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ON THE KÜNNETH FORMULA FOR INTERSECTION COHOMOLOGY

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ABSTRACT. We find the natural perversity functions for which intersection cohomology satisfies the Künneth formula.

1. INTRODUCTION

It was first shown by J. Cheeger [Ch] that the Künneth formula holds for the "middle" intersection cohomology, i.e.,

$$I^{\overline{m}}H^i(X\times Y;\mathbb{R})\cong\bigoplus_{a+b=i}I^{\overline{m}}H^a(X;\mathbb{R})\otimes I^{\overline{m}}H^b(Y;\mathbb{R})$$

where X and Y are compact pseudomanifolds with even codimension strata. It has been observed that for any perversity $\overline{p}$ and for any compact pseudomanifold X, there is a short exact sequence for intersection cohomology with integer coefficients,

$$0\to\bigoplus_{a+b=i}I^{\overline{p}}H^a(X)\otimes H^b(M)\to I^{\overline{p}}H^i(X\times M)\to\bigoplus_{a+b=i-1}\operatorname{Tor}(I^{\overline{p}}H^a(X),H^b(M))\to 0$$

provided M is a compact manifold (compare [Ki]). On the other hand, for most perversities $\overline{p}$, the intersection cohomology groups $I^{\overline{p}}H^i(X\times Y)$ do not satisfy the Künneth formula (see counterexamples in §5). In this note we find the natural perversity functions $\overline{p}$ for which $I^{\overline{p}}H^*$ satisfies the Künneth formula.

Fix a coefficient ring R which is a principal ideal domain and suppose X and Y are compact pseudomanifolds. Let $\overline{p}$ be a perversity. Recall ([GS]) that the pseudomanifold X is locally $\overline{p}$-torsion free (over R) if, for each stratum S of X, the link L of S satisfies

$$I^{\overline{p}}T^{c-p(c)-2}(L;R)=0\quad\text{where } c=\operatorname{codim}(S),$$

and $I^{\overline{p}}T^i(L;R)$ denotes the torsion subgroup of $I^{\overline{p}}H^i(L;R)$.

We consider two cases:

Case (A). Suppose that the perversity $\overline{p}$ satisfies $p(a)+p(b)\le p(a+b)\le p(a)+p(b)+1$ for all a and b.

Case (B). Suppose that the perversity $\overline{p}$ satisfies $p(a)+p(b)\le p(a+b)\le p(a)+p(b)+2$ for all a and b, and assume that either X or Y is locally $\overline{p}$-torsion free (over R).

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Theorem 1. *In either case* (A) *or* (B) *above, there is a split short exact sequence for intersection cohomology with perversity* $\overline{p}$ *and with coefficients in* R,

$$0 \to \bigoplus_{a+b=i} IH^a(X)\otimes IH^b(Y) \to IH^i(X\times Y) \to \bigoplus_{a+b=i-1} \mathrm{Tor}^R(IH^a(X), IH^b(Y)) \to 0.$$

Remarks. 1. If R is a field then all pseudomanifolds are locally $\overline{p}$-torsion free over R, and the theorem says that the intersection cohomology of $X\times Y$ is the tensor product of the intersection cohomology of X with that of Y, provided that the perversity satisfies the inequalities in condition (B) above.

2. If Y is a manifold then it is locally $\overline{p}$-torsion free for any perversity $\overline{p}$, and its intersection cohomology is just the usual cohomology, so we recover the case studied in [Ki].

3. All pseudomanifolds are locally $\overline{t}$-torsion free, where $\overline{t}$ denotes the "top" perversity, and Theorem 1 is the Künneth formula in ordinary homology.

4. The condition on the perversity $\overline{p}$ means that the graph of the perversity function does not deviate far from some straight line through the origin. Perversities with this property also arise in connection with L^q-cohomology [BGM].

