

***N*-BODY OBSERVABLES IN THE CALKIN ALGEBRA**

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ABSTRACT. The commutators of many operators which are used in the phase space analysis of the N -body scattering are compact. This fact makes it possible to give a description of certain classes of such operators in terms of commutative C^* -algebras inside the Calkin algebra.

1. INTRODUCTION

The quantum evolution of a system of N particles is usually described by selfadjoint operators on $L^2(\mathbb{R}^{Nn})$ of the form

$$(1) \quad H = - \sum_{i=1}^N \frac{1}{2} \Delta_i + \sum_{i,j=1}^N v_{ij}(x_i - x_j)$$

[RS1, CFKS]. Such operators are called N -body *Schrödinger operators*. If we try to study the scattering theory for such operators it is very useful to look at them “microlocally” [SigSof, De1]. A typical tool used in the “phase space analysis” of N -body systems are operators of the form

$$(2) \quad h(H)Q(x/|x|)g(D),$$

where $h(H)$ is a cutoff in the energy, $Q(x/|x|)$ is a cutoff in the configuration space, and $g(D)$ is a cutoff in the momentum. The phase space analysis of N -body systems is in a sense more difficult and more interesting than the usual microlocal analysis [Hö] because the potentials do not decay at infinity in certain directions of the configuration space. In particular, the position and momentum cutoffs that appear in operators of the form (2) used in the study of N -body systems usually satisfy certain geometric requirements reflecting the geometry of N -body potentials.

It turns out that the commutator of two operators of the form (2) (with certain natural conditions on h , Q , g) is compact. Consequently, the images of such operators inside the Calkin algebra form a commutative C^* -algebra. (Recall that the Calkin algebra is the algebra of bounded operators on a Hilbert space factored by the compact operators.)

It is possible to give a fairly detailed description of this commutative algebra inside the Calkin algebra. Namely, it is isomorphic to the space of continuous functions on a certain locally compact set ES vanishing at infinity. The set ES

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is a certain geometric object whose definition takes into account the geometry of the N -body phase space and the spectra of all the subsystems. (The letters ES stand for the energy shell.)

Another class of N -body observables that we study in this paper are operators of the form (2) multiplied by an operator that belongs to the pure point spectrum of cluster Hamiltonians. We can also describe the images of such operators inside the Calkin algebra in terms of C^* -algebras.

Originally, the formalism of this paper was intended as a tool in our study of asymptotic observables [De2]. Theorem 2.1 was an important preliminary step in our original proof of the main result of [De2]. This proof was based on Sigal's and Soffer's propagation theorem [SigSof, De1]. It turned out, however, that by using more recent ideas of [Graf] one can simplify the proof of the main result of [De2] significantly and avoid the application of Theorem 2.1. Nevertheless, we think that the formalism of the Calkin algebra presented in this paper reveals a certain nontrivial natural structure of N -body systems which is interesting in itself.

The proof of Theorem 2.1 has two major ingredients. One of them is the compactness of certain observables related to N -body systems. Facts of this sort are well known and are usually associated with the so-called geometric method. The geometric method consists roughly in looking at the N -body Schrödinger operators locally in the configuration space. An early result where this kind of thinking proved useful was the Hunziker-Van Winter-Zhislin theorem [H, VW, Zh, RS2]. More applications of the geometric method can be found in [Sig, E1, Sim, CFKS]. The usefulness of cutoffs both in the configuration space and in the momentum was exploited by V. Enss (eg. [E2, E3]) and also in [SigSof].

The second ingredient of the proof of Theorem 2.1 is the use of certain basic properties of C^* -algebras, especially of the theorem on the tensor product of C^* -algebras [Ta].

2. NOTATION AND THE MAIN RESULT

$B(\mathcal{H})$ will denote the C^* -algebra of bounded operators on a Hilbert space \mathcal{H} and $CB(\mathcal{H})$ will denote the C^* -algebra of compact operators on \mathcal{H} . $CB(\mathcal{H})$ is a closed ideal in $B(\mathcal{H})$. Thus we can define the factor algebra $B(\mathcal{H})/CB(\mathcal{H})$, which is called the *Calkin algebra* [Di]. If $B \in B(\mathcal{H})$ then $[B]_{\text{Cal}}$ will denote the image of B in $\text{Cal}(\mathcal{H})$. We will often drop the subscript Cal if it will not lead to a misunderstanding.

If C is a selfadjoint operator then $\sigma(C)$ will denote its spectrum.

From the mathematical point of view it is convenient to replace the class of N -body Schrödinger operators with a more general class of operators first used by S. Agmon [A, ABG]. In order to introduce this class we need some definitions.

A partially ordered set (\mathcal{A}, \subset) with an operation \cup such that if $a_1 \subset b$ and $a_2 \subset b$ then $a_1 \cup a_2 \subset b$ is called a *semilattice*. Throughout this paper we fix a certain finite semilattice $(\mathcal{A}, \subset, \cup)$. We will assume that there exists an element $a_{\min} \in \mathcal{A}$ such that $a_{\min} \subset a$ for all $a \in \mathcal{A}$. The element $\bigcup_{a \in \mathcal{A}} a$ will be denoted a_{\max} . We can also define the operation \cap inside \mathcal{A} . Namely, we set $a_1 \cap a_2 := \bigcup_{b \subset a_1, b \subset a_2} b$. (There is at least one such b , namely a_{\min} .) In this way \mathcal{A} becomes a lattice.

Throughout the paper X will denote a fixed Euclidean space isomorphic to \mathbb{R}^n . We also fix a certain family $\{X_a : a \in \mathcal{A}\}$ of subspaces of X indexed with the elements of the semilattice \mathcal{A} . We will assume that $a_1 \subset a_2$ whenever $X_{a_1} \supset X_{a_2}$, $a_1 \cup a_2 = b$ whenever $X_{a_1} \cap X_{a_2} = X_b$, $X_{a_{\min}} = X$, and $X_{a_{\max}} = \{0\}$. (In other words, $\{X_a : a \in \mathcal{A}\}$ is a family of subspaces closed wrt the intersection, and containing X and $\{0\}$.) Note that $X_{a \cap b} \supset X_a + X_b$ (but sometimes the inclusion may be strict).

