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PSEUDO-ISOTOPIES OF IRREDUCIBLE 3-MANIFOLDS

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ABSTRACT. It is shown that a certain subspace of the space of all pseudo-isotopies of any irreducible 3-manifold is connected. This subspace consists of those pseudo-isotopies corresponding to 1-parameter families of functions which have nondegenerate critical points of index 1 and 2 only and which contain no slides among the 2-handles.

Some of the techniques developed are used to prove a weak four-dimensional h-cobordism theorem.

INTRODUCTION

Let P be any compact irreducible 3-manifold and let $\mathscr{P}$ be the space of pseudo-isotopies of P. In this paper a subset $\mathscr{P}_0$ of $\mathscr{P}$ is studied. The elements of $\mathscr{P}_0$ correspond roughly to those pseudo-isotopies, in high-dimensional pseudo-isotopy theory, whose Wh_2 invariant vanishes. More precisely, a pseudo-isotopy $F: P \times I \to P \times I$ is in $\mathscr{P}_0$ if there is a 1-parameter family $f_t: P \times I \to I$ of smooth functions joining $p \circ F$ to p (p denotes the projection map $P \times I \to I$) such that the nondegenerate critical points of the f_t have index 1 and 2 only and such that there are no slides among the 2-handles in the graphic of f_t. Slides among the 1-handles, however, are allowed. (To be in $\mathscr{P}_0$, F must also satisfy some somewhat technical properties that are stated in §1.) The main theorem is that such a 1-parameter family f_t can be deformed, rel endpoints, to a 1-parameter family of functions without critical points. Thus $\pi_0(\mathscr{P}_0) = 0$, in marked contrast to the results of the high-dimensional theory for manifolds which are not simply connected.

It should be mentioned that the irreducibility of P^3 in the theorem is essential. Examples where $\pi_0(\mathscr{P}_0) \neq 0$ for reducible 3-manifolds have been known for some time, since the high-dimensional obstructions are sometimes realizable for 3-manifolds.

A family f_t as above gives rise to a 1-parameter family of handlebody structures for $P \times I$ which is the starting point of the proof of the theorem. But beyond this, the proof does not follow the usual approach to pseudo-isotopy theory as developed by Cerf [C] and Hatcher and Wagner [HW]. Their high-dimensional methods would have us focus on the intersection of the attaching circles of the 2-handles with the transverse 2-spheres of the 1-handles in the three-dimensional level sets of the f_t. But, as usual, troubles arise with high-dimensional techniques in dimension 3. So, instead, 2-spheres $\Sigma(t)$ are defined

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(in §2) which move by isotopy with the attaching circles, capturing their motion. (The attaching circles themselves are moving only by isotopy since there are no slides among the 2-handles.) The focus is then on the intersection of these 2-spheres $\Sigma(t)$ with the transverse 2-spheres of the 1-handles.

The idea of the proof is to move the $\Sigma(t)$ about (letting the attaching circles follow along with them) until they are disjoint from the transverse 2-spheres of the 1-handles. The main tool used for this is a disjunction technique of Hatcher [H1] and its subsequent elaboration by Hendriks and Laudenbach [HL]. In §§2 and 3 this technique is described and adapted to the setting of this paper.

Once disjunction has been achieved, the focus then does shift (in §4) to the attaching circles of the 2-handles and the transverse 2-spheres of the 1-handles and to how these sets intersect. After an additional deformation, the punctured copies of $S^1 \times S^2$ bounded by the 2-sphere components of $\Sigma(t)$ each contain one attaching circle and one transverse 2-sphere. Arguments in §4.2 show how to deform these latter two sets within the punctured copies of $S^1 \times S^2$ until they intersect transversely in one point, thus making cancellation of the handles easy. These arguments all rely on Cerf's theorem $\Gamma_4 = 0$ (the π_0 version of the Smale conjecture). Finally, the disjunction technique used in the proof yields a deformation which moves one endpoint of the family of functions under consideration. This problem is dealt with in §4.1.

In §5 some of the methods used in previous sections are applied, without making use of the parameter, to prove a weak four-dimensional h-cobordism theorem. Section 1 contains groundwork for the rest of the paper as well as a definition of the space $\mathscr{P}_0$.

1. The space $\mathscr{P}_0$

The space $\mathscr{P}_0$ of pseudo-isotopies that is studied in this paper is defined in this section, and some properties of the elements of $\mathscr{P}_0$ are developed. These properties are all mentioned in Proposition 1.5. Not much is lost if they are included as part of the definition of $\mathscr{P}_0$, except that §1.3, which is used in the proof of Proposition 1.5, is also used a number of times later on. This section begins with some remarks on notation and conventions that will be used throughout the paper. A more complete discussion of these can be found in Chapter 1 of [HW].

Much of this section applies to any compact smooth manifold, whereas later sections apply only to irreducible 3-manifolds. But, for convenience, we assume, from the start, that P is any compact smooth irreducible 3-manifold, with or without boundary. A diffeomorphism $F: P \times I \to P \times I$ is called a pseudo-isotopy of P if F restricted to $P \times 0$ is the identity, and, if P has boundary, $F(x, t) \in P \times \{t\}$ for all $x \in \partial P$. $\mathscr{P}$ denotes the space of all pseudo-isotopies of P with the C^∞ topology. $\widehat{\mathscr{F}}$ denotes the space of all triples (η, f, μ), where $f: (P \times I, P \times 0, P \times 1) \to (I, 0, 1)$ is a smooth function with no critical points near $P \times 0$ and $P \times 1$, and η is a gradient-like vector field for f with respect to the Riemannian metric μ on $P \times I$. (If the manifold P has boundary then f must also satisfy $f(x, t) = t$ for all $x \in \partial P$.) $\widehat{\mathscr{E}}$ denotes the space of all those triples $(\eta, f, \mu) \in \widehat{\mathscr{F}}$ in which f has no critical points. There is an isomorphism $\pi_1(\widehat{\mathscr{F}}, \widehat{\mathscr{E}}) \to \pi_0(\mathscr{P})$ (see [HW, pp. 17 and 33]). So, as usual, we work with elements of $\pi_1(\widehat{\mathscr{F}}, \widehat{\mathscr{E}})$ instead of directly with elements of $\mathscr{P}_0$.

If $(\eta, f, \mu) \in \widehat{\mathscr{F}}$ and p is an isolated critical point of f then we write $W(p)$ for $\{x \in P^3 \times I: \lim_{t \to -\infty} \varphi_t(x) = p\}$. (Here φ_t is the 1-parameter family of diffeomorphisms generated by η.) $W(p)$ is called the stable set and $W^*(p)$ the unstable set of p. If $L \in I$ we use $W_L(p)$ to denote $W(p) \cap f^{-1}(L)$. If p is nondegenerate, then $W_L(p)$ is a sphere for some $L \in I$. For such L, $W_L(p)$ is called the stable sphere of p. Similarly $W_L^*(p)$ denotes $W^*(p) \cap f^{-1}(L)$ and, if this set is a sphere, is called the unstable sphere of p. Or sometimes, since the triple (η, f, μ) determines a handlebody structure for $P^3 \times I$, the set $W_L(p)$ is called the attaching sphere and $W_L^*(p)$ the transverse sphere of the handle associated with p. If p is a critical point of f of index i and q is another of index j with $f(p) < L < f(q)$ then any nonempty intersection of the form $W_L^*(p) \cap W_L(p)$ is called a j/i intersection. Transverse j/j intersections are often referred to as handle slides. Another convention we use is that the letter p (resp. q, x) with subscripts denotes index 1 (resp. index 2, index 0) critical points.

If $l, L \in I$ and $A \subset f^{-1}(l)$ then the notation $\varphi_l^L(A)$ is used to stand for the intersection of $f^{-1}(L)$ with the set of integral curves which meet A. This notation is usually only used when $\varphi_l^L(\{a\})$ is nonempty for each $a \in A$. In this case we have a diffeomorphism $\varphi_l^L: A \to \varphi_l^L(A)$. Sometimes the subscript l is dropped from the notation φ_l^L.

According to [HW, Chapter 1] each element of $\pi_1(\widehat{\mathscr{F}}, \widehat{\mathscr{E}})$ is represented by a 1-parameter family (η_t, f_t, μ_t) which satisfies the following four properties:

(1) For all but finitely many values of t, either f_t has no critical points or the critical points of f_t are all nondegenerate with distinct critical values. At each of the finitely many exceptional values of t either f_t has one birth-death critical point or all the critical points of f_t are nondegenerate with exactly two having the same critical value.

(2) If p is any birth-death critical point occurring, say, at time t_0 then there is a small open interval Δ containing t_0 and, for some open ball U in $\mathbb{R}^4$ centered at the origin, a family of embeddings $\varphi_t: U \to P^3 \times I$, $t \in \Delta$, such that $\varphi_{t_0}(0) = p$ and

$$f_t \circ \varphi_t(x_1, \ldots, x_4) = \pm x_1^2 \pm x_2^2 \pm x_3^2 \pm (t - t_0)x_4 + x_4^3.$$

(The choice of signs here is determined by the index of p.) Moreover the metric μ_t restricted to $\varphi_t(U)$ is the metric induced by φ_t from the standard metric on $\mathbb{R}^4$ and, on $\varphi_t(U)$, η_t is the gradient of f_t computed using the metric μ_t.

(3) If p is a nondegenerate critical point of index 1 (resp. index 2) of any f_t then $f_t(p)$ is less than (resp. greater than) $L = 1/2$. If p is a birth or death point of index 1 then $f_t(p) = L$. Similar statements can be made about other pairs of critical points of consecutive index and their corresponding birth-death points, but we will only need the index 1 and 2 cases.

(4) The stable and unstable sets of the nondegenerate critical points vary smoothly with t except at handle slides.

Definition. A pseudo-isotopy $F: P^3 \times I \to P^3 \times I$ is in $\mathscr{P}_0$ if the element in $\pi_1(\widehat{\mathscr{F}}, \widehat{\mathscr{E}})$ corresponding to F under the isomorphism $\pi_1(\widehat{\mathscr{F}}, \widehat{\mathscr{E}}) \to \pi_0(\mathscr{P})$ has a representative (η_t, f_t, μ_t) which satisfies properties (1) through (4) above and, in addition, satisfies the following two properties:

(5) All the nondegenerate critical points of the f_t have index 1 or index 2, and there are no slides among the 2-handles.

(6) All the birth-death critical points of the f_t are independent.

To say that a birth-death critical point p is independent means that all trajectories leading to or coming from p meet either $P^3 \times \{0\}$ or $P^3 \times \{1\}$. Property (6) is included in the definition of $\mathscr{P}_0$ since the standard way [HW, p. 62] of achieving (6) might introduce 2-handle slides.

The main result of this paper is

Theorem 1. $\pi_0(\mathscr{P}_0) = 0$ *if* P^3 *is compact and irreducible.*

Most of the rest of the paper is devoted to proving this theorem. The rest of this section is mainly preparation for the proof of Proposition 1.5.

1.1. **Lemma.** *If an element of* $\pi_1(\widehat{\mathscr{F}}, \widehat{\mathscr{E}})$ *has a representative* (η_t, f_t, μ_t), $-1 \leq t \leq 2$, *satisfying properties* (1) *through* (6) *above then it has a representative which satisfies the following property:*

(7) *All births occur during the interval* $(-1, 0)$ *and all deaths during* $(1, 2)$. *Moreover there are no* j/i *intersections of any kind during* $[-1, 0]$ (*resp.* $[1, 2]$) *except for those involving critical points of a pair that are born together* (*resp. die together*).

This representative still satisfies properties (1) *through* (6) *and for the embeddings* φ_t, $t \in \Delta$, *provided near each birth-death critical point* p *by property* (2) *we have* $0 \in \Delta$ *if* p *is a birth point and* $1 \in \Delta$ *if* p *is a death point.*

Proof. Reparametrize so that there is only one death point of f_t after $t = 1$. The idea of the proof is to delay the other deaths, one at a time, until they occur after $t = 1$. Deformations which do this can be obtained by "moving" the death points along paths ω_i in $P \times \{0.5\} \times I \subset P \times I \times I$. Using general position along with properties (4) and (5) (and, if it is necessary to move death points past birth points in the graphic, using independence of the birth points as well) we may assume that each of the paths ω_i is disjoint from the attaching circles of the index 2 critical points. Thus no 2-handle slides are introduced. By induction we assume that, for each family of embeddings φ_t, $t \in \Delta$, associated with one of the delayed deaths, we have $1 \in \Delta$. Thus, for $t > 1$, we can take the paths ω_i to be disjoint from the images of the embeddings φ_t. So there are no j/i intersections of any kind during [1, 2] except for those involving pairs of critical points that die together. Proceed similarly with the birth points to complete the proof.

For the next few lemmas let $f_t \colon P^3 \times I \to I$, $t \in [0, 1]$, be a 1-parameter family of Morse functions. Assume $p \colon [0, 1] \to P \times I$ is a smooth function with each $p(t)$ an index 1 critical point of f_t.

1.2. **Parametrized Morse Lemma.** *There is a* 1*-parameter family* $g_t \colon U \to P \times I$, $t \in [0, 1]$, *of Morse parametrizations around* $p(t)$ *meaning that*

(a) *there is some neighborhood* $U \subset \mathbb{R}^4$ *of* 0 *such that each* g_t *is a diffeomorphism from* U *onto an open subset of* $P \times I$ *containing* $p(t)$, *and*

(b) *for* $t \in [0, 1]$, $f_t \circ g_t(x_1, \dots, x_4) = f_t(p(t)) - x_1^2 + x_2^2 + x_3^2 + x_4^2$, *and* $g_t(0) = p(t)$.

Proof. A parametrized version of the proof of the Morse Lemma in [M2, p. 6] can be worked out. The only difficulty comes in finding, at each step of the diagonalization process, an entry in the matrix being diagonalized which is nonzero for all values of the parameter t. To overcome this difficulty, start the proof using coordinates x_i^t whose axes at time t are tangent to the stable and unstable sets of the critical point $p(t)$. Such coordinates exist since the stable and unstable sets give a smoothly varying splitting of the fibers of the tangent bundle over the critical points $p(t)$. This in turn gives a bundle map $H: [0, 1] \times \mathbb{R}^4 \to T(P \times I)$ covering p with $H_t(e_1) \subset T_{p(t)}(W(p(t)))$ and $H_t(e_i) \subset T_{p(t)}(W^*(p(t)))$, $i = 1, 2, 3$. (Here the e_i are the standard basis vectors of $\mathbb{R}^4$, $H_t(x)$ means $H(t, x)$, and T means tangent space with T_x the fiber over x.) H, suitably restricted, followed by the exponential map (computed using any Riemannian metric on $P \times I$) gives the desired coordinates x_i^t.

1.3. **Lemma** (Relative version of 1.2). *Assume that* 1*-parameter families* $h_t^1: U_1 \to P^3 \times I$, $0 \le t \le \varepsilon$, *and* $h_t^2: U_2 \to P \times I$, $1 - \varepsilon \le t \le 1$, *of Morse parametrizations around* $p(t)$ *are given for some small* $\varepsilon > 0$. *Then there is, up to sign, a* 1*-parameter family* $g_t: U \to P \times I$, $0 \le t \le 1$, *of Morse parametrizations around* $p(t)$ *agreeing with the given ones near* 0 *and* 1. (*The "up to sign" means that at some value of* t, *say* $t = 1/2$, *it may be necessary to change the sign of one or two of the coordinates given by* g_t. *This sign change is not needed if the signs of the coordinates given by the* h_t^i *are compatible.*)

Proof. h_t^1 viewed as a map $U_1 \times [0, \varepsilon) \to P \times I \times [0, \varepsilon)$ induces a (Riemannian) metric on its image from the standard metric on $U_1 \times [0, \varepsilon) \subset \mathbb{R}^4 \times \mathbb{R}^1$; similarly for h_t^2. Using a partition of unity "extend" these two metrics to all of $P \times I \times [0, 1]$. (Shrink the domains U_i of the h_t^i slightly if necessary to get an honest extension.) By restricting to slices $P \times I \times \{t\}$ we obtain a 1-parameter family μ_t, $0 \le t \le 1$, of metrics on $P \times I$. In the rest of the proof use the gradient of f_t with respect to μ_t to compute $W(p(t))$ and $W^*(p(t))$, and let $\exp_t$ denote the exponential map with range $P \times I$ equipped with Riemannian metric μ_t.

Let $H: [0, 1] \times \mathbb{R}^4 \to T(P \times I)$ be a bundle map (possibly just up to sign as explained below) covering p such that

(i) for $t \in [0, \varepsilon/2) \cup (1 - \varepsilon, 1]$, H_t equals the composite $\{t\} \times \mathbb{R}^4 = T_0(U_i) \to T_{p(t)}(P \times I)$, where the arrow is the map $(h_t^i)_*$, and

(ii) for $t \in [0, 1]$, H_t respects, as in the previous proof, the splitting determined by the stable and unstable manifolds $W(p(t))$ and $W^*(p(t))$.

Note that (ii) follows from (i) if $t \in [0, \varepsilon/2) \cup (1 - \varepsilon, 1]$. The family H_t can be assumed to be smooth in t except possibly at one point, say $t = \varepsilon/2$, where $H_t(e_1)$ may change sign to $-H_t(e_1)$ and $H_t(e_2)$ may change to $-H_t(e_2)$. (As before e_1 and e_2 are unit basis vectors in $\mathbb{R}^4$.) Now as in the previous proof use the coordinates given by $\exp_t \circ H_t|$ to start with diagonalization process. ($H_t|$ denotes H_t suitably restricted.) Observe that for $t \in [0, \varepsilon/2) \cup (1 - \varepsilon, 1]$, $\exp_t \circ H_t|$ equals h_t^i. Thus the diagonalization process does nothing for t near 0 and 1 and so yields the desired parametrizations g_t. (The possible change in sign which occurs in the family g_t at $t = \varepsilon/2$ can be pushed to any value of t; however, if the change in sign of the family H_t occurred outside of $[0, \varepsilon/2) \cup (1 - \varepsilon, 1]$, say at $t = 1/2$, then the resulting family g_t would have a discontinuity at $t = 1/2$, which would be worse than just a change in sign.)

1.4 **(1.3 with addition of metrics and gradient-like vector fields).** Assume that in addition to the embeddings h_t^i in Lemma 1.3, a 1-parameter family μ_t, $0 \le t \le 1$, of metrics on $P \times I$ is given such that for t near 0 and 1 the metric induced by h_t^i from the standard metric on $\mathbb{R}^4$ agrees with μ_t on the image of h_t^i. Use the family μ_t in the previous proof (in which the partition of unity argument is now superfluous) to obtain the Morse parametrizations g_t. Then the family μ_t can be deformed to another family of metrics μ_t', $0 \le t \le 1$, which is compatible with the metrics induced by the g_t for all t as follows. First, using a partition of unity, define μ_t' so as to agree with μ_t except near $p(t)$, where μ_t' should agree with the metric induced by g_t. Then deform μ_t to μ_t'. Arguments of this sort are easy (since the space of metrics on $\mathbb{R}^n$ is convex) and so are never made explicit in what follows.

Similarly the space of gradient-like vector fields for a fixed function $f\colon P^3 \times I \to I$ and metric μ on $P^3 \times I$ is convex. Thus gradient-like vector fields can be defined using partitions of unity and can be deformed from one to another with straight line homotopies. So, for example, assume that in 1.3 we are also given a 1-parameter family of vector fields which are gradient-like with respect to the family of metrics μ_t given above. Then this family of vector fields can be deformed along with the metrics so as to remain gradient-like. Arguments of this sort, as with those involving metrics, are omitted from now on.

1.5. **Proposition.** *Let F be any pseudo-isotopy in $\mathscr{P}_0$. Then the element in $\pi_1(\widehat{\mathscr{F}}, \widehat{\mathscr{E}})$ corresponding to F has a representative (η_t, f_t, μ_t), $-1 \le t \le 2$, which satisfies properties* (1) *through* (7) *above and also satisfies the following two properties*:

(8) *For each death point d_i with corresponding paths $p_i(t)$ and $q_i(t)$ of index* 1 *and* 2 *critical points meeting at d_i, the unstable set $W^*(p_i(t))$ and the stable set $W(q_i(t))$ intersect transversely in one point for $1 \le t \le t(d_i)$. (Here $t(d_i)$ is the time of the death d_i.) Similarly we have one point transverse intersection from the time of each birth to time* 0.

(9) *For each path $p_i(t)$, $0 \le t \le 1$, of index* 1 *critical points there is a* 1-*parameter family $g_t^i\colon U_i \to P \times I$, $0 \le t \le 1$, of Morse parametrizations around $p_i(t)$. Moreover, for $0 \le t \le 1$ the metric induced by the g_t^i agrees with μ_t on the sets $g_t^i(U_i)$.*

Proof. Given $F \in \mathscr{P}_0$ let (η_t', f_t', μ_t') be the corresponding 1-parameter family provided by Lemma 1.1. Since each of the sets Δ mentioned in Lemma 1.1 contain either 0 or 1, property (7) is automatically satisfied by the family (η_t', f_t', μ_t'). The idea of the proof is to use 1.3 (and 1.4) to achieve property (8) without disturbing the stable and unstable sets near 0 and 1 so that property (7) will continue to hold. In order to use 1.3, Morse parametrizations are needed near 0 and 1 which join smoothly with the parametrizations φ_t of property (2). But these can be obtained by changing only the coordinates x_4 given by the φ_t. (We may assume that such deformations do not change the stable and unstable sets.) So 1.3 and 1.4 can be applied to complete the proof.

2. Disjunction theory without a parameter

Included in this section and §3 is a description of the Hendriks-Laudenbach disjunction technique. Much of this is taken from §3 of [HL]. Many of the ideas in [HL] in turn come from Hatcher [H1]. So the reader may want to look at these two papers for somewhat different descriptions of the technique as well as different applications of it. Much notation is also borrowed from these two papers.

Very roughly this disjunction technique is a tool for making all the 2-spheres in a given collection of disjoint 2-spheres in a 3-manifold disjoint from all the 2-spheres in another such collection. In our application one of the collections will consist of the unstable 2-spheres of the index 1 critical points of a Morse function $f_t\colon P^3 \times I \to I$. These 2-spheres correspond to the $N_{ji}(t)$ in [HL] and in fact we often denote them here by $N_{ji}(t)$. The other collection of 2-spheres that we will use is $\Sigma(t)$, which is defined below. $\Sigma(t)$ corresponds to the set $f_c(\Sigma)$ in [HL] except that $\Sigma(t)$ contains half as many 2-spheres as does $f_c(\Sigma)$. The $N_{ji}(t)$ and the 2-spheres in $\Sigma(t)$ are contained in a level set of f_t lying between the index 1 and index 2 critical points of f_t. An important purpose of §§2 and 3 is to show that in this level set the Hendriks-Laudenbach disjunction technique can be realized by deforming f_t.

The two operations—special handle moves and regular moves—used in the disjunction technique are described in this section without a parameter, while the parametrized versions are given in §3. Much of the present section, however, is written with the parametrized theory in mind.

Given a pseudo-isotopy in $\mathscr{P}_0$ of a compact irreducible manifold P^3, let (η_t, f_t, μ_t) be the corresponding 1-parameter family provided by Proposition 1.5. Thus, (η_t, f_t, μ_t) satisfies properties (1) through (9) of §1. Let $p_1, \dots, p_g$ be the paths of index 1 critical points of f_t, and $q_1, \dots, q_g$ those of index 2 labeled so that $W_L(q_i(0))$ and $W_L^*(p_i(0))$ meet transversely in one point (property (8)). Because of property (7), $W(q_j(0)) \cap W^*(p_i(0)) = \varnothing$ whenever $i \neq j$. So we can pick disjoint 2-spheres $\Sigma_1, \dots, \Sigma_g$ in $f_0^{-1}(L)$ so that the connected sum $f_0^{-1}(L) = P^3 \# (\#_i (S^1 \times S^2)_i)$ is formed along the Σ_i and so that the sets $W_L(q_i(0)) \cup W_L^*(p_i(0))$ are disjoint from the Σ_i. Each Σ_i could be obtained, for instance, by taking two parallel copies of $W_L^*(p_i(0))$, one on either side of $W_L^*(p_i(0))$, and joining them with a tube which runs along the arc contained in $W_L(q_i(0))$ which joins the two copies of $W_L^*(p_i(0))$ and is disjoint from $W_L^*(p_i(0))$.

Given a Riemannian metric on P there is a canonical way to identify the level surfaces $f_t^{-1}(L)$, $0 \le t \le 1$, with $f_0^{-1}(L)$. Namely define $f\colon M \times I \times I \to I$ by $f(x, y, t) = f_t(x, y)$ and define $p\colon M \times I \times I \to I$ by $p(x, y, t) = t$. Then p restricted to the manifold $f^{-1}(L)$ is a Morse function without critical points. Use the orthogonal trajectories of this Morse function as done in [M2, Theorem 3.1] to get the desired diffeomorphisms $h_t\colon f_0^{-1}(L) \to f_t^{-1}(L)$.

Since there are no 2-handle slides, the h_t can be used to view the attaching circles $W_L(q_i(t))$, $0 \le t \le 1$, as moving around by isotopy in $f_0^{-1}(L)$. Use isotopy extension to pick an isotopy $g_t\colon f_0^{-1}(L) \to f_0^{-1}(L)$ with $h_t \circ g_t(W_L(q_i(0))) = W_L(q_i(t))$ for $0 \le t \le 1$.

Definition. Let $\Sigma_i(t) = h_t \circ g_t(\Sigma_i)$, $0 \le t \le 1$, and let $\Sigma(t) = \bigcup \Sigma_i(t)$.

Pick base points $b_i \in \Sigma_i$ and let $b_i(t) = h_t \circ g_t(b_i)$ be the corresponding basepoints in $\Sigma_i(t)$.

For each path $p_i(t)$, $0 \le t \le 1$, of index 1 critical points of f_t, pick, in level sets slightly above the critical values $f_t(p_i(t))$, smoothly varying bicollared neighborhoods $B_i(t)$, $0 \leq t \leq 1$, of the unstable 2-spheres of the $p_i(t)$. This is easy using the Morse parametrizations provided by property (9). We may assume that the $B_i(t)$ are contained in disjoint neighborhoods of the $p_i(t)$ where each of these neighborhoods is the image of a Morse parametrization. The $B_i(t)$ are identified with $S^2 \times (-1, 1)$ with $S^2 \times \{0\}$ corresponding to the unstable 2-spheres of the critical points $p_i(t)$.

A subset of a level set of f_t is said to be visible if it is disjoint from the stable sets of all the index 1 critical points of f_t. In general, for instance if there are slides among the 1-handles, no one slice $S^2 \times \{x\}$ of some $B_i(t)$ is visible for all $t \in [0, 1]$; however, a visible slice remains so for small changes in t. Using this, Sard's Theorem, and compactness of I, choose a finite cover of $[0, 1]$ with open intervals U_j, $j \in J$, and, for each $j \in J$, choose numbers $c_{ji} \in (-1, 1)$, $i = 1, \dots, g$, such that

(i) the slices $N_{ji}(t) = S^2 \times \{c_{ji}\} \subset B_i(t)$, $i = 1, \dots, g$, are visible for all $t \in \overline{U}_j$,
(ii) $\Sigma(t)$ is transverse to $\bigcup_i \varphi^L(N_{ji}(t))$ for all $t \in \overline{U}_j$,
(iii) $N_{ji}(t) \neq N_{kl}(t)$ for $j \neq k$, and
(iv) $b_i(t) \notin \bigcup_i N_{ji}(t)$ for all $t \in \overline{U}_j$.

If $0 \in U_a$ and $1 \in U_b$ we may assume that $c_{ai} = c_{bi} = 0$ for $i = 1, \dots, g$. We may also assume that each $t \in I$ is contained in at most two of the sets in the cover U_j, $j \in J$.

Use the Morse parametrizations of property (9) again to deform f_t, $0 \leq t \leq 1$, rel endpoints and within the images of the Morse parametrizations as follows. For each intersecting pair U_j, U_k of sets in the cover U_j, $j \in J$, introduce g dovetails into the graphic of f_t so that each $N_{ji}(t)$ is the unstable 2-sphere of an index 1 critical point. These dovetails should consist of index 1 and index 0 critical points with the lifespans of the index 0 ones making up small neighborhoods of $\overline{U_j \cap U_k}$. In addition, the stable sets of the index 1 critical points after the deformation should be contained in the stable sets of the index 1 critical points computed before the deformation. We also require, for all $t \in [0, 1]$ and all index 1 critical points p, that $f_t(p) < L_1$ for some fixed number $L_1 < L$. This requirement is easily met because of property (3) in §1. Essentially, the effect this deformation has on the graphic of f_t is to replace all 1-handle slides with dovetails.

Let $p_{ji}(t)$ denote the (perhaps newly created) index 1 critical point with $W_L^*(p_{ji}(t)) = N_{ji}(t)$. Here, since $N_{ji}(t)$ is visible, we use $N_{ji}(t)$ to denote $\varphi^L(N_{ji}(t))$. This will often be done from now on, with context left to determine the intended meaning of the symbol $N_{ji}(t)$.

For $t \in U_j$ let $\Gamma_j(t) = \Sigma(t) \cap (\bigcup_i N_{ji})$. $\Gamma_j(t)$ consists of disjoint circles which we think of as the elements of $\Gamma_j(t)$. Let $\Gamma(t) = \bigcup \Gamma_j(t)$, where the union is taken over those j with $t \in U_j$. Each circle $\gamma(t)$ in $\Gamma(t)$ bounds a unique disk $D_\Sigma(\gamma(t))$ in $\Sigma(t) - \bigcup\{b_i(t)\}$. The inclusion of these disks defines a partial order on $\Gamma(t)$. Let $\gamma(t)$ be minimal in $\Gamma(t)$ with respect to this order with, say, $\gamma(t) \subset N_{jc}(t)$. Cut the manifold $f_t^{-1}(L)$ along the $N_{ji}(t)$, $1 \leq i \leq g$ (but along none of the $N_{ki}(t)$), to obtain a compact manifold Q' whose boundary consists of ∂P^3 together with two copies of each of the $N_{ji}(t)$. Regard $D_\Sigma(\gamma(t))$ as a subset of Q'. From now on let N_{jc} denote the copy of $N_{jc}(t)$ which intersects $D_\Sigma(\gamma(t))$, and let N'_{jc} denote the other copy. Let Q be the manifold obtained by gluing $2g-1$ 3-balls to Q', one along each of the copies of the $N_{ji}(t)$ except for N_{jc}. $Q' \subset Q$ so subsets of Q' are also thought of as subsets of Q.

Q is homeomorphic to the irreducible manifold P^3 with an open 3-ball removed. Therefore, if $P^3 \neq S^3$, there is a unique disk $D_N(\gamma(t)) \subset N_{jc} = \partial Q - \partial P^3$ such that $D_N(\gamma(t)) \cup D_\Sigma(\gamma(t))$ is a 2-sphere (with corners) which bounds a ball $B(\gamma(t)) \subset Q$. (If $P^3 = S^3$ we add that the ball $B(\gamma(t))$ be the one that does not contain N'_{jc}.) The circle of intersection $\gamma(t)$ will be eliminated using one of two possible moves. If $N'_{jc} \subset B(\gamma(t))$, a special handle move, which will be described in §2.2, is used. (So, if $P^3 = S^3$, then special handle moves are not used.) If $N'_{jc} \not\subset B(\gamma(t))$ then a regular move is used.

2.1. **Regular moves without a parameter.** Assume that N'_{jc} is not contained in $B(\gamma(t))$. The description of a regular move depends somewhat on whether $t \in U_j \cap U_k$ or $t \in U_j - \bigcup_{i\neq j} U_i$. For the moment assume $t \in U_j \cap U_k$. Remove from $B(\gamma(t))$ any of the N_{ki} and any of the copies of the N_{ji} that may be contained in its interior, and let $\widehat{B}_L(\gamma(t))$ be the closure of that component of the resulting set which contains $D_N(\gamma(t)) \cup D_\Sigma(\gamma(t))$. $\widehat{B}_L(\gamma(t))$ may be thought of as a subset of $f_t^{-1}(L)$. Let $\widehat{S}(t) = \{wv\colon N_{wv}(t)$ is a boundary component of $\widehat{B}_L(\gamma(t))\}$.

In case $t \in U_j - \bigcup_{i\neq j} U_i$ define $B'_L(\gamma(t)) \subset Q$ to be $B(\gamma(t))$ minus the open 3-balls in $B(\gamma(t))$ bounded by the copies of the $N_{ji}(t)$ in $B(\gamma(t))$. Unlike $\widehat{B}_L(\gamma(t))$, $B'_L(\gamma(t))$ may contain both copies of some of the $N_{ji}(t)$ and so cannot be thought of as a subset of $f_t^{-1}(L)$; however, there is a natural map $B'_L(\gamma(t)) \to f_t^{-1}(L)$ which is injective off these copies of the $N_{ji}(t)$. Let $B_L(\gamma(t)) \subset f_t^{-1}(L)$ denote the image of this map. $B_L(\gamma(t))$ is either a ball with perhaps some open 3-balls removed or the same connected sum with copies of $S^1 \times S^2$. Let $S(t) = \{ji\colon N_{ji}(t) \subset B_L(\gamma(t))\} = \{ji\colon$ a copy of $N_{ji}(t)$ in $Q(t)$ is contained in $B(\gamma(t))\}$. (The wv in $S(t)$ and $\widehat{S}(t)$ correspond to 1-handles which will slide over N_{jc} during the regular move.)

Now we assume $t \in U_j \cap U_k$, except for statements in brackets, which refer to the case where $t \in U_j - \bigcup_{i\neq j} U_i$. Choose numbers l and l_2 such that $L_1 < l < l_2 < L$. (Recall that the value of f_t on each index 1 critical point is less than L_1.) Without changing the gradient-like vector field, use the independent trajectories principle [HW, p. 64] to raise the level of the p_s, $s \in \widehat{S}(t)$ $[S(t)]$, by deforming f_t through Morse functions to a Morse function f_{tu} with $f_{tu}(p_s) = l_2$, $s \in \widehat{S}(t)$ $[S(t)]$. Observe that $\varphi^l_L(\widehat{B}_L(\gamma(t)) - \bigcup_{s\in\widehat{S}(t)} N_s(t))$ $[\varphi^l_L(B_L(\gamma(t)) - \bigcup_{s\in S(t)} N_s(t))] \subset f_{tu}^{-1}(l)$ is a punctured ball with the punctures filled by one point (or perhaps two in case $t \in U_j - \bigcup_{i\neq j} U_i$) from each of the 0-spheres $W_l(p_s)$, $s \in \widehat{S}(t)$ $[S(t)]$. Let $\widehat{B}_l(\gamma(t))$ $[B_l(\gamma(t))] \subset f_{tu}^{-1}(l)$ be the corresponding unpunctured ball. Since $jc \notin \widehat{S}(t)$ $[S(t)]$, the boundary of $\widehat{B}_l(\gamma(t))$ $[B_l(\gamma(t))]$ is $\varphi^l_L(D_N(\gamma(t))) \cup \varphi^l_L(D_\Sigma(\gamma(t)))$. Pick a disjunction isotopy of $f_{tu}^{-1}(l)$, with support on a small neighborhood of $\widehat{B}_l(\gamma(t))$ $[B_l(\gamma(t))]$ which pushes $\varphi^l_L(D_\Sigma(\gamma(t)))$ through $\widehat{B}_l(\gamma(t))$ $[B_l(\gamma(t))]$ and just across $\varphi^l_L(D_N(\gamma(t)))$. In the level set $f_{tu}^{-1}(l)$, realize this isotopy (in the sense of [M1, Lemma 4.7]) by deforming, in levels slightly above l, the gradient-like vector field η_t to another denoted by η_{tu}. Redefine $N_{jc}(t)$ on $f_t^{-1}(L) = f_{tu}^{-1}(L)$ by setting

$N_{jc}(t) = W_L^*(p_{jc})$ computed using η_{tu}. The other $N_{wv}(t) = W_L^*(p_{wv})$ are not changed by η_{tu}. To complete the regular move use the independent trajectory principle again to deform f_{tu} back to a Morse function which assigns values less than L_1 to all index 1 critical points.

The result of this regular move is to remove $\gamma(t)$ from $\Gamma(t)$ as well as any other circles in $\Gamma(t)$ that are contained in $D_N(\gamma(t))$. Otherwise $\Gamma(t)$ is unchanged. To help see how the move affects the handlebody structure of $P^3 \times I$ when $t \in U_j \cap U_k$, we introduce the graph $G(t)$ which has one vertex corresponding to each index 0 critical point and one corresponding to P, and which has one edge for each $N_{wi}(t)$. A particular edge $N_{wi}(t)$ is to attach to the two vertices which correspond to the two endpoints of $W(p_{wi}(t))$. Let $G_j(t)$ (resp. $G_k(t)$) be the subgraph of $G(t)$ which has the same vertices as $G(t)$ and which has only the $N_{ji}(t)$ (resp. only the $N_{ki}(t)$) as edges.

We would like to use regular moves (and special handle moves) repeatedly to empty $\Gamma(t)$. As an inductive hypothesis we assume, in the case where $t \in U_j \cap U_k$, that both the subgraphs $G_j(t)$ and $G_k(t)$ are trees. With this assumption each edge of $G_j(t)$ (and similarly for $G_k(t)$) can be oriented so that it "points towards" the vertex P. Call the two endpoints of each edge in $G_j(t)$ the tip and tail of the edge, depending on the direction which the edge points. More precisely, the (shortest) edgepath in the tree $G_j(t)$ joining the tail of any given edge to the vertex P should contain the given edge. For $w = j, k$ define bijections $C_w : \{N_{wi}(t)\} \to \{\text{index 0 critical points}\}$ such that $C_w(N_{wi}(t))$ is the index 0 critical point which corresponds to the tail of the edge $N_{wi}(t)$. Note that, in intermediate level surfaces, $W^*(C_w(N_{wi}(t))) \cap W(p_{wi}(t))$ consists of exactly one point. Furthermore the pairs of critical points $\{p_{ji}(t), C_j(N_{ji}(t))\}$ (or the pairs $\{p_{ki}(t), C_k(N_{ki}(t))\}$) can be cancelled in any order without changing the level set $f_t^{-1}(L)$ or the unstable sets of the remaining index 1 critical points. Thus, at each step of the induction, the manifold Q is homeomorphic to P^3 minus an open 3-ball. To verify this last statement (which also follows directly from the induction hypothesis), it is not necessary to cancel the critical points in the pairs given by the bijection C_j (or the bijection C_k), but then there is some restriction on the order in which the pairs of critical points can be cancelled. Such restrictions are unwanted in §3.

We now check that the subgraph $G_j(t)$ is still a tree after a regular move. Of course we assume $t \in U_j \cap U_k$ since $G_j(t)$ is not defined if $t \in U_j - \bigcup_{i \neq j} U_i$. (We continue to use the notation used in the above description of a regular move.) Observe that all trajectories through $\widehat{B}_l(\gamma(t)) - \varphi_L^l(D_N(\gamma(t)))$ if followed downward lead to the same index 0 critical point x_0 or they all lead to $P^3 \times \{0\}$. We say $\widehat{B}_l(\gamma(t))$ flows to x_0 or to $P^3 \times \{0\}$ accordingly. In terms of the graph $G(t)$ this means that one endpoint of the edge N_{jc} is either the vertex x_0 or the vertex P, and that all the edges N_{wv}, $wv \in \widehat{S}(t)$, attach to this vertex. A regular move causes these edges to slide across the edge N_{jc} and attach to the other endpoint of N_{jc}. This clearly does not disconnect the subgraph $G_j(t)$ since N_{jc} is an edge of $G_j(t)$. The Euler characteristic of $G_j(t)$ is 1. (It has $g+1$ vertices and g edges.) It follows that $G_j(t)$ is a tree after the move.

A regular move will change the bijection $C_j : \{N_{ji}\} \to \{\text{index 0 critical points}\}$ only if the move causes the tail (x_0 say) of an edge N_{jd} of the subgraph $G_j(t)$ to slide across N_{jc}. If this is the case then, before the move, the endpoint of N_{jc} other than x_0 is $C_j(N_{jc})$. Let $x_1 = C_j(N_{jc})$. Then, after the regular move, $C_j(N_{jc}(t)) = x_0$ and $C_j(N_{jd}(t)) = x_1$. Otherwise the bijection C_j stays the same.

Next the tree $G_k(t)$ is considered. Each vertex x of $G_k(t)$ determines, in $G_k(t)$, a (shortest) edgepath from x to the vertex P. The two such edgepaths determined by the two endpoints of N_{jc} together with the edge N_{jc} itself form what we call the edgeloop determined by N_{jc}. (This edgeloop essentially identifies the 1-handle which is determined by p_{jc} if all the p_{ki} are cancelled with the index 0 critical points.)

Lemma 2.1. *If the edge $N_{kl}(t)$ is in the edgeloop determined by $N_{jc}(t)$, then $kl \notin \widehat{S}(t)$.*

Proof. Suppose $kl \in \widehat{S}(t)$. Cut the manifold Q along $N_{kl}(t)$ and glue 3-balls to the two boundary components created. Since the subgraph $G_k(t)$ is a tree, the resulting manifold has two components. Assuming the edge $N_{kl}(t)$ is in the edgeloop determined by $N_{jc}(t)$, one component, say K, is a ball with boundary $\partial Q - \partial P^3$. K contains $D_\Sigma(\gamma(t))$, so $(N_{jc}(t) - D_N(\gamma(t))) \cup D_\Sigma(\gamma(t))$ bounds a ball in K. Since $kl \in \widehat{S}(t)$, there is a path in $\widehat{B}_L(\gamma(t)) \subset K$ joining $N_{kl}(t)$ to $D_N(\gamma(t))$, so this ball does not contain $N_{kl}(t)$. Thus this ball, bounded by $(N_{jc}(t) - D_N(\gamma(t))) \cup D_\Sigma(\gamma(t))$, is contained in Q. (If $P^3 = S^3$ then this ball does not contain the copy of N_{jc} in Q different from $\partial Q - \partial P^3$.) This contradicts the choice of $D_N(\gamma(t))$ and proves the lemma.

It follows easily from this lemma that $G_k(t)$ is connected after a regular move, hence a tree. It also follows that a regular move does not cause the tail of any edge of $G_k(t)$ to slide over N_{jc}. So the bijection $C_k\colon \{N_{ki}\} \to \{$index 0 critical points$\}$ is not changed by a regular move.

2.2. **Special handle moves without a parameter.** Recall that $N'_{jc}(t)$ denotes the copy of $N_{jc}(t)$ in Q different from $\partial Q - \partial P^3$. We now address the case where the ball $B(\gamma(t))$ in Q contains $N'_{jc}(t)$. Remove from $B(\gamma(t))$ the open ball in $B(\gamma(t))$ bounded by $N'_{jc}(t)$ and let $A(\gamma(t))$ denote the resulting annulus. Assume that $t \in U_j \cap U_k$. (The case $t \in U_j - \bigcup_{i \neq j} U_i$ is treated later.) $D_\Sigma(\gamma(t))$ flows to an index 0 critical point or to $P^3 \times \{0\}$. Let v be the vertex in the graph $G(t)$ corresponding to whichever of these it is. Let $N_{ki(1)}(t), \dots, N_{ki(n)}(t)$ be the edgepath in the tree $G_k(t)$ from v to the other endpoint of the edge $N_{jc}(t)$. Then the 2-spheres $N_{ki(1)}(t), \dots, N_{ki(n)}(t)$ are contained in $A(\gamma(t))$ and are parallel in $A(\gamma(t))$ to $N'_{jc}(t)$. Let $A_0, \dots, A_n$ be the annuli in $A(\gamma(t))$ with $\partial A_0 = (D_N(\gamma(t)) \cup D_\Sigma(\gamma(t))) \cup N_{ki(1)}$, $\partial A_z = N_{ki(z)} \cup N_{ki(z+1)}$ for $1 \le z \le n-1$, and $\partial A_n = N_{ki(n)} \cup N'_{jc}$. For $z = 0, 1, \dots, n$, remove from A_z any of the N_{ki} and any of the copies of the N_{ji} contained in its interior and let $\widehat{A}'_z(\gamma(t))$ denote the closure of that component of the resulting set which contains ∂A_z. Think of the $\widehat{A}'_z(\gamma(t))$ as subsets of $f_t^{-1}(L)$, and let $\widehat{S}_z(t) = \{wv\colon N_{wv}(t)$ is a boundary component of $\widehat{A}'_z(\gamma(t))$ different from both components of $\partial A_z \subset f_t^{-1}(L)\}$. Let $\widehat{S}(t) = \bigcup \widehat{S}_z(t)$, where the union is taken over $z = 0, 1, \dots, n$.

We now describe the changes in the Morse function f_t and gradient-like vector field η_t which make up a handle move. Just as in the description of a regular move, begin by deforming f_t to a Morse function f_{tu} with $f_{tu}(p_s(t)) =$

l_2, $s \in \widehat{S}(t)$. (Recall: level of index 1 critical points $< L_1 < l < l_2 < L$.) Then for each $z = 0, 1, \dots, n$, $\varphi_L^l(\widehat{A}'_z(\gamma(t)) - \bigcup_{s\in\widehat{S}(t)} N_s(t)) \subset f_{tu}^{-1}(l)$ is a punctured annulus with each puncture point corresponding to an element of $\widehat{S}_z(t)$. Denote the corresponding unpunctured annuli in $f_{tu}^{-1}(l)$ by $\widehat{A}_z(\gamma(t))$. Let $\psi_t: S^2 \times [-1, 1] \to f_{tu}^{-1}(l)$ be an embedding satisfying

(1) $\psi_t(S^2 \times \{0\}) = \varphi_L^l(N_{jc}(t)) = W_l^*(p_{jc}(t))$,
(2) $\psi_t(S^2 \times [0, 1]) \subset \bigcup \widehat{A}_z(\gamma(t))$,
(3) $\psi_t(S^2 \times \{1\})$ is close to $\varphi_L^l(D_N(\gamma(t)) \cup D_\Sigma(\gamma(t)))$, and
(4) for each $N_{ki(z)}(t)$, $\psi_t^{-1}(N_{ki(z)}(t)) = S^2 \times \{y_z\}$ for some $y_z \in (0, 1)$.

Choose $\varepsilon > 0$ so that for $h \in [1-\varepsilon, 1]$, $\psi_t(S^2 \times \{h\})$ is visible and transverse to $\Sigma(t)$. Let α_t, $0 \le t \le 1$, be an isotopy of $S^2 \times [-1, 1]$ which sends S^2 slices to S^2 slices at all times and for which $\alpha_1(S^2 \times [1-\varepsilon, 1]) = S^2 \times [0, 1]$. Identify the image of ψ_t with $S^2 \times [-1, 1]$, and, in $f_{tu}^{-1}(l)$, realize the isotopy α_t by deforming the gradient-like vector field in levels just above l. The effect this has in $f_{tu}^{-1}(l)$ is that $\varphi_L^l(\Sigma)$ and $W_l(p_s)$, $s \in \widehat{S}$, are moved by α_t, but N_{jc} and the $N_{ki(z)}$ are not. Thus, $\gamma(t)$ and any curves in $\Sigma(t) \cap (N_{jc}(t) - D_N(\gamma(t)))$ are eliminated from $\Gamma_j(t)$; however, $\Gamma_k(t)$ is changed in a chaotic way. This is remedied after the following preliminary paragraph.

If $\widehat{A}_0$ flows to $P^3 \times \{0\}$ (i.e., if all trajectories through $\widehat{A}_0 - \partial A_0$ meet $P^3 \times \{0\}$), then, in order that the bijection $C_k: \{N_{ki}\} \to$ {index 0 critical points} of the inductive hypothesis can exist, $\widehat{A}_1, \dots, \widehat{A}_n$ must flow to distinct index 0 critical points, which we label $x_1, \dots, x_n$ respectively. If $\widehat{A}_0$ flows to an index 0 critical point x_0 then we claim that $\widehat{A}_1, \dots, \widehat{A}_n$ flow to distinct index 0 critical points, again labeled $x_1, \dots, x_n$. To see this, first suppose that some $\widehat{A}_m$, where $1 \le m \le n$, flows to $P^3 \times \{0\}$. Then cut the manifold Q along $N_{ki(1)}$ and argue just as in the proof of Lemma 2.1 to obtain a contradiction.

For $z = 1, \dots, n$ cancel $p_{ki(z)}$ with x_z with a deformation which changes the gradient-like vector field only in a small neighborhood of the trajectory leading from $p_{ki(z)}$ to x_z. Then introduce near p_{jc}, and on the side of p_{jc} opposite the $p_{ki(z)}$, pairs of cancelling critical points of index 0 and 1, also labelled x_z and $p_{ki(z)}$ respectively. Do this so that the order of the new $p_{ki(z)}$ is the same as that of the old, and so that $W(\text{new}\, p_{ki(z)})$ is contained in the $W(p_{jc})$ computed before the introduction of any of the cancelling pairs of critical points. By changing ψ_t we may assume that

$$W_l^*(\text{new}\, p_{ki(z)}) = \alpha_1(W_l^*(\text{old}\, p_{ki(z)})).$$

Let $N_{ki(z)}$ now denote $W_L^*(\text{new}\, p_{ki(z)})$. Then $\Gamma_k(t) = \Sigma(t) \cap (\bigcup_i N_{ki})$ is unchanged. Finish the special handle move by lowering the index 1 critical points that were raised at the start of the move to heights less than L_1.

Now we discuss the effect a special handle move has on the trees $G_j(t)$ and $G_k(t)$. Let A denote the set of edges $\{N_{wv}: wv \notin \widehat{S}$, and N_{wv} attaches (before the move) to the endpoint of the edge N_{jc} different from $x_n\}$. By letting, for $i = 1, \dots, n-1$, the vertex x_i of the graph $G(t)$ before the move correspond to the vertex x_{i+1} of $G(t)$ after the move, and by letting x_0 (or the vertex P) correspond to x_1, and x_n to x_0 (or P), one can see that the graph $G(t)$ after the move is the same as $G(t)$ before it except that the edges in A have slid across the edge N_{jc}. (The direction that these slide is opposite that which the handles in the above description of a special handle move slide.) Thus in the subgraph $G_j(t)$ (resp. $G_k(t)$) the edges N_{ji} (resp. N_{ki}) in A slide across N_{jc}. This clearly does not disconnect $G_j(t)$, and does not disconnect $G_k(t)$ since the edgepath in $G_k(t)$ joining the two endpoints of N_{jc} stays fixed during the move.

Now we show how the bijections C_j and C_k are affected by a special handle move. If $\widehat{A}_0$ flows to $P \times \{0\}$ then clearly, before the move, $C_k(N_{ki(z)}) = x_z$ for $z = 1, \dots, n$. Next we prove that this is also true if $\widehat{A}_0$ flows to an index 0 critical point, say x_0. Let N_{kd} be the edge (other than N_{jc}) incident with the vertex x_0 in the edgeloop determined by N_{jc}. Suppose $kd \in \widehat{S}$ or $N_{kd} = N_{ki(1)}$. Start at x_0 and move along the edgeloop away from N_{jc}. Let N_{kf} be the first edge encountered which is different from any of the $N_{ki(z)}$. Cut the manifold Q along N_{kf} and argue just as in the proof of Lemma 2.1 to obtain a contradiction. Since $C_k(N_{kd}) = x_0$ and since $N_{kd} \neq N_{ki(1)}$, $C_k(N_{ki(1)})$ must equal x_1. This forces each $C_k(N_{ki(z)})$ to be x_z. It follows from this and the given choice of notation (and in case $\widehat{A}_0$ flows to x_0 from the fact that $kd \notin \widehat{S}$) that a handle move does not change the bijection $C_k: \{N_{ki}\} \to \{$index 0 critical points$\}$.

The only changes in the bijection $C_j: \{N_{ji}\} \leftrightarrow \{$index 0 critical points$\}$ caused by a handle move involve $x_1, \dots, x_n$ and, if $\widehat{A}_0$ flows to x_0, x_0 as well. For $z = 1, \dots, n-1$, if $C_j(N_{ji}) = x_z$ before the move then $C_j(N_{ji}) = x_{z+1}$ after it. In case $\widehat{A}_0$ flows to $P^3 \times \{0\}$, so that $C_j(N_{jc}) = x_1$, this determines C_j. So assume $\widehat{A}_0$ flows to x_0. For a number of cases the corresponding bijections C_j are specified by saying what $C_j^{-1}(x_0)$ or $C_j^{-1}(x_1)$ is. If $C_j(N_{jc}) = x_0$ before the move, then $C_j(N_{jc}) = x_1$ afterwards. For another case assume $C_j(N_{jc}) = x_n$ before the move and say $C_j(N_{jd}) = x_0$ before the move. If $jd \in \widehat{S}$, then $C_j(N_{jc}) = x_0$ afterwards, and if $jd \notin \widehat{S}$, then $C_j(N_{jc}) = x_1$ afterwards.

We next give the description of a special handle move in the case where $t \in U_j - \bigcup_{i \neq j} U_i$. It is similar to, but simpler than, the one above since large portions of the above description involve the N_{ki} and so are not relevant here. For $t \in U_j$ let $A'_L(\gamma(t))$ denote $A(\gamma(t))$ minus the open 3-balls in $A(\gamma(t))$ bounded by copies of the $N_{ji}(t)$ in $A(\gamma(t))$, and let $A_L(\gamma(t)) \subset f_t^{-1}(L)$ be the image of the natural map from $A'_L(\gamma(t))$ to $f_t^{-1}(L)$. Begin the special handle move by raising to level l_2 the critical points p_s, $s \in S(t)$, where $S(t) = \{ji: ji \neq jc$ and a copy of $N_{ji}(t)$ in $Q(t)$ is contained in $A(\gamma(t))\}$. Let $A_l(\gamma(t)) \subset f_t^{-1}(l)$ be the annulus (with punctures filled) corresponding to $A_L(\gamma(t))$ via the trajectories. Let $\psi_t: S^2 \times [-1, 1] \to f_{tu}^{-1}(l)$ be an embedding satisfying properties (1) and (3) above and also satisfying $\psi_t(S^2 \times [0, 1]) \subset A_l(\gamma(t))$. Now, as in the above case where $t \in U_j \cap U_k$, deform the gradient-like vector field in levels just above l so that $\psi_t(S^2 \times \{1 - \varepsilon\})$ becomes N_{jc}. Finish the move by lowering the critical points raised at the start of the move.

3. Disjunction theory with a parameter

Let (η_t, f_t, μ_t) be the 1-parameter family of §2. In the parametrized theory we essentially view this family as constant on each of the sets U_i in the cover of the parameter space. Thus we can perform the same unparametrized move for all t in any one of the U_i. Just as in [H1] these moves must be tapered off near the boundary points of the U_i to ensure that they give deformations which are continuous as t leaves each U_i. With some effort the resulting tapered deformations can be made to fit together on the intersections of the U_i provided that the unparametrized moves are performed in an appropriate order. An idea of Hatcher [H1] gives such an order in the form of a function λ which we now discuss.

Let U_j, $j \in J$, be the finite open cover of §2 which satisfies properties (i) through (iv) of that section. After perhaps refining this cover, there is, according to [HL] and [H1], a function $\lambda: \Gamma \to (0, 1)$ satisfying

(v) for each U_j and each circle $\gamma(t) \in \Gamma_j(t)$, the function $U_j \to (0, 1)$ defined by $t \to \lambda(\gamma(t))$ is constant, and

(vi) $\lambda(\gamma_1(t)) < \lambda(\gamma_2(t))$ whenever $D_\Sigma(\gamma_1(t)) \subset D_\Sigma(\gamma_2(t))$, and $\lambda(\gamma_1(t)) \neq \lambda(\gamma_2(t))$ whenever $\gamma_1(t) \neq \gamma_2(t)$.

Let $u(1) < u(2) < \cdots < u(p)$ denote the values of λ. To facilitate the tapering process alluded to above, set $U_{j0} = U_j$, and choose open intervals $U_{j1}, \ldots, U_{jp}$ such that $U_{ji} \supset U_{ji+1}$ and such that the U_{jp}, $j \in J$, still cover I.

We are now ready to define a deformation $(\eta_{tu}, f_{tu}, \mu_{tu})$, $0 \leq u \leq 1$, of the 1-parameter family $(\eta_{t0}, f_{t0}, \mu_{t0}) = (\eta_t, f_t, \mu_t)$, $-1 \leq t \leq 2$, provided by Proposition 1.5 and used in §2, which eliminates all circles from Γ. Usually though, to keep notation down, no explicit mention of η_{tu} and μ_{tu} is made. (This should cause no trouble. See §1.4.) Assume that f_{tu} has been defined for $0 \leq u \leq u(m)$ so that the following inductive assumptions hold:

(a) Near each intersection $U_{jm} \cap U_{km}$ the part of the graphic involving the index 1 and index 0 critical points consists of g dovetails, where if p_{ja}, p_{kb}, and x are the critical points making up any one of these, then the lifespan of the index 0 critical point x makes up a small neighborhood of $\overline{U_{jm} \cap U_{km}}$. Moreover if l satisfies $f_{tu(m)}(x) < l < f_{tu(m)}(p_\alpha)$ for $\alpha = ja, kb$, then throughout the life of x, both $W_l^*(x) \cap W_l(p_{ja})$ and $W_l^*(x) \cap W_l(p_{kb})$ consist of one point.

(b) Each $N_{ji}(t)$ is visible for all $t \in U_{jm}$. More precisely, for $t \in U_{jm}$, the unstable set $W_L^*(p_{ji}(t))$ (computed using $(\eta_{tu(m)}, f_{tu(m)}, \mu_{tu(m)})$) of each index 1 critical point $p_{ji}(t)$ does not meet the stable set of any other index 1 critical point.

(c) $\Sigma(t)$ is transverse to $\bigcup_i N_{ji}(t)$ for $t \in U_{jm}$. Moreover, at least the circles $\gamma(t) \in \Gamma$ with $\lambda(\gamma(t)) \leq u(m-1)$ have been eliminated from Γ and no new circles of intersection have been introduced.

Let $J_m = \{j \in J :$ there is a $\gamma(t) \in \Gamma_j(t)$ with $\lambda(\gamma(t)) = u(m)\}$. The U_j, $j \in J_m$, are disjoint by property (vi) of λ and the fact that $N_{ji}(t) \neq N_{kl}(t)$ for $ji \neq kl$. For t outside neighborhoods of the sets $\overline{U}_j$, $j \in J_m$, we will define the deformation f_{tu}, $u(m) \leq u \leq u(m+1)$, to be constant. So assume $j \in J_m$ and let $\gamma(t)$ be the unique circle in $\Gamma(t)$ with $\lambda(\gamma(t)) = u(m)$. Assume that $\gamma(t)$ is in $\Sigma(t) \cap N_{jc}(t)$ so that the subscript jc plays the same role here as it did in §2.

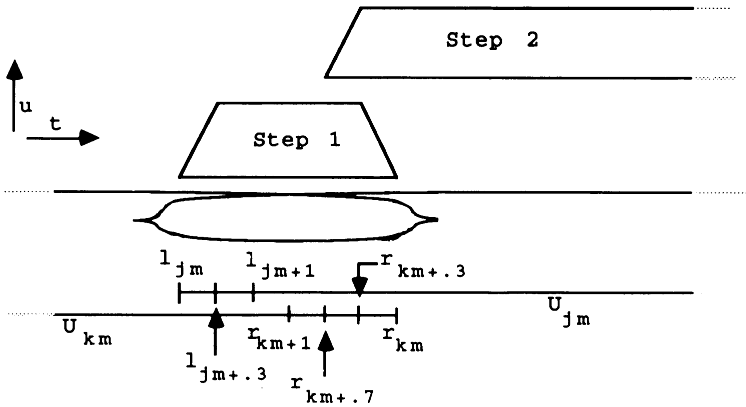

FIGURE 1

Let l_{wm} and r_{wm} denote the left- and right-hand endpoints of the sets $\overline{U_{wm}}$. Choose numbers $l_{wm+0.3}$, $l_{wm+0.7}$, and $r_{wm+0.3}$, $r_{wm+0.7}$ so that $l_{wm} < l_{wm+0.3} < l_{wm+0.7} < l_{wm+1}$ and $r_{wm} > r_{wm+0.3} > r_{wm+0.7} > r_{wm+1}$. Assume $l_{jm} \in U_{km}$ and $r_{jm} \in U_{rm}$. Then $U_{jm} \cap U_{km} = (l_{jm}, r_{km})$ and $U_{jm} \cap U_{rm} = (l_{rm}, r_{jm})$.

The descriptions of both regular and special handle moves with a parameter are divided into four steps. Figure 1 shows the portions of the parameter space where Steps 1 and 2 are carried out. Slanted lines indicate where deformations are tapered off. At these spots there are difficulties which are dealt with in Steps 3 and 4. The figure also shows part of the graphic of $f_{tu(m)}$.

Throughout the remainder of this section notation introduced in §2 is freely used.

3.1. **Regular moves with a parameter.** We first assume that a regular move is required to remove $\gamma(t)$ from $\Gamma_j(t)$ for one, and hence all, t in U_{jm}. The case in which a special handle move is required is dealt within §3.2.

For $t \in U_{jm}$ let $N''_{jc}(t) \subset f^{-1}_{tu(m)}(L)$ be a parallel copy of $N_{jc}(t)$ lying just to the side of $N_{jc}(t)$ opposite from $D_\Sigma(\gamma(t))$. Let $\gamma''(t) \subset N''_{jc}(t) \cap \Sigma(t)$, $D_\Sigma(\gamma''(t))$, and $D_N(\gamma''(t)) \subset N''_{jc}(t)$ be the sets corresponding to $\gamma(t)$, $D_\Sigma(\gamma(t))$, and $D_N(\gamma(t))$ respectively, but defined using $N''_{jc}(t)$ in place of $N_{jc}(t)$. Let $B_L(\gamma''(t)) \subset f^{-1}_{tu(m)}(L)$ be the set bounded by $D_\Sigma(\gamma''(t)) \cup D_N(\gamma''(t))$ corresponding (at least before doing Step 1 below) to $B_L(\gamma(t))$.

We do not bother to keep careful track of the parameter u in the following definition of the deformation f_{tu}, $u(m) \le u \le u(m+1)$ (or $u(m) = u(p) \le u \le 1$). We will speak of the level l to indicate $f^{-1}_{tu}(l)$ for appropriate u. Similarly, a point $x \in P^3 \times I$ has height l means $f_{tu}(x) = l$.

Step 1. Use the independent trajectories principle [HW, p. 64] to construct a deformation with support on a small neighborhood of $[l_{jm}, r_{km}]$ which puts, for $t \in [l_{jm}, r_{km}]$ the index 1 critical points p_s, $s \in \widehat{S}(t)$, at height l_2. For one $t \in (l_{jm}, r_{km})$ pick a diffeomorphism θ_t from a neighborhood of the ball $\widehat{B}_l(\gamma(t))$ containing $\varphi^l_L(D_N(\gamma''(t)) \cup D_\Sigma(\gamma''(t)))$ onto a standard model. Let $\widehat{B}$, D_Σ, D_N, and D''_N be the sets in this model corresponding under θ_t

to $\widehat{B}_l(\gamma(t))$, $\varphi_L^l(D_\Sigma(\gamma(t)))$, $\varphi_L^l(D_N(\gamma(t)))$, and $\varphi_L^l(D_N(\gamma''(t)))$ respectively. Use isotopy extension to obtain similar diffeomorphisms θ_t for all $t \in [l_{jm}, r_{km}]$. Just as in §2.1 let D_u be a disjunction isotopy of the standard model which pushes D_Σ through $\widehat{B}$ and just across D_N. In addition, D_u should have support contained in the complement of a neighborhood of D_N''. Use θ_t and D_u to perform a regular move for $t \in [l_{jm+0.3}, r_{km+0.3}]$, and to perform less and less of the move as t increases from $r_{km+0.3}$ to r_{km}, until none at all is performed when $t = r_{km}$. Taper off the move, in a similar fashion, as t decreases from $l_{jm+0.3}$ to l_{jm}. Finish by lowering the critical points which were raised at the start of this step to a height less than L_1. (If $\widehat{S}(t) \neq \varnothing$ then $N_{jc}(t)$ will not be visible for some values of t in $(l_{jm}, l_{jm+0.3})$ and in $(r_{km+0.3}, r_{km})$, and so not all index 1 critical points can be made to have height that of p_{jc}.)

Also perform a regular move on $U_{jm} \cap U_{rm}$ similar to the one just described. In the rest of this section more deformations are defined only on $U_{km} \cap U_{jm}$ with the understanding that analogous deformations are to be carried out on $U_{jm} \cap U_{rm}$.

Step 2. As in Step 1, for $t \in [r_{km+1}, l_{rm+1}]$ put the index 1 critical points $p_{ji}(t)$, $ji \in \{ji\colon N_{ji}(t) \subset B_L(\gamma''(t))\}$ $(= S(t)$ before doing Step 1), at height l_2. Then for $t \in [r_{km+1}, l_{rm+1}]$ there is (after filling punctures) a ball $B_l(\gamma''(t))$ in level l bounded by the 2-sphere $\varphi_L^l(D_\Sigma(\gamma''(t)) \cup D_N(\gamma''(t)))$ which corresponds, via the trajectories, to the set $B_L(\gamma''(t))$. Just as in Step 1 identify small neighborhoods of $B_l(\gamma''(t))$, $t \in [r_{km+1}, l_{rm+1}]$, with a standard model. Then perform a regular move for $t \in [r_{km+0.3}, l_{rm+0.3}]$, tapered on $[r_{km+0.7}, r_{km+0.3}]$, and $[l_{rm+0.3}, l_{rm+0.7}]$, by using a disjunction isotopy of a standard model which pushes $D_\Sigma(\gamma'')$ through $B_l(\gamma'')$ and just across $D_N(\gamma'')$. Finish by lowering the critical points which were raised at the start of this step to heights less than L_1.

Inductive assumption (c) holds after performing Steps 1 and 2 for all $j \in J_n$ and after defining the deformation to be constant for values of t and u not covered by Steps 1 and 2.

Step 3. As suggested at the end of Step 1, inductive assumption (b) may not hold. Specifically, for each $ki \in S(t)$ at the start of Step 1 there is now a value of $t \in (r_{km+0.3}, r_{km})$ for which $W^*(p_{jc}(t)) \cap W(p_{ki}(t)) \neq \varnothing$. Inductive assumption (a) may also not hold since in Step 2 some of the stable sets $W(p_{ji}(t))$ may slide across some of the unstable sets $W^*(p_{ki}(t))$. Thus, for some i, $W(p_{ji}(t)) \cap W^*(C_j(N_{ji}(t)))$ may be empty for some $t \in (r_{km+0.7}, r_{km+0.3})$. To fix these problems hasten the deaths of all the p_{ki} so that they occur between r_{km+1} and $r_{km+0.7}$. These deformations should alter the gradient-like vector field only in small neighborhoods of the unique trajectories from $p_{ki}(t)$ to $C_k(N_{ki}(t))$. (Such trajectories exist since, as shown in §2.1, the bijection C_k does not change during the regular move.) Then since the N_{ji}, $i \neq c$, were (and are) visible, the sets Γ_{ji}, $i \neq c$, are unchanged by these last deformations. Γ_{jc} is also unchanged since, in Step 2, $\varphi_L^l(\Sigma(t))$ is swept out of $B_l(\gamma''(t))$. Thus inductive assumptions (b) and (c) now hold.

Step 4. Step 1 causes essentially the same regular move to be carried out for all $t \in [l_{jm+0.3}, r_{km+0.3}]$. Thus there are bijections C_j and C_k associated with $U_{km+0.3} \cap U_{jm+0.3}$ like those in §2. For $t \in U_{km+0.3} \cap U_{jm+0.3}$ recall that if $\widehat{B}_l(\gamma(t))$ flows to the index 0 critical point $x_0(t)$ and there is a $ji \in \widehat{S}(t)$

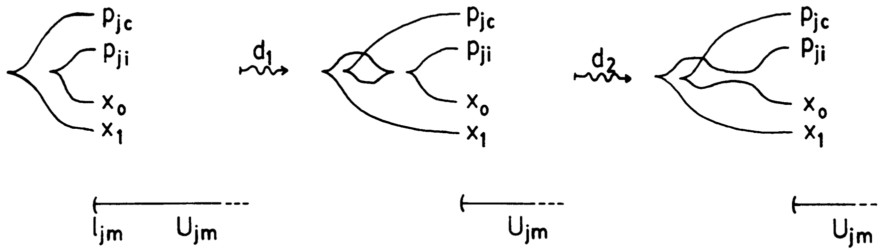

FIGURE 2. Exchanging destinies

with $C_j(N_{ji}) = x_0$ then the regular move causes C_j to change. If this is the case then induction assumption (a) does not now hold. To remedy this perform a deformation $d_2 \circ d_1$ which is constant for $t \in U_{jm}$ and which has the effect on the graphic illustrated in Figure 2. d_1 can be carried out in small neighborhoods of the $p_{jc}(t)$ and d_2 exists by the uniqueness of birth lemma [C] or [HW]. The birth critical point created by d_1 should lie on $W(p_{jc})$ between p_{jc} and x_1. After $d_2 \circ d_1$ we may assume that, in an intermediate level surface, $W(p_{jc}) \cap W^*(x_0)$ consists of one point throughout the life of x_0. However, $W(p_{ji}) \cap W^*(x_1) = \varnothing$ during much of the life of x_1. Fix this with a deformation similar to those in Step 3 which delays the brith of p_{jc} until it occurs just after $l_{jm+0.3}$. (In fact, whether or not C_j changes, such a deformation together with those in Step 3 makes N_{jc} visible throughout the life of p_{jc}. Because of this and Step 3 again, inductive assumption (b) could be strengthened to: each N_{ji} is visible throughout the life of p_{ji}.) Inductive assumption (a) now holds as do (b) and (c).

In case $1 \in U_j$ modify Step 2 so that there is tapering only near $r_{km+0.3}$. The result is that f_{1u}, $u(m) \leq u \leq u(m+1)$, is a regular move of the sort described in §2.1 for $t \in U_j - \bigcup_{i \neq j} U_i$.

3.2. **Special handle moves with a parameter.** Assume now that for $t \in U_{jm}$ a special handle move is required to remove $\gamma(t)$ from $\Gamma(t)$. First some notation is introduced which will be of help in fitting together the moves in Steps 1 and 2 below. For $t \in [l_{jm}, r_{jm}]$ let $Q''(t)$ be the manifold obtained by cutting the level L (at time t) along all the $N_{ji}(t)$ except $N_{jc}(t)$ and then glueing 3-balls to the $2(g-1)$ boundary components created by the cuts. Choose four 1-parameter families $N''_{jc}(t)$, $S_{\pm 1}(t)$, and $S_{1-\varepsilon}(t)$, $l_{jm} \leq t \leq r_{jm}$, of 2-spheres in $Q''(t)$ such that for each t the following assertions are satisfied.

(a) $S_1(t)$ is close to and disjoint from $D_N(t) \cup D_\Sigma(t)$, and $S_1(t)$ is contained in $A(\gamma(t))$.

(b) There is an embedding $f\colon S^2 \times [-1, 1] \to Q''(t)$ satisfying

(i) $f(S^2 \times [0, 1]) \subset A(\gamma(t))$,

(ii) $f(S^2 \times \{\pm 1\}) = S_{\pm 1}(t)$, $f(S^2 \times \{0\}) = N_{jc}(t)$,
$f(S^2 \times \{-1/2\}) = N''_{jc}(t)$, and $f(S^2 \times \{0.9\}) = S_{1-\varepsilon}(t)$, and

(iii) for $x \in [0.9, 1]$, $f(S^2 \times \{x\})$ is transverse $\Sigma(t)$.

View these four families as contained in the levels L. The following steps correspond roughly to like numbered steps in §2.2.

Step 1a. Raise the critical points $p_s(t)$, $s \in \widehat{S}(t)$, $t \in [l_{jm}, r_{km}]$, to height l_2.

Use isotopy extension to choose a 1-parameter family of embeddings $\psi_t: S^2 \times [-1, 1] \to \text{level}_t\, l$, $l_{jm} \le t \le r_{km}$, such that $\psi_t(S^2 \times \{-1/2\}) = \varphi_L^l(N''_{jc}(t))$, $\psi_t(S^2 \times \{\pm 1\}) = \varphi_L^l(S_{\pm 1}(t))$, $\psi_t(S^2 \times \{0.9\}) = \varphi_L^l(S_{1-\varepsilon}(t))$, and $\psi_t(S^2 \times \{0\}) = \varphi_L^l(N_{jc}(t)) = W_l^*(p_{jc}(t))$. Thus ψ_t satisfies properties (1), (2), and (3) of §2.2.

If $C_j(N_{jc}) \ne x_n$ then adjust the graphic as in Step 4 of §3.1 so that p_{jc} cancels with x_n and $C_j^{-1}(x_n)$ with $x_0 = C_j(N_{jc})$. Now, whether or not $C_j(N_{jc}) = x_n$, the graphic contains a dovetail involving x_n, p_{jc}, and $p_{ki(n)}$. Also each of the sets $W^*(x_n) \cap W(p_{jc})$ and $W^*(x_n) \cap W(p_{ki(n)})$ consist of one point throughout the life of x_n. Eliminate this dovetail with a deformation which has support near the trajectories joining $p_{jc}(t)$ to $x_n(t)$ and $p_{ki(n)}(t)$ to $x_n(t)$ and which, furthermore, does not change either of $W^*(p_{jc}(t))$ or $W^*(p_{ki(n)}(t))$ for $t \notin (l_{jm}, r_{km+0.3})$. In place of the dovetail the graphic now contains a path $p'_{ki(n)}(t)$ of index 1 critical points joining $p_{ki(n)}$ to p_{jc}. We further require of this deformation that, for $t \in (l_{jm}, r_{km+0.3})$, $W_l^*(p'_{ki(n)}(t))$ is contained in the annulus in level l that is bounded by $W_l^*(p_{ki(n)}(t))$ and $W_l^*(p_{jc}(t))$ as computed before the deformation. For a small $\delta > 0$ let $\alpha = l_{jm} - \delta$ and let $\beta = r_{km} + \delta$. Let f_{tv}, $\alpha \le t \le \beta$, be the portion of the path in $\widehat{\mathscr{F}}$ with which the last deformation finishes. With the help of the canonical coordinates near $p'_{ki(n)}(t)$, as provided by §1.3 (and §1.4) define a path g_t, $\alpha \le t \le \beta$, in $\widehat{\mathscr{F}}$ from $f_{\alpha v}$ to $f_{\beta v}$ so that the following hold:

(a) As t varies from α to l_{jm} a cancelling pair of index 0 and 1 critical points, labeled $x'_n(t)$ and $p'_{jc}(t)$ respectively, is born on $W(p'_{ki(n)}(t)) = W(p_{ki(n)}(t))$ close to and on the side of $p'_{ki(n)}(t)$ so that the order of $W_l^*(p'_{jc})$ followed by $W_l^*(p'_{ki(n)})$ is opposite that of $W_l^*(p_{jc})$ followed by $W_l^*(p_{ki(n)})$ before the dovetail was eliminated.

(b) As t varies from l_{jm} to $r_{km+0.3}$, $p'_{jc}(t)$ slides along $W(p'_{ki(n)}(t))$ (computed using f_{tv}) pushing $p'_{ki(n)}(t)$ ahead of it until, at time $r_{km+0.3}$, $p'_{jc}(t)$ is the point which was $p'_{ki(n)}(t)$, and $W^*(p'_{jc}(t))$ equals what was $W^*(p_{jc}(t)) = W^*(p'_{ki(n)}(t))$.

(c) As t varies from $r_{km+0.3}$ to β, $p'_{jc}(t)$ persists as $p_{jc}(t)$, and, sometime after r_{km}, $p'_{ki(n)}(t)$ is cancelled with the index 0 critical point $x'_n(t)$ that was introduced in statement (a) above. We also require that at all times $W_l^*(p'_{jc}(t))$ and $W_l^*(p'_{ki(n)}(t))$ are close enough together so that $W_l^*(p_{ki(n)}(t)) \subset \varphi_t((-1/3, 1])$.

Since $\widehat{\mathscr{F}}$ is contractible this serves to define a deformation from f_{tv} to g_t, $\alpha \le t \le \beta$. (This deformation could also be defined by introducing a dovetail in the graphic with enough control to achieve (a), (b), and (c).) Now drop the primes from the notation p'_{jc}, x'_n, and $p'_{ki(n)}$.

The graphic is now the same as it was at the start of this step: one dovetail has been replaced by another one with critical points labelled in the same way. But the order of the unstable spheres $W_l^*(p_{jc})$ and $W_l^*(p_{ki(n)})$ of the two index 1 critical points in this dovetail has been switched.

Next switch the order of $W_l^*(p_{jc})$ and $W_l^*(p_{ki(n-1)})$, while preserving the order of all the $W_l^*(p_{ki(m)})$, with deformations similar to those above, but using $p_{ki(n-1)}$ and x_{n-1} in place of $p_{ki(n)}$ and x_n. Continue with this until each of the $W_l^*(p_{ki(m)})$, $m = 1, \dots, n$, has been "moved past" $W_l^*(p_{jc})$.

Step 1b. Use the embeddings φ_t defined at the start of Step 1a to identify $S^2 \times [-1, 1]$ with the image of φ_t. Choose an isotopy F_t, $l_{jm} \le t \le r_{km}$, of $S^2 \times [-1, 1]$, fixed on $S^2 \times \{[-1, -1/2] \cup \{1\}\}$, such that

(1) $F_s = \mathrm{id}$ for $s = l_{jm}$ and r_{km},

(2) for $m = 1, \dots, n$, and for $t \in [l_{jm+0.3}, r_{km+0.3}]$, $F_t(W_l^*(p_{ki(m)}(t)))$ computed before Step 1a $= W_l^*(p_{ki(m)}(t))$ computed after Step 1a, and

(3) for $t \in [l_{jm+0.3}, r_{km+0.3}]$, $F_t(S^2 \times \{0.9\}) = W_l^*(p_{jc}(t))$ computed after Step 1a.

Again using the fact that $\widehat{\mathscr{F}}$ is contractible deform the gradient-like vector field in levels just above l to a path realizing F_t. Then $\Gamma_k(t)$ is unchanged for $t \in [l_{jm+0.3}, r_{km+0.3}]$, and $\gamma(t)$ has been eliminated from $\Gamma_j(t)$ for $t \in [l_{jm+0.3}, r_{km+0.3}]$. Lower the critical points that were raised at the start of Step 1a to heights less than L_1.

Step 2. Raise the critical points $p_s(t)$, $s \in S(t)$, $t \in [r_{km+0.7}, l_{rm+0.7}]$, to height l_2. Choose a 1-parameter family of embeddings $\theta_t \colon S_2 \times [-1, 1] \to \mathrm{level}_t\, l$, $t \in [r_{km+0.7}, l_{rm+0.7}]$, which for each t satisfies

(a) $\theta_t(S^2 \times \{0\}) = \varphi_L^l(N_{jc}''(t))$,

(b) $\theta_t(S^2 \times \{\pm 1\}) = \varphi_L^l(S_{\pm 1}(t))$, and

(c) $\theta_t(S^2 \times \{0.9\}) = \varphi_L^l(S_{1-\varepsilon}(t))$.

Use θ to perform a special handle move, like that in §2.2 in the case where $t \in U_j - \bigcup_{i \neq j} U_i$, by deforming the gradient-like vector field in levels just above l so that, for $t \in [r_{km+0.3}, l_{rm+0.3}]$, $\theta_t(S^2 \times \{0.9\})$ is moved to $\theta_t(S^2 \times \{0\})$. The move should be tapered off on the rest of $[r_{km+0.7}, l_{rm+0.7}]$.

Step 3. Just as in Step 3 of §3.1 hasten the deaths of all the p_{ki} until they occur just before $r_{km+0.7}$. This makes $N_{jc}(t)$ visible for all $t \in U_{jm+1}$. Then deform the gradient-like vector field in levels just below (or above) l so that, for $t \in [r_{km+0.7}, l_{rm+0.7}]$, $W_l^*(p_{jc}(t)) = \varphi_L^l(S_{1-\varepsilon}(t))$. Then, for t in the larger interval $[l_{jm+0.3}, r_{jm+0.3}]$, $W_l^*(p_{jc}(t)) = \varphi_L^l(S_{1-\varepsilon}(t))$, and so $\gamma(t)$, together with any other circle of $\Gamma_{jc}(t)$ in $N_{jc}(t) - D_N(t)$, have been eliminated from $\Gamma_{jc}(t)$ while no new circles of intersection have been introduced. Otherwise Γ is unchanged. Here $\Gamma(t)$ is defined to be $\bigcup \Gamma_j(t)$, where the union is over those j such that $t \in U_{jm+1}$. Lower the critical points which were raised at the start of Step 2 to a height near L_1.

Step 4. In the graphic, p_{jc} now cancels with x_1. Because of the way the bijection C_j changes as described at the end of §2.2 this is desired unless, before Step 1a, we had $C_j(p_{jc}) = x_n$ and $jd \in \widehat{S}$, where $C_j(p_{jd}) = x_0$. If so perform a deformation like that in Step 4 of §3.1 so that p_{jc} cancels with x_0 and p_{jd} with x_1, and then delay the birth of p_{jc}. This ensures that inductive assumption (a) holds. Thus all the inductive assumptions now hold.

In case $1 \in U_j$ then, just as at the end of §3.1, modify the above description so that f_{1u}, $u(m) \le u \le u(m+1)$, is a special handle move of the sort described in §2.2 for $t \in U_j - \bigcup_{i \neq j} U_i$.

4. Concluding arguments

We say that a Morse function $f \colon P^3 \times I \to I$ is in $\mathscr{E}'$ if all the critical points of f have index 1 or 2, if the stable manifold of each index 2 critical point intersects the unstable manifold of only one index 1 critical point, if each such intersection is transverse and consists of exactly one point in intermediate levels, and if these are the only intersections of the stable and unstable manifolds of distinct critical points. In §3 a deformation, rel endpoints, was constructed between f_t, $0 \le t \le 1$, and the 1-parameter family h_s defined to be $f_{0u} = f_0$, $0 \le u \le 1$, followed by f_{t1}, $0 \le t \le 1$, followed by f_{1u}, $1 \ge u \ge 0$. Since the family $f_{t0} = f_t$, $-1 \le t \le 2$, satisfies properties (7) and (8) in §1, the two endpoints f_{00} and f_{10} of h_s are in $\mathscr{E}'$. The goal of this section is to complete the proof of Theorem 1 by deforming h_s, rel endpoints, to a 1-parameter family in $\mathscr{E}'$. In §4.1, f_{1u}, $0 \le u \le 1$, is deformed, rel endpoints, to a family in $\mathscr{E}'$ and in §4.2 the same is done for f_{t1}, $0 \le t \le 1$.

4.1. For $0 \le u \le 1$, let $N_i(u)$ denote $W_L^*(p_i(u))$ computed using f_{1u}. Here the $p_i(u)$ are the g index 1 critical points of f_{1u}. Let q_i, $i = 1, \dots, g$, be the index 2 critical points of f_{1u} labelled so that $N_i(0)$ intersects $S_i^1 = W_L(q_i)$ transversely in one point. According to the ends of §§3.1 and 3.2 the deformation f_{1u}, $0 \le u \le 1$, consists of a sequence of (unparametrized) regular and special handle moves of the sort described in §§2.1 and 2.2 for the case where $t \in U_j - \bigcup_{i \ne j} U_i$. We show that after each of these moves $f_{1u(k)} \in \mathscr{E}'$ and that, for $0 \le u \le 1$, each $N_i(u)$ intersects S_i^1 transversely in one point. (Actually it is necessary to modify special handle moves slightly for this to be true.) During the moves, however, we may have, for $i \ne j$, $N_i(u) \cap S_j^1 \ne \varnothing$. These unwanted intersections will be eliminated.