5. More generally, one may start with a local system $\mathcal{L}_X$ of R-modules on the nonsingular part of X, and a local system $\mathcal{L}_Y$ of R-modules on the nonsingular part of Y. Then the same result holds if we replace $IH^*(X)$ by $IH^*(X;\mathcal{L}_X)$, $IH^*(Y)$ by $IH^*(Y;\mathcal{L}_Y)$, and $IH^*(X\times Y)$ by $IH^*(X\times Y;\mathcal{L}_X\boxtimes\mathcal{L}_Y)$. Note however, that the definition of locally $\overline{p}$-torsion free involves the group $I^{\overline{p}}H^{c-p(c)-2}(L;\mathcal{L}_X|_L)$, which may vary as the local system $\mathcal{L}_X$ varies. Since the proof is the same, we will omit further mention of local systems.

6. If X or Y is not compact, the same result holds provided we use compactly supported intersection cohomology throughout.

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2. The Künneth theorem in sheaf cohomology

Suppose X and Y are pseudomanifolds, but are not necessarliy compact. Let $\mathbf{I^{\overline{p}}C^\bullet}(X;R)$ denote the complex of sheaves of intersection cochains on X, with coefficients in the ring R [GM2]. Let π_1 and π_2 denote the projections of $X\times Y$ to X and Y respectively. It follows from sheaf theory [Bo, V, Theorem 10.19, p. 170; Iv, VII, 2.7, p. 323] that the compactly supported cohomology of the complex of sheaves

$$\mathbf{S}^\bullet = \pi_1^*(\mathbf{I^{\overline{p}}C}^\bullet(X)) \otimes \pi_2^*(\mathbf{I^{\overline{p}}C}^\bullet(Y))$$

satisfies the Künneth formula, i.e., it fits into the short exact sequence

$$0 \to \bigoplus_{a+b=i} IH_c^a(X)\otimes IH_c^b(Y) \to H_c^i(X\times Y;\mathbf{S}^\bullet)$$
$$\to \bigoplus_{a+b=i-1} \mathrm{Tor}^R(IH_c^a(X), IH_c^b(Y)) \to 0$$

where IH_c denotes intersection cohomology with perversity $\overline{p}$, compact supports, and coefficients in R. Therefore it suffices to prove the following result:

Proposition 2. *In either case* (A) *or* (B) *above, the multiplication map which is defined on the nonsingular part* $X^\circ \times Y^\circ$ *of* $X \times Y$,

$$\pi_1^*(\mathbf{R}_{X^\circ}) \otimes \pi_2^*(\mathbf{R}_{Y^\circ}) \to \mathbf{R}_{X^\circ \times Y^\circ},$$

extends to a unique quasi-isomorphism,

$$\pi_1^*(\mathbf{I}^{\bar{\mathbf{p}}}\mathbf{C}^\bullet(X; R)) \otimes \pi_2^*(\mathbf{I}^{\bar{\mathbf{p}}}\mathbf{C}^\bullet(Y; R)) \cong \mathbf{I}^{\bar{\mathbf{p}}}\mathbf{C}^\bullet(X \times Y; R).$$

3. Intersection cohomology of a join

In this section we assume that X and Y are compact. As a corollary to Theorem 1, we compute $I^{\bar{p}}H^*(X * Y)$, where $X * Y$ denotes the join of X and Y.

Proposition 3. *Let* X *and* Y *be pseudomanifolds of dimensions* $m-1$ *and* $n-1$. *In either case* (A) *or* (B) *above, we have* $I^{\bar{p}}H^i(X * Y) =$

$$\begin{cases} \bigoplus\limits_{\substack{a+b=i \\ a \le q(m) \\ b \le q(n)}} IH^a(X) \otimes IH^b(Y) \oplus \bigoplus\limits_{\substack{a+b=i-1 \\ a \le q(m) \\ b \le q(n)}} \operatorname{Tor}^R(IH^a(X), IH^b(Y)), & i \le q(m)+q(n), \\ \operatorname{Tor}^R(IH^{q(m)}(X), IH^{q(n)}(Y)), & i = q(m)+q(n)+1, \\ 0, & i = q(m)+q(n)+2, \\ IH^{q(m)+1}(X) \otimes IH^{q(n)+1}(Y), & i = q(m)+q(n)+3, \\ \bigoplus\limits_{\substack{a+b=i-1 \\ a > q(m) \\ b > q(n)}} IH^a(X) \otimes IH^b(Y) \oplus \bigoplus\limits_{\substack{a+b=i-2 \\ a > q(m) \\ b > q(n)}} \operatorname{Tor}^R(IH^a(X), IH^b(Y)), & i \ge q(m)+q(n)+4, \end{cases}$$

