TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 332, Number 1, July 1992

EXCLUDING SUBDIVISIONS OF INFINITE CLIQUES

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ABSTRACT. For every infinite cardinal κ we characterize graphs not containing a subdivision of K_κ.

1. INTRODUCTION

In this paper graphs may be infinite, and may have loops and multiple edges. A graph G is a *subdivision* of a graph H if G can be obtained from H by replacing the edges of H by internally disjoint paths joining the same ends. Let G be a graph. A *tree-decomposition* of G is a pair (T, W), where T is a tree (a connected graph with no circuits) and $W = (W_t : t \in V(T))$ is such that

(W1) $\bigcup_{t\in V(T)} W_t = V(G)$, and every edge of G has both ends in some W_t, and

(W2) if t' lies on the path of T between t and t'', then $W_t \cap W_{t''} \subseteq W_{t'}$.

If κ is a cardinal, we say that (T, W) has *width* $< \kappa$ if $|W_t| < \kappa$ for every $t \in V(T)$, and

$$\left|\bigcup_{i=1}^{\infty} \bigcap_{j\geq i} W_{t_j}\right| < \kappa$$

for every infinite path $t_1, t_2, \ldots$ in T. We shall prove the following result for excluding a subdivision of $K_{\aleph_0}$, the countable clique.

(1.1) *A graph G contains no subgraph isomorphic to a subdivision of $K_{\aleph_0}$ if and only if G admits a tree-decomposition of width $< \aleph_0$.*

One cannot hope for an analogous theorem for uncountable cardinals because of the following [4].

(1.2) *For every cardinal κ there exists a graph G with no subgraph isomorphic to a subdivision of $K_{\aleph_1}$ such that for every tree-decomposition (T, W) of G there exists $t \in V(T)$ with $|W_t| \geq \kappa$.*

However, the next best weakening works, namely "well-founded tree-decomposition," which we now introduce. A *well-founded tree* is a nonempty partially ordered set $T = (V, \leq)$ such that for every two elements $t_1, t_2 \in V$ their infimum exists and such that the set $\{t' \in V : t' < t\}$ is well-ordered for every

Received by the editors March 30, 1990.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 05C75.
The first and third authors' research was performed under a consulting agreement with Bellcore. The first author was also supported by NSF under Grant No. DMS-8903132.
The third author was also supported by DIMACS Center, Rutgers University, New Brunswick, New Jersey.

$t \in V$. It follows that T has a minimum element, called the *root* and denoted by $\mathrm{root}(T)$. We write $V(T) = V$ and call the elements of $V(T)$ the *vertices* of T. If λ is an ordinal we say that T is $< \lambda$ *high* if every chain in T has order type $< \lambda$. For $t_1, t_2 \in V(T)$ we define $T[t_1, t_2]$ to be the set of all $t \in V(T)$ such that $\inf(t_1, t_2) \le t$, and either $t \le t_1$ or $t \le t_2$. For $t, t' \in V(T)$ we say that t' is a *successor* of t, and that t is a *predecessor* of t', if $t < t'$ and there is no $t'' \in V(T) - \{t, t'\}$ with $t \le t'' \le t'$.

A *well-founded tree-decomposition* of a graph G is a pair (T, W), where T is a well-founded tree and $W = (W_t : t \in V(T))$ satisfies

(W1) $\bigcup_{t \in V(T)} W_t = V(T)$, and every edge of G has both ends in some W_t,
(W2) if $t' \in T[t, t'']$ then $W_t \cap W_{t''} \subseteq W_{t'}$, and
(W3) if $C \subseteq V(T)$ is a chain and $c = \sup C \in V(T)$, then $\bigcap_{t \in C} W_t \subseteq W_c$.

We say that (T, W) has *width* $< \kappa$ if $|\bigcup_{t \in C} \bigcap \{W_{t'} : t' \in C, t' \ge t\}| < \kappa$ for every chain $C \subseteq V(T)$. It follows that if (T, W) has width $< \kappa$ then $|W_t| < \kappa$ for every $t \in V(T)$. We say that (T, W) is $< \kappa$ *high* if T is $< \kappa$ high.

We say that a well-founded tree T is *graph-theoretic* if every chain in T has order type $\le \omega$. Let R be a tree and let $r \in V(R)$. We define $t_1 \le t_2$ for $t_1, t_2 \in V(R)$ to mean that t_1 lies on the path between r and t_2. It is easily seen that $T = (V(R), \le)$ is a graph-theoretic well-founded tree and that every graph-theoretic well-founded tree arises this way. Moreover, $T[t_1, t_2]$ is the set of all vertices of R which lie on the path between t_1 and t_2 in R. We say that a well-founded tree-decomposition (T, W) is *graph-theoretic* if T is graph-theoretic. Thus we have proved

(1.3) *Let κ be a cardinal. A graph G admits a graph-theoretic well-founded tree-decomposition of width $< \kappa$ if and only if G admits a tree-decomposition of width $< \kappa$.*

The true version of (1.1) for larger cardinals is the following.

(1.4) *Let κ be an infinite cardinal and let G be a graph. Then the following two conditions are equivalent:*

(i) *G contains no subgraph isomorphic to a subdivision of K_κ,*
(ii) *G admits a well-founded tree-decomposition of width $< \kappa$.*

If $\kappa = \aleph_0$ then the tree-decomposition in (ii) *can be chosen graph-theoretic. If κ is regular and uncountable, then* (i) *and* (ii) *are equivalent to*

(iii) *G admits a well-founded tree-decomposition of width $< \kappa$ which is $< \kappa$ high.*

By (1.3), (1.4) implies (1.1). Notice that if κ is regular and (T, W) is a tree-decomposition of a graph G which is $< \kappa$ high, then (T, W) has width $< \kappa$ if and only if $|W_t| < \kappa$ for every $t \in V(T)$. The equivalence of (i) and (iii) for regular uncountable cardinals is similar to (and interderivable with) a result independently obtained by Diestel [2]. Diestel's theorem generalizes a theorem of Halin [3].

There are other conditions that are equivalent to the conditions of (1.4) when κ is regular uncountable. We now introduce two of them; others can be found in [7].

A *linear decomposition* of a graph G is a pair (L, X), where L is a (Dedekind) complete linearly ordered set and $X = (X_l : l \in L)$ is such that

(L1) $\bigcup_{l \in L} X_l = V(G)$, and every edge of G has both ends in some X_l,

(L2) if $l \leq l' \leq l''$, then $X_l \cap X_{l''} \subseteq X_{l'}$, and
(L3) $\bigcap_{i \in I} X_i \subseteq X_{\inf(I)} \cap X_{\sup(I)}$ for every nonempty interval $I \subseteq L$.

We say that (L, X) has *width* $< \kappa$ if $|X_l| < \kappa$ for every $l \in L$. Let us remark that the requirement that L be complete is not restrictive, because any "incomplete" decomposition can be completed in the obvious way. Linear decompositions are motivated by path-decompositions from [5], and their relation to excluding infinite trees is studied in [8].

Finally, we introduce the following generalization of stoppages from [1]. Let G be a graph. A *cut* in G is an ordered pair (A, B) of subsets of $V(G)$ such that $A \cup B = V(G)$ and there is no edge between $A - B$ and $B - A$. The *order* of (A, B) is $|A \cap B|$. Now let κ be a cardinal. A *stoppage of order* κ in a graph G is a set $\mathscr{S}$ of cuts, all of order $< \kappa$, such that

(i) if (A, B) is a cut in G of order $< \kappa$, then $\mathscr{S}$ contains one of (A, B), (B, A),
(ii) if (A_1, B_1), $(A_2, B_2) \in \mathscr{S}$, then $(G|A_1) \cup (G|A_2) \neq G$ (where $G|A$ is the restriction of G to A), and
(iii) if $\mathscr{M} \subseteq \mathscr{S}$ is a chain of cuts (that is, for (A_1, B_1), $(A_2, B_2) \in \mathscr{M}$ either $A_1 \subseteq A_2$ and $B_1 \supseteq B_2$, or $A_1 \supseteq A_2$ and $B_1 \subseteq B_2$) and for

$$A = \bigcup \{A' : (A', B') \in \mathscr{M}\}, \qquad B = \bigcap \{B' : (A', B') \in \mathscr{M}\}$$

the order of (A, B) is $< \kappa$, then $(A, B) \in \mathscr{S}$.

We will refer to (i), (ii), (iii) as the stoppage axioms.

The following result extends (1.4) for κ regular uncountable.

(1.5) *Let G be a graph and let κ be a regular uncountable cardinal. Then the following conditions are equivalent*:

(i) *G has no subgraph isomorphic to a subdivision of K_κ,*
(ii) *G admits a $< \kappa$ high well-founded tree-decomposition of width $< \kappa$,*
(iii) *G admits a linear decomposition of width $< \kappa$,*
(iv) *G has no stoppage of order $\geq \kappa$.*

Theorem (1.5) is false for $\kappa = \aleph_0$, because it is shown in [8] that if $\kappa = \aleph_0$ then (iii) is equivalent to not containing a subgraph isomorphic to a subdivision of the $\aleph_0$-branching tree. The assumption that κ is regular cannot be dropped either, because of the following (we do not know whether there is a similar counterexample when $\mathrm{cf}(\kappa) > \omega$).

(1.6) *There exists a graph which contains no subgraph isomorphic to a subdivision of $K_{\aleph_\omega}$ and which has no linear decomposition of width $< \aleph_\omega$.*

The paper is organized as follows: In §2 we prove (1.4) and in §3 we prove (1.5) and (1.6). We end this section with the following lemma, a relative of (3.4) from [6].

(1.7) *Let G be a graph, let F be the vertex-set of a connected subgraph of G, let (T, W) be a well-founded tree-decomposition of G and let $t_1, t_2 \in V(T)$ be such that $W_{t_1} \cap F \neq \varnothing \neq W_{t_2} \cap F$. If $t, t' \in T[t_1, t_2]$ are such that $t \leq t'$ and there is no $t'' \in V(T)$ with $t < t'' < t'$ then $W_t \cap W_{t'} \cap F \neq \varnothing$. In particular, if $t \in T[t_1, t_2]$ then $W_t \cap F \neq \varnothing$.*

Proof. We proceed by induction on $|V(P)|$, where P is the shortest path connecting W_{t_1} and W_{t_2} with $V(P) \subseteq F$. If $|V(P)| = 1$ the result follows from (W2). If $|V(P)| \geq 2$ let $u \in V(P) \cap W_{t_1}$ be an endvertex of P, and let $v \in V(P)$ be the neighbor of u in P. By (W1) there is $r \in V(T)$ such that $u, v \in W_r$. It is easy to see that either $t, t' \in T[t_1, r]$ or $t, t' \in T[r, t_2]$. In the former case $u \in W_t \cap W_{t'} \cap F$ by (W2), and in the latter one the result follows from the induction hypothesis applied to r, t_2 and F. □

2. Greedy method

In this section we prove (1.4). First we shall prove that (ii) implies (i).

(2.1) *Let κ be a cardinal, let G be a graph and let (T, W) be a well-founded tree-decomposition of G of width $< \kappa$. Then G contains contains no subgraph isomorphic to a subdivision of K_κ.*

Proof. Let (T, W) be as stated in (2.1). Suppose for a contradiction that G contains a subgraph H isomorphic to a subdivision of K_κ, and let $V \subseteq V(H)$ be the set of vertices corresponding to vertices of K_κ. For each $v \in V$, let $t(v)$ be the minimal $t \in V(T)$ with $v \in W_t$ (it is unique by (W2)).

(1) *If $v_1, v_2 \in V$ then either $t(v_1) \leq t(v_2)$ or $t(v_2) \leq t(v_1)$.*

For let $t = \inf(t(v_1), t(v_2))$. There are κ paths of H and hence of G, between v_1 and v_2, mutually disjoint except for v_1 and v_2, and all passing through W_t by (1.7). Since $|W_t| < \kappa$ it follows that one of $v_1, v_2 \in W_t$, and so either $t = t(v_1)$ or $t = t(v_2)$. The claim follows.

(2) *If $v_1, v_2 \in V$ and $t(v_1) \leq t(v_2)$ then $v_1 \in W_t$ for all t with $t(v_1) \leq t \leq t(v_2)$.*

For suppose not. Choose t_2 with $t(v_1) \leq t_2 \leq t(v_2)$ minimal such that $v_1 \notin W_{t_2}$. Since $v_1 \in W_t$ for all t with $t(v_1) \leq t < t_2$, it follows that $t_2 \neq \sup\{t : t(v_1) \leq t < t_2\}$ by (W3), and so there exists a predecessor t_1 of t_2. By (1.7), each of the κ internally disjoint paths of H between v_1 and v_2 passes through $W_{t_1} \cap W_{t_2}$, and so one of v_1, v_2 belongs to $W_{t_1} \cap W_{t_2}$. But $v_1 \notin W_{t_2}$, and $v_2 \notin W_{t_1}$ since $t_1 < t_2 \leq t(v_2)$. This is a contradiction, and (2) follows.

Let $C = \{t \in V(T) : t \leq t(v)$ for some $v \in V\}$.

(3) *C is a chain, and $V \subseteq \bigcup_{t \in C} \bigcap \{W_{t'} : t' \in C, t' \geq t\}$.*

For C is a chain by (1). Let $v \in V$, and let $t' \in C$, $t' \geq t(v)$. Choose $v' \in V$ with $t' \leq t(v')$. Then $t(v) \leq t(v')$ and by (2), $v \in W_{t'}$. Hence $v \in \bigcap\{W_{t'} : t' \in C, t' \geq t(v)\}$ and the claim follows.

But $|V| = \kappa$, and so (3) contradicts the fact that (T, W) has width $< \kappa$. □

Next we prove the rest of (1.4). We first prove that every graph G admits a "standard decomposition" and then prove that if G is as in (1.4)(i) then this decomposition satisfies the conclusion of (1.4)(ii). If X is a set we put $\|X\| = 0$ if X is finite, and $\|X\| = |X|$ otherwise. If G is a graph and $K \subseteq V(G)$, then $N(K)$ is the set of all vertices in $V(G) - K$ which are adjacent to a vertex in K, $G \backslash K$ is the graph obtained from G by deleting the vertices of K and all edges incident with these vertices, and a *K-flap* is the set of vertices of a component of $G \backslash K$.

Let (T, W) be a well-founded tree-decomposition of a graph G. We say that (T, W) is a *standard decomposition* of G if

(S1) $W_{\mathrm{root}(T)} = \varnothing$,
(S2) if t' is a successor of t in T, then $\|W_{t'}\| \leq \|W_t\|$ and $W_{t'} - W_t \neq \varnothing$,
(S3) for every chain C in T which does not have a maximal element, any two nonadjacent vertices in $W_C := \bigcup_{t \in C} \bigcap \{W_{t'} : t' \in C, t' \geq t\}$ are joined in G by $\|W_C\|$ internally disjoint paths, and if C has a supremum, say t, then $W_t = W_C$ and t has at least one successor,
(S4) for every $t \in V(T)$, every W_t-flap of G which intersects $\bigcup_{t \leq t'} W_{t'}$ is intersected by $W_{t'}$ for some successor t' of t.

(2.2) *Every graph admits a standard decomposition.*

Proof. Let G be a graph. Let $u, v \in X \subseteq V(G)$ and let μ be a cardinal. If there exists a set $Y \subseteq X - \{u, v\}$ with $\|Y\| \leq \mu$ which meets every path P joining u, v with $V(P) \subseteq X$, then we let $\Theta(u, v, X, \mu)$ be one such set Y, and if not we define $\Theta(u, v, X, \mu) = \varnothing$.

For some ordinal λ, we shall construct a transfinite sequence W_α $(\alpha \leq \lambda)$ of subsets of $V(G)$ satisfying

(1) $\varnothing = W_0 \subseteq W_1 \subseteq \cdots \subseteq W_\alpha \subseteq \cdots \subseteq W_\lambda = V(G)$.

Also, for each $\alpha < \lambda$ we shall construct a partition $\mathscr{K}_\alpha$ of $V(G) - W_\alpha$ into nonempty sets satisfying

(2) for all $\alpha < \lambda$ if $K, K' \in \mathscr{K}_\alpha$ are distinct then no vertex of K has a neighbor in K', and

(3) for $\alpha \leq \beta < \lambda$, every member of $\mathscr{K}_\beta$ is a subset of a member of $\mathscr{K}_\alpha$.

Third, for each $\alpha < \lambda$ and each $K \in \mathscr{K}_\alpha$ we shall construct a subset $W_{(K, \alpha)}$ of $K \cup W_\alpha$.

The inductive definition is as follows. Let $W_0 = \varnothing$, $\mathscr{K}_0 = \{V(G)\}$, and $W_{(V(G), 0)} = \varnothing$. Suppose that for some ordinal α we have defined W_β, $\mathscr{K}_\beta$ and $W_{(K, \beta)}$ $(K \in \mathscr{K}_\beta)$ for all $\beta < \alpha$. Now we wish to define W_α, $\mathscr{K}_\alpha$, $W_{(K, \alpha)}$ $(K \in \mathscr{K}_\alpha)$. Let

(4) $W_\alpha = \bigcup \{W_{(K, \beta)} : \beta < \alpha, K \in \mathscr{K}_\beta\}$.

If $W_\alpha = V(G)$ we set $\lambda = \alpha$ and stop. Otherwise, there are two cases.

If α is a successor ordinal, say $\alpha = \beta + 1$, let $\mathscr{K}_\alpha$ be the set of all W_α-flaps of G. For each $K \in \mathscr{K}_\alpha$ let $L \in \mathscr{K}_\beta$ include K, let $w \in K$ be arbitrary, let $Z = \{w\} \cup N(K)$, and let

$$W_{(K, \alpha)} = Z \cup \bigcup \{\Theta(u, v, K \cup N(K), \|W_{(L, \beta)}\|) : u, v \in Z\}.$$

Now, let α be a limit ordinal. For each $u \in V(G) - W_\alpha$ and each $\beta < \alpha$, let $K_\beta(u)$ be the member of $\mathscr{K}_\beta$ containing u; then, by (3),

$$K_0(u) \supseteq K_1(u) \supseteq \cdots \supseteq K_\beta(u) \supseteq \cdots.$$

Let $K(u) = \bigcap_{\beta < \alpha} K_\beta(u)$, and let $\mathscr{K}_\alpha = \{K(u) : u \in V(G) - W_\alpha\}$. For $u \in V(G) - W_\alpha$, let

$$W_{(K(u), \alpha)} = \bigcup_{\beta < \alpha} \bigcap_{\beta < \gamma < \alpha} W_{(K_\gamma(u), \gamma)}.$$

This completes the inductive definition. It is easy to see that the ordinal λ indeed exists, that $W_{(K, \alpha)} \subseteq K \cup W_\alpha$ for every $\alpha < \lambda$ and every $K \in \mathscr{K}_\alpha$, and that (1), (2), (3) are satisfied.

We put $V(T) = \{(K, \alpha) : K \in \mathscr{K}_\alpha, \alpha \leq \lambda\}$ and define $(K, \alpha) \leq (K', \alpha')$ if $K \supseteq K'$ and $\alpha \leq \alpha'$. We put $T = (V(T), \leq)$. It follows easily that T is a well-founded tree. Let $W = (W_{(K,\alpha)} : (K, \alpha) \in V(T))$. We shall show that (T, W) is the desired standard decomposition, but we need several observations before we do so.

(5) *$N(K) \subseteq W_{(K,\alpha)}$ for every $K \in \mathscr{K}_\alpha$ and every ordinal $\alpha < \lambda$.*

We prove this by transfinite induction on α. The statement is obviously satisfied when $\alpha = 0$, so let α be an ordinal and assume that (5) holds for all $\beta < \alpha$. Let $K \in \mathscr{K}_\alpha$ and let $v \in N(K)$. If α is a successor ordinal then $v \in W_{(K,\alpha)}$ by the definition of $W_{(K,\alpha)}$ so let α be a limit ordinal. Let u be a neighbor of v in K. Define $K_\beta(u)$ $(\beta < \alpha)$ as before; then $K = \bigcap_{\beta<\alpha} K_\beta(u)$. Since $v \notin K$, there exists $\beta < \alpha$ with $v \notin K_\beta(u)$, and hence with $v \notin K_\gamma(u)$ for $\beta < \gamma < \alpha$. Since u, v are adjacent and $u \in K_\gamma(u)$, it follows that $v \in N(K_\gamma(u)) \subseteq W_{(K_\gamma(u),\gamma)}$ for $\beta < \gamma < \alpha$, by the inductive hypothesis; and thus $v \in W_{(K,\alpha)}$ by the definition of $W_{(K,\alpha)}$. This proves (5).

(6) *For $\alpha \leq \beta < \lambda$, if $K \in \mathscr{K}_\alpha$ and $L \in \mathscr{K}_\beta$ and $L \subseteq K$ then $W_{(L,\beta)} \subseteq W_{(K,\alpha)} \cup K$.*

We prove this by transfinite induction on β. We may assume that $\alpha < \beta$. If β is a successor, $\beta = \gamma + 1$ say, let (L, β) be a successor of (M, γ). Then

$$W_{(L,\beta)} \subseteq L \cup N(L) \subseteq M \cup N(M) \subseteq M \cup W_{(M,\gamma)}$$

by (5), and $W_{(M,\gamma)} \subseteq W_{(K,\alpha)} \cup K$ by the induction hypothesis. It follows that $W_{(L,\beta)} \subseteq M \cup W_{(K,\alpha)} \cup K \subseteq W_{(K,\alpha)} \cup K$, as required. Now assume that β is a limit ordinal. Let $v \in W_{(L,\beta)}$. By definition of $W_{(L,\beta)}$ there exists (M, γ) with $(K, \alpha) \leq (M, \gamma) < (L, \beta)$ with $v \in W_{(M,\gamma)}$. By the induction hypothesis, $W_{(M,\gamma)} \subseteq W_{(K,\alpha)} \cup K$, and so $v \in W_{(K,\alpha)} \cup K$. The result follows.

Let $v \in V(G)$. A pair $(K, \alpha) \in V(T)$ is called a *nest* of v if $v \in W_{(K,\alpha)}$ and for every ordinal $\beta < \alpha$ there is no $L \in \mathscr{K}_\beta$ such that $v \in W_{(L,\beta)}$. Since $W_\lambda = V(G)$, it follows from (4) with $\alpha = \lambda$ that there is a nest of v, for every $v \in V(G)$.

(7) *Let (K, α) be a nest of $v \in V(G)$. Then $v \in K$.*

For it follows from the minimality of α and (4) that $v \notin W_\alpha$. Since $v \in W_{(K,\alpha)} \subseteq K \cup W_\alpha$, the claim follows.

(8) *If (K, α) is a nest of $v \in W_{(L,\beta)}$, then $(K, \alpha) \leq (L, \beta)$.*

For $\alpha \leq \beta$, by the minimality of α. Let $K' \in \mathscr{K}_\alpha$ with $L \subseteq K'$. Since $v \in K$ by (7), and $v \in W_{(L,\beta)} \subseteq W_{(K',\alpha)} \cup K' \subseteq W_\alpha \cup K'$ by (6), we deduce that $K \cap (W_\alpha \cap K') \neq \varnothing$, and so $K = K'$. The result follows.

From (8) we deduce that the nest of v is unique.

(9) *(T, W) is a well-founded tree-decomposition.*

We must verify (W1)–(W3). We start with (W1). Certainly $\bigcup_{t \in V(T)} W_t = V(T)$, by (4) with $\alpha = \lambda$. Let e be an edge of G with endvertices u, v, say. Let (K, α) be the nest of u and let (L, β) be the nest of v. We may assume that $\alpha \leq \beta$. It follows from (7) that $u \in K$ and $v \in L$, and since they are adjacent we deduce that $L \subseteq K$. If $\alpha = \beta$ then $K = L$ and we are done, and so we assume that $\alpha < \beta$. Then $u \notin L$ since $W_{(K,\alpha)} \subseteq W_{\alpha+1} \subseteq W_\beta$ and $L \cap W_\beta = \varnothing$. Hence $u \in N(L)$ and so $u \in W_{(L,\beta)}$ by (5), as desired. This proves (W1).

To verify (W2) let $v \in W_{(K,\alpha)} \cap W_{(L,\beta)}$. First let $(K, \alpha) < (M, \gamma) < (L, \beta)$. Then $L \subseteq M \subseteq K$ and $v \notin M$ because M is disjoint from $W_{(K,\alpha)}$. Therefore,

$v \in W_{(L,\beta)} - M \subseteq W_{(M,\gamma)}$ by (6). Hence (W2) holds for t, t', t'' if $t < t' < t''$. Now let $K \cap L = \varnothing$. Let (N, δ) be the nest of v; then $(N, \delta) \leq (K, \alpha)$ and $(N, \delta) \leq (L, \beta)$ by (8). This together with the above shows that $v \in W_{(M,\gamma)}$ for every $(M, \gamma) \in T[(K, \alpha), (L, \beta)]$.

Finally, condition (W3) follows directly from the construction. This completes the proof of (9).

(10) *If* (K, α), $(L, \alpha+1) \in V(T)$ *and* $(L, \alpha+1)$ *is a successor of* (K, α), *then* $N(L) \subseteq W_{(K,\alpha)}$.

For let $v \in N(L)$ and let (M, β) be the nest of v. Since $v \in N(L) \subseteq W_{\alpha+1}$ by (2), it follows from (4) that $\beta \leq \alpha$. If $\beta = \alpha$ then $K = M$ and $v \in W_{(M,\beta)} = W_{(K,\alpha)}$, while if $\beta < \alpha$ then $v \notin K$ because $K \cap W_{(M,\beta)} = \varnothing$, and hence $v \in N(K) \subseteq W_{(K,\alpha)}$ by (5).

(11) (T, W) *is a standard decomposition.*

We must verify (S1)–(S4). Condition (S1) is clear. To see (S2) let $(K, \alpha+1)$ be a successor of (L, α). Then $N(K) \subseteq W_{(L,\alpha)}$ by (10), and by the construction

$$W_{(K,\alpha+1)} = Z \cup \bigcup \{\Theta(u, v, K \cup N(K), \|W_{(L,\alpha)}\|) : u, v \in Z\},$$

where $Z = \{w\} \cup N(K)$ and $w \in K$. Thus $w \in W_{(K,\alpha+1)} - W_{(L,\alpha)} \neq \varnothing$, and

$$\|W_{(K,\alpha+1)}\| \leq \|Z\| + \|Z\|^2 \cdot \|W_{(L,\alpha)}\| \leq \|W_{(L,\alpha)}\|,$$

because $\|Z\| \leq \|W_{(L,\alpha)}\|$. To prove (S3) let C be a chain in T which does not have a maximal element, and let $W_C = \bigcup_{t \in C} \bigcap \{W_{t'} : T' \in C, t' \geq t\}$. Suppose for a contradiction that u, $v \in W_C$ are separated by a cutset, say A, of cardinality $< \|W_C\|$. It follows that $\|W_C\| = |W_C| \geq \aleph_0$. Then there exists $(K, \alpha) \in C$ such that $|W_{(K,\alpha)}| \geq |A|$ and u, $v \in W_t$ for every $t \in C$ with $t \geq (K, \alpha)$. Let $(L, \alpha+1) \in C$ be the successor of (K, α), and let $(M, \alpha+2) \in C$ be the successor of $(L, \alpha+1)$. It follows that u, $v \in N(L) \cap N(M)$. By construction $W_{(L,\alpha+1)} - \{u, v\}$ meets every path P joining u and v with $V(P) \subseteq L \cup N(L)$, but that contradicts the fact that u, $v \in N(M)$ since $M \cap W_{(L,\alpha+1)} = \varnothing$ and M is a $W_{\alpha+2}$-flap. Hence u, v are not separated by a cutset of size $< \|W_C\|$, and the first part of (S3) follows. Now if C has a supremum, say $t = (K, \beta)$, then $W_t = W_C$ follows directly from the construction. It also follows that β is a limit ordinal, and $\beta < \lambda$. Since $W_{(M,\beta)} \subseteq W_\beta$ for all $M \in \mathscr{K}_\beta$ from the construction, and hence $W_{\beta+1} = W_\beta$, it follows that K includes a $W_{\beta+1}$-flap L since $K \neq \varnothing$, and hence $(L, \beta+1)$ is a successor of (K, β). This proves (S3).

To prove (S4) let $t = (K, \alpha)$ and let F be a W_t-flap which intersects $\bigcup_{t \leq t'} W_{t'}$. We claim that $F \cap W_{\alpha+1} = \varnothing$. For if, say, $F \cap W_{(L,\beta)} \neq \varnothing$, where $\beta \leq \alpha$ then $t \not\leq (L, \beta)$ and so $F \cap W_t \neq \varnothing$ by (1.7), a contradiction. Hence $F \cap W_{\alpha+1} = \varnothing$. It follows that F is a $W_{\alpha+1}$-flap and thus $F \in \mathscr{K}_{\alpha+1}$. But $F \cap W_{(F,\alpha+1)} \neq \varnothing$ by the construction, as desired. □

The following is an easy lemma and the proof is left to the reader.

(2.3) *Let* G *be a graph and let* κ *be an infinite cardinal. If there exists a set* $X \subseteq V(G)$ *with* $|X| \geq \kappa$ *such that every two nonadjacent vertices of* X *are joined by* κ *internally disjoint paths, then* G *contains a subgraph* H *isomorphic to a subdivision of* K_κ *in such a way that every vertex of* H *which corresponds to a vertex of* K_κ *belongs to* X.

The following result implies (i) $\Rightarrow$ (ii) of (1.4), because (1.4) obviously holds for finite graphs.

(2.4) *Let κ be an infinite cardinal, let G be an infinite graph which contains no subgraph isomorphic to a subdivision of K_κ and let (T, W) be a standard decomposition of G. Then $|V(T)| \leq |V(G)|$ and (T, W) has width $< \kappa$.*

Proof. Let $f : V(T) - \{\text{root}(T)\} \to V(G)$ be a mapping satisfying

(i) if t' is a successor of t then $f(t') \in W_{t'} - W_t$, and
(ii) if $t \neq \text{root}(T)$ does not have a predecessor then $f(t) \in W_{t'} - W_t$, where t' is a successor of t.

Such a mapping exists by (S2) and (S3). It follows from (W2) that $|f^{-1}(v)| \leq 2$ for every $v \in V(G)$, and hence $|V(T)| \leq |V(G)|$ because G is infinite.

To prove the assertion about width we must prove that

(1) *for every chain $C \subseteq V(T)$, the set $W_C = \bigcup_{t\in C} \bigcap\{W_{t'} : t' \in C,\, t' \geq t\}$ has cardinality $< \kappa$.*

We prove (1) by transfinite induction on the ordinal type of the set $\{t' \in V(T) : t' \leq t$ for some $t \in C\}$, which we denote by $h(C)$. From (S1) we deduce that (1) is true if $h(C) \leq 1$. So assume that (1) is true for all C with $h(C) < \alpha$ and let C with $h(C) = \alpha > 1$ be given. If C has a maximal element, say t, and t does not have a predecessor, then $W_C = W_t = W_{C-\{t\}}$ by (S3), which has cardinality $< \kappa$ by the induction hypothesis. If t has a predecessor, say t', then $W_C = W_t$, $W_{C-\{t\}} = W_{t'}$ and $|W_t| < \kappa$ by (S2) and the induction hypothesis. Finally, if C does not have a maximal element, then $|W_C| < \kappa$ by (S3) and (2.3).

This proves (1) and hence (2.4). □

If $T = (V, \leq)$ is a well-founded tree and $t \in V$, then the order type of the set $\{t' \in V : t' < t\}$ is called the *height* of t and is denoted by $\text{ht}(t)$. The following implies that if $\kappa = \aleph_0$ then the well-founded tree-decomposition in (ii) of (1.4) can be chosen graph-theoretic.

(2.5) *Let κ be an infinite regular cardinal, and let G and (T, W) be as in (2.4). Then every chain in T has order type $\leq \kappa$.*

Proof. Suppose not. Then there is a vertex $t_0 \in V(T)$ with $\text{ht}(t_0) = \kappa$. By (S3) there exists a successor t_0' of t_0, and by (S2) there exists a W_{t_0}-flap F with $F \cap W_{t_0'} \neq \varnothing$. For every $v \in W_{t_0}$ there exists, by (S3), $t_v < t_0$ such that $v \in W_t$ for all t with $t_v \leq t \leq t_0$. Let $t_1 = \sup\{t_v : v \in W_{t_0}\}$. Since $|W_{t_0}| < \kappa$ by (2.4) and since κ is regular, we deduce that t_1 is well-defined and that $t_1 < t_0$. Since $W_{t_0} \subseteq W_{t_1}$, it follows that either $F \cap W_{t_1} \neq \varnothing$, or F is a W_{t_1}-flap in which case $F \cap W_{t_1'} \neq \varnothing$ for some successor t_1' of t_1, by (S4). In either case, $F \cap W_{t_0} \neq \varnothing$ by (1.7), a contradiction. □

Our next result implies (i) $\Rightarrow$ (iii) of (1.4) for regular uncountable cardinals and thus completes the proof of (1.4).

(2.6) *Let κ be a regular uncountable cardinal, and let G and (T, W) be as in (2.4). Then T is $< \kappa$ high.*

Proof. Suppose for a contradiction that C is a chain in T of order type κ. We may assume that C has the property that if $t \in C$ and $t' \leq t$ then $t' \in C$.

Let $X = \bigcup_{t \in C} \bigcap \{W_{t'} : t' \in C, t' \geq t\}$. From (2.4), $|X| < \kappa$, and since κ is regular it follows that there exists $t_0 \in C$ such that $X \subseteq W_t$ for every $t \in C$ with $t_0 \leq t$.

(1) *For every* $t_1 \in C$ *with* $t_1 \geq t_0$ *there exists* $t \in C$ *with* $t \geq t_1$ *such that* $X = W_t$.

To prove (1) we shall construct a sequence $t_1 < t_2 < \cdots$ of elements of C as follows. Assume that $n \geq 1$ and that t_n has already been constructed. We may assume that $W_{t_n} - X \neq \varnothing$, for otherwise we are done. For every $v \in W_{t_n} - X$ there exists $t_v \in C$ with $t_v \geq t_n$ and such that $v \notin W_{t_v}$. Since $|W_{t_n}| < \kappa$ and κ is regular, it follows that $t_{n+1} := \sup\{t_v : v \in W_{t_n} - X\}$ exists and belongs to C, and that $(W_{t_n} - X) \cap W_{t_{n+1}} = \varnothing$. Since $W_{t_n} - X \neq \varnothing$ we deduce that $t_{n+1} > t_n$. Since κ is regular uncountable it follows that $t = \sup t_n$ exists and belongs to C. We claim that t is as desired. For suppose for a contradiction that $v \in W_t - X$. Then $v \in W_{t'}$ for some $t' < t$ by (S3), and hence $v \in W_{t_n}$ for some n. But then $v \notin W_{t_{n+1}}$, contrary to (W2). Hence such a choice of v is impossible, which proves (1).

From (1) we deduce that there exist t_1, $t_2 \in C$ with $t_0 \leq t_1 < t_2$ such that $W_{t_1} = W_{t_2} = X$. Let $t_2' \in C$ be the successor of t_2 in C. By (S2), $W_{t_2'} \cap F \neq \varnothing$ for some X-flap F of G. By (S4), $W_{t_1'} \cap F \neq \varnothing$ for some successor t_1' of t_1. But $X \cap F = W_{t_2} \cap F \neq \varnothing$ by (1.7), a contradiction. □

A natural question arises whether a highly connected graph can be decomposed into highly connected pieces. This turns out to be true, as follows. If μ is a cardinal we say that a graph G is *μ-connected* if $|V(G)| \geq \mu$ and $G \backslash X$ is connected for every $X \subseteq V(G)$ with $|X| < \mu$. We say that a subset $A \subseteq V(G)$ is *μ-connected* if the graph induced by A is μ-connected.

(2.7) *Let* κ *be a regular uncountable cardinal, let* $\mu < \kappa$ *be an infinite cardinal, and let* G *be a* μ*-connected graph which contains no subgraph isomorphic to a subdivision of* K_κ. *Then there exists a well-founded tree-decomposition* (T, W) *of* G *such that*

(i) $|V(T)| \leq |V(G)|$,
(ii) *every chain of* T *has order type* $< \kappa$,
(iii) $|W_t| < \kappa$ *and* W_t *is* μ*-connected for every* $t \in V(T)$, *and*
(iv) *for* t, $t' \in V(T)$, *if* $t' \leq t$ *then* $W_{t'} \subseteq W_t$, *and if* $\mathrm{ht}(t)$ *is a limit ordinal then* $W_t = \bigcup_{t' < t} W_{t'}$.

Proof. Let (T, W'') be a standard decomposition of G. Then by (2.4), $|V(T)| \leq |V(G)|$ and (T, W'') has width $< \kappa$, and by (2.6) every chain in T has order type $< \kappa$. Since G is μ-connected there exists a μ-connected set $M \subseteq V(G)$ with $|M| = \mu$. For $t \in V(T)$ let $W_t = W_t'' \cup M$, and let $W = (W_t : t \in V(T))$. Then (T, W) is a well-founded tree-decomposition of G of width $< \kappa$.

(1) *Let* $t_1 \in V(T)$ *be a successor of* $t_0 \in V(T)$. *There is a subset* $D_{t_1} \subseteq \bigcup_{t \geq t_1} W_t$ *such that*

(i) $W_{t_1} \subseteq D_{t_1}$ *and* $|D_{t_1}| < \kappa$, *and*
(ii) *there is no cut* (A, B) *of the restriction of* G *to* D_{t_1} *such that* $|A \cap B| < \mu$, $A \nsubseteq B \nsubseteq A$, *and* $W_{t_0} \cap W_{t_1} \subseteq A$.

For since G is μ-connected there is a μ-connected $F \subseteq V(G)$ with $W_{t_1} \subseteq F$ and with $|F| = \max(\mu, |W_{t_1}|)$. Set $D_{t_1} = F \cap \bigcup_{t \geq t_1} W_t$; then (i) is clearly satisfied. To prove that D_{t_1} satisfies (ii) suppose for a contradiction that (A, B) is a cut as in (ii). Since the restriction of G to F is μ-connected there exists a path P in G joining a vertex of $A - B$ to a vertex of $B - A$ with $V(P) \subseteq F - (A \cap B)$. Let us choose such a path with $|V(P)|$ minimum. Since (A, B) is a cut of the restriction of G to D_{t_1} it follows that $V(P) - \bigcup_{t \geq t_1} W_t \neq \varnothing$. From (W1) and (1.7) it follows that P contains a proper subpath joining a vertex $u \in W_{t_0} \cap W_{t_1} \subseteq A$ to a vertex in $B - A$. But $u \notin B$ and this subpath is shorter than P, a contradiction. Thus D_{t_1} satisfies both (i) and (ii) and the proof of (1) is complete.

Let $D_{\mathrm{root}(T)} = M$, and for each $t \in V(T) - \{\mathrm{root}(T)\}$ with no predecessor, let $D_t = \varnothing$. For each $t \in V(T)$ let $W'_t = \bigcup_{t' \leq t} D_{t'}$.

(2) *For each* $t \in V(T)$, $|W'_t| < \kappa$ *and* W'_t *is μ-connected.*

For since every chain of T has order type $< \kappa$ and each $|D_{t'}| < \kappa$ and κ is regular, it follows that each $|W'_t| < \kappa$. We prove that W'_t is μ-connected by transfinite induction on $\mathrm{ht}(t)$. Certainly $|W'_t| \geq \mu$ since $D_{\mathrm{root}(T)} \subseteq W'_t$. If t is a successor of some t_0 then $W'_t = W'_{t_0} \cup D_t$. Suppose that (A, B) is a cut of the restriction of G to W'_t with $|A \cap B| < \mu$ and $A - B$, $B - A \neq \varnothing$. Now from the inductive hypothesis, W'_{t_0} is μ-connected, and so not both $(A - B) \cap W'_{t_0}$, $(B - A) \cap W'_{t_0} \neq \varnothing$. We assume that $W'_{t_0} \subseteq A$. Now $(A \cap D_t, B \cap D_t)$ is a cut of the restriction of G to D_t, and certainly $B \cap D_t \subseteq A \cap D_t$ since $B \not\subseteq A$ and $W'_{t_0} \subseteq A$. From the choice of D_t it follows that $A \cap D_t \subseteq B \cap D_t$, and so $D_t \subseteq B$. Thus $W_{t_0} \cap W_t \subseteq W'_{t_0} \cap D_t \subseteq A \cap B$, and so $|W_{t_0} \cap W_t| < \mu$. But $M \subseteq W_{t_0} \cap W_t$, a contradiction. Finally, suppose that t has no predecessor. Since $W'_{\mathrm{root}(T)}$ is μ-connected we may assume that $t \neq \mathrm{root}(T)$. Let $C = \{t' \in V(T) : t' < t\}$. Then $W'_t = \bigcup_{t' \in C} W'_{t'}$ since $D_t = \varnothing$, and so W'_t is μ-connected since it is the union of a nested sequence of μ-connected subsets. This proves (2).

Let $W' = (W'_t : t \in V(T))$.

(3) (T, W') *is a well-founded tree-decomposition.*

We observe first that $W_t \subseteq W'_t$ for each $t \in V(T)$. This is clear if $t = \mathrm{root}(T)$ or t is a successor. Otherwise let C be the set of all $t' < t$ which are successors. Then $W_t \subseteq \bigcup_{t' \in C} W_{t'}$ by (S3); each such $W_{t'} \subseteq W'_{t'}$, and $\bigcup_{t' \in C} W'_{t'} \subseteq W'_t$ by the definition of W'. This proves that $W_t \subseteq W'_t$ for each $t \in V(T)$ and (W1) follows. To see (W2), let t_1, t_2, $t_3 \in V(T)$ with $t_2 \in T[t_1, t_3]$; and let $v \in W'_{t_1} \cap W'_{t_3}$. Choose $t'_1 \leq t_1$ with $v \in D_{t'_1}$, and $t'_3 \leq t_3$ similarly. If $t'_1 \leq t_2$ or $t'_3 \leq t_2$ then $v \in W'_{t_2}$ as required, and so we assume not. Hence $t_2 \in T[t'_1, t'_3]$. Now $v \in D_{t'_1} \subseteq \bigcup_{t \geq t'_1} W_t$, and similarly $v \in \bigcup_{t \geq t'_3} W_t$. Since $t_2 \in T[t'_1, t'_3]$ and $t_2 \leq t'_1$, t'_3 it follows from (W2) (for (T, W)) that $v \in W_{t_2} \subseteq W'_{t_2}$ as required. Condition (W3) is clear because $W'_t \subseteq W'_{t'}$ for $t \leq t'$.

Finally, we observe that if $t \in V(T) - \{\mathrm{root}(T)\}$ has no predecessor (i.e., $\mathrm{ht}(t)$ is a limit ordinal) then since $D_t = \varnothing$,

$$W'_t = \bigcup_{t' < t} D_{t'} = \bigcup_{t' < t} W_{t'}.$$

The result follows. □

For an application of (2.7) see [9].

3. Linear decompositions and stoppages

Now we prove (1.5). By (1.4), (i) $\Rightarrow$ (ii). We shall prove (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i). We start with (ii) $\Rightarrow$ (iii).

(3.1) *Let κ be an uncountable regular cardinal, let G be a graph and let (T, W) be a $< \kappa$ high well-founded tree-decomposition of G of width $< \kappa$. Then G admits a linear decomposition of width $< \kappa$.*

Proof. Let $\leq'$ be a well-ordering of $V(T)$ with the property that

(1) *for every $t \in V(T)$, the order type of the set of successors of t is not a limit ordinal.*

Let L be the set of maximal chains of T and let $\leq$ be defined on L as follows. For $l_1, l_2 \in L$ we say that $l_1 \leq l_2$ if either $l_1 = l_2$, or $l_1 \neq l_2$ and $t_1 \leq' t_2$, where $t_i = \min\{t \in l_i : t \notin l_1 \cap l_2\}$. It follows that L is linearly ordered by $\leq$.

(2) *$(L, \leq)$ is complete.*

We must verify that every nonempty subset of L has a supremum and an infimum. We shall do it for supremum, for the infimum case is analogous and in fact easier. So let $C \subseteq L$. Let $t_0 = \text{root}(T)$ and assume that we have already constructed $t_0, \ldots, t_\beta, \ldots (\beta < \alpha)$, where $\text{ht}(t_\beta) = \beta$ and $t_0 \leq t_1 \leq \cdots \leq t_\beta \leq \cdots$. If $\{t_\beta\}_{\beta<\alpha}$ is a maximal chain in T we stop, and otherwise we choose $t_\alpha \in V(T)$ such that

- (i) $\text{ht}(t_\alpha) = \alpha$,
- (ii) $t_\beta \leq t_\alpha$ for every $\beta < \alpha$,
- (iii) $t \leq' t_\alpha$ for every t such that $\text{ht}(t) = \alpha$, $t_\beta \leq t$ for every $\beta < \alpha$, and $t \in l$ for some $l \in C$,
- (iv) t_α is $\leq'$-minimal subject to (i)–(iii).

Such a choice is possible, because it follows from (1) that there exists at least one vertex satisfying (i)–(iii). (We remark that this vertex is unique when α is a limit ordinal, because any two vertices of T have an infimum.) Let α be the least ordinal such that t_α is undefined, and let $l = \{t_\beta : \beta < \alpha\}$. Then $l = \sup C$, as is easily seen. This proves (2).

We define, for $l \in L$, $X_l = \bigcup_{t\in l} W_t$, and put $X = (X_l : l \in L)$. We claim that (L, X) is a linear decomposition of G of width $< \kappa$. For condition (L1) is obviously satisfied. To prove (L2) let $l_1 < l < l_2$ and let $v \in X_{l_1} \cap X_{l_2}$. Then $v \in W_{t_1} \cap W_{t_2}$ for some $t_1 \in l_1$ and some $t_2 \in l_2$. Let $t = \inf(t_1, t_2)$; then $t \in l$ and so $W_t \subseteq X_l$, but $v \in W_t$ by (W2) and so $v \in X_l$, as required. To prove (L3) let $I \subseteq L$ be a nonempty interval. Again, it is enough to show that $\bigcap_{i\in I} X_i \subseteq X_{\sup(I)}$. So let $l = \sup I$ and let $i_1 \in I$; we may assume that $i_1 \neq l$. Then there exists $i_2 \in I$ such that $i_1 \cap i_2 \subseteq l$. Let $t_j = \inf\{t \in V(T) : t \in i_j - i_{3-j}\}$ $(j = 1, 2)$ and let $t = \inf(t_1, t_2)$. Then $\bigcap_{i\in I} X_i \subseteq X_{i_1} \cap X_{i_2} \subseteq \bigcup_{t'\leq t} W_{t'} \subseteq X_l$ using (W2), as desired.

Hence (L, X) is a linear decomposition. The statement about width follows easily, because κ is regular and each X_l is the union of $< \kappa$ sets, each of cardinality $< \kappa$. □

(3.2) *Let κ be a cardinal and let G be a graph. If G admits a linear decomposition of width $< \kappa$, then G has no stoppage of order $\geq \kappa$.*

Proof. Suppose for a contradiction that (L, X) is a linear decomposition of

G of width $< \kappa$ and that $\mathscr{S}$ is a stoppage in G of order $\geq \kappa$. For $l \in L$ let $C_l = (\bigcup_{l' \leq l} X_{l'}, \bigcup_{l' \geq l} X_{l'})$. We observe that C_l is a cut in G of order $< \kappa$. Let A be the set of all $l \in L$ such that $C_l \in \mathscr{S}$. We deduce from the second stoppage axiom that if $l \in A$ and $l' \leq l$, then $l' \in A$.

(1) *A has a maximum element.*

For let $a = \sup A$. We deduce from the third stoppage axiom that

$$\left(\bigcup_{l<a} X_l, \bigcup_{l \geq a} X_l\right) \in \mathscr{S}.$$

If $(\bigcup_{l \leq a} X_l, \bigcup_{l \geq a} X_l) \notin \mathscr{S}$ then $(\bigcup_{l \geq a} X_l, \bigcup_{l \leq a} X_l) \in \mathscr{S}$ by the first stoppage axiom, and yet $(G|\bigcup_{l<a} X_l) \cup (G|\bigcup_{l \geq a} X_l) = G$, contrary to the second axiom. Hence $(\bigcup_{l \leq a} X_l, \bigcup_{l \geq a} X_l) \in \mathscr{S}$ and thus $a \in A$.

Similarly let B be the set of all $l \in L$ such that $(\bigcup_{l' \geq l} X_{l'}, \bigcup_{l' \leq l} X_{l'}) \in \mathscr{S}$. Analogously, B has a minimum element, say b. It follows that $a < b$ and that there is no $l \in L$ with $a < l < b$. Hence $(G|\bigcup_{l \leq a} X_l) \cup (G|\bigcup_{l \geq b} X_l) = G$, contrary to the second stoppage axiom. □

(3.3) *Let κ be an infinite cardinal, let G be a graph and let H be a subgraph of G isomorphic to a subdivision of K_κ. Then G has a stoppage of order κ.*

Proof. Let V be the set of vertices of H which correspond to a vertex of K_κ. If (A, B) is a cut in G of order $< \kappa$, then exactly one of A, B contains V. Let $\mathscr{S}$ be the set of all cuts (A, B) in G of order $< \kappa$ such that $V \subseteq B$. Then it is easy to verify that the stoppage axioms are satisfied. □

Thus we have proved (1.5). Now we prove (1.6) which we restate.

(3.4) *There exists a graph which contains no subgraph isomorphic to a subdivision of $K_{\aleph_\omega}$ and which has no linear decomposition of width $< \aleph_\omega$.*

Proof. Let T be the well-founded tree with every chain of order type $\leq \omega$ in which every element $t \in V(T)$ has $\aleph_\omega$ successors. For every $t \in V(T)$ we choose a set M_t and an element $m_t \in M_t$ in such a way that $|M_t| = \aleph_{\mathrm{ht}(t)}$ and $M_t \cap M_{t'} = \varnothing$ for distinct t, $t' \in V(T)$. Let G be the simple graph with vertex set $\bigcup_{t \in V(T)} M_t$ and such that for u, $v \in V(G)$ there exists an edge in G with endvertices u, v if and only if $u \neq v$, and either u, $v \in M_t$ for some $t \in V(T)$, or $u \in M_t$ and $v = m_{t'}$ for a successor $t' \in V(T)$ of $t \in V(T)$, or $v \in M_t$ and $u = m_{t'}$ for a successor $t' \in V(T)$ of $t \in V(T)$. For $t \in V(T)$ let G_t be the subgraph of G induced by the set $\bigcup_{t' \geq t} M_{t'}$. Let $W_{\mathrm{root}(T)} = M_{\mathrm{root}(T)}$, and if $t' \in V(T)$ is a successor of $t \in V(T)$ let $W_{t'} = M_t \cup M_{t'}$. Let $W = (W_t : t \in V(T))$. It is easy to see that (T, W) is a well-founded tree-decomposition of G of width $< \aleph_\omega$, and hence G has no subgraph isomorphic to a subdivision of $K_{\aleph_\omega}$ by (1.4).

It remains to be shown that G has no linear decomposition of width $< \aleph_\omega$ either. So suppose for a contradiction that G has a linear decomposition (L, X) of width $< \aleph_\omega$. A *clique* in G is a subset $M \subseteq V(G)$ such that every pair of distinct members of M are adjacent.

(1) *For every clique M, the set $I(M) = \{l \in L : M \subseteq X_l\}$ is a nonempty closed interval in L.*

For $v \in M$, $I(\{v\})$ is a closed interval by (L2) and (L3), and by (L1) any two of these intervals meet. We deduce that $\{l \in L : M \subseteq X_l\} = \bigcap_{v \in M} I(\{v\})$ is as desired.

For $t \in V(T)$ let $a(t) \leq b(t)$ be the endvertices of $I(M_t)$. Let $t_0 = \text{root}(T)$ and assume that we have already constructed $t_0, \ldots, t_n$. Let t_{n+1} be a successor of t_n such that $M_{t_{n+1}} \cap (X_{a(t_n)} \cup X_{b(t_n)}) = \varnothing$. Such a choice is possible since $|X_{a(t_n)} \cup X_{b(t_n)}| < \aleph_\omega$ and there are $\aleph_\omega$ such successors. This completes the inductive definition of $t_0, t_1, \ldots$.

Now for each $n \geq 0$, $M = M_{t_n} \cup \{m_{t_{n+1}}\}$ is a clique, and so $\varnothing \neq I(M) = I(M_{t_n}) \cap I(\{m_{t_{n+1}}\})$. Yet $a(t_n)$, $b(t_n) \notin I(\{m_{t_{n+1}}\})$, and hence $I(\{m_{t_{n+1}}\}) \subseteq I(M_{t_n})$. But $I(M_{t_{n+1}}) \subseteq I(\{m_{t_{n+1}}\})$, and so $I(M_{t_{n+1}}) \subseteq I(M_{t_n})$. By the completeness of L there exists $l \in L$ such that $l \in \bigcap_{n \geq 0} I(M_{t_n})$, that is, $M_{t_n} \subseteq X_l$ for all $n \geq 0$. But $|M_{t_0} \cup M_{t_1} \cup \cdots| = \aleph_\omega > |X_l|$, a contradiction. □

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