By induction assume that before one of the moves, say when $u = u(k)$, we have $f_{1u(k)} \in \mathscr{E}'$ with $S_i^1 \cap N_i(u(k)) \ne \varnothing$ for each i. Let $\gamma \subset N_c(u(k)) \cap \Sigma(1)$ be the circle in $\Gamma(1)$ with $\lambda(\gamma) = u(k)$ which the move will eliminate. First assume that the move is a regular move. Then $D_N(\gamma) \cap S_c^1 = \varnothing$ because if not then S_c^1 must leave $B_L(\gamma)$ through one of its 2-sphere boundary components, none of which is a copy of $N_c(u(k))$. Thus S_c^1 would meet an $N_i(u(k))$ for some $i \ne c$, contradicting the inductive assumption. (Alternatively observe that, after the regular move, $D_N(\gamma)$ does not intersect any of the S_i^1. So if given the inductive assumption and $D_N(\gamma) \cap S_c^1 \ne \varnothing$, then after the move we would have $N_c(u(k+1)) \cap S_i^1 = \varnothing$ for all i, a contradiction.) Since S_c^1 misses $\partial B_L(\gamma)$, it misses $B_L(\gamma)$; and thus $\varphi_L^l(S_c^1)$ misses $B_l(\gamma)$. Therefore, since the isotopy used in defining the move has support near $B_l(\gamma)$, the one point transverse intersection of S_c^1 with $N_c(u(k))$ is maintained throughout the move (as are the other one point transverse intersections of the S_i^1 with the $N_i(u(k))$). Of course the S_s^1, $s \in S(u(k))$, intersect $B_L(\gamma)$ before the move, in fact they are contained in $B_L(\gamma)$ then. This causes the above-mentioned unwanted intersections since these S_s^1 intersect N_c at some time during the move (but not after it is completed).

Assume now that the move is a special handle move. Then, since a copy of $N_c(u(k))$ is contained in $B_L(\gamma)$, $D_N(\gamma)$ intersects S_c^1 transversely in one point. In order to maintain this one point transverse intersection throughout the move the following additional step is required as part of the move. First note that $N''_{jc}(t)$, the displaced copy of $N_{jc}(t)$ as defined at the start of §3.2, is only of real use for values of t near portions of the parameter space contained in more than one of the U_i. Thus we may alter the definition of the family $N''_{jc}(t)$ so that $N''_{jc}(t) = N_{jc}(t)$ for t near 1. Next, for t near 1, let θ_t be the embedding defined in Step 2 of §3.2. (It is assumed that Step 1 of §3.2 has been performed and that the critical points have been raised as instructed at the start of Step 2.) As usual, identify $S^2 \times [0, 0.9]$ with $\theta_t(S^2 \times [0, 0.9])$. Since $D_N(\gamma)$ intersects S_c^1 transversely in one point, $S_c^1 \cap (S^2 \times [0, 0.9])$ is an arc. Choose an isotopy g_s, $0 \leq s \leq 1$, of $S^2 \times [0, 0.9]$ (rel a neighborhood of $S^2 \times \{0, 0.9\}$) which deforms, for $0 \leq s \leq 1/2$, this arc so that it lies unknotted in $D^2 \times [0, 0.9] \subset S^2 \times [0, 0.9]$ for some 2-disk $D^2 \subset S^2$. Use (the π_0 part of) the Smale Conjecture (version 6 in the appendix of [H2]) to choose g_s, $1/2 \leq s \leq 1$, to further deform the arc until it intersects each S^2 slice of $S^2 \times [0, 0.9]$ transversely in one point. Then for t near 1, say for $t \in [1 - \delta, 1]$, deform the gradient-like vector field in levels just above l so that all of the isotopy g_s is realized at $t = 1$, and so that less and less of it is realized as t decreases until there is no deformation when $t = 1 - \delta$. Note that when $t = 1$ the one point transverse intersection of S_c^1 and N_c is maintained during this deformation and, in addition, this one point transverse intersection is now maintained during the rest of the special handle move.

Just as in the case of a regular move, the S_s^1, $s \in S(u(k))$, will intersect N_c at some time during the special handle move but not after it has been completed. Observe however that if S_d^1 causes one of these unwanted intersections then, before the move, S_d^1 must leave $A'_L(\gamma)$ through the copy of $N_i(u(k))$ different from the one through which it entered. Thus $S_d^1 \subset A_L(\gamma)$ and the 1-disk $W_l(q_d(u(k))) \cup W_l(p_d(u(k)))$ is contained in $A_l(\gamma)$. (In the case of a regular move, if S_d^1 causes an unwanted intersection then $W_l(q_d(u(k))) \cup W_l(p_d(u(k)))$ is contained in $B_l(\gamma)$.) This makes it easy, using ideas like those in the exchange lemma [HW, pp. 132–143], to eliminate the unwanted intersections without the usual introduction of slides among the 2-handles. Details follow.

As above let f_{1u}, $u(k) \leq u \leq u(k+1)$, be the 1-parameter family corresponding to a regular or special handle move. Thus $f_{1u(k)}$ and $f_{1u(k+1)}$ are in $\mathscr{E}'$. So for $s \in S(u(k))$, $W_l(q_c) \cap W_L^*(p_s) = \varnothing$ throughout the move. Deform f_{1u}, $u(k) \leq u \leq u(k+1)$, rel endpoints, to another family, also denoted by f_{1u}, with graphic like that in Figure 3. (Besides p_c and q_c, Figure 3 shows only one pair p_s, q_s of critical points. All the p_s, $s \in S$, should be above q_c during the move.) Observe that $W_l^*(p_c)$ is an open 2-disk whenever q_c is below level l since, throughout the move, there is a unique trajectory joining p_c and q_c. If the family f_{1u}, $u(k) \leq u \leq u(k+1)$, consists of a regular move then

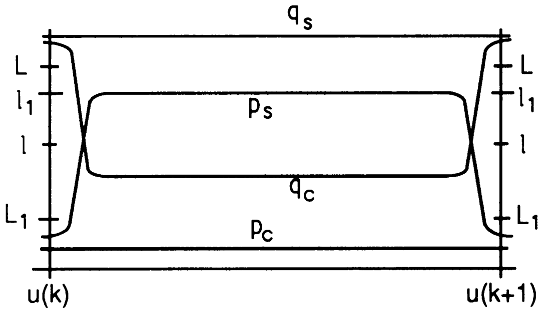

FIGURE 3. Preparation for a 1/2 intersection

the unwanted intersections can be eliminated by deforming this family, rel endpoints, so that the contents of the ball $B_l(\gamma)$ are swept around $W_l^*(p_c)$ instead of through $\varphi_L^l(D_N(\gamma)) \subset W_l^*(p_c)$. ($\varphi_L^l(D_N(\gamma)) \subset W_l^*(p_c)$ since $D_N(\gamma) \cap S_c^1 = \varnothing$ for a regular move.) More precisely this deformation can be defined as follows. First, for values of u slightly greater than $u(k)$, deform the 2-spheres $\varphi_L^l(\Sigma(1, u))$ in small neighborhoods of $W_l^*(p_c)$ so that $\varphi_L^l(D_N(\gamma))$ slides off $W_l^*(p_c)$. Then use these Σ to define a 1-parameter family of regular moves beginning with f_{1u}, $u(k) \leq u \leq u(k+1)$, and finishing with a path in $\mathscr{E}'$. Of course this deformation should keep f_{1u}, $u = u(k), u(k+1)$, fixed. If f_{1u}, $u(k) \leq u \leq u(k+1)$, consists of a special handle move then the paths of the 1-disks $W_l(q_s) \cup W_l(p_s)$, $s \in S$, miss $W_l(q_c)$ during the move, and so these paths can be deformed to go around $W_l^*(p_c)$. All this shows that the 1-parameter family f_{1u}, $0 \leq u \leq 1$, can be deformed (rel $u = 0, 1$) to a family in $\mathscr{E}'$.

4.2. In this section a homotopy is defined between the 1-parameter family f_{t1}, $0 \leq t \leq 1$, and a family in $\mathscr{E}'$. Let $\widetilde{K}_i(t)$ be the component of $f_t^{-1}(L) - \Sigma(t)$ containing $S_i^1(t) = W_L(q_i)$, and let $K_i(t)$ denote the closure of $\widetilde{K}_i(t)$ in $f_t^{-1}(L)$. Each $S_i^1(t)$ intersects the unstable set of at least one index 1 critical point, and each such unstable set intersects at least one $S_i^1(t)$. Since $\Sigma(t) \cap N_{ji}(t, 1) = \varnothing$, it follows that if $t \in U_j - \bigcup_{i \neq j} U_i$, then each $K_i(t)$ contains exactly one $N_{ji}(t, 1)$. Also if $t \in U_j \cap U_k$, then, since the $p_{ji}(t)$ can be cancelled without affecting the $N_{ki}(t)$, each $K_i(t)$ contains one $N_{ji}(t, 1)$ and, by a similar argument, one $N_{ki}(t, 1)$. Since $\widetilde{K}_i(t)$ is diffeomorphic to $S^1 \times S^2$ with one point removed, these last two 2-spheres bound a unique annulus $A_i(t)$ in $K_i(t)$. If $P^3 \neq S^3$ this annulus $A_i(t)$ flows to an index 0 critical point, say x_i. (The case where $P^3 = S^3$ will be treated §4.3.) The rest of $K_i(t)$ flows to P^3 since the $A_i(t)$ are disjoint. Thus if the boundary of $A_i(t)$ is $N_{ja}(t) \cup N_{kb}(t)$, we have $C_j(N_{ja}) = x_i = C_k(N_{kb})$. In view of this and induction hypothesis (a) in §3 we can deform f_{t1}, $0 \leq t \leq 1$ (rel $t = 0, 1$), to a 1-parameter family f_{t2} by eliminating all the dovetails in the graphic of f_{t1} in such a way that if $p_i(t)$, $0 \leq t \leq 1$, $i = 1, \ldots, g$, denote (paths of) index 1 critical points of f_{t2} then each of the unstable sets $W_L^*(p_i(t))$ is contained in some $K_i(t)$, say $W_L^*(p_i(t)) \subset K_i(t)$. With the following two lemmas it is easy to deform f_{t2} to a family in $\mathscr{E}'$.

Lemma 1. *Let* $(x, y) \in S^1 \times S^2$ *and* $(x_*, y_*) \in S^1 \times S^2 - (\{x\} \times S^2 \cup S^1 \times \{y\})$. *Let* $f\colon S^2 \to S^1 \times S^2 - (x_*, y_*)$ *be an embedding such that* $f(S^2)$ *intersects* $S^1 \times \{y\}$ *transversely in one point. Then there is an isotopy* F_t, $0 \leq t \leq 1$, *of* $S^1 \times S^2 - (x_*, y_*)$ *such that* $F_0 = \mathrm{id}$ *and* $F_1(f(S^2)) = \{x\} \times S^2$, *and such that, for all* t

(a) $F_t(f(S^2))$ *intersects* $S^1 \times \{y\}$ *transversely in one point, and*
(b) F_t *is the identity on a neighborhood of* (x_*, y_*).

Proof. Deform $f(S^2)$ near the point $(p, y) = f(S^2) \cap S^1 \times \{y\}$ so that some portion of $f(S^2)$ near (p, y) is contained in the meridian sphere $\{p\} \times S^2$. Let $S^1 \times D^2 \subset S^1 \times S^2$ be the complement of a small neighborhood of $S^1 \times \{y\}$, and let W be the 2-disk $f(S^2) \cap S^1 \times D^2$ with boundary $\{p\} \times S^1$. For the moment ignore the point (x_*, y_*) and deform W in $S^1 \times D^2$ to the meridian disk $\{p\} \times D^2$. (This can be done by first using a standard disjunction argument to make W disjoint from some meridian disk $\{z\} \times D^2$, and then using the Alexander-Schoenflies theorem to further deform W to $\{p\} \times D^2$.) We may assume that this deformation of W extends by the identity to an isotopy g_t of $S^1 \times S^2$ with $g_1(f(S^2)) = \{p\} \times S^2$ and with $g_t(f(S^2))$ intersecting $S^1 \times \{y\}$ transversely in one point.

Next an isotopy is needed which accomplishes what g_t does, and, in addition, keeps a neighborhood of (x_*, y_*) fixed. Let $\pi_i\colon S^1 \times S^2 \to S^i$, $i = 1, 2$, be the projection maps. Choose $T_t^1 \in \mathrm{SO}(2) \subset \mathrm{Diff}(S^1)$ such that $T_t^1(x_*) = (\pi_1 \circ g_t)(x_*, y_*)$ for all t. Choose an isotopy T_t^2 of S^2 which keeps a neighborhood of y fixed and satisfies $T_t^2(y_*) = (\pi_2 \circ g_t)(x_*, y_*)$. Then with $T_t = T_t^1 \times T_t^2$ we have $T_t(x_*, y_*) = g_t(x_*, y_*)$. Let $N(x_*, y_*) \subset S^1 \times S^2 - (S^1 \times \{y\})$ be a neighborhood of (x_*, y_*) small enough so that $T_t(\overline{N(x_*, y_*)}) \cap g_t(f(S^2)) = \varnothing$. Let G_t be an isotopy of $S^1 \times S^2$ such that $G_t = g_t$ on $f(S^2) \cup S^1 \times \{y\}$ and $G_t = T_t$ on $N(x_*, y_*)$. Then

(a) $T_t^{-1} \circ G_t$ is the identity near (x_*, y_*), and (we may assume that) $T_0^{-1} \circ G_0 = \mathrm{id}$,
(b) $(T_t^{-1} \circ G_t)(f(S^2))$ intersects $S^1 \times \{y\}$ transversely in one point, and
(c) $(T_1^{-1} \circ G_1)(f(S^2)) = \{T_1^{-1}(p)\} \times S^2$.

Since $T_1^{-1}(p) \neq x_*$, it is now easy to deform $(T_1^{-1} \circ G_1)(f(S^2))$ to get an isotopy F_t which satisfies the conclusions of the lemma. □

Lemma 2. *Let* $(x_*, y_*) \in S^1 \times S^2$ *and let* $x \in S^1$ *with* $x \neq x_*$. *Let* $f_t\colon S^2 \to S^1 \times S^2 - (x_*, y_*)$, $0 \leq t \leq 2$, *be a* 1*-parameter family of embeddings with* $f_0(S^2) = f_2(S^2) = \{x\} \times S^2$. *Then there is a* 1*-parameter family* F_{tu}, $0 \leq t \leq 2$, $0 \leq u \leq 1$, *of isotopies of* $S^1 \times S^2 - (x_*, y_*)$ *such that*

(1) F_{tu} *is the identity on a neighborhood of* (x_*, y_*),
(2) $F_{t0} = F_{0u} = F_{2u} = \mathrm{id}$, *and*
(3) $F_{t1}(f_t(S^2)) = \{x\} \times S^2$.

Proof. View f_t as a family of embeddings of S^2 in $S^1 \times S^2$. Use the techniques in [H1] to obtain a 1-parameter family g_{tu}, $0 \leq t \leq 2$, $0 \leq u \leq 1$, of isotopies of $S^1 \times S^2$ such that

(a) $g_{t0} = g_{0u} = g_{2u} = \mathrm{id}$, and
(b) $g_{t,1}(f_t(S^2)) = \{x_t\} \times S^2$, where $x_t \in S^1$ depends smoothly on t and $x_0 = x_2 = x$.

Let $\mathrm{SO}(2) \times \mathrm{SO}(3) \to S^1 \times S^2$ be the fibration given by evaluation at (x_*, y_*), and let $g\colon [0, 2] \times I \to S^1 \times S^2$ be defined by $g(t, u) = g_{t,u}(x_*, y_*)$. Lift g up the fibration to a map $T\colon [0, 2] \times I \to \mathrm{SO}(2) \times \mathrm{SO}(3)$ with $T_{t0} = T_{0u} = T_{2u} = \mathrm{id}$. Then, with $T(t, u)$ denoted by T_{tu}, we have $T_{tu}(x_*, y_*) = g_{tu}(x_*, y_*)$. As in the proof of the previous lemma, we may modify g_{tu} so that $T_{tu} = g_{tu}$ near (x_*, y_*). Then $F_{tu} = T_{tu}^{-1} \circ g_{tu}$ satisfies the conclusions of the lemma except for condition (3). But $F_{t1}(f_t(S^2)) = T_{t1}^{-1}(\{x_t\} \times S^2) = \{u_t\} \times S^2$, where $u_t \in S^1 - \{x_*\}$ and $u_0, u_2 = x$. So condition (3) can easily be achieved as well. □

Since $f_{0u} = f_{00}$ for $u \in [0, 2]$ and since property (2) in §1 provides standard coordinates around each pair $p_i(0)$, $q_i(0)$ of critical points, the sets $W_L^*(p_i(0))$

and $W_L(q_i(0)) = S_i^1$ in $K_i(0)$ correspond, via some diffeomorphism of $\widetilde{K}_i(0)$ with $S^1 \times S^2 - (x_*, y_*)$, to slices $\{x\} \times S^2$ and $S^1 \times \{y\}$ in $S^1 \times S^2 - \{(x_*, y_*)\}$. Fix smoothly varying diffeomorphisms $K_i(0) \to K_i(t)$ such that $S_i^1(0)$ maps to $S_i^1(t) = W_L(q_i(t))$. Use these to identify $K_i(0)$ with $K_i(t)$. Then the slices $\{x\} \times S^2$ and $S^1 \times \{y\} = S_i^1(t)$ are defined in $K_i(t)$ for all $t \in [0, 1]$. Recall from §4.1 that $f_{12} = f_{11} \in \mathscr{E}'$ so that the sets $W_L^*(p_i(1))$ and $S^1 \times \{y\}$ in $\widetilde{K}_i(1)$ intersect transversely in one point. Thus Lemma 1 provides an isotopy of $K_i(1)$ which is the identity near $\partial K_i(1)$ and which deforms $W_L^*(p_i(1))$ to $\{x\} \times S^2$ while maintaining the one point transverse intersection at all times. Realize these isotopies, one for each i, with a deformation of the gradient-like vector fields in levels just below L to obtain a family of Morse functions f_{t2}, $1 \leq t \leq 2$. Note that this family is in $\mathscr{E}'$. Also the set $W_L^*(p_i(2))$, computed using $f_{2,2}$, equals $\{x\} \times S^2$ in $K_i(2)$. ($K_i(t) = K_i(1)$ for $1 \leq t \leq 2$.) Thus we can apply Lemma 2 to obtain a family of isotopies F_{tu} which deforms the family of 2-spheres $W_L^*(p_i(t))$, $0 \leq t \leq 2$ (viewed as a family in $S^1 \times S^2 - \{(x_*, y_*)\}$), to $\{x\} \times S^2$. The F_{tu} give isotopies of $K_i(t)$ which are the id near $\partial K_i(t)$. Realize these isotopies by deforming the gradient-like vector fields of the f_{t2}, $0 \leq t \leq 2$, in levels just below L. Let f_{tu}, $2 \leq u \leq 3$, denote the resulting deformation (rel endpoints) of f_{t2}, $0 \leq t \leq 2$. Observe that the family f_{t3}, $0 \leq t \leq 2$, is in $\mathscr{E}'$, as are $f_{1u} = f_{11}$, $1 \leq u \leq 2$; f_{t2}, $1 \leq t \leq 2$; $f_{2u} = f_{22}$, $2 \leq u \leq 3$; and $f_{0u} = f_{00}$, $0 \leq u \leq 3$. This together with the result of §4.1 shows that any family f_t in $\pi_1(\widehat{\mathscr{F}}, \widehat{\mathscr{E}})$ coming from an element of $\mathscr{P}_0$ can be deformed, rel endpoints, to a family in $\mathscr{E}'$. So, assuming $P^3 \neq S^3$, the theorem now follows easily from the parametrized Smale cancellation lemma [HW, p. 172].

4.3. Now assume $P^3 = S^3$. Then on each of the intersections $U_j \cap U_k$ one, but at most one, of the (families of) annuli $A_i(t)$, defined in §4.2, might flow to P^3 rather than to an index 0 critical point. If none flow to P^3 proceed as in §4.2. Otherwise let $A_c(t) \subset K_c(t)$ denote the annulus which flows to P^3 at time t, and let $x_c(t)$ be the index 0 critical point which the set $K_c(t) - A_c(t)$ flows to. Since the $A_i(t)$, $i \neq c$, all flow to distinct index 0 critical points, it follows that the sets $K_i(t) - A_i(t)$, $i \neq c$, all flow to $x_c(t)$.

Fix a t_j in each of the sets $U_j - \overline{\bigcup_{i \neq j} U_i}$. Let $p_i(t_j)$, $i = 1, \ldots, g$, denote the index 1 critical points of $f_{t_j 1}$ labeled so that $W_L^*(p_i(t_j)) \subset K_i(t_j)$. In each $K_i(t_i)$ we have the sets $S^1 \times \{y\} = W_L(q_i(t_j))$ and $\{x\} \times S^2$ as defined in the §4.2. Using, for instance, an unparametrized version of Lemma 2, choose, for each i, an isotopy of $K_i(t_j)$ which deforms $W_L^*(p_i(t_j))$ to $\{x\} \times S^2$. Then deform the gradient-like vector field of the family f_{t1}, $0 \leq t \leq 1$, in levels just below L so that these isotopies are realized when $t = t_j$. Taper off these deformations on small neighborhoods of the t_j. Now for each t_j, we have $f_{t_j 1} \in \mathscr{E}'$. Next, on small neighborhoods of the t_j, use the Smale lemma [HW, p. 172] to further deform the family f_{t1}, $0 \leq t \leq 1$, to a family f_{t2}, $0 \leq t \leq 1$, with $f_{t_j 2} \in \mathscr{E}$ and with independent birth and death points near the t_j. Components of the graphic that do not involve any of the index 0 critical points x_c can now be eliminated following the same steps used in §4.2. (Here by a component of the graphic we mean a subset of the graphic which becomes, after some deformation of the sort given by the independent trajectories principle, an actual connected component of the graphic.) Specifically, first eliminate the dovetails in these components in such a way that the ascending manifolds of the index 1 critical points never intersect Σ. Then use the two lemmas to get the desired one point transverse intersection. Finish by using the Smale lemma. The level set between any pair of critical points that remain after this is $S^1 \times S^2$. Thus these remaining critical points can be eliminated using an argument similar to, but easier than, the one just sketched since now we need not worry about Σ and can work with $S^1 \times S^2$ instead of a punctured $S^1 \times S^2$. This completes the proof of Theorem 1 when $P^3 = S^3$.

5. A cobordism theorem

In this section the following theorem is proved as an application of some of the (unparametrized) techniques developed above. (When $P = S^3$, this theorem is a result of Laudenbach and Poénaru [LP].)

Theorem 2. *Let $(W^4; P^3, N^3)$ be a smooth compact cobordism, where P is irreducible. Assume $H_*(W, P; \mathbb{Z}) = 0$. Also assume that W has a handlebody structure consisting of 2- and 3-handles attached to $N \times I$, where the attaching circles of the 2-handles bound disjoint disks in N. Then W is diffeomorphic to $P \times I$.*

Proof. Pick a Morse function $f\colon (W; P, N) \to (I; 0, 1)$ and a gradient-like vector field for f which together give a handlebody structure dual to that given by the hypothesis of the theorem. Let $p_1, \ldots, p_g$ denote the index 1 critical points of f and $q_1, \ldots, q_g$ the index 2 ones. As usual we may assume that $f(p_i) < L < f(q_i)$. By assumption the circles $W_1^*(q_i) \subset N = f^{-1}(1)$ bound disjoint 2-disks in N. Choose disjoint 3-disks in N with each containing one of these 2-disks. Let Σ_i' denote the boundary of the 3-disk which contains $W_1^*(q_i)$. Let Σ_i denote $\varphi_1^L(\Sigma_i') \subset f^{-1}(L)$, and let K_i be the closure of the component of $f^{-1}(L) - \Sigma_i$ which contains the circle $S_i^1 = W_L(q_i)$. Now, using (an unparametrized version of) the disjunction procedure used to prove Theorem 1, make the Σ_i disjoint from the $W_1^*(p_i)$ by deforming f and the gradient-like vector field. More specifically, first slightly deform the unstable 2-spheres $N_i = W_L^*(p_i)$ so that they intersect the 2-spheres Σ_i transversely in circles. Then eliminate all these circles of intersection by repeatedly using regular and special handle moves of the sort described in §2 in the case where $t \in U_j - \bigcup_{i \neq j} U_i$. Proceed in the order given in §2, that is, eliminate innermost circles on the Σ_i first.

Since $H_*(W, P) = 0$, it now follows that each K_i contains exactly one of the $N_i = W_1^*(p_i)$, say $N_i \subset K_i$, and that the algebraic intersection number of S_i^1 and N_i is ± 1. Let $\overline{K}_i$ denote the closed manifold obtained from K_i by attaching a 3-ball to ∂K_i. So we have $f^{-1}(L) = N \# (\#_i \overline{K}_i)$. Since $N_i \subset K_i$, $\overline{K_i} = S^1 \times S^2 \# M_i$ for some manifold M_i. But since $\overline{K_i}$ is the result of doing (integer framed) surgery on the trivial knot in S^3, $\overline{K}_i = S^1 \times S^2$. Thus, using for instance an unparametrized version of Lemma 2 in §4, each N_i can be deformed within K_i until it intersects S_i^1 transversely in one point. So all the critical points of f can be eliminated. This proves that W is a product.

Remark. In the above theorem if $N = P$ or if $N \neq S^3$, then it is not necessary to assume that the attaching circles of the 2-handles are unknotted since this follows from Property R [G]. Of course the assumption that these circles are unlinked (meaning that there are disjoint 3-balls in N with each containing one of the circles) is still needed. To see this observe that the proof of the theorem goes through with this altered hypothesis up to the point where $\overline{K}_i = S^1 \times S^2 \# M_i$. Then on the one hand

$$f^{-1}(L) = N \# \left(\#_i K_i \right) = N \# \left(\#_i (S^1 \times S^2 \# M_i) \right),$$

while on the other hand $f^{-1}(L) = P \# (\#_i S^1 \times S^2)$. So by the prime decomposition theorem for 3-manifolds we have $P = N \# (\#_i M_i)$ even if P is not orientable. Thus if $N = P$ or if $N \neq S^3$, all the $M_i = S^3$. Hence $\overline{K}_i = S^1 \times S^2$ and so Property R implies that the S^1_i are unknotted.

Other such variants of the theorem are possible.

References

[C] J. Cerf, *La stratification naturelle des espaces de fonctions différentiables réeles et le théorème de la pseudo-isotopie*, Inst. Hautes Études Sci. Publ. Math. **39** (1970), 5–173.

[G] D. Gabai, *Foliations and the topology of 3-manifolds*, J. Differential Geom. **18** (1983), 445–503.

[H1] A. Hatcher, *On the diffeomorphism group of* $S^1 \times S^2$, Proc. Amer. Math. Soc. **83** (1981), 427–430.

[H2] ——, *A proof of the Smale conjecture*, $\mathrm{Diff}(S^3) \simeq O(4)$, Ann. of Math. (2) **117** (1983), 553–607.

[HW] A. Hatcher and J. Wagoner, *Pseudo-isotopies of compact manifolds*, Astérisque **6** (1973).

[HL] H. Hendriks and F. Laudenbach, *Difféormorphismes des sommes connexes en dimension trois*, Topology **23** (1984), 423–443.

[LP] F. Laudenbach and V. Poénaru, *A note on 4-dimensional handlebodies*, Bull. Soc. Math. France **100** (1972), 337–344.

[M1] J. Milnor, *Lectures on the h-cobordism theorem*, Princeton mathematical notes, 1965.

[M2] ——, *Morse theory*, Princeton mathematical notes, 1969.

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