where $q(m) = m - p(m) - 2$ *and* $q(n) = n - p(n) - 2$.

Notice that the group $\operatorname{Tor}^R(IH^{q(m)}(X), IH^{q(n)}(Y))$ vanishes in case (B).

Proof. Cover the join $X * Y$ by two subsets $U \cong X \times cY$ and $V \cong cX \times Y$, whose intersection is $U \cap V \cong X \times Y \times [\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon]$. We shall compute $IH^*(X * Y)$ using Mayer-Vietoris and the Künneth formula for each of the three subsets. Consider the Mayer-Vietoris sequence in intersection cohomology

$$\cdots \xrightarrow{\Delta_{i-1}} IH^i(X * Y) \xrightarrow{\Phi_i} IH^i(X \times cY) \oplus IH^i(cX \times Y) \xrightarrow{\Psi_i} IH^i(X \times Y) \xrightarrow{\Delta_i} \cdots.$$

By Theorem 1, the Künneth formula holds for the intersection cohomology groups of $X \times Y$, $X \times cY$, and $cX \times Y$. So we have a diagram with split short exact rows

$$
\begin{array}{ccccccccc}
0 & \to & \displaystyle\bigoplus_{a+b=i} IH^a(X)\otimes IH^b(Y) & \to & IH^i(X\times Y) & \to & \displaystyle\bigoplus_{a+b=i-1} \operatorname{Tor}^R(IH^a(X),\, IH^b(Y)) & \to & 0 \\
 & & & & \uparrow \Psi_i & & & & \\
 & & \displaystyle\bigoplus_{a+b=i} IH^a(X)\otimes IH^b(cY) & & IH^i(X\times cY) & & \displaystyle\bigoplus_{a+b=i-1} \operatorname{Tor}^R(IH^a(X),\, IH^b(cY)) & & \\
0 & \to & \oplus & \to & \oplus & \to & \oplus & \to & 0 \\
 & & \displaystyle\bigoplus_{a+b=i} IH^a(cX)\otimes IH^b(Y) & & IH^i(cX\times Y) & & \displaystyle\bigoplus_{a+b=i-1} \operatorname{Tor}^R(IH^a(cX),\, IH^b(Y)) & & \\
 & & & & \uparrow \Phi_i & & & & \\
 & & & & IH^i(X * Y) & & & &
\end{array}
$$

Using the computation of the intersection cohomology of a cone [GM1], we observe that, for instance

$$
\bigoplus_{a+b=i} IH^a(X)\otimes IH^b(cY) = \bigoplus_{\substack{a+b=i\\ b\le q(n)}} IH^a(X)\otimes IH^b(Y)
$$

is a subgroup of $\bigoplus_{a+b=i} IH^a(X)\otimes IH^b(Y)$. Thus there are homomorphisms α_i and β_i so that the following diagram commutes:

$$
\begin{array}{ccccccccc}
0 & \to & \displaystyle\bigoplus_{a+b=i} IH^a(X)\otimes IH^b(Y) & \to & IH^i(X\times Y) & \to & \displaystyle\bigoplus_{a+b=i-1} \operatorname{Tor}^R(IH^a(X),\, IH^b(Y)) & \to & 0 \\
 & & \uparrow \alpha_i & & \uparrow \Psi_i & & \uparrow \beta_i & & \\
 & & \displaystyle\bigoplus_{\substack{a+b=i\\ b\le q(n)}} IH^a(X)\otimes IH^b(Y) & & IH^i(X\times cY) & & \displaystyle\bigoplus_{\substack{a+b=i-1\\ b\le q(n)}} \operatorname{Tor}^R(IH^a(X),\, IH^b(Y)) & & \\
0 & \to & \oplus & \to & \oplus & \to & \oplus & \to & 0 \\
 & & \displaystyle\bigoplus_{\substack{a+b=i\\ a\le q(m)}} IH^a(X)\otimes IH^b(Y) & & IH^i(cX\times Y) & & \displaystyle\bigoplus_{\substack{a+b=i-1\\ a\le q(m)}} \operatorname{Tor}^R(IH^a(X),\, IH^b(Y)) & & \\
 & & & & \uparrow \Phi_i & & & & \\
 & & & & IH^i(X * Y) & & & &
\end{array}
$$

Check that:

$$
\ker(\alpha_i) = \begin{cases} \displaystyle\bigoplus_{\substack{a+b=i\\ a\le q(m)\\ b\le q(n)}} IH^a(X)\otimes IH^b(Y), & i\le q(m)+q(n), \\ 0, & i > q(m)+q(n), \end{cases}
$$

$$
\operatorname{coker}(\alpha_i) = \begin{cases} 0, & i\le q(m)+q(n)+1, \\ \displaystyle\bigoplus_{\substack{a+b=i\\ a>q(m)\\ b>q(n)}} IH^a(X)\otimes IH^b(Y), & i > q(m)+q(n)+1, \end{cases}
$$

$$
\ker(\beta_i) = \begin{cases} \displaystyle\bigoplus_{\substack{a+b=i-1\\ a\le q(m)\\ b\le q(n)}} \operatorname{Tor}^R(IH^a(X),\, IH^b(Y)), & i\le q(m)+q(n)+1, \\ 0, & i > q(m)+q(n)+1, \end{cases}
$$

$$\operatorname{coker}(\beta_i) = \begin{cases} 0, & i \leq q(m) + q(n) + 2, \\ \displaystyle\bigoplus_{\substack{a+b=i-1 \\ a>q(m) \\ b>q(n)}} \operatorname{Tor}^R(IH^a(X), IH^b(Y)), & i > q(m) + q(n) + 2. \end{cases}$$

By splicing together short exact sequences involving the kernels and cokernels of the maps α_i and β_i, we obtain the following diagram with everywhere exact columns and split exact rows:

$$\begin{array}{ccccccccc}
0 & \to & \displaystyle\bigoplus_{a+b=i} IH^a(X) \otimes IH^b(Y) & \to & IH^i(X \times Y) & \to & \displaystyle\bigoplus_{a+b=i-1} \operatorname{Tor}^R(IH^a(X), IH^b(Y)) & \to & 0 \\
 & & \uparrow \alpha_i & & \uparrow \Psi_i & & \uparrow \beta_i & & \\
 & & \displaystyle\bigoplus_{\substack{a+b=i \\ b\leq q(n)}} IH^a(X) \otimes IH^b(Y) & & IH^i(X \times cY) & & \displaystyle\bigoplus_{\substack{a+b=i-1 \\ b\leq q(n)}} \operatorname{Tor}^R(IH^a(X), IH^b(Y)) & & \\
0 & \to & \oplus & \to & \oplus & \to & \oplus & \to & 0 \\
 & & \displaystyle\bigoplus_{\substack{a+b=i \\ a\leq q(m)}} IH^a(X) \otimes IH^b(Y) & & IH^i(cX \times Y) & & \displaystyle\bigoplus_{\substack{a+b=i-1 \\ a\leq q(m)}} \operatorname{Tor}^R(IH^a(X), IH^b(Y)) & & \\
 & & \uparrow & & \uparrow \Phi_i & & \uparrow & & \\
 & & \ker(\alpha_i) \oplus \operatorname{coker}(\alpha_{i-1}) & & IH^i(X * Y) & & \ker(\beta_i) \oplus \operatorname{coker}(\beta_{i-1}) & &
\end{array}$$

For $i \leq q(m) + q(n) + