollary 2.5. *Let M be an irreducible, closed, orientable 3-manifold. Let g be a nonnegative integer. Suppose that $\pi_1(M)$ has a subgroup isomorphic to $\pi_1(S_g)$, and that for some prime p we have*

$$\operatorname{rank} H_1(M, \mathbb{Z}_p) \geq 2g + 2.$$

Then M is almost sufficiently large. □

The *torus theorem* [Jo, JaS] asserts that for a sufficiently large, closed, irreducible, orientable 3-manifold M, if $\pi_1(M)$ has a rank-two free abelian subgroup, then either M contains an incompressible torus or M is a Seifert fibered space. The result is conjectured to remain true if we remove the hypothesis that M is sufficiently large. It is shown in [Sh, Theorem 3.1] that this is true if we assume that $\pi_1(M)$ has infinitely many distinct subgroups of finite index. We shall reprove this fact in order to clarify the connection with Jaco's Theorem 2.1 above.

Proposition 2.6. *Let M be a closed, irreducible, orientable 3-manifold. Suppose that $\pi_1(M)$ has a free abelian subgroup of rank 2. Suppose in addition that $\pi_1(M)$ has infinitely many distinct subgroups of finite index. Then either M contains an incompressible torus or M is a Seifert fibered space.*

The proof uses

Lemma 2.7. *Under the hypotheses of 2.6, M is almost sufficiently large.*

Proof. We fix a rank-2 free abelian subgroup A of $\pi_1(M)$. If A is contained in infinitely many distinct finite-index subgroups of $\pi_1(M)$, then by Corollary 2.3, M is almost sufficiently large. Now suppose that A is contained in only finitely many distinct finite-index subgroups of $\pi_1(M)$. Then the intersection of all finite-index subgroups of $\pi_1(M)$ which contain A is itself a finite-index subgroup Δ of $\pi_1(M)$.

The hypothesis of the theorem implies that Δ has infinitely many distinct finite-index subgroups; since Δ must be finitely generated, it has finite quotients of arbitrarily large order. On the other hand the construction of Δ gives that no proper finite-index subgroup of Δ contains A. Hence any homomorphism of Δ onto a finite group G must map A onto G. Thus all finite quotients of Δ are abelian. In particular, Δ has finite abelian quotients of arbitrarily large order and hence it has infinite commutator quotient. Thus if M_Δ denotes the covering space of M corresponding to Δ, we have $H_1(\Delta; \mathbb{Q}) \neq 0$. By [He, Lemma 6.6] this means that M_Δ is sufficiently large, and hence M is almost sufficiently large. □

Proof of 2.6. By [Sc2], either M is sufficiently large or $\pi_1(M)$ has an infinite cyclic normal subgroup. Assume that the second alternative holds. By Lemma 2.7, M has a sufficiently large finite-sheeted cover $\widetilde{M}$. Now $\pi_1(\widetilde{M})$ itself has an infinite cyclic normal subgroup, and by [MeeSY, Theorem 3], $\widetilde{M}$ is irreducible. So by [JaS, Corollary II.6.4], $\widetilde{M}$ is a Seifert fibered space. It now follows from [Sc3] that M is also a Seifert fibered space. □

Combining Proposition 2.6 with Corollary 1.7, we obtain the following result, which was first proved in [Tur] and can also be deduced from the results of [Mes1].

Proposition 2.8 (Turaev). *Let M be a closed, irreducible, orientable 3-manifold. Suppose that $\pi_1(M)$ has a free abelian subgroup of rank 2. Suppose in addition that* $\operatorname{rank} H_1(M; \mathbb{Z}_p) \geq 3$ *for some prime p. Then either M contains an incompressible torus or M is a Seifert fibered space.* □

In [EM], Evans and Moser address the question of when a 3-manifold group contains a free subgroup of rank 2. If the closed, orientable 3-manifold M

is irreducible and sufficiently large they show that $\pi_1(M)$ usually contains a rank-2 free subgroup; in fact they classify all the exceptions. (For some closely related results treated from a different point of view, see [Wa].) If one drops the hypothesis that M is sufficiently large, it is not known when such a free subgroup exists. The following result gives the answer when M has enough $\bmod p$ homology.

Theorem 2.9. *Let M be a closed, orientable, irreducible 3-manifold such that $\operatorname{rank} H_1(M, \mathbb{Z}_p) \geq 4$ for some prime p. Then $\pi_1(M)$ has a free subgroup of rank 2.*

Proof. We write $\Gamma = \pi_1(M)$, and we set $r = \operatorname{rank} H_1(M; \mathbb{Z}_p) = \operatorname{rank} H_1(\Gamma; \mathbb{Z}_p)$.

The hypothesis $\operatorname{rank} H_1(M, \mathbb{Z}_p) \geq 4$ implies first of all that Γ is nonabelian; we shall prove this well-known fact by the following argument due to Turaev [Tur]. If Γ_1 denotes the subgroup of Γ generated by all commutators and pth powers, it follows from Lemma 1.3 that $\operatorname{rank} H_1(\Gamma_1; \mathbb{Z}_p) \geq \frac{1}{2}r(r-1)$. Since $r \geq 4$ it follows that $\operatorname{rank} H_1(\Gamma_1; \mathbb{Z}_p) > r$. But if Γ were abelian we would have $\operatorname{rank} H_1(\Delta; \mathbb{Z}_p) \leq r$ for every subgroup Δ of Γ.

To show that $\pi_1(M)$ actually has a free subgroup of rank 2, we distinguish two cases. First suppose that $\pi_1(M)$ has no free abelian subgroup of rank 2. Then Corollary 1.9 guarantees that any two noncommuting elements of Γ generate a free group of rank 2; and two such elements do exist since Γ is nonabelian.

Now suppose that Γ has a free abelian subgroup of rank 2. By Proposition 2.8, M is either a sufficiently large manifold or a Seifert fibered space.

If M is sufficiently large then according to [EM, Corollary 4.10], Γ has a rank-2 free subgroup unless it is solvable. Furthermore, in [EM] a complete list is given of those solvable groups that occur as fundamental groups of sufficiently large 3-manifolds; an examination of the list reveals that all these groups are generated by at most three elements. Hence if M is sufficiently large and $\operatorname{rank} H_1(M; \mathbb{Z}_p) \geq 4$ for some prime p then Γ has a free subgroup of rank 2.

If M is Seifert fibered, then according to [He, p. 118], Γ has a normal cyclic subgroup $\langle f \rangle$ such that $\Phi = \Gamma/\langle t \rangle$ is a Fuchsian group. If Φ is hyperbolic then Φ has a rank-2 free subgroup, and hence so does Γ. If Φ is not hyperbolic then it is generated by at most two elements; in this case, Γ is generated by at most three elements, so that $\operatorname{rank} H_1(M; \mathbb{Z}_p) \leq 3$. □

3. Some remarks on growth series of groups

Let G be a finitely generated group. By a *weighted (finite) generating set* for G we mean a pair (S, W), where S is a finite generating set for G and W is a function that assigns a positive integer to each element of S. We call such a function W a *weight system.*

If (S, W) is a weighted finite generating set, we define the (S, W)*-length* of an element γ of G to be the smallest integer n such that γ can be expressed in the form $s_1^{\varepsilon_1} \cdots s_m^{\varepsilon_m}$, where $s_i \in S$ and $\varepsilon_i = \pm 1$ for $i = 1, \ldots, m$, and $W(s_1) + \cdots + W(s_m) = n$. The growth power series of G with respect to (S, W) is defined to be the formal power series $\sum_{n=0}^{\infty} a_n t^n$, where a_n is the number of elements of G having (S, W)-length equal to n. We shall denote this series by $p_{S,W}(t)$. Note that $p_{S,W}(t) = (1-t)\sum_{n=0}^{\infty} b_n t^n$, where $b_n = \sum_{i=0}^{n} a_i$ is the number of elements of G having (S, W)-length at most n.

Proposition 3.1. *Suppose G_1 and G_2 are groups with weighted generating sets (S_1, W_1) and (S_2, W_2). Let G denote the free product $G_1 * G_2$ and S the generating set $S = S_1 \cup S_2$. Let W be the weight system for S defined by $W|S_i = W_i$. Then*

$$\frac{1}{p_{S,W}(t)} = \frac{1}{p_{S_1,W_1}(t)} + \frac{1}{p_{S_2,W_2}(t)} - 1.$$

Proof. Let $p_{S_i,W_i}(t) = \sum_{n=0}^{\infty} a_{i,n}t^n$. Any element of G can be written uniquely as an alternating product of elements in G_1 and elements in G_2. For $n > 0$ and for $i = 1, 2$, define $c_{i,n}$ to be the number of elements of G of (S, W)-length n that end (on the right) with an element of G_i. Then

$$c_{2,n} = c_{1,n-1}a_{2,1} + c_{1,n-2}a_{2,2} + \cdots + c_{1,1}a_{2,n-1} + a_{2,n},$$
$$c_{1,n} = c_{2,n-1}a_{1,1} + c_{2,n-2}a_{1,2} + \cdots + c_{2,1}a_{1,n-1} + a_{1,n}$$

for $n \geq 1$. Define $c_{1,0} = c_{2,0} = 1$ and let $f_i(t) = \sum_{n=0}^{\infty} c_{i,n}t^n$. Then the equations above imply that

$$f_1(t) = f_2(t)(p_{S_1,W_1}(t) - 1) + 1,$$
$$f_2(t) = f_1(t)(p_{S_2,W_2}(t) - 1) + 1.$$

This gives us two linear equations in two unknowns, f_1 and f_2, with coefficients in the power series ring in one variable. Using Cramer's rule to solve, we get

$$f_i = \frac{p_{S_i,W_i}}{p_{S_1,W_1} + p_{S_2,W_2} - p_{S_1,W_1}p_{S_2,W_2}}.$$

Now $p_{S,W} = f_1 + f_2 - 1$, which gives us the desired result. □

Example 3.2. Let G be the free group on two generators s_1 and s_2. Set $S = \{s_1, s_2\}$, and let W be a weight system that assigns weight n_i to s_i. Then

$$p_{S,W}(t) = \frac{(1 + t^{n_1})(1 + t^{n_2})}{1 - t^{n_1} - t^{n_2} - 3t^{n_1+n_2}}.$$

Given a finite generating set S for the group G, we may consider the standard weight system W defined by $W(s) = 1$ for all $s \in S$. In this case we write p_S for $p_{S,W}$. We have $p_S(t) = (1-t)\sum_{n=0}^{\infty} b_n t^n$, where $b_n = \sum_{i=0}^{n} a_i$ is the number elements of G that can be expressed as words of length at most n in the generating set S.

It is clear from the definition of the b_n that $b_{m+n} \leq b_m b_n$ for any m, $n \geq 0$. Hence the numbers $c_n = \log b_m$ form a subadditive sequence, i.e. we have $c_{m+n} \leq c_m + c_n$ whenever $m, n \geq 0$. It is a standard exercise to show that $\lim_{n\to\infty} c_n/n$ exists for any subadditive sequence (c_n). Hence $\lim b_n^{1/n}$ exists. This limit is called the *exponential growth rate* of the group G with respect to (S, W), and is denoted $\omega(s, W)$. In the case where W is the standard weight system we write $\omega(S)$ for $\omega(S, W)$.

Note that $\omega(S, W)$ can be interpreted as the reciprocal of the radius of convergence of $\sum_{n=0}^{\infty} b_n t^n = (1-t)^{-1}p_{S,W}(t)$. It follows that if G is infinite then $\omega(S, W)$ is also the reciprocal of the radius of convergence of $p_{S,W}(t)$. Alternatively we may think of $\omega(S, W)$ as the supremum of all numbers B such that $b_n > B^n$ for all large enough positive integers n. We say that G has *exponential growth* if $\omega(S, W) > 1$; this condition is easily seen to be independent of the generating set S and the weight system W (cf. [Mi2, Lemma 1]).

If G is a finitely generated group, we set $\omega(G) = \inf \omega(S)$, where the infimum is taken over all finite generating sets S for the group G. We call $\omega(G)$ the *minimal growth rate* of G, and we say that G has *uniformly exponential growth* if $\omega(G) > 1$.

The following result will be useful.

Proposition 3.3. *Let G be a finitely generated group, and let H be a subgroup of index $d < \infty$. Then $\omega(G) \geq \omega(H)^{1/(2d-1)}$.*

The proof of Proposition 3.3 will depend on the following elementary fact, for which we have included a proof because we do not know a reference.

Lemma 3.4. *Let G be a group with a finite generating set S. Let H be a subgroup of index d in H. Then H has a generating set consisting of elements which can be expressed as words of length at most $2d-1$ in the generating set S.*

Proof. We may assume without loss of generality that G is free on the generating set S. We identify G with $\pi_1(X, x)$, where X is a graph with a single vertex x; let us make the identification in such a way that each element of X is represented by a loop that traverses a single edge once in some direction. Then H has the form $p_\#(\pi_1(\widetilde{X}, \tilde{x}))$ where $\widetilde{X}$ is some d-sheeted covering graph of X and $\tilde{x}$ is some base vertex of $\widetilde{X}$. The graph $\widetilde{X}$ has d vertices, and hence a maximal tree T in $\widetilde{X}$ also has d vertices.

Let E denote the set of all edges of $\widetilde{X}$ that are not contained in T. For each $e \in E$, let β_e be a path that traverses e once in some direction. For each vertex v of $\widetilde{X}$, let α_v denote the embedded edge path in T that begins at $\tilde{x}$ and ends at v. For each $e \in E$ let γ_e denote the loop $\alpha_v * \beta_e * \overline{\alpha_w}$, where v and w denote the initial and terminal points of β_e. The elements of $\pi_1(\widetilde{X}, \tilde{x})$ defined by the γ_e constitute a generating set for $\pi_1(\widetilde{X}, \tilde{x})$. But each α_v is an embedded edge path in a tree with d vertices and hence has length at most $d-1$; hence each γ_e is an edge path of length at most $2d-1$. It follows that the image of γ_e in $G = \pi_1(X, x)$ is a word of length at most $2d-1$. □

Remark 3.5. The estimate $2d-1$ in the statement of Lemma 3.4 is sharp. Indeed, let X be a graph with one vertex $\tilde{x}$ and two edges s and t. For any integer $d \geq 1$, let $\widetilde{X}$ be a graph with d vertices $v_1, \ldots, v_d$ and edges $e_0, e_1, \ldots, e_{d-1}$, $f_1, \ldots, f_{d-1}$, f_d, where e_i and f_i each join v_i to v_{i+1} for $i = 1, \ldots, d-1$, while e_0 has both endpoints at v_1 and f_d has both endpoints at v_d. Let us assign, arbitrarily, a preferred orientation to each of the edges s, t, e_0 and f_d; and for $i = 1, \ldots, d-1$, let us assign to e_i the preferred orientation for which v_i is the initial point, and to f_i the preferred orientation for which v_i is the terminal point. Let $p : \widetilde{X} \to X$ be the map that sends each v_i to x, sends e_i and f_i to s when i is odd and to t when i is even, and maps the preferred orientations of the e_i and f_i to those of s and t. Then p is a d-sheeted covering map and hence maps $\pi_1(\widetilde{X}, v_1)$ onto an index-d subgroup H of $G = \pi_1(X, x)$. Any edge path in $\widetilde{X}$ that begins and ends at v_1 and traverses f_d at least once must clearly have length at least

$2d-1$; it follows that some element of any generating set for H must have length $\geq 2d-1$ in the natural generators of G.

Proof of Proposition 3.3. Let S be any finite generating set for G. Let T denote the set of all elements of H that are expressible as words of length $\leq 2d-1$ in S. It follows from Lemma 3.4 that T is a finite generating set for H. Any element of H that can be expressed as a word of length n in T can clearly be expressed as a word of length $\leq (2d-1)n$ in S. Hence if we write $(1-t)^{-1}p_S(t) = \sum_{n=0}^{\infty} b_n t^n$ and $(1-t)^{-1}p_T(t) = \sum_{n=0}^{\infty} c_n t^n$, we have $b_{(2d-1)n} \geq c_n$ for any nonnegative integer n. It follows that $\omega(S) \geq \omega(T)^{1/(2d-1)} \geq \omega(H)^{1/(2d-1)}$. Since S was an arbitrary finite generating set for G, this implies that $\omega(G) \geq \omega(H)^{1/(2d-1)}$. □

Proposition 3.3 immediately implies

Corollary 3.6. *If a finitely generated group G has a finite-index subgroup of uniformly exponential growth, then G itself has uniformly exponential growth.* □

The next result shows how the property of being k-free influences the minimal growth rate of a group.

Proposition 3.7. *Let k be a positive integer, and let G be a finitely generated group which is k-free. Then either*

(i) *G is a free group of rank $< k$, or*

(ii) *for every generating set S of G there are elements $\gamma_1, \dots, \gamma_k \in S$ which freely generate a rank-k free subgroup of G.*

If (ii) *holds then $\omega(G) \geq 2k-1$.*

Proof. Let S be a finite generating set. Let us say that a set $T \subset S$ is *small* if the subgroup generated by T is free of some rank $< k$. The empty set is of course small. Let T_0 be a maximal small subset of S; that is, T_0 is small but there is no small T with $T_0 \not\subseteq T \subset S$. If $T_0 = S$ then (i) obviously holds. Now suppose that $T_0 \neq S$; we shall show that (ii) holds.

Choose an element $\gamma \in S - T_0$. Set $T_1 = T_0 \cup \{\gamma\}$, and let G_i denote the subgroup of G generated by T_i for $i = 1, 2$. Since T_0 is small, G_0 is free on a generating set $\{x_1, \dots, x_t\}$, where $t < k$. Thus G_1 is generated by $t+1 \leq k$ elements; since G is k-free, G_1 is free. The maximality of T_0 implies that G_1 has rank k.

Now since T_1 generates G_1 there are elements $\gamma_1, \dots, \gamma_k \in T_0$ whose images generate $H_1(G_1; \mathbb{Q}) = (G_1/G_1') \otimes \mathbb{Q}$. (Here G_1' denotes the commutator subgroup of G_1.) Thus the group H generated by the γ_i, which is free by the Nielsen-Schreier theorem, has rank k. By [K, p. 59], H is freely generated by the γ_i. Thus (ii) holds.

To prove the last assertion, note that if S is any generating set for G, and if $\gamma_1, \dots, \gamma_k \in S$ freely generate a free subgroup of G, then for any $n \geq 1$, the $(2k)\cdot(2k-1)^{n-1}$ reduced words of length n in the γ_i represent distinct elements of G. It follows at once that $\omega(S) \geq 2k-1$. Thus condition (ii) implies that $\omega(S) \geq 2k-1$ for any generating set S, and hence that $\omega(G) \geq 2k-1$. □

Corollary 3.8. *Let G be a finitely generated group which is virtually 2-free. Then either G has a cyclic subgroup of finite index, or G has uniformly exponential growth.*

Proof. Let H be a finite-index subgroup of G which is 2-free. By Proposition 3.7, H is either free of rank < 2 (and hence cyclic) or of uniformly exponential growth. The conclusion now follows from Corollary 3.6. □

4. Growth rates of 3-manifold groups

In this section we apply the material from §§1–3 to study the growth rate of a 3-manifold group. Our first result is an easy consequence of Theorem 2.9.

Proposition 4.1. *Let M be a closed, orientable, irreducible 3-manifold such that* $\operatorname{rank} H_1(M, \mathbb{Z}_p) \geq 4$ *for some prime p. Then $\pi_1(M)$ has exponential growth.*

Proof. By 2.9, $\pi_1(M)$ has a free subgroup F of rank 2. Since F has exponential growth, it follows from [Mi2, proof of Lemma 1] that $\pi_1(M)$ does as well. □

The above result should be compared with the result on growth of 3-manifold groups stated by Turaev in [Tur]: if $\operatorname{rank} H_1(M, \mathbb{Z}_p) \geq 3$ for some prime p then either $\pi_1(M)$ has a nilpotent subgroup of finite index, or for any set of generators of $\pi_1(M)$ there is a constant $C > 1$ such that the nth term of the growth power series is at least $C^{n/\log n}$. Our methods do not give Turaev's result when $\operatorname{rank} H_1(M, \mathbb{Z}_p) = 3$, but for rank $H_1(M, \mathbb{Z}_p) \geq 4$ our conclusion is much stronger.

For hyperbolic 3-manifolds we obtain a stronger result than 4.1.

Theorem 4.2. *The fundamental group of any closed hyperbolic 3-manifold has uniformly exponential growth.*

Proof. Let M be a closed hyperbolic 3-manifold. By Proposition 1.13, $G = \pi_1(M)$ is virtually 2-free; thus by Corollary 3.8, either G has uniformly exponential growth, or there is a cyclic subgroup H of finite index in G. But the latter alternative is impossible, for H would be the fundamental group of a finite-sheeted covering of M; and the quotient of H^3 by a discrete cyclic group cannot be closed. □

With suitable restrictions on the homology of M we get explicit estimates for $\omega(\pi_1(M))$.

Proposition 4.3. *Let M be an irreducible, closed, orientable 3-manifold, and let k be a nonnegative integer. Suppose that $G = \pi_1(M)$ has no subgroup isomorphic to $\pi_1(S_g)$ for any g with $0 < g < k$. Suppose furthermore that either*

(a) $\operatorname{rank} H_1(M; \mathbb{Q}) \geq k + 1$, *or*
(b) $\operatorname{rank} H_1(M; \mathbb{Z}_p) \geq k + 2$ *for some prime p.*

Then $\omega(G) \geq 2k - 1$.

Proof. By Proposition 1.8, G is k-free. Furthermore, since (a) or (b) holds, G is certainly not free of rank $< k$. Hence the conclusion follows from Proposition 3.7. □

Proposition 4.4. *Let M be a closed, orientable, hyperbolic 3-manifold. Set* $G = \pi_1(M)$.

(i) *If* $\operatorname{rank}(H_1(M, Z_p) \geq 4$ *for some prime p, or* $\operatorname{rank} H_1(M, \mathbb{Q}) \geq 3$, *then* $\omega(G) \geq 3$.

(ii) *If* $\operatorname{rank} H_1(M, \mathbb{Q}) \geq 2$, *then* $\omega(G) \geq 1/\rho = 1.8105\ldots$, *where* ρ *is the unique real root of* $3t^3 - 2t^2 + 2t - 1$.

(iii) *If* $\operatorname{rank} H_1(M, \mathbb{Q}) \geq 1$, *then* $\omega(G) \geq 3^{1/4}$.

Proof. Since M is closed and hyperbolic, $G = \pi_1(M)$ has no subgroup isomorphic to $\mathbb{Z} \oplus \mathbb{Z} = \pi_1(S_1)$. Thus assertion (i) is included in the case $k = 2$ of Proposition 4.3.

To prove (ii) and (iii), first note that a closed hyperbolic 3-manifold M cannot have an abelian fundamental group. Hence an arbitrary generating set S for G contains noncommuting elements x and y.

Suppose that $\operatorname{rank} H_1(M, \mathbb{Q}) \geq 2$. In this case it follows from Proposition 1.11 that x and $z = yxy^{-1}$ freely generate a rank-2 free subgroup F of G. The generating set $T = \{x, z\}$ of F has a weight system W defined by $W(x) = 1$, $W(z) = 3$. If an element γ of F has (T, W)-length n, it is clear that γ, regarded as an element of G, has S-length $\leq n$. In particular we have $\omega(S) \geq \omega(T, W)$. But by Example 3.2, $p_{T,W}(t)$ is a rational function with denominator $3t^3 - 2t^2 + 2t - 1$, and therefore $\omega(T, W) = 1/\rho$. Thus $\omega(S) \geq 1/\rho$, and (ii) is proved.

If $\operatorname{rank} H_1(M, \mathbb{Q}) \geq 1$, then by Proposition 1.12 the elements $u = xyx^{-1}y^{-1}$ and $v = yx^{-1}y^{-1}x$ freely generate a rank-2 free subgroup F of G. The generating set $T = \{u, v\}$ for F has a weight system W defined by $W(u) = W(v) = 4$, and arguing as in the proof of (ii) we see that $\omega(S) \geq \omega(T, W)$. But an application of 3.1, or an easy direct argument, shows that $\omega(T, W) = 3^{1/4}$. □

Remark. If in the argument used to prove 4.4(ii) one uses the elements xy and yx in place of x and yxy^{-1}, then in place of the estimate $\omega(G) \geq 1/\rho$ one obtains the somewhat weaker estimate $\omega(G) \geq 3^{1/2}$. The same comment applies to Proposition 5.4(ii) below.

5. Margulis numbers, injectivity radii and volumes

In this section we shall give applications of the results and methods of the previous sections to the study of the geometry of closed hyperbolic 3-manifolds. The arguments can be adapted to more general classes of Riemannian 3-manifold, but we have concentrated on the hyperbolic case as it seems to be the most interesting.

Let M be a closed hyperbolic n-manifold. We regard M as the quotient of H^n by a discrete, torsion-free group Γ of isometries. We define the *Margulis number* of M to be the supremum $\mu = \mu_M$ of all nonnegative numbers ε which satisfy the following condition:

($*$) If x and y are elements of Γ, and if there is a point $p \in H^n$ such that $d(x \cdot p, p) < \varepsilon$ and $d(y \cdot p, p) < \varepsilon$, then $xy = yx$.

(It is known that for any positive integer n there is a number $\varepsilon > 0$ which satisfies ($*$) for all closed hyperbolic n-manifolds. Thus $\mu(n) = \inf \mu_M$, where M ranges over all n-manifolds of curvature ≤ -1, is a strictly positive constant. For proofs and explicit estimates when $n = 3$, see [Mey1]. We will not use these results.)

The Margulis number of a closed hyperbolic manifold plays a key role in studying its geometry. We shall illustrate this by giving a proof of the following well-known fact.

Proposition 5.1. *If M is a closed hyperbolic n-manifold then some point of M has a neighborhood which is isometric to a hyperbolic ball of radius $\mu_M/2$.*

Proof. Set $\Gamma = \pi_1(M)$; we identify Γ with a discrete torsion-free group of isometries of H^n. Every element $x \in \Gamma - \{1\}$ has a well-defined axis $A_x \subset H^n$. The centralizer $C(x)$ leaves A_x invariant; thus $C(x)$ is isomorphic to a discrete torsion-free group of isometries of a line, and is therefore cyclic. It follows that $C(x)$ is the unique maximal cyclic subgroup containing x. Thus each nontrivial element of Γ lies in a unique maximal cyclic subgroup $C(x)$, and two nontrivial elements x and y commute if and only if $C(x) = C(y)$.

For each $x \in \Gamma - 1$ and each $\varepsilon > 0$, define $T(x, \varepsilon)$ to be the open subset of H^n consisting of all points p such that $d(p, x \cdot p) < \varepsilon$. Set $T(C) = \bigcup_{1 \neq x \in C} T(x, \varepsilon)$. It is an exercise in hyperbolic geometry to show that $T(C) \neq H^n$.

(In general we do not have $T(x^m, \varepsilon) \subset T(x, \varepsilon)$ for $m > 0$. For example, if x is the composition of a $180°$ rotation about its axis A with a sufficiently small translation along A then $T(x^2, \varepsilon)$ is not contained in $T(x, \varepsilon)$.)

The definition of $\mu = \mu_M$ implies that whenever x and y are noncommuting elements of Γ we have $T(x, \mu) \cap T(y, \mu) = \phi$. If C_1 and C_2 are distinct maximal cyclic subgroups, and if x_i is a nontrivial element of C_i, then by the above discussion x_1 and x_2 do not commute, and so $T(x_1, \mu) \cap T(x_2, \mu) = \phi$. Hence we have $T(C_1, \mu) \cap T(C_2, \mu) = \phi$.

Thus the sets of the the form $T(C)$, where C ranges over the maximal cyclic subgroups of Γ, are pairwise disjoint. Since these sets are proper open subsets of the connected space H^n, they cannot cover H^n. Let p be a point of H^n which does not lie in any of the sets $T(C)$. Then for any $x \in \Gamma - \{1\}$ we have $p \notin T(x, \mu)$.

Thus for every $x \in \Gamma - \{1\}$ we have $d(p, x \cdot p) \geq \mu$. Hence if B denotes the ball of radius $\mu/2$ about p, we have $x \cdot B \cap B = \phi$ for every $x \in \Gamma - \{1\}$. It follows that B is embedded in M via the covering projection $H^n \to M$. □

Margulis numbers are related to growth rates via the following result. The proof of this result is essentially due to Gromov [Gr].

Proposition 5.2. *Let c be a positive real number, and let M be a closed hyperbolic manifold of dimension $n \geq 2$. Suppose that for any two noncommuting elements x, y of $\Gamma = \pi_1(M)$, the group generated by $S = \{x, y\}$ has exponential growth rate $\omega(S) \geq c$ with respect to the generating set S. Then*

$$\mu_M \geq \frac{\log c}{n-1}.$$

Proof. We must show that if x, $y \in \Gamma$ and $p \in H^n$ satisfy $d(xp, p) < (\log c)/(n-1)$ and $d(yp, p) < (\log c)/(n-1)$ then $xy = yx$. Assume to the contrary that x and y do not commute. Then by the hypothesis, if $f(m)$ denotes the number of elements of Γ that can be expressed as words of length $\leq m$ in x and y, the function $f(m)$ has exponential growth rate at least c; that is, for any constant $c' < c$, there is a constant K such that

$$f(m) \geq K \cdot (c')^m \tag{5.2.1}$$

for all sufficiently large m.

Choose $\varepsilon < (\log c)/(n-1)$ so that $d(xp, p) < \varepsilon$ and $d(yp, p) < \varepsilon$. Choose a neighborhood U of p such that U has a finite, positive volume and $\gamma U \cap U = \varnothing$ for all $\gamma \in \Gamma - \{1\}$. Set $v = \operatorname{vol} U$. If $\gamma \in \Gamma$ is expressible as a word of length $\leq m$ in x and y then $d(\gamma p, p) < m\varepsilon$. Thus for any m there are $f(m)$ elements $\gamma \in \Gamma$ such that $\gamma \cdot p$ lies in the ball of radius $m\varepsilon$ about p. Hence if $\Delta = \operatorname{diam} U$, there are $f(m)$ disjoint sets of the form $\gamma \cdot U$, $\gamma \in \Gamma$, in a ball of radius $m\varepsilon + \Delta$ about p.

There is a constant K' depending on the dimension n such that the volume of any ball of sufficiently large radius r in hyperbolic n-space is bounded above by $K'e^{(n-1)r}$. Hence for large m we have

$$f(m) \cdot v \leq K' \cdot \exp((n-1)(m\varepsilon + \Delta)). \tag{5.2.2}$$

From (5.2.1) and (5.2.2) we have

$$vK \cdot (c')^m \leq K' \cdot \exp((n-1)(m\varepsilon + \Delta)) \tag{5.2.3}$$

for large m. Now taking logarithms of both sides of (5.2.3), dividing by m and taking the limit as $m \to \infty$ we get

$$(n-1)\varepsilon \geq \log c'$$

for any $c' < c$. Hence $(n-1)\varepsilon \geq \log c$. But this is a contradiction, since $\varepsilon < (\log c)/(n-1)$. Hence x and y must commute. □

Corollary 5.3. *Let M be a closed hyperbolic manifold of dimension $n \geq 2$. Suppose that $\pi_1(M)$ is 2-free. Then*

$$\mu_M \geq \frac{\log 3}{n-1}. \quad \square$$

In particular, a 2-free closed hyperbolic 3-manifold group has Margulis number at least $\frac{\log 3}{2}$.

Proposition 5.4. *Let M be a closed, orientable hyperbolic 3-manifold.*

(i) *If $\operatorname{rank} H_1(M, Z_p) \geq 4$ or $\operatorname{rank} H_1(M, \mathbb{Q}) \geq 3$, then $\mu_M \geq (\log 3)/2$.*
(ii) *If $\operatorname{rank} H_1(M, \mathbb{Q}) \geq 2$, then $\mu_M \geq \frac{1}{2}\log(1/\rho) = 0.2968\ldots$ where ρ is the unique real root of $3t^3 - 2t^2 + 2t - 1$.*
(iii) *If $\operatorname{rank} H_1(M, \mathbb{Q}) \geq 1$, then $\mu_M \geq (\log 3)/8$.*

Remark. In the case where $H_1(M, \mathbb{Q}) \geq 3$, Culler and Shalen [CS] have recently shown that $\mu_M \geq \log 3$. Unlike the rather simple and elementary arguments used here, their arguments are involved and use some very heavy machinery.

Proof of 5.4. Assertion (i) follows from Corollaries 1.9 and 5.3. To prove assertions (ii) and (iii), set $\Gamma = \pi_1(M)$. Let x and y be noncommuting elements of Γ. Set $S = \{x, y\}$; let G denote the subgroup of Γ generated by x and y, and as usual let $\omega(S)$ denote the growth rate of G with respect to the generating set S. If the hypothesis of (ii) holds then the proof of Proposition 4.4(ii) shows that $\omega(S) \geq 1/\rho$; thus the conclusion of (ii) follows from 5.2. Similarly, assertion (iii) follows from 5.2 and the proof of 4.4(iii). □

It follows from Propositions 5.1 and 5.4 that a hyperbolic 3-manifold which satisfies the hypothesis of 5.4(i), (ii) or (iii) contains an isometric copy of a hyperbolic ball of radius $(\log 3)/4$, $\frac{1}{4}\log(1/\rho) = 0.1484\ldots$ or $(\log 3)/16$, respectively. Meyerhoff [Mey2] shows how, using work by Böröczky [Bö], one can obtain a lower bound for the volume of a hyperbolic manifold which is known to contain a hyperbolic ball of a given radius. The relevant calculations have been carried out by Marc Culler and yield the following corollary.

Corollary 5.5. *Let M be a closed, orientable hyperbolic 3-manifold.*

(i) *If* $\operatorname{rank} H_1(M, Z_p) \geq 4$ *or* $\operatorname{rank} H_1(M, \mathbb{Q}) \geq 3$, *then M has volume at least* 0.1124.

(ii) *If* $\operatorname{rank} H_1(M, \mathbb{Q}) \geq 2$, *then M has volume at least* 0.0176.

(iii) *If* $\operatorname{rank} H_1(M, \mathbb{Q}) \geq 1$, *then M has volume at least* 0.0017.

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Department of Mathematics, Statistics, and Computer Science (m/c 249), University of Illinois, Box 4348, Chicago, Illinois 60680

E-mail address, Peter Shalen: u10123@uicvm.bitnet

E-mail address, Philip Wagreich: u11788@uicvm.bitnet