The orthogonal complement of X_a in X will be denoted X^a . The dual spaces to X , X_a , and X^a will be denoted by K , K_a , and K^a resp. π_a will stand for the projection of X onto X_a and of K onto K_a . π^a will stand for the projection of X onto X^a and of K onto K^a . We will often write x_a instead of $\pi_a x$, x^a instead of $\pi^a x$, etc.

S will denote the unit sphere in X . There will also be special symbols for the following sets:

$$Z_a := \left(X_a \setminus \bigcup_{b \not\subset a} X_b \right) \cap S, \quad Y_a := \left(X \setminus \bigcup_{b \not\subset a} X_b \right) \cap S.$$

Note that $Y_a = \bigcup_{b \subset a} Z_b$ and $Y_{a_{\max}} = S$.

D , D_a , and D^a will denote the operators $\frac{1}{i}\nabla$, $\frac{1}{i}\nabla_a$, and $\frac{1}{i}\nabla^a$ on $L^2(X)$. Likewise, Δ , Δ_a , and Δ^a denote the Laplacians corresponding to the variables x , x_a , and x^a .

The Euclidean norm of a vector $x \in X$ will be denoted $|x|$.

We will often abuse notation by identifying functions on the sphere with functions on X homogeneous of degree zero.

We assume that for any $a \in \mathcal{A}$ we are given a real function v_a on X^a (which we will call the potential corresponding to the subsystem $a \in \mathcal{A}$). We will assume that $v_a(x^a)(1 - \Delta^a)^{-1}$ is compact on $L^2(X^a)$.

We set

$$V(x) := \sum_{a \in \mathcal{A}} v_a(\pi^a x).$$

Note that V is bounded relatively to $-\Delta$ with the infinitesimal bound. We define H to be the selfadjoint operator on $L^2(X)$ such that $\mathcal{D}(H) = \mathcal{D}(-\Delta)$ and $H = -\frac{1}{2}\Delta + V$.

If $a \in \mathcal{A}$ then we set

$$V_a(x) := \sum_{b \subset a} v_b(\pi^b x).$$

We define H_a to be the selfadjoint operator on $L^2(X)$ such that $\mathcal{D}(H_a) = \mathcal{D}(-\Delta)$ and $H_a = -\frac{1}{2}\Delta + V_a$.

We may identify $L^2(X)$ with $L^2(X_a) \otimes L(X^a)$. Then we can write

$$H_a = -\frac{1}{2}\Delta_a \otimes 1 + 1 \otimes H^a,$$

where H^a is a certain selfadjoint operator on $L^2(X^a)$ that describes the inner degrees of freedom of the subsystem $a \in \mathcal{A}$. Clearly, $H^{a_{\max}} = H$.

\mathcal{B}_a will denote the pure point spectrum of H^a . $\mathcal{B}_a^{\text{disc}}$ will denote the discrete spectrum of H^a . Note that $H^{a_{\min}}$ is defined on the one-dimensional space $L^2(X^{a_{\min}})$ and equals zero. Therefore $\mathcal{B}_a = \{0\}$.

$\mathcal{H}_{a\tau}$ will denote the subspace of eigenvectors of H^a with eigenvalue τ . $P_{a\tau}$ will denote the projections onto $\mathcal{H}_{a\tau}$.

m will denote a certain fixed real number such that $-m \notin \sigma(H)$.

We will need a number of elementary concepts taken from the theory of commutative C^* -algebras [Di, Ta]. If Λ is a topological space then $C(\Lambda)$ and $C_0(\Lambda)$ denote the $*$ -algebras of continuous functions on Λ and continuous functions on Λ of compact support resp. If Λ is a locally compact space then $C_\infty(\Lambda)$ denotes the C^* -algebra of continuous functions on Λ that vanish at infinity. Note that if Λ is compact then $C_\infty(\Lambda) = C(\Lambda)$.

It is well known that any commutative C^* -algebra with identity is isomorphic to $C(\Lambda)$ for some compact Λ . Any commutative C^* -algebra without identity is isomorphic to $C_\infty(\Lambda)$ for some locally compact but noncompact Λ .

If Λ is locally compact but noncompact then Λ^{comp} will denote the one-point compactification of Λ , that is, $\Lambda \cup \{\infty\}$ where ∞ is the "point at infinity." Note that if we adjoin the identity to $C_\infty(\Lambda)$ then we obtain an algebra canonically isomorphic to $C(\Lambda^{\text{comp}})$. If Λ is compact then we set $\Lambda^{\text{comp}} := \Lambda$.

If Λ_1 and Λ_2 are compact spaces then any continuous map $\sigma: \Lambda_1 \rightarrow \Lambda_2$ induces a $*$ -homomorphism $\sigma^*: C(\Lambda_2) \rightarrow C(\Lambda_1)$ such that if $\psi \in C(\Lambda_2)$ then $\sigma^*(\psi)(\lambda) := \psi(\sigma(\lambda))$. More generally, if Λ_1 and Λ_2 are locally compact spaces then any continuous map $\sigma: \Lambda_1^{\text{comp}} \rightarrow \Lambda_2^{\text{comp}}$ such that $\sigma(\infty) = \infty$ induces a $*$ -homomorphism $\sigma^*: C_\infty(\Lambda_2) \rightarrow C_\infty(\Lambda_1)$ in the same fashion.

If Ξ is a closed subset of a locally compact space Λ then $\Lambda \setminus \Xi$ is also locally compact. The C^* -algebra $C(\Lambda \setminus \Xi)$ can be identified with the subalgebra of $C_\infty(\Lambda)$ consisting of those $\psi \in C_\infty(\Lambda)$ that vanish at Ξ . Any closed ideal inside $C_\infty(\Lambda)$ is of the form of $C_\infty(\Lambda \setminus \Xi)$ for some closed Ξ .

For any $a \neq a_{\max}$ we define

$$es_a := \{(k_a, \lambda) \in K_a \times \mathbb{R} : \lambda - \frac{1}{2}k_a^2 \in \sigma(H^a)\}$$

and

$$ES := \bigcup_{a \neq a_{\max}} Z_a \times es_a.$$

If $a \neq a_{\max}$ and $\tau \in \mathcal{B}_a$ then we define

$$ES_{a\tau} := Z_a \times \{(k_a, \lambda) \in K_a \times \mathbb{R} : \lambda - \frac{1}{2}k_a^2 = \tau\}.$$

We also define

$$ES^{\text{disc}} := \bigcup_{a \neq a_{\max}} \bigcup_{\tau \in \mathcal{B}_a^{\text{disc}}} ES_{a\tau}.$$

Next we define the map

$$\mathcal{K}_a : ES \rightarrow (Y_a \times K_a \times \mathbb{R})^{\text{comp}}$$

as follows. If $(x_b, k_b, \lambda) \in Z_b \times es_b$ then

$$\mathcal{K}_a(x_b, k_b, \lambda) := \begin{cases} (x_b, \pi_a k_b, \lambda) & \text{if } b \subset a, \\ \infty & \text{otherwise.} \end{cases}$$

We set $\mathcal{K}_{a\tau}$ to be the restriction of \mathcal{K}_a to $ES_{a\tau}$ (or equivalently, the imbedding of $ES_{a\tau}$ in $(Y_a \times K_a \times \mathbb{R})^{\text{comp}}$).

We endow ES with the weakest topology such that the maps \mathcal{K}_a are continuous. Note that ES becomes in this way a locally compact space. Let us

also remark that if $(x_a, k_a, \lambda) \in Z_a \times es_a$ then an example of a basis of neighborhoods of (x_a, k_a, λ) in ES can be given by $(\mathcal{K}_a)^{-1}(\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3)$, where $\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3$, are neighborhoods of x_a, k_a, λ in Y_a, K_a, \mathbb{R} resp.

The following theorem is the main result of this paper.

Theorem 2.1. (a) Let $a \neq a_{\max}$. Then there exists a unique $*$ -homomorphism $\tilde{\rho}_a$ from $C_\infty(Y_a, K_a, \mathbb{R})$ to $\text{Cal}(L^2(X))$ that satisfies the following condition. Let $Q \in C_\infty(Y_a)$, $g \in C_\infty(K_a)$, and $h \in C_\infty(\mathbb{R})$. Then

$$\tilde{\rho}_a(Q \otimes g \otimes h) = [Q(x)g(D_a)h(H)]_{\text{Cal}} = [Q(x)g(D_a)h(H_a)]_{\text{Cal}}.$$

(b) There exists a unique isometric $*$ -homomorphism ρ from $C_\infty(ES)$ to $\text{Cal}(L^2(X))$ such that if $\psi \in C_\infty(Y_a \times K_a \times \mathbb{R})$ then $\rho(\mathcal{K}_a^*(\psi)) = \tilde{\rho}_a(\psi)$.

(c) Now assume additionally that $\tau \in \mathcal{B}_a$. Then there exists a unique $*$ -homomorphism $\tilde{\rho}_{a\tau}$ from $CB(\mathcal{H}_{a\tau}) \otimes C_\infty(Y_a, K_a, \mathbb{R})$ to $\text{Cal}(L^2(X))$ that satisfies the following condition. Let $B \in CB(\mathcal{H}_{a\tau})$, $Q \in C_\infty(Y_a)$, $g \in C_\infty(K_a)$, and $h \in C_\infty(\mathbb{R})$. Then

$$\begin{aligned} \tilde{\rho}_{a\tau}(B \otimes Q \otimes g \otimes h) &= [BQ(x)g(D_a)h(H)]_{\text{Cal}} \\ &= [BQ(x)g(D_a)h(H_a)]_{\text{Cal}} \\ &= [BQ(\pi_a x)g(D_a)h(H_a)]_{\text{Cal}}. \end{aligned}$$

(d) There exists a unique isometric $*$ -homomorphism $\rho_{a\tau}$ from $CB(\mathcal{H}_{a\tau}) \otimes C_\infty(ES_{a\tau})$ to $\text{Cal}(L^2(X))$ such that if $B \in CB(\mathcal{H}_{a\tau})$ and $\psi \in C_\infty(Y_a \times K_a \times \mathbb{R})$ then $\rho_{a\tau}(B \otimes \mathcal{K}_{a\tau}^*(\psi)) = \tilde{\rho}_{a\tau}(B \otimes \psi)$.

(e) If $C_i \in \text{Ran } \rho_{a_i \tau_i}$, where $i = 1, 2$ and $(a_1, \tau_1) \notin (a_2, \tau_2)$, then $C_1 C_2 = 0$.

(f) Let $\psi \in C_\infty(ES)$, $B \in CB(\mathcal{H}_{a\tau})$, and $\omega \in C_\infty(ES_{a\tau})$. Then

$$\rho(\psi)\rho_{a\tau}(B \otimes \omega) = \rho_{a\tau}(B \otimes \omega)\rho(\psi) = \rho_{a\tau}(B \otimes \omega\psi|_{ES_{a\tau}}).$$

(g) Let $\psi \in C_\infty(ES^{\text{disc}})$. Then

$$(1) \quad \rho(\psi) = \bigcup_{a \neq a_{\max}} \bigcup_{\tau \in \mathcal{B}_a^{\text{disc}}} \rho_{a\tau}(P_{a\tau} \otimes \psi|_{ES_{a\tau}}).$$

We can summarize (a), (b), (c), and (d) by drawing the following commutative diagrams:

$$(2) \quad \begin{array}{ccc} C_\infty(Y_a \times K_a \times \mathbb{R}) & \xrightarrow{\mathcal{K}_a^*} & C_\infty(ES) \\ & \searrow \tilde{\rho}_a & \swarrow \rho \\ & \text{Cal}(L^2(X)) & \end{array}$$

$$(3) \quad \begin{array}{ccc} CB(\mathcal{H}_{a\tau}) \otimes C_\infty(Y_a \times K_a \times \mathbb{R}) & \xrightarrow{\mathcal{K}_{a\tau}^*} & CB(\mathcal{H}_{a\tau}) \otimes C_\infty(ES_{a\tau}) \\ & \searrow \tilde{\rho}_{a\tau} & \swarrow \rho_{a\tau} \\ & \text{Cal}(L^2(X)) & \end{array}$$

3. COMMUTATIVE C^* -ALGEBRAS

This section contains certain facts about commutative C^* -algebras that we will use in our paper. We start with a number of well-known results which we state without a proof [Di, Ta]. In the sequel we will assume that Λ is a locally compact set and Ξ is its closed subset.

Proposition 3.1. Let $\tilde{\rho}$ be a $*$ -homomorphism of $C_\infty(\Lambda)$ into a C^* -algebra \mathcal{E} . Suppose that $C_\infty(\Lambda \setminus \Xi) \subset \text{Ker } \tilde{\rho}$. Then there exists a unique $*$ -homomorphism $\rho : C_\infty(\Xi) \rightarrow \mathcal{E}$ such that the following diagram commutes:

$$\begin{array}{ccc} C_\infty(\Lambda) & \rightarrow & C_\infty(\Xi) \\ \tilde{\rho} \searrow & & \swarrow \rho \\ & C & \end{array}$$

Proposition 3.2. If \mathcal{E} is a closed ideal in $C_\infty(\Lambda)$ and for every $\lambda \in \Lambda \setminus \Xi$ there exists $\psi \in \mathcal{E}$ such that $\psi(\lambda) \neq 0$ then $C_\infty(\Lambda \setminus \Xi) \subset \mathcal{E}$.

Proposition 3.3. If for $i = 1, \dots, n$ we are given $*$ -homomorphisms $\rho_i : C_\infty(\Lambda_i) \rightarrow \mathcal{E}$ with commuting images then the formula

$$\rho_1 \otimes \dots \otimes \rho_n(\psi_1 \otimes \dots \otimes \psi_n) := \rho_1(\psi_1) \dots \rho_n(\psi_n)$$

defines a $*$ -homomorphism $\rho_1 \otimes \dots \otimes \rho_n : C_\infty(\Lambda_1 \times \dots \times \Lambda_n) \rightarrow \mathcal{E}$.

The remaining part of this section will be devoted to a certain construction which will be used in the proof of our main result. Here we will describe this construction in general terms. The symbols \mathcal{A} , S , Z_a , and Y_a that will be used in this section need not have the meaning that we ascribed to them in the previous section.

Let \mathcal{A} be a finite lattice and S be a compact set. We will assume that $S = \bigcup_{a \in \mathcal{A}} Z_a$, where the Z_a are certain fixed disjoint subsets of S . We set $Y_a := \bigcup_{b \subset a} Z_b$ and assume that the Y_a are open in S and noncompact if $a \neq a_{\max}$. For any $a \in \mathcal{A}$ we assume that Ω_a is a compact set and for any $b \subset a$ we are given a continuous map $\sigma_{ab} : \Omega_b \rightarrow \Omega_a$ such that if $a \subset b \subset c$ then $\sigma_{ab}\sigma_{bc} = \sigma_{ac}$. We set $\Lambda := \bigcup_{a \in \mathcal{A}} Z_a \times \Omega_a$ and $\Lambda_a := (Y_a \times \Omega_a)^{\text{comp}}$. We define a family of continuous maps $\mathcal{K}_a : \Lambda \rightarrow \Lambda_a$ such that if $(x, \omega) \in Z_b \times \Omega_b$ then

$$\mathcal{K}_a(x, \omega) := \begin{cases} (x, \sigma_{ab}(\omega)) & \text{if } b \subset a, \\ \infty & \text{otherwise.} \end{cases}$$

We equip Λ with the weakest topology such that the \mathcal{K}_a are continuous.

Theorem 3.4. Suppose that \mathcal{E} is a C^* -algebra, and both $\alpha : C(S) \rightarrow \mathcal{E}$ and $\zeta_a : C(\Omega_a) \rightarrow \mathcal{E}$ are unital $*$ -homomorphisms that satisfy the following conditions:

- (1) If $Q \in C_\infty(Y_a)$ and $\psi \in C(\Omega_a)$ then $\alpha(Q)\zeta_a(\psi) = \zeta_a(\psi)\alpha(Q)$.
- (2) If $b \subset a$, $Q \in C_\infty(Y_b)$, and $\psi \in C(\Omega_a)$ then

$$\alpha(Q)\zeta_a(\psi) = \alpha(Q)\zeta_b(\sigma_{ab}^*(\psi)).$$

Then there exists a unique $*$ -homomorphism $\rho : C(\Lambda) \rightarrow \mathcal{E}$ such that the following diagram commutes:

$$\begin{array}{ccc} C_\infty(Y_a \times \Omega_a) & \xrightarrow{\mathcal{K}_a^*} & C(\Lambda) \\ \alpha \otimes \zeta_a \searrow & & \swarrow \rho \\ & \mathcal{E} & \end{array}$$

Proof. By (1) there exists a unique $*$ -homomorphism $\alpha \otimes \zeta_a : C_\infty(Y_a \times \Omega_a) \rightarrow \mathcal{E}$ such that $\alpha \otimes \zeta_a(Q \otimes \psi) = \alpha(Q)\zeta_a(\psi)$. We extend $\alpha \otimes \zeta_a$ to a $*$ -homomorphism $\rho_a : C_\infty(\Lambda_a) \rightarrow \mathcal{E}$ by setting $\rho_a(1) := 1$.

Now let $a_1, a_2 \in \mathcal{A}$, $b = a_1 \cap a_2$, $Q_i \in C_\infty(Y_{a_i})$, and $\psi_i \in C(\Omega_{a_i})$. Then

$$(1) \quad \begin{aligned} \rho_{a_1}(Q_1 \otimes \psi_1) \rho_{a_2}(Q_2 \otimes \psi_2) &= \zeta_{a_1}(\psi_1) \alpha(Q_1) \alpha(Q_2) \zeta_{a_2}(\psi_2) \\ &= \zeta_{a_1}(\psi_1) \alpha(Q_1 Q_2) \zeta_{a_2}(\psi_2). \end{aligned}$$

Note that $Y_{a_1} \cap Y_{a_2} = Y_b$. Therefore $Q_1 Q_2 \in C_\infty(Y_{a_1}) \cap C_\infty(Y_{a_2}) = C_\infty(Y_b)$. Thus (1) equals

$$(2) \quad \zeta_b(\sigma_{a_1 b}^*(\psi_1)) \alpha(Q_1 Q_2) \zeta_b(\sigma_{a_2 b}^*(\psi_2)) = \alpha(Q_1 Q_2) \zeta_b(\sigma_{a_1}^*(\psi_1) (\sigma_{a_2 b}^*(\psi_2))).$$

Clearly, $\rho_{a_2}(Q_2 \otimes \psi_2) \rho_{a_1}(Q_1 \otimes \psi_1)$ equals (2) as well. Therefore the images of ρ_{a_1} and ρ_{a_2} commute. Hence there exists a unique $*$ -homomorphism $\bigotimes_{a \in \mathcal{A}} \rho_a : C(\times_{a \in \mathcal{A}} \Lambda_a) \rightarrow \mathcal{C}$. We denote it by $\tilde{\rho}$. Clearly, $\text{Ker } \tilde{\rho} = C_\infty(\times_{a \in \mathcal{A}} \Lambda_a \setminus \Xi)$ for some closed $\Xi \subset \times_{a \in \mathcal{A}} \Lambda_a$. Let us fix our attention at a certain $\lambda = \times_{a \in \mathcal{A}} \lambda_a \in \times_{a \in \mathcal{A}} \Lambda_a$. The remaining part of our proof will be broken into a number of steps.

Step 1. If $\lambda_{a_{\max}} = (y_1, \omega_1)$, $\lambda_b = (y_2, \omega_2)$, and $y_1 \neq y_2$ then $\lambda \notin \Xi$.

To see this choose open neighborhoods \mathcal{U}_1 and \mathcal{U}_2 of y_1 and y_2 in $Y_{a_{\max}}$ and Y_b resp. such that $\overline{\mathcal{U}_1} \cap \overline{\mathcal{U}_2} = \emptyset$. Let $Q_1 \in C_\infty(S)$ and $Q_2 \in C_\infty(Y_b)$ such that $\text{supp } Q_i \subset \mathcal{U}_i$ and $Q_i(y_i) \neq 0$ for $i = 1, 2$. We can treat Q_1 as an element of $C(\Lambda_{a_{\max}})$ and Q_2 as an element of $C(\Lambda_b)$. Now, $\tilde{\rho}(Q_1 \otimes Q_2) = \rho_{a_{\max}}(Q_1) \rho_b(Q_2) = \alpha(Q_1) \alpha(Q_2) = \alpha(Q_1 Q_2) = 0$.

Thus $Q_1 \otimes Q_2 \in \text{Ker } \tilde{\rho}$ and $Q_1 \otimes Q_2(\lambda) \neq 0$. By Proposition 3.2 this implies the statement of Step 1.

Step 2. If $\lambda_{a_{\max}} = (y_1, \omega_1)$, $y_1 \in Z_b$, and $\lambda_a = \infty$ for some $a \supset b$, then $\lambda \notin \Xi$.

In fact, note that $y_1 \in Y_a$. Thus we can find neighborhoods \mathcal{U}_1 and \mathcal{U}_2 of y_1 and ∞ in S and Y_a resp. such that $\overline{\mathcal{U}_1} \cap \overline{\mathcal{U}_2} = \emptyset$. Henceforth we argue as above.

So far we know that if $\lambda \in \Xi$ then there exists $b \in \mathcal{A}$ and $y \in Z_b$ such that

$$(3) \quad \lambda_a = \begin{cases} (y, \omega_a) & \text{for } a \supset b, \\ \infty & \text{otherwise.} \end{cases}$$

Step 3. If λ is of the form (3) and $\omega_a \neq \sigma_{ab}(\omega_b)$ then $\lambda \notin \Xi$.

In fact, let \mathcal{U}_1 and \mathcal{U}_2 be neighborhoods of ω_a and $\sigma_{ab}(\omega_b)$ in Ω_a such that $\overline{\mathcal{U}_1} \cap \overline{\mathcal{U}_2} = \emptyset$. Let $\psi_1, \psi_2 \in C(\Omega_a)$ such that $\psi_1(\omega_a) \neq 0$, $\psi_2(\sigma_{ab}(\omega_b)) \neq 0$, and $\text{supp } \psi_i \subset \mathcal{U}_i$. Let $Q \in C_\infty(Y_b)$ such that $Q(y) \neq 0$. We treat $Q \otimes \psi_1$ as an element of $C(\Lambda_a)$ and $Q \otimes \sigma_{ab}^*(\psi_2)$ as an element of $C(\Lambda_b)$. Now,

$$\begin{aligned} \tilde{\rho}(Q \otimes \psi_1 \otimes Q \otimes \sigma_{ab}^*(\psi_2)) \\ &= \rho_a(Q \otimes \psi_1) \rho_b(Q \otimes \sigma_{ab}^*(\psi_2)) = \alpha(Q) \zeta_a(\psi_1) \zeta_b(\sigma_{ab}^*(\psi_2)) \alpha(Q) \\ &= \alpha(Q) \zeta_a(\psi_1) \zeta_a(\psi_2) \alpha(Q) = \alpha(Q) \zeta_a(\psi_1 \psi_2) \alpha(Q) = 0. \end{aligned}$$

Thus $Q \otimes \psi_1 \otimes Q \otimes \sigma_{ab}^*(\psi_2) \in \text{Ker } \tilde{\rho}$ and $Q \otimes \psi_1 \otimes Q \otimes \sigma_{ab}^*(\psi_2)(\lambda) \neq 0$. This ends the proof of Step 3.

We conclude that if $\lambda \in \Xi$ then there exist $b \in \mathcal{A}$, $y \in Z_b$ and $\omega \in \Omega_b$ such that

$$(4) \quad \lambda_a = \begin{cases} (y, \sigma_{ab}(\omega)) & \text{for } a \supset b, \\ \infty & \text{otherwise.} \end{cases}$$

Now consider the map $\mathcal{K} : \Lambda \rightarrow \times_{a \in \mathcal{A}} \Lambda_a$ such that $\mathcal{K}(\lambda) := \times_{a \in \mathcal{A}} \mathcal{K}_a(\lambda)$. Clearly, \mathcal{K} is continuous and injective. (4) means that $\Xi \subset \mathcal{K}(\Lambda)$.

To complete the proof we draw the following diagram:

$$\begin{array}{ccccc}
 C(\Lambda_a) & \xrightarrow{\mathcal{K}_a^*} & C(\times_{a \in \mathcal{A}} \Lambda_a) & \longrightarrow & C_\infty(\mathcal{K}(\Lambda)) & \xleftarrow{2} & C(\Lambda) \\
 & \searrow \rho_a & \downarrow \hat{\rho} & & \downarrow 1 & & \swarrow \rho \\
 & & \mathcal{C} & & & &
 \end{array}$$

Arrow 1 exists by the previous reasoning and Proposition 3.1. Arrow 2 is an isomorphism. This proves the existence of ρ . Q.E.D.

4. PROOF OF THE MAIN THEOREM

Theorem 2.1 can be viewed as a fancy way of saying that certain operators are compact. Hence it should not be surprising that its proof is based on the following lemma, which belongs to the standard folklore of the geometric method.

Lemma 4.1. *The following operators are compact.*

- (a) $[Q(x), g(D_a)]h(H)$ if $Q \in C(S)$, $g \in C_\infty(K_a)$, and $h \in C_\infty(\mathbb{R})$.
- (b) $Q(x)(h(H) - h(H_a))$ if $Q \in C_\infty(Y_a)$ and $h \in C_\infty(\mathbb{R})$.
- (c) $[Q(x), h(H_a)]$ if $Q \in C(S)$ and $h \in C_\infty(\mathbb{R})$.

Proof. (a) By the approximation argument we can assume that $Q \in C^\infty(S)$, $g \in C_0^\infty(K_a)$, and $h \in C_0(\mathbb{R})$. Then $[Q(x), g(D_a)]\langle x \rangle$ is bounded by a standard result from the calculus of pseudodifferential operators and $\langle x \rangle^{-1}h(H)$ is compact.

(b) We may assume that $Q \in C_0^\infty(Y_a)$ and $h \in C_0^\infty(\mathbb{R})$. Let $h_1 \in C_0^\infty(\mathbb{R})$ such that $h_1((s+m)^{-1}) = h(s)$. We will use the following representation:

$$h(H) = \frac{1}{2\pi} \int ds \hat{h}_1(s) e^{is(H+m)^{-1}}.$$

The operator that we investigate is equal to

$$\begin{aligned}
 & \frac{1}{2\pi} \int ds \hat{h}_1(s) s \int_0^1 d\tau e^{is\tau(H+m)^{-1}} \\
 & \times Q(x)((H+m)^{-1} - (H_a+m)^{-1}) e^{is(1-\tau)(H_a+m)^{-1}} \\
 & + \frac{1}{2\pi} \int ds \hat{h}_1(s) s^2 \int_0^1 d\tau \int_0^1 d\tau_1 e^{is\tau(H+m)^{-1}} [(H+m)^{-1}, Q(x)] \\
 & \times e^{is(\tau_1-\tau)(H_a+m)^{-1}} ((H+m)^{-1} - (H_a+m)^{-1}) e^{is(1-\tau_1)(H_a+m)^{-1}}.
 \end{aligned}$$

It is easy to show that $((H+m)^{-1} - (H_a+m)^{-1})$ and $[(H+m)^{-1}, Q(x)]$ are compact. This implies the compactness of the above expression.

(c) We can assume that $Q \in C^\infty(S)$ and $h \in C_0^\infty(\mathbb{R})$. Then we argue similarly as above. Q.E.D.

Proof of Theorem 2.1(a) and (b). The proof is based on Theorem 3.4. We assume that the symbols S , \mathcal{A} , Y_a and Z_a have the meaning defined in §2. We

set $\mathcal{E} := \text{Cal}(L^2)$ and $\Omega_a := es_a^{\text{comp}}$, and if $b \subset a$ then $\sigma_{ab} : es_b^{\text{comp}} \rightarrow es_a^{\text{comp}}$ is defined by $\sigma_{ab}(k_b, \lambda) := (\pi_a k_b, \lambda)$ and $\sigma_{ab}(\infty) := \infty$. (The map is well defined by the HZV theorem [RS2].) We also define

$$C(S) \ni Q \rightarrow \alpha(Q) := [Q(x)]_{\text{Cal}} \in \text{Cal}(L^2(X)).$$

Since es_a is the joint spectrum of the commuting operators D_a and H_a , we can define

$$C(es_a^{\text{comp}}) \ni \psi \rightarrow \zeta_a(\psi) := [\psi(D_a, H_a)]_{\text{Cal}} \in \text{Cal}(L^2(X)).$$

Note that if $g \in C_\infty(K_a)$ and $h \in C_\infty(\mathbb{R})$ then

$$\zeta_a(g \otimes h|_{es_a}) = [g(D_a)h(H_a)]_{\text{Cal}}.$$

We need to check the conditions of Theorem 3.4. It is enough to assume that ψ is of the form $\psi = g \otimes h|_{es_a}$, where $g \in C_\infty(K_a)$ and $h \in C_\infty(\mathbb{R})$. By Lemma 4.1(a) and (c) if $Q \in C(S)$ then $[Q(x), g(D_a)h(H_a)]$ is compact. Thus $\alpha(Q)\zeta_a(\psi) = \zeta_a(\psi)\alpha(Q)$.

Furthermore, if $b \subset a$ and $Q \in C_\infty(Y_b)$ then by Lemma 4.1(a), (b) the operator $Q(x)g(D_a)h(H_a) - Q(x)g(D_a)h(H_b)$ is compact. Hence $\alpha(Q)\zeta_a(\psi) = \alpha(Q)\zeta_b(\sigma_{ab}^*(\psi))$.

To complete the verification of the hypotheses of Theorem 3.5 it remains to check them for $\psi = 1$, which is trivial.

Now let

$$\Lambda := \bigcup_{a \neq a_{\max}} Z_a \times es_a^{\text{comp}}.$$

Clearly, ES is a subset of Λ . Recall that we defined \mathcal{K}_a to be a map from ES to $(Y_a \times K_a \times \mathbb{R})^{\text{comp}}$. Let \mathcal{K}_a^1 denote the map from Λ to $\Lambda_a = (Y_a \times es_a^{\text{comp}})^{\text{comp}}$ which coincides with \mathcal{K}_a on ES and is equal to ∞ on $\Lambda \setminus ES$.

Now Theorem 3.4 guarantees the existence of a $*$ -homomorphism ρ from $C(\Lambda)$ to $\text{Cal}(L^2(X))$ with the properties that can be summarized by the following diagram:

$$\begin{array}{ccc} & & k_a^* \\ & \nearrow & \searrow \\ C_\infty(Y_a \times K_a \times \mathbb{R}) & \longrightarrow & C((Y_a \times es_a^{\text{comp}})^{\text{comp}}) \xrightarrow{\mathcal{K}_a^{1*}} C(\Lambda) \\ & \searrow \rho_a & \nearrow \rho \\ & & \text{Cal}(L^2(X)) \end{array}$$

Next note that the images $\mathcal{K}_a^*(C_\infty(Y_a \times K_a \times \mathbb{R}))$ are contained in $C_\infty(ES)$ (which is a subalgebra of $C(\Lambda)$). Therefore if we restrict ρ to $C_\infty(ES)$ then we can draw the diagram (2) of §2. Thus defined ρ is unique because the images of \mathcal{K}_a^* generate $C_\infty(ES)$.

Now let us prove that ρ is isometric. Let $(y_a, p_a, \mu) \in Z_a \times es_a$. Fix $Q \in C_\infty(Y_a)$, $g \in C(K_a)$, and $h \in C_\infty(\mathbb{R})$ such that $Q(y_a)g(p_a)h(\mu) \neq 0$. Clearly,

$$\text{s-lim}_{t \rightarrow \infty} e^{ity_a D_a} Q(x) e^{-ity_a D_a} = Q(y_a).$$

Hence

$$\text{s-lim}_{t \rightarrow \infty} e^{ity_a D_a} Q(x) g(D_a) h(H_a) e^{-ity_a D_a} = Q(y_a) g(D_a) h(H_a).$$

Therefore

$$\| [Q(x)g(D_a)h(H_a)]_{\text{Cal}} \| \geq Q(y_a)g(p_a)h(\mu).$$

This means that $\|\rho(\mathcal{K}_a^*(Q \otimes g \otimes h))\| > 0$. Consequently $\text{Ker } \rho = \{0\}$. This implies the isometricity of ρ and ends the proof of (a) and (b). Q.E.D.

To proceed further with the proof of Theorem 2.1 we will need the following lemma.

Lemma 4.2. *Let $B \in CB(\mathcal{K}_{a\tau})$, $Q \in C(S)$, and $h \in C_\infty(\mathbb{R})$. Then*

$$[BQ(x)h(H_a)] = [BQ(\pi_a x)h(H_a)] = [h(H_a)BQ(\pi_a x)].$$

Proof. We set $\chi_T(s) = 1$ if $|s| < T$, and $\chi_T(s) = 0$ if $|s| \geq T$. If $\varepsilon > 0$ then we will find $T > 0$ such that $\|B(1 - \chi_T(|x^a|))\| < \varepsilon$. We will also find T_1 such that $\|\chi_T(|x^a|)(1 - \chi_{T_1}(|x_a|))(Q(x) - Q(x_a))\| < \varepsilon$. Therefore up to an arbitrarily small error $B(Q(x) - Q(x_a))$ is equal to $B(Q(x) - Q(x_a))\chi_T(|x^a|)\chi_{T_1}(|x_a|)$. But $\chi_T(|x^a|)\chi_{T_1}(|x_a|)h(H_a)$ is compact. This proves the first equality.

To prove the second equality note that by Lemma 4.1(c) it is enough to show that

$$(1) \quad h(H_a)B(Q(x) - Q(x_a))$$

is compact. As before we can replace (1) up to an arbitrarily small error with

$$h(H_a)B(Q(x) - Q(x_a))\chi_T(|x^a|)\chi_{T_1}(|x_a|).$$

Now

$$h(H_a)B\chi_{T_1}(|x_a|) = h(H_a)(-\Delta_a + 1)(-\Delta_a + 1)^{-1}\chi_{T_1}(|x_a|)B$$

is compact. Q.E.D.

Proof of Theorem 2.1(c) and (d). Consider the homomorphisms

$$CB(\mathcal{K}_{a\tau}) \ni B \rightarrow [B] \in \text{Cal}(L^2(X))$$

and

$$C_\infty(Y_a, K_a, \mathbb{R}) \ni \psi \rightarrow \tilde{\rho}_a(\psi) \in \text{Cal}(L^2(X)).$$

First note that their images commute. To see this it is enough to consider ψ of the form $Q \otimes g \otimes h$, where $Q \in C_\infty(Y_a)$, $g \in C(K_a)$, and $h \in C_\infty(\mathbb{R})$. Then by Lemma 4.2

$$\begin{aligned} [B]\tilde{\rho}_a(\psi) &= [BQ(x)g(D_a)h(H_a)] = [BQ(x_a)h(H_a)g(D_a)] \\ &= [Q(x_a)Bh(H_a)g(D_a)] = [Q(x)Bh(H_a)g(D_a)] = \tilde{\rho}_a(\psi)[B]. \end{aligned}$$

This proves the existence of $\tilde{\rho}_{a\tau}$.

Next note that if $g(k_a)h(\frac{1}{2}k_a^2 + \tau) = 0$ for $k_a \in K_a$ then $Bg(D_a)h(H_a) = 0$. Likewise, if $Q = 0$ on Z_a then by Lemma 4.2

$$[BQ(x)h(H_a)] = [BQ(x_a)h(H_a)] = 0.$$

Hence if $\psi = 0$ on $ES_{a\tau}$ then $B \otimes \psi \in \text{Ker } \tilde{\rho}_{a\tau}$. This implies the existence of $\rho_{a\tau}$. \square

Proof of Theorem 2.1(e). If $a_1 = a_2$ and $\tau_1 \neq \tau_2$ then the statement follows from the spectral theorem. Assume that $a_1 \not\subset a_2$. It is enough to consider $B_i \in$

$CB(\mathcal{H}_{a_i\tau_i})$ and $\omega_i = \ell_{a_i\tau_i}^*$ where $Q_i \in C_\infty(Y_{a_i})$, $g_i \in C(K_{a_i})$, and $h_i \in C_\infty(\mathbb{R})$. Now

$$(2) \quad \begin{aligned} & \rho_{a_1\tau_1}(B_1 \otimes \omega_1) \rho_{a_2\tau_2}(B_2 \otimes \omega_2) \\ &= [B_1 h_1(H) g(D_{a_1}) Q_1(x) Q_2(x) g(D_{a_2}) h_2(H) B_2]. \end{aligned}$$

Note that $Q_2 = 0$ on Z_{a_2} . For any $\varepsilon > 0$ we can change Q_1 outside Z_{a_1} such that $|Q_1 Q_2| \leq \varepsilon$. Therefore the norm of (2) can be made arbitrarily small. Consequently, (2) is zero. Q.E.D.

Proof of (f). It is enough to assume that $\psi = \ell_b^*(Q \otimes g \otimes h)$, where $Q \in C_\infty(Y_b)$, $g \in C_\infty(K_b)$, and $h \in C_\infty(\mathbb{R})$. Likewise we can assume that $\omega = \ell_{a\tau}^*(Q_1 \otimes g_1 \otimes h_1)$, where $Q_1 \in C_\infty(Y_a)$, $g_1 \in C_\infty(K_a)$, and $h_1 \in C_\infty(\mathbb{R})$. Now,

$$(3) \quad \rho_{a\tau}(B \otimes \omega) \rho(\psi) = B Q_1(x) Q(x) g_1(D_a) g(D_b) h_1(H) h(H).$$

If $b \supset a$ then this is equal to

$$\tilde{\rho}_{a\tau}(B \otimes Q_1 Q \otimes g_1 \pi_a^*(g) \otimes h_1 h) = \rho_{a\tau}(B \otimes \omega \psi|_{ES_{a\tau}}).$$

If $b \not\supset a$ then (3) is zero because $Q = 0$ on Z_a . Q.E.D.

Proof of (g). By (f) the set of those $\psi \in C_\infty(ES)$ that satisfy (1) of §2 is an ideal in $C_\infty(ES)$. It is obviously closed. We will show that $C_\infty(ES^{\text{disc}})$ is contained in this ideal.

Let $(y_a, p_a, \mu) \in Z_a \times es_a \cap ES^{\text{disc}}$. Choose $Q \in C_\infty(Y_a)$, $g \in C(K_a)$, and $h \in C_\infty(\mathbb{R})$ such that $Q(y_a)g(p_a)h(\mu) \neq 0$, and if $(k_a, \lambda) \in \text{supp } g \times \text{supp } h$ then $\lambda - \frac{1}{2}k_a^2 \notin \sigma_{\text{ess}}(H^a)$. Set $\psi =: \ell_a^*(Q \otimes g \otimes h)$. Then

$$\begin{aligned} \rho(\psi) &= [Q(x)g(D_a)h(H_a)] \\ &= \sum_{\tau \in \mathcal{B}_a^{\text{disc}}} [Q(x)g(D_a)h(\frac{1}{2}D_a^2 + \tau)P_{a\tau}] = \sum_{\tau \in \mathcal{B}_a^{\text{disc}}} \rho_{a\tau}(P_{a\tau} \otimes \psi|_{ES_{a\tau}}). \end{aligned}$$

Q.E.D.

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