HARMONIC LOCALIZATION OF ALGEBRAIC K-THEORY SPECTRA

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ABSTRACT. The Lichtenbaum-Quillen conjectures hold for the harmonic localization of the K-theory spectrum of a nice scheme. Various consequences of this fact are explored; for example, the harmonic localization of the K-theory of the integers at a regular prime is explicitly identified.

In [Mit] the author proved the following theorem:

Theorem. *Let KX denote the algebraic K-theory spectrum of an arbitrary ring or scheme. Then the higher Morava K-theories of KX vanish: $K(n)_* KX = 0$, $n \geq 2$. Here we have a fixed prime l, and all spectra should be regarded as localized at l.*

In this paper our primary goal is to explore some of the consequences of this theorem. We start from the simple observation that if E is any spectrum such that $K(n)_* E = 0$ for $n \geq 2$, then the Bousfield localization of E with respect to ordinary K-theory—$L_1 E$—coincides with its harmonic localization $L_\infty E$. Here L_∞ is Bousfield localization with respect to the wedge of *all* the Morava K-theories $K(n)$, $0 \leq n < \infty$, while L_1 is the same as localizing with respect to $K(0) \vee K(1)$ or (rational homology) $\vee$ (mod l complex K-theory). Thus any theorem about $L_1 KX$ holds also for $L_\infty KX$. For example, Waldhausen [W] observed that deep work of Thomason [T] shows that the Lichtenbaum-Quillen conjectures on the mod l^v K-theory of X hold for $L_1 KX$, X a "reasonable" scheme. Hence:

Theorem (4.4). *The Lichtenbaum-Quillen conjectures hold for $L_\infty KX$, X a reasonable scheme.*

The significance of this result lies in the fact that L_∞ is a priori a much weaker form of localization than L_1 and hence any theorem about it is a priori much stronger than its L_1 analogue. To put it another way, Thomason's theorem shows that if the Lichtenbaum-Quillen conjectures are false then the "error term" is a spectrum that is acyclic for complex K-theory, whereas (4.3) says the error term is actually acyclic for *all* the Morava K-theories—a far more stringent restriction.

In §4 we collect some further consequences for $L_\infty KX$. These are all results that can be shown to be true for KX itself if the Lichtenbaum-Quillen

conjectures hold. With one exception (Theorem (4.5)) the proofs amount to simply combining the author's theorem above with the work of Thomason and Dwyer-Friedlander. For example, the beautiful theorems of [DF2, DF3], which explicitly identify the étale K-theory space for certain rings of algebraic integers R, yield corresponding results for harmonic localizations. Let $L_\infty^\wedge KX = (L_\infty KX)^\wedge$, where $^\wedge$ denotes l-adic completion. For example:

Theorem. *Let l be an odd regular prime. Then there is a map $K\mathbb{Z}[\frac{1}{l}] \to j^\wedge \vee \Sigma bo^\wedge$ that induces an equivalence $L_\infty^\wedge K\mathbb{Z}[\frac{1}{l}] \cong J^\wedge \vee \Sigma KO^\wedge$.*

Here J is the J-spectrum at l with connective cover j; KO is the real K-theory spectrum with connective cover bo.

As already noted, Thomason's work is a crucial ingredient in all these results. Now Thomason's theorem was originally stated in terms of the so-called "Bott-periodic" K-theory introduced by Snaith. That is, it applied to the homotopy groups of the mapping telescope $\beta^{-1}KX \wedge M\mathbb{Z}/l^\nu$, where $M\mathbb{Z}/l^\nu$ is the mod l^ν Moore spectrum and $\beta \in \pi_* KX \wedge M\mathbb{Z}/l^\nu$ is a "Bott element." On the other hand, by a theorem of Bousfield $L_1(KX \wedge M\mathbb{Z}/l^\nu) = A^{-1}KX \wedge M\mathbb{Z}/l^\nu$, where A is an "Adams element." The link between the two is provided by a theorem of Snaith, which says that up to isogeny A and B coincide and so define the same mapping telescope. The results of §§1–3 were originally motivated by a desire to better understand Snaith's theorem, which although not difficult is crucial for our program.

One point here is that β originates as an element of $\pi_2 B\mathbb{Z}/l_+ \wedge M\mathbb{Z}/l$ (l odd) and yet in the ring $\pi_* B\mathbb{Z}/l_+ \wedge M\mathbb{Z}/l$, β and A are linearly independent, even modulo isogeny. Hence one would like to understand more clearly why β and A coalesce when $B\mathbb{Z}/l_+$ is mapped into $K\mathbb{Z}[\xi_l]$. Thus in §1 we explicitly compute (1) $\pi_* B\mathbb{Z}/l_+ \wedge M\mathbb{Z}/l$ modulo nilpotent elements; this is an amusing application of the nilpotence theorem of [DHS]; and (2) both the β-periodic and the A-periodic homotopy of $B\mathbb{Z}/l_+$. We also review Snaith's theorem in §2, and in §3 we study the $K(1)$-localization of $B\mathbb{Z}/l_+$. The reason for including these results here is that they are needed to explain why the natural map $B\mathbb{Z}/l_+ \to K\mathbb{Z}[\xi_l]$ should factor through $K\mathbb{F}_q$ for suitable q—and indeed does so factor after harmonic localization; see Theorem 4.6. However many readers will prefer to begin with §4, referring to the earlier sections as needed.

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0. NOTATION AND PRELIMINARIES

Throughout this paper, all spectra are localized at a fixed prime l. The prime l is *odd* except where $l = 2$ is explicitly allowed. The completion of a spectrum X at l is denoted $X^\wedge$. When the prime p occurs it is always distinct from l; q stands for a power of p. $J(q)$ denotes the fibre of $\psi^q - 1: KU \to KU$. The "Image of J" spectrum at l is $J \equiv J(p)$, where p generates $(\mathbb{Z}/l^2)^*$. The connective covers of various spectra are written with lower case letters—e.g. $j(q)$ is the connective cover of $J(q)$—except that we use the traditional notation bu, bo for the connective covers of KU, KO. $L_E X$ denotes Bousfield localization of X with respect to E. If $E = K(0) \vee K(1) \vee \cdots \vee K(n)$ we write $L_n X$ for $L_E X$. In particular, $L_1 X$ is the same as $L_{KU} X$.

The "nilpotence theorem" refers to

(0.1) **Theorem** [DHS]. *Let E be a ring spectrum, $\alpha \in \pi_* E$. If $BP_* \alpha = 0$, then α is nilpotent.*

Here E is an *arbitrary* ring spectrum; in particular E need not be associative. The nonassociative case of (0.1) requires some interpretation, left implicit in [DHS]. The point is that if $\pi_* E$ is not power-associative then the term "nilpotent" is ambiguous. However it is clear from the proof of (0.1) (see [DHS]), that if $BP_* \alpha = 0$ then there is an $n > 0$ such that $\alpha^n = 0$ with any placement of parentheses.

There is also the following variant, using the Morava K-theories $K(n)$.

(0.2) **Theorem** [HS]. *If E is a ring spectrum, $\alpha \in \pi_* E$, and $K(n)_* \alpha = 0$ for all n, $0 \leq n \leq \infty$, then α is nilpotent.*

We recall here that $K(\infty) = H\mathbb{Z}/l$ (mod l homology).

If l is odd, the Moore spectrum $M\mathbb{Z}/l$ is a commutative ring spectrum. Unfortunately, it is not associative for $l = 3$; this is a theorem of Toda that was also proved by Neisendorfer [N]. However, the following self-contained analysis will suffice for our purposes. Let $M = M\mathbb{Z}/l$, l odd. It is trivial to show that $M \wedge M \cong M \vee \Sigma M$ and $[\Sigma M, M] = 0$. Hence maps $M \wedge M \to M$ are uniquely determined by the induced map on π_0. It follows that M has a unique ring spectrum structure, and this structure is commutative. Next we have $M \wedge M \wedge M \cong M \vee \Sigma M \vee \Sigma^2 M$. If $l > 3$, $[\Sigma^2 M, M] = 0$ and the same argument shows the multiplication is associative. However if $l = 3$, $[\Sigma^2 M, M] = \mathbb{Z}/l$, generated by the composite $\overline{\overline{\alpha}}_1 : \Sigma^2 M \to S^3 \overset{\alpha_1}{\to} S^0 \to M$ where α_1 generates $\pi_3 S^0$. Let $\varphi : M \wedge M \wedge M \to M$ denote the "universal associator" given by the difference of the two 3-fold multiplications. We conclude that $\varphi = c\eta$ for some $c \in \mathbb{Z}/3$, where η is the composite $M \wedge M \wedge M \overset{\delta \wedge \delta \wedge \delta}{\longrightarrow} S^1 \wedge S^1 \wedge S^1 = S^3 \overset{\alpha_1}{\to} S^0 \to M$. Here $\delta : M \to S^1$ is the integral Bockstein or pinch map. In fact $c = \pm 1$ by the Toda/Neisendorfer theorem, but we will not need this. Let E be a commutative associative ring spectrum; thus $E \wedge M\mathbb{Z}/3$ is a commutative but possibly nonassociative ring spectrum. If $\alpha, \beta, \gamma \in \pi_* E \wedge M\mathbb{Z}/3$ write $\langle \alpha, \beta, \gamma \rangle$ for the associator $(\alpha\beta)\gamma - \alpha(\beta\gamma)$. We say that α *associates* if for all β, γ the associator of α, β, γ in any order is zero. Then the formula $\varphi = c\eta$ shows at once:

(0.3) **Proposition.** (a) *If α is in the image of the reduction map $\pi_* E \to \pi_* E \wedge M\mathbb{Z}/3$, then α associates.*

(b) *Suppose $\alpha, \beta \in \pi_* E \wedge M\mathbb{Z}/3$ and both α and β have even dimension. Then the subalgebra generated by α and β is associative. In particular $\pi_* E \wedge M\mathbb{Z}/3$ is power-associative.*

(c) *Let $i : S^0 \to E$ denote the unit map. If $i_* \alpha_1 = 0$, $E \wedge M\mathbb{Z}/3$ is associative.*

Note (c) holds trivially for BP, KU, etc. However $i_* \alpha_1$ is definitely nonzero for many of the spectra considered in this paper, e.g. $K\mathbb{Z}$. In any case properties (a), (b), (c) will suffice (barely!) for our purposes; for the sake of clarity we will write as though $l > 3$ and leave the modifications for $l = 3$ to the reader.

(0.4) *Abuse of notation.* In order to avoid a tedious proliferation of constants we will often write $\alpha = \beta$ when we really mean $\alpha = c\beta$, c a unit in $\mathbb{Z}/l$ or

$\mathbb{Z} l^\nu$. For example in §1 the equation $\beta^l = A\beta$ should be interpreted this way. Indeed we only define A as "a" generator of a certain group of order l, without specifying which one it is. Certainly one could choose A so that $\beta^l = A\beta$ holds on the nose, but it seems pointless to fuss.
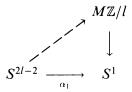
1. On the $\bmod l$ stable homotopy ring of $B\mathbb{Z}/l_+$

The $\bmod l^\nu$ homotopy of a spectrum E is $\pi_* E \wedge M\mathbb{Z}/l^\nu$ where $M\mathbb{Z}/l^\nu$ is the $\bmod l^\nu$ Moore spectrum. Occasionally we write this as $\pi_*(E; \mathbb{Z}/l^\nu)$. If $l^\nu \neq 2$, $M\mathbb{Z}/l^\nu$ is a ring spectrum; if l is odd it is commutative for all ν and associative for $l^\nu \neq 3$. In this section we will frequently restrict our attention to the case $\nu = 1$, and write as though $l > 3$. However all the results of this paper are perfectly valid for $l = 3$; the reader can make the necessary adjustments using (0.3). Then if E is a commutative associative ring spectrum, so is $E \wedge M\mathbb{Z}/l$, and $\pi_* E \wedge M\mathbb{Z}/l$ is a commutative graded ring. Since l is odd, $\pi_* E \wedge M\mathbb{Z}/l$ is a $\mathbb{Z}/l$ vector space.

Now consider commutative graded rings R such that $l^\nu \cdot R = 0$ for some ν. A homomorphism $\varphi : R \to S$ between two such rings is an *isogeny* if $\mathrm{Ker}\, \varphi$ consists of nilpotent elements and for every $s \in S$ there is an integer $n \geq 0$ such that $s^{l^n} \in \mathrm{Im}\, \varphi$. Elements x, y in R will be called *isogenous* if $x^m = y^n$ for some m, n. For example, since l is odd every odd-dimensional element of R is isogenous to zero. Clearly isogenous elements define the same localization: $x^{-1} R = y^{-1} R$. Any ring homomorphism $\varphi : R \to S$ induces a map on isogeny classes, which is bijective if φ is an isogeny. Now consider a commutative ring spectrum E and the tower of ring spectra

$$\to E \wedge M\mathbb{Z}/l^\nu \xrightarrow{r_\nu} E \wedge M\mathbb{Z}/l^{\nu-1} \to \cdots \to E \wedge M\mathbb{Z}/l.$$

Each reduction map r_ν is an isogeny on π_*, since (a) $x \in \mathrm{Ker}(r_\nu)_* \Rightarrow x = 0 \bmod l$ and (b) the boundary maps associated to the cofibre sequences $M\mathbb{Z}/l \to M\mathbb{Z}/l^\nu \to M\mathbb{Z}/l^{\nu-1}$ are derivations, and so annihilate lth powers. Hence each r_ν is bijective on isogeny classes, and any given element x of some $\pi_* E \wedge M\mathbb{Z}/l^{\nu_0}$ uniquely determines a compatible family $[x_\nu]$ of isogeny classes in π_* of the tower. It will be convenient to refer to such a family as an *l-adic isogeny class*.

Example. Let α_1 be a generator of $\pi_{2l-3} S^0$. Then there is unique lift

$$
\begin{array}{ccc}
 & & M\mathbb{Z}/l \\
 & \nearrow & \downarrow \\
S^{2l-2} & \xrightarrow{\alpha_1} & S^1
\end{array}
$$

Furthermore $BP_*(A) = v_1 \in BP_* M\mathbb{Z}/l = BP_*/l$. Since $\mathrm{Ext}^0(BP_*/l) = \mathbb{Z}/l[v_1]$ [L], we conclude from the nilpotence theorem that $\mathbb{Z}/l[A] \hookrightarrow \pi_* M\mathbb{Z}/l = \pi_*(S^0; \mathbb{Z}/l)$ is an isogeny. As noted above, A determines an l-adic isogeny class $[A_\nu] \in \pi_* M\mathbb{Z}/l^\nu$, where in fact we can take $A_\nu \in \pi_{2(l-1)l^{\nu-1}} M\mathbb{Z}/l^\nu$, with $BP_*(A_\nu) = v_1^{l^{\nu-1}}$. Thus the mapping telescope $A_\nu^{-1} M\mathbb{Z}/l^\nu$ is independent of the choice of A_ν. If E is a ring spectrum with unit map $S^0 \xrightarrow{i} E$ the class $[i_* A_\nu]$ will also be denoted A_ν. Any of these elements $A_\nu \in \pi_* E \wedge M\mathbb{Z}/l^\nu$ will be

called an *Adams element*. The corresponding self-map $\overline{A} : S^r \wedge M\mathbb{Z}/l^\nu \to M\mathbb{Z}/l^\nu$ given by multiplication by A is an isomorphism on K-theory and will be called an Adams map.

Now consider the commutative ring spectra $B\mathbb{Z}/l_+^\nu$. Here "+" denotes a disjoint basepoint, and the ring structure on the suspension spectrum comes from the H-space structure on $B\mathbb{Z}/l^\nu$ (see also §2). Note that it is an augmented ring spectrum: the natural map of spaces $B\mathbb{Z}/l_+^\nu \to S^0$ induces a map of ring spectra $\varphi \colon B\mathbb{Z}/l_+^\nu \to S^0$ such that ε composed with the unit map is the identity. The fibre of ε is of course $B\mathbb{Z}/l^\nu$. Let $\alpha \colon S^1 \to B\mathbb{Z}/l^\nu$ be a generator of π_1. The natural map $\varphi \colon \pi_2(B\mathbb{Z}/l^\nu \, ; \mathbb{Z}/l^\nu) \to \pi_1 B\mathbb{Z}/l^\nu$ is an isomorphism (stably and unstably) and we let $\beta = \varphi^{-1}(\alpha)$. We regard β as an element of $\pi_2(B\mathbb{Z}/l_+^\nu \, ; \mathbb{Z}/l^\nu)$, with $\varepsilon_*\beta = 0$. Following Snaith we call β a *Bott element*. The terminology comes from the fact that on the space level the natural map $B\mathbb{Z}/l^\nu \to \mathbb{C}P^\infty \to BU$ takes β to a generator of $\pi_2(BU \, ; \mathbb{Z}/l^\nu)$, and similarly on the spectrum level the natural map of ring spectra $B\mathbb{Z}/l_+^\nu \to bu$ takes β to a generator x of $\pi_2 bu \wedge M\mathbb{Z}/l^\nu$. Since $\pi_*(bu \, ; \mathbb{Z}/l^\nu) = \mathbb{Z}/l^\nu[x]$, this shows β is nonnilpotent. Note that the isogeny class of β is independent of the choice of α. Returning to the case $\nu = 1$, we now have nonnilpotent elements $A \in \pi_{2l-2} B\mathbb{Z}/l_+ \wedge M\mathbb{Z}/l$ and $\beta \in \pi_2 \beta\mathbb{Z}/l_+ \wedge M\mathbb{Z}/l$. Note that A and β are not isogenous, since $\varepsilon_*A = A$ and $\varepsilon_*\beta = 0$. In fact:

(1.1) **Theorem.** *The Bott element β and the Adams element A generate a subring $\mathbb{Z}/l[\beta, A](\beta^l - A\beta)$ of $\pi_*(B\mathbb{Z}/l_+ \, ; \mathbb{Z}/l)$. Furthermore this subring is a retract, and the kernel of the retraction $r \colon \pi_*(B\mathbb{Z}/l_+, \mathbb{Z}/l) \to \mathbb{Z}/l[\beta, A]/(\beta^l - \beta A)$ is precisely the ideal of nilpotent elements.*

Proof. Let $h \colon \pi_*(B\mathbb{Z}/l_+, \mathbb{Z}/l) \to BP_*(B\mathbb{Z}/l_+, \mathbb{Z}/l)$ denote the mod l BP-Hurewicz map. From the known structure of $BP_*B\mathbb{Z}/l$ it follows easily that $BP_{ev}(B\mathbb{Z}/l \, ; \mathbb{Z}/l) = \bigoplus_{i=1}^{l-1} \Sigma^{2i} BP_*/l$ as BP_*BP-comodules, where the ith copy of BP_*/l is generated by $h(\beta^i)$. In fact the map of ring spectra $B\mathbb{Z}/l_+ \to \mathbb{C}P_+^\infty$ maps $BP_{ev}(B\mathbb{Z}/l \, ; \mathbb{Z}/l)$ isomorphically onto $BP_*(\mathbb{C}P^{l-1}, \mathbb{Z}/l)$. If $b \in BP_2\mathbb{C}P^\infty$ is the standard generator, inspection of the formal group law for BP shows $b_1^l = v_1 b_1$. Hence $(h(\beta))^l = v_1 h(\beta)$; in particular $\beta^l \neq 0$. On the other hand the image of β^l in ordinary mod l homology is zero, so β^l factors through the inclusion of the 2-skeleton $\Sigma M\mathbb{Z}/l \to B\mathbb{Z}/l$. But clearly the same is true of $A\beta$: $h(A\beta) = v_1 h(\beta)$, and $A\beta$ factors through the 2-skeleton. Since $\pi_{2l}\Sigma M\mathbb{Z}/l \cong \mathbb{Z}/l$ by direct calculation, it follows that $\beta^l = A\beta$. Now let Ext^0 denote $\text{Ext}^0_{BP_*BP}(BP_*, -)$. Since $\text{Ext}^0 BP_*/l = \mathbb{F}_l[v_1]$, we have $\text{Ext}^0(BP_{ev}(B\mathbb{Z}/l_+ \, ; \mathbb{Z}/l)) = \mathbb{Z}/l[v_1, h(\beta)]/(h(\beta)^l - v_1 h(\beta))$. Since $h(A) = v_1$, the map h provides the retraction r of the theorem. Any element not in the kernel of r is nonnilpotent. Finally, any odd-dimensional class is automatically nilpotent, since l odd, while any element in the kernel of h nilpotent by the nilpotence theorem. $\square$

(1.2) *Remarks.* (a) Let $\partial \colon \pi_n(E \, ; \mathbb{Z}/l) \to \pi_{n-1}(E \, ; \mathbb{Z}/l)$ denote the Bockstein. Then the relation $\beta^l = A\beta$ shows $(\partial A)\beta + A(\partial \beta) = 0$. Here ∂A generates $\pi_{2l-3} M\mathbb{Z}/l$ and $\partial \beta$ generates $\pi_1 B\mathbb{Z}/l \wedge M\mathbb{Z}/l$. (b) An easy calculation shows $K(1)_* B\mathbb{Z}/l_+ = K(1)_*[\beta]/(\beta^l - v_1\beta)$, where β is identified with its Hurewicz image in $K(1)_*$. Thus $\pi_*(B\mathbb{Z}/l_+ \, ; \mathbb{Z}/l) \to K(1)_* B\mathbb{Z}/l_+$ is a retraction with

kernel the nilpotent elements, as above. But to prove the theorem this way requires analysis of the $K(n)_*$-Hurewicz maps, $2 \leq n \leq \infty$.

We now turn to the v_1-periodic homotopy of $B\mathbb{Z}/l_+ \wedge M$, $M = M\mathbb{Z}/l$. Here the v_1-periodic homotopy (mod l) of a spectrum X is defined to be $(1_X \wedge \overline{A})^{-1}\pi_* X \wedge M$. If X is a commutative ring spectrum this is just $A^{-1}\pi_* X \wedge M$, regarding $X \wedge M$ as an algebra spectrum over M. The terminology comes from the fact that $BP_*(A) = v_1 \in BP_* M$. The v_1-periodic homotopy of M itself was first computed by Mahowald when $l = 2$. For l odd we have

(1.3) **Theorem** [Mi]. *If l is odd, $A^{-1}\pi_* M = \mathbb{F}_l[A, A^{-1}]\langle \partial A \rangle$.* $\square$

Using this one can show

(1.4) **Theorem** [Bou]. *$L_1 M\mathbb{Z}/l^v = A^{-1}M\mathbb{Z}/l^v$. More generally for any spectrum E, $L_1(E \wedge M\mathbb{Z}/l^v) = A^{-1}E \wedge M\mathbb{Z}/l^v$.*

(1.5) **Corollary.** *A map of spectra $X \to Y$ is an isomorphism on $K(1)$ if and only if it is an isomorphism on $A^{-1}\pi_*(\ ; \mathbb{Z}/l)$.*

(1.6) **Theorem.** *$A^{-1}\pi_* B\mathbb{Z}/l_+ \wedge M \cong (A^{-1}\pi_* M)[\beta]/(\beta^l - A\beta)$.*

Proof. Let $\varphi_i : \Sigma^{2i}M \to B\mathbb{Z}/l_+ \wedge M$ be given by

$$S^{2i} \wedge M \xrightarrow{\beta^i \wedge 1} B\mathbb{Z}/l_+ \wedge M \wedge M \xrightarrow{1 \wedge m} B\mathbb{Z}/l_+ \wedge M$$

where $m : M \wedge M \to M$ is multiplication, $0 \leq i \leq l - 1$. Then $\varphi = \bigvee \varphi_i : \bigvee \Sigma^{2i}M \to B\mathbb{Z}/l_+ \wedge M$ is an isomorphism on $K(1)_*$ and hence on $E(1)_*$. By (1.4), $A^{-1}\varphi$ is an equivalence. Thus $A^{-1}\pi_1 B\mathbb{Z}/l_+ \wedge M$ is a free $A^{-1}\pi_* M$-module on $1, \beta, \ldots, \beta^{l-1}$. The ring structure follows from (1.1). $\square$

(1.7) *Remarks.* (a) The class $\partial\beta$ is nonzero in $A^{-1}\pi_* B\mathbb{Z}/l_+ \wedge M$. In terms of the above isomorphism, $\partial\beta$ corresponds to $(A^{-1}\partial A)\beta$—see Remark 1.2.

(b) The proof shows $L_1 B\mathbb{Z}/l_+ \wedge M \cong \bigvee_{0 \leq i \leq l-1} \Sigma^{2i}L_1 M$. We will consider the K-theory localization of $B\mathbb{Z}/l_+$ itself later.

(c) Consider the spectrum $E = \beta^{-1}B\mathbb{Z}/l_+ \wedge M$. The relation $\beta(\beta^{l-1} - A) = 0$ shows $A = \beta^{l-1}$ in $\pi_* E$. In particular A is a unit and E is $E(1)$-local. In fact we clearly have $\pi_* E = \mathbb{F}_l[\beta, \beta^{-1}]\langle \partial\beta \rangle$, and $E \cong \bigvee_{0 \leq i \leq l-2} L_1 M$. If we regard $L_1(B\mathbb{Z}/l_+ \wedge M)$ as a free module spectrum over $\overline{L}_1 M$ with "basis" $1, \beta, \beta^{l-2}, \beta^{l-1} - A$, the natural map $\bigvee_{0 \leq i \leq l-1} L_1 M = L_1(B\mathbb{Z}/l \wedge M) = A^{-1}B\mathbb{Z}/l_+ \wedge M \to \beta^{-1}B\mathbb{Z}/l_+ \wedge M = \bigvee_{0 \leq i \leq l-2} L_1 M$ is just projection on the first $l - 1$ summands.

2. BOTT ELEMENTS IN ALGEBRAIC K-THEORY

In this section we review Snaith's theorem [S]. It will be convenient to use the method of permutative categories [Ma] to define various spectra and maps between them. There are the following permutative categories with associated spectra:

Category	Spectrum
S = finite sets	S^0
SG = finite free G-sets	$\Sigma^\infty BG_+$
PR = finitely generated projective R-modules	KR

If R is commutative **PR** is bipermutative under tensor product and KR is a commutative ring spectrum. Similarly if G is abelian the product $S \times_G T$ of free G-sets makes **SG** bipermutative and $\Sigma^\infty BG_+$ is a commutative ring spectrum—indeed one can check the multiplication agrees with the usual one used in §1. A permutative functor θ induces a map on the corresponding spectra, for which the same notation will be used. We have the following list of such functors, where we have also indicated the corresponding map on plus constructions:

Functor	Map of spectra	Map on plus construction
$\mathbf{SG} \xrightarrow{\varepsilon} \mathbf{S}$	$\Sigma^\infty BG_+ \xrightarrow{\varepsilon} S^0$	Projection $\Sigma_n \wr G \to \Sigma_n$
$S \mapsto S/G$	(augmentation)	
$\mathbf{SG} \xrightarrow{t} \mathbf{S}$	$\Sigma^\infty BG_+ \xrightarrow{t} S^0$	Inclusion $\Sigma_n \wr G \hookrightarrow \Sigma_{n \cdot |G|}$
$S \mapsto S$	(usual transfer)	
$\mathbf{S} \xrightarrow{i} \mathbf{PR}$	$S^0 \xrightarrow{i} KR$	Inclusion $\Sigma_n \subset GL_n R$
$S \mapsto RS$	(unit map of the ring spectrum KR, R commutative)	
$\mathbf{SG} \xrightarrow{\psi} \mathbf{PRG}$	$\Sigma^\infty BG_+ \xrightarrow{\psi} KRG$	$\Sigma_n \wr G \subset GL_n RG$
$S \mapsto RS$		
$\mathbf{PR'} \xrightarrow{\tau} \mathbf{PR'}$	$KR' \xrightarrow{\tau} KR$	$GL_n R' \subseteq GL_{nk} R$
$M \mapsto M$	(transfer)	(if R' free of rank k)
$R \subset R'$, R'		

f.g. projective over R

Remark. If R, G are commutative, ε, i, and ψ (but not t and τ) are bipermutative and therefore yield maps of ring spectra. Of course we also have functors: $\varphi_\# : \mathbf{PR} \to \mathbf{PR'}$ for any ring homomorphism $R \to R'$, where $\varphi_\# M = R' \otimes_R M$. Now let $R = \mathbb{Z}[\frac{1}{l}]$, $R' = R[\xi_l]$, and $G = \mathbb{Z}/l$. Let $\varphi : RG \to R[\xi_l]$ be the obvious map (identifying $\mathbb{Z}/l$ with the torsion subgroup of $R[\xi_l]^*$ by taking $\xi_l = e^{2\pi i/l}$). Let η denote the composite functor $\mathbf{SG} \xrightarrow{\psi} \mathbf{PRG} \xrightarrow{\varphi_\#} \mathbf{PR}$.

(2.1) **Theorem** (Snaith). *The following diagram commutes*

$$
\begin{array}{ccc}
\Sigma^\infty BG_+ & \xrightarrow{\ \eta\ } & KR' \\
{\scriptstyle t-\varepsilon}\downarrow & & \downarrow{\scriptstyle \tau} \\
S^0 & \xrightarrow{\ i\ } & KR
\end{array}
$$

Proof. We show that it and $\tau\eta + i\varepsilon$ are isomorphic functors. Clearly $i \circ t$ is the functor $S \mapsto RS$. Now the ring RG is canonically isomorphic to $R \times R'$ and indeed the category $\mathbf{PRG} \cong \mathbf{PR} \times \mathbf{PR'}$. Explicitly the isomorphism is given by $M \mapsto (\pi M, (1-\pi)M)$, where $\tau = \frac{1}{|G|}\sum_{g \in G} g$. Thus $it = \pi\psi + \tau(1-\pi)\psi$, where we have regarded π and $(1-\pi)$ as functors. But one can easily check that $i\pi\psi \cong i\varepsilon$ and $1-\pi \cong \varphi_\#$ and hence $it \cong \tau\eta + i\varepsilon$. □

Now consider the action of $\operatorname{Aut}\mathbb{Z}/l = \operatorname{Gal}(R'/R)$ on KR'. In the group ring $\mathbb{Z}_{(l)}\operatorname{Aut}\mathbb{Z}/l$ write $1 = \pi_0 + \cdots + \pi_{l-2}$, where the π_i are primitive orthogonal idempotents and $\pi_0 = \frac{1}{l-1}\sum_{g\in\operatorname{Aut}\mathbb{Z}/l} g$. Then KR' (localized at l) splits as a wedge sum of "eigenspectra": $KR' \cong \bigvee_{i=0}^{l=2} X_i$, where $X_i = \pi_i KR'$. Straightforward manipulations of functors as in (2.1) then yield

(2.2) Proposition. *The transfer* $\tau : KR' \to KR$ *factors through an equivalence* $X_0 \xrightarrow[\cong]{f} KR$. *In fact* $f^{-1} = \frac{1}{l-1}\pi_0 i_\#$, *where* $i : R \hookrightarrow R'$.

In particular τ annihilates X_i for $i > 0$. Similarly, $B\mathbb{Z}/l_+$ splits into eigenspectra Y_i, with $Y_0 \cong (B\Sigma_l)_+$, and t factors through Y_0.

(2.3) Theorem (Snaith). $\eta_*\beta^{l-1} = A$ *in* $\pi_*(KR';\mathbb{Z}/l)$.

Proof. Since β^{l-1} is fixed by $\operatorname{Aut}\mathbb{Z}/l$, and η is $\operatorname{Aut}\mathbb{Z}/l$-equivariant, by (2.2) it is enough to show $\tau\eta_*\beta^{l-1} = A$ or equivalently $(t-\varepsilon)_*\beta^{l-1} = A$, using (2.1). But $\varepsilon_*\beta = 0$, and it is a very well-known fact that $t_*\beta^{l-1} = A$ (which can be proved in a number of interesting ways using Hopf invariants, e-invariants, the Kahn-Priddy theorem, framed cobordism, or whatever the reader prefers). $\square$

Now β^{l-1} defines an l-adic isogeny class—the Bott class—in $\pi_*K\mathbb{Z}\wedge M\mathbb{Z}/l^v$. Using the unique ring homomorphism $\mathbb{Z} \to R$ (R any ring) we obtain a Bott class for the K-theory of any ring or more generally any scheme. If R happens to have l^vth roots of unity, this class coincides with the obvious one obtained from $\eta : B\mathbb{Z}/l^v_+ \to KR$. Thus we have a well-defined Bott class for any algebraic K-theory spectrum (see [DFST]). But Snaith's theorem shows the Bott isogeny class coincides with the Adams isogeny class. This is convenient since the Adams class is obviously defined and functorial for all spectra. More importantly, by (1.4) we have

(2.4) Corollary. *Let* X *be any scheme with Bott element* $\beta \in \pi_*KX \wedge M\mathbb{Z}/l^v$. *Then* $\beta^{-1}KX \wedge M\mathbb{Z}/l^v = A^{-1}KX \wedge M\mathbb{Z}/l^v = L_1 KX \wedge M\mathbb{Z}/l^v$.

(2.5) Remark. In (2.1) we used very little about the ring $R = \mathbb{Z}[\frac{1}{l}]$. We will not bother to state a general version of (2.1), but we note for future reference that if $l\nmid p-1$, $\mathbb{F}_q = \mathbb{F}_p[\xi_l]$ and $[\mathbb{F}_q : \mathbb{F}_p] = l-1$, the same argument shows that the diagram

$$\begin{array}{ccc} B\mathbb{Z}/l_+ & \xrightarrow{\ \eta\ } & K\mathbb{F}_q \\ {\scriptstyle t-\varepsilon}\downarrow & & \downarrow{\scriptstyle \tau} \\ S^0 & \xrightarrow[\ i\ ]{} & K\mathbb{F}_p \end{array}$$

commutes.

3. $K(1)$-LOCALIZATION

In this section we discuss the $K(1)$-localization of $B\mathbb{Z}/l_+$ and related spectra. The following remarks are useful:

(3.1) Proposition. *If* X *is bounded above,* $K(1)_*X = 0$. *Hence* $K(1)_*$ *is invariant under passage to connective covers.* $\square$

This is true more generally for $K(n)_*$, $1 \leq n < \infty$, see [R1, (4.8)]. Here there is a trivial proof: $A^{-1}\pi_*X \wedge M = A^{-1}[M, X] = 0$ since X bounded above. Hence by the easy part of (1.5), $K(1)_*X = 0$.

(3.2) Proposition. *Let E be any spectrum and let $M = M\mathbb{Z}/l$. Then $L_{E \wedge M}(X) = (L_E X)^\wedge$ for all X. In particular $L_{K(1)}X = (L_1 X)^\wedge$.*

Proof. Consider $X \xrightarrow{i} L_E X \xrightarrow{j} (L_E X)^\wedge$. Clearly $E \wedge M \wedge i$ is an equivalence, and $M \wedge j$ is an equivalence by definition of the completion. Hence $E \wedge M \wedge ji$ is an equivalence. Now suppose Y is a spectrum with $E \wedge M \wedge Y \sim *$; we must show $[Y, (L_E X)^\wedge] = 0$. Certainly $[M \wedge Y, L_E X] = 0$ and hence $[Y, L_E X \wedge M] = 0$. By induction on ν, $[Y, L_E X \wedge M\mathbb{Z}/l^\nu] = 0$. But $(L_E X)^\wedge = \text{holim}_\nu L_E X \wedge M\mathbb{Z}/l^\nu$ so $[Y, (L_E X)^\wedge] = 0$. □

(3.3) Corollary. *Suppose $f \colon X \to Y$ with the fibre of f bounded above, and Y is $E(1)$-local. Then $X \to Y^\wedge$ is $K(1)$-localization.* □

(3.4) Examples. (a) $L_{K(1)}bu = KU^\wedge$. Note however that $L_1 bu$ is not KU; see [R1, 9.21]. Similarly $L_{K(1)}e(1) = E(1)^\wedge$.

(b) $L_{K(1)}S^0 = L_{K(1)}j = J^\wedge$. The first equality comes from the fact that $S^0 \to j$ is a $K(1)$ isomorphism. This fact can be proved directly, but also follows from Miller's theorem by computing $A^{-1}\pi_*(\ ,\ \mathbb{Z}/l)$.

(c) $L_{K(1)}K\mathbb{F}_q = L_{K(1)}j(q) = J(q)^\wedge$.

(3.5) Theorem. *Let $t : B\mathbb{Z}/l^+ \to S^0$ denote the transfer, and let q be a prime power such that $l|q - 1$ but $l^2 \nmid q - 1$. Let $\theta : B\mathbb{Z}/l^+ \to K\mathbb{F}_q$ be the map of ring spectra induced by an embedding $\mathbb{Z}/l \hookrightarrow \mathbb{F}_q^*$. Then $t \vee \theta : B\mathbb{Z}/l_+ \to S^0 \vee K\mathbb{F}_q$ is a $K(1)_*$ isomorphism.*

Proof. It would be possible to calculate $K(1)_*$ directly, but instead we will show $(t \vee \theta)_*$ is an isomorphism on v_1-periodic homotopy. In the source we have a free $A^{-1}\pi_*M$-module on $1, \beta, \ldots, \beta^{l-1}$. In the target we have a free $A^{-1}\pi_*M$-module on $i_0, 1, \ldots, \beta^{l-2}$ where i_0 generates the S^0 summand and $1, \beta, \ldots, \beta^{l-2}$ generate the $K\mathbb{F}_q$ summand since $\beta^{l-1} = A$ in $\pi_*K\mathbb{F}_q \wedge M$ by Snaith's theorem. We claim that the matrix of $t \vee \theta$ with respect to these bases is

$$\begin{pmatrix} 0 & 0 & & 0 & A \\ 1 & 0 & & \cdot & A \\ 0 & 1 & \cdots & \cdot & 0 \\ \vdots & 0 & & \vdots & 0 \\ \vdots & \vdots & & 0 & \vdots \\ 0 & 0 & & 1 & 0 \end{pmatrix}$$

and hence is invertible.

For the first column we have $t_*1 = l \cdot i_0$ integrally and hence $t_*1 = 0 \mod l$, while obviously $\theta_*1 = 1$. For the middle columns we need to check $t_*\beta^i = 0$ for $1 \leq i \leq l - 2$, but in fact $\pi_k M = 0$ for $0 < k < 2l - 3$ (alternatively, this follows from the $\text{Aut}\,\mathbb{Z}/l$-action). Since $\theta_*\beta^{l-1} = A$ by Snaith's theorem (see Remark (2.5)) and $t_*\beta^{l-1} = A$, the proof is complete. □

(3.6) Corollary. $L_{K(1)}B\mathbb{Z}/l_+ \cong J^\wedge \vee J(q)^\wedge$.

Remark. When $l = 2$, $t : \mathbb{R}P^\infty \to S^0$ is a $K(1)$-isomorphism, so $L_{K(1)}\mathbb{R}P^\infty = L_{K(1)}S^0 = J^\wedge$; see [R1, 9.1], for $L_1\mathbb{R}P^\infty$.

(3.7) **Corollary.** $K(1)_*K\mathbb{F}_q \cong \mathbb{F}_l[\beta]/(\beta^{l-1} - v_1)$.

Now let $B\mathbb{Z}/l_+ \cong Y_0 \vee Y_1 \vee \cdots \vee Y_{l-2}$ denote the splitting into $\operatorname{Aut}\mathbb{Z}/l$-eigenspectra as in §2. Let $F \xrightarrow{h} B\mathbb{Z}/l_+$ denote the fibre of $B\mathbb{Z}/l_+ \to K\mathbb{F}_q$.

(3.8) **Corollary.** $L_{K(1)}F \to L_{K(1)}B\mathbb{Z}/l_+$ *factors through* $L_{K(1)}Y_0$.

Proof. We must show $L_{K(1)}F \to L_{K(1)}B\mathbb{Z}/l_+ \to L_{K(1)}Y_i$ is null for $i > 0$. But inspection of the $K(1)$-equivalence $t \vee \theta$ shows that the restriction of θ to $L_{K(1)}Y_i$ is inclusion of a wedge summand for $i > 0$. Since θh is null by definition, the result follows. □

4. HARMONIC LOCALIZATION

Let $E(\infty) = \bigvee_{0 \le n < \infty} K(n)$, where $K(n)$ is the nth Morava K-theory. The *harmonic localization* $L_\infty X$ of a spectrum X is the Bousfield localization $L_{E(\infty)}X$. X is *harmonic* if $X \underset{\cong}{\to} L_\infty X$; i.e. X is already $E(\infty)$-local. Let $L_\infty^\wedge X = (L_\infty X)^\wedge$ denote the l-completion of the harmonic localization.

(4.1) **Proposition.** *Let* $F(\infty) = \bigvee_{1 \le n \le \infty} K(n)$. *Then* $L_\infty^\wedge X = L_{F(\infty)}X$.

Proof. By (3.2) $L_\infty^\wedge X = L_{E(\infty)\wedge M}X$. But $K(0) \wedge M \sim *$ and $K(n) \wedge m \cong K(n) \vee \Sigma K(n)$ is Bousfield equivalent to $K(n)$, and the proposition follows. □

(4.2) **Proposition.** *Let* X *be any spectrum such that* $K(n)_*X = 0$ *for* $n \ge 2$. *Then the natural maps* $L_\infty X \to L_1 X$ *and* $L_\infty^\wedge X \to L_1^\wedge X$ *are equivalences.*

Proof. $L_1 X$ is harmonic, so it is enough to show that $f : L_\infty X \to L_1 X$ is an isomorphism on $K(n)$, $0 \le n < \infty$. Certainly this is true for $n = 0, 1$. For $n \ge 2$ we need only check that $K(n)_*L_1 X = 0$. This can be seen in many different ways, one of which is the following: Since $L_1 X = X \wedge L_1 S^0$ [R1], by the Künneth isomorphism it is enough to check the case $X = S^0$. But $L_1 S^0$ can be constructed by cofibrations from KU and $H\mathbb{Q}$ [R1], and $K(n)_*KU = 0 = K(n) * H\mathbb{Q}$ if $n \ge 2$. □

(4.3) **Corollary.** *Let* X *be any ring (possibly noncommutative) or any scheme. Then* $L_1 KX \cong L_\infty KX$ *and* $L_1^\wedge KX \cong L_\infty^\wedge KX$.

Proof. $K(n)_*KX = 0$ for $n \ge 2$ by [Mit]. □

Now let X be a scheme. By "Thomason's hypotheses" we mean the hypotheses of Theorem (4.1) in [T]. In particular X should be a regular scheme of finite Krull dimension, with $\frac{1}{l} \in X$ and also $\sqrt{-1} \in X$ if $l = 2$. The hypotheses are very general, and include for example smooth varieties over a field of characteristic $\ne l$ which is finite or separably closed, and $\operatorname{Spec}\mathcal{O}_F[\frac{1}{l}]$, where $\mathcal{O}_F$ is the ring of integers in a number field F (with $\sqrt{-1} \in F$ if $l = 2$).

(4.4) **Theorem.** *Let* X *be a scheme satisfying Thomason's hypotheses and let* KX *denote the associated K-theory spectrum. Then the Lichtenbaum-Quillen conjectures hold for the harmonic localization of KX: there is a spectral sequence with* $E_2^{p,q} = H_{\text{ét}}^p(X; \mathbb{Z}/l^\nu(q/2))$, *converging to* $\pi_{q-p}(L_\infty KX; \mathbb{Z}/l^\nu)$.

Remarks. (a) The notation means $E_2^{p,q} = 0$ if q odd; the indexing is Thomason's, so that d_r has bidegree $(r, r-1)$. Note q and $q-p$ can be negative. For further discussion, see [T].

(b) As usual the hypothesis X regular can be dropped provided K-theory is replaced by G-Theory.

Proof. By Thomason's theorem there exists such a spectral sequence converging to $\pi_*(L_1KX; \mathbb{Z}/l^\nu)$, but $L_1KX = L_\infty KX$ by [Mit]. $\square$

(4.5) **Theorem.** *The unit map* $S^0 \to L_\infty K\mathbb{Z}$ *factors through the connective J-spectrum*:

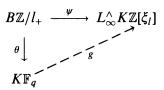$$S^0 \longrightarrow L_\infty K\mathbb{Z}$$
$$\downarrow$$
$$j$$

Proof. Since $L_\infty K\mathbb{Z} = L_1 K\mathbb{Z}$, this follows from the fact that $S^0 \to j$ is an isomorphism on K-theory. $\square$

Remarks. (a) If Theorem (4.5) holds without localizing—e.g. if the Lichtenbaum-Quillen conjectures are true—it follows easily that every algebraic K-theory spectrum KX, indeed every module spectrum over $K\mathbb{Z}$, is a module spectrum over j.

(b) Theorem (4.5) is true on the zero-space level without localizing [Mit].

More generally one may conjecture that the natural map $B\mathbb{Z}l_+^\nu \to K\mathbb{Z}[\xi_{l^\nu}]$ factors through $K\mathbb{F}_q$ for a suitable finite field $\mathbb{F}_q$; this is known to be true on the zero-space level [DFM]. It seems very likely that this is at least true after harmonic localization, but here we will be content with the following special case:

(4.6) **Theorem.** *Let p be a prime generating $(\mathbb{Z}/l^2)^*$, and let $q = p^{l-1}$. Then there is a factorization g in the diagram*

$$B\mathbb{Z}/l_+ \xrightarrow{\ \psi\ } L_\infty^\wedge K\mathbb{Z}[\xi_l]$$
$$\theta \downarrow \qquad \nearrow g$$
$$K\mathbb{F}_q$$

Proof. As usual $L_\infty^\wedge$ can be replaced by $L_1^\wedge$. However θ is not an isomorphism on $K(1)$, so the trivial argument of (4.5) does not apply. But consider the diagram

$$B\mathbb{Z}/l_+ \xrightarrow{\ \theta\ } K\mathbb{F}_q$$
$$t-\varepsilon \downarrow \qquad\qquad \downarrow \tau$$
$$S^0 \xrightarrow{\ i\ } K\mathbb{F}_p$$

as in Remark (2.5); and let $F \xrightarrow{h} B\mathbb{Z}/l_+$ denote the fibre of θ. Then $i \circ (t-\varepsilon) \circ h$

is nullhomotopic. Note $K\mathbb{F}_p = j$. Now consider the diagram

$$
\begin{array}{ccc}
B\mathbb{Z}/l_+ & \xrightarrow{\psi} & L_1 K\mathbb{Z}[\xi_l] \\
{\scriptstyle t-\varepsilon}\Big\downarrow & & \Big\downarrow{\scriptstyle L_1\tau} \\
S^0 & \longrightarrow & L_1 K\mathbb{Z} \\
{\scriptstyle i}\Big\downarrow & & \\
j & &
\end{array}
$$

We conclude that $(L_1\tau) \circ \psi \circ h$ is null.

Now as noted in §2, τ can be identified with projection on the fixed spectrum of the $\operatorname{Aut}\mathbb{Z}/l$ action on $K\mathbb{Z}[\xi_l]$. Hence it remains to show that the projections of ψh on the other $l - 2$ eigenspectra are all trivial. But by (3.8) $L_1^\wedge F \to L_1^\wedge B\mathbb{Z}/l_+$ factors through the fixed spectrum $L_1^\wedge B\Sigma_{l_+}$, and ψ is $\operatorname{Aut}\mathbb{Z}/l$-equivariant, which completes the proof. $\square$

(4.7) **Theorem.** *Let F be a number field with ring of integers $\mathscr{O}_F$, and let $\mathscr{P}$ be a nonzero prime ideal in $\mathscr{O}_F$ with residue field $\mathbb{F}_q$, $\operatorname{char}\mathbb{F}_q \neq l$. Suppose that $[F(\xi_l) : F] = [\mathbb{F}_q(\xi_l) : \mathbb{F}_q]$ and that the l-torsion subgroups of $F(\xi_l)^*$ and $\mathbb{F}_q(\xi_l)^*$ have the same order. If $l = 2$, assume $\sqrt{-1} \in F$. Then the natural map $L_\infty^\wedge K\mathscr{O}_F \to L_\infty^\wedge K\mathbb{F}_q$ is a retraction. In other words, $K\mathbb{F}_q$ is a wedge summand of $K\mathscr{O}_F$ after completed harmonic localization.*

Proof. Once more, we can replace $L_\infty^\wedge$ by $L_{K(1)}$. By a theorem of Bousfield [Bou2], the functor $L_{K(1)}$ factors through Ω^∞: spectra $\to$ spaces. By a theorem of Harris and Segal [HS], $\Omega^\infty K\mathscr{O}_F \to \Omega^\infty K\mathbb{F}_q$ is a retraction under the above hypotheses, and the result follows. $\square$

(4.8) **Theorem.** *Suppose l is an odd regular prime. Then there are maps*

$$
K\mathbb{Z}\left[\xi_l, \frac{1}{l}\right] \xrightarrow{f} K\mathbb{F}_q^\wedge \vee \left(\bigvee_{1 \le i \le (l-1)/2} \Sigma bu^\wedge\right)
$$

and

$$
K\mathbb{Z}\left[\frac{1}{l}\right] \xrightarrow{g} K\mathbb{F}_p^\wedge \vee \Sigma bo^\wedge
$$

which induce equivalences on completed harmonic localizations. Here $\mathbb{F}_q$, $\mathbb{F}_p$ are suitable residue fields as in (4.7); in particular $K\mathbb{F}_p^\wedge$ is the l-adic connective J-spectrum.

Proof. Write $Y^\wedge$ for the target of the proposed map f, and let $R = \mathbb{Z}[\xi_l, \frac{1}{l}]$. Let $K^{\text{ét}}R$ denote the étale K-theory spectrum of Dwyer and Friedlander [DF1]. There is a natural map of ring spectra $j : KR \to K^{\text{ét}}R$ and we claim $L_\infty^\wedge(j)$ is an equivalence; i.e. j_* is an isomorphism on $K(n)$ for $n \ge 1$. If $n \ge 2$ we first show $K(n)_* K^{\text{ét}}R = 0$. This follows from the argument of [Mit], but a much easier proof is available here: Theorem (5.6) of [DF1] implies that multiplication by the Adams map induces an isomorphism $\pi_n(K^{\text{ét}}R; \mathbb{Z}/l) \xrightarrow{\cong}$ $\pi_{n+2(l-1)}K^{\text{ét}}(R; \mathbb{Z}/l)$ for all $n \ge 0$, and hence $(K^{\text{ét}}R)^\wedge$ is the (-1)-connected

cover of its $K(1)$-localization. Hence $K(n)_*K^{\text{ét}}R = 0$, $n \geq 2$ (in fact all of this would be true for the étale K-theory of any scheme). By [Mit] again, this proves the claim for $n \geq 2$. For $n = 1$, Thomason's theorem implies that j is an isomorphism on $A^{-1}\pi_*(-;\mathbb{Z}/l)$. Hence j is an isomorphism on $K(1)$, which completes the proof of the claim. On the other hand in [DF2] Dwyer and Friedlander construct an equivalence $h : K^{\text{ét}}R \to Y^\wedge$ (this is where the assumption l regular is used). Then hj is the desired map. The construction of g is the same, appealing to [DF3] for the final step. $\square$

Remarks. (1) In [DF2] it is never explicitly shown that there is a map of *spectra* $K^{\text{ét}}R \to Y^\wedge$; the map is constructed on the space level. However, according to Dwyer and Friedlander there is no difficulty in carrying out the construction on the spectrum level. Nevertheless it may be of interest to note that the spectrum level result actually *follows* from the space level result: for both $K^{\text{ét}}R$ and $Y^\wedge$ are connective covers of their $K(1)$-localizations (the proof for $Y^\wedge$ is the same as the proof of $K^{\text{ét}}R$ given above). Hence if $\Omega^\infty K^{\text{ét}}R \cong \Omega^\infty Y^\wedge$, Bousfield's theorem [Bou2] implies $K^{\text{ét}}R \cong Y^\wedge$.

(2) If $l = 2$, similar results hold for rings such as $\mathbb{Z}[\xi_{2^n}, \frac{1}{2}]$ if $n \geq 2$, using the remarks on p. 144 of [DF2]. In the case $\mathbb{Z}[\frac{1}{2}]$ one would like an analogous result with $Y^\wedge$ the homotopy pullback of the diagram

$$b_0$$
$$\downarrow$$
$$K\mathbb{F}_3 \longrightarrow bu;$$

see [DF2, (4.2)]. The trouble lies in identifying the $K(1)$-localization of $K\mathbb{Z}[\frac{1}{2}]$, since Thomason's theorem does not apply when $l = 2$ and $\sqrt{-1} \notin R$.

(3) Let Z be any spectrum which is local with respect to $F(\infty) = K(1) \vee K(2) \vee \cdots$. Then for R, Y as above $[Y^\wedge, Z] \xrightarrow{\cong} [KR, Z]$. In particular we could take $Z = E^\wedge$, where E is harmonic; for example $E = BP$ [R1, 4.2]. So, for example, $(BP^\wedge)^*(K\mathbb{Z}[\frac{1}{l}]) \cong (BP^\wedge)^*(j_l \vee \Sigma bo)$—when l is an odd regular prime. The groups $(BP^\wedge)^*(j_l)$ and $(BP^\wedge)^*(bo)$ are well within the range of current technique and certainly could be computed. This leads to a rather peculiar state of affairs: we can compute the l-adic BP-cohomology of $K\mathbb{Z}[\frac{1}{l}]$, but not yet the ordinary cohomology! It is important to note here, however, that the BP-cohomology of a spectrum does not determine its BP-homology; as the example of a $\mathrm{mod}\, l$ Eilenberg-Mac Lane spectrum shows.

Of course we can also compute the $K(1)$-homology in (4.8). Let $S = K(1)_*[t_1, t_2, \ldots]/(t_n^p - v_1^{p^{n-1}} t_n)$, where $|t_n| = 2(p^n - 1)$, and regard S as merely a graded vector space.

(4.9) **Theorem.** *Let l be an odd regular prime. Then as $K(1)$-modules*

$$K(1)_*K\mathbb{Z}\left[\xi_l, \frac{1}{l}\right] \cong K(1)_*[\beta]/(\beta^{l-1} - v_1) \oplus K(1)_* \otimes \left(\bigoplus_{i=0}^{l-2}(l-1)\cdot\Sigma^{2i+1}S\right)$$

and

$$K(1)_*K\mathbb{Z}\left[\frac{1}{l}\right] \cong K(1)_* \oplus K(1)_* \otimes \left(\bigoplus_{i=0}^{l-2/2}\Sigma^{4i+1}S\right).$$

Proof. By (4.8) $K(1)_*KR = K(1)_*Y^\wedge$ (this uses only the work of Thomason and Dwyer-Friedlander, since only $K(1)$ is involved). Since $K(1)_*Y^\wedge = K(1)_*Y$ for any spectrum Y, we have only to compute $K(1)_*$ of $K\mathbb{F}_q$, bu, and bo. The first case is described in §3. Now let $e(1)$ denote the connective Adams summand of complex K-theory, thus $\pi_*e(1) = \mathbb{Z}_{(l)}[v_1]$. Then $bu \cong \bigvee_{i=0}^{l-2} \Sigma^{2i}e(1)$ and $bo \cong \bigvee_{i=0}^{l-2} \Sigma^{4ii}e(1)$ (since l odd). On the other hand $K(1)_*e(1) = K(1)_* \otimes S$ (see [R2], and beware that its author writes $K(1)_*K(1)$ for the even-dimensional part of $K(1)_*K(1)$, which coincides with $K(1)_*e(1)$.) □

Remark. (a) The first summand in each of the isomorphisms of (4.9) is clearly a retract as $K(1)_*$-algebras. The second summand is concentrated in odd dimensions and hence consists of elements with square zero. Presumably the full ring structure could be worked out. (b) In both cases the spectrum Y has the same rational homotopy groups as KR, by Borel's theorem [Bor]. Naturally, one would like to combine this fact with (4.8) to conclude $L_\infty KR = L_\infty Y$; unfortunately, this seems to be difficult.

We conclude with some remarks on the application of the nilpotence theorem to $K_*(R, \mathbb{Z}/l)$. If the Lichtenbaum-Quillen conjectures hold for a scheme X, and $\beta \in K_*(X, \mathbb{Z}/l)$ is a Bott element, then $\mathbb{Z}/l[\beta] \to K_*(X, \mathbb{Z}/l)$ is an isogeny. In particular this is true for $L_\infty^\wedge KX$, if X is as in (4.3). In any case the nilpotence theorem together with the fact that $K(n)_*KX = 0$ for $n \geq 2$ implies that any nonnilpotent element of $K_*(X, \mathbb{Z}/l)$ is detected by either $K(1)_*$ or $H_*(\ ,\mathbb{Z}/l)$ (or both). For example, let $X = \operatorname{spec} R$ with R as in (4.8). Then by combining (0.2), [Mit], and (4.9) we see that if $\alpha \in \pi_*KX \wedge M$ exactly one of the following holds: either α is isogenous to β, α is nilpotent, or α has nonnilpotent Hurewicz image in $\operatorname{mod} l$ homology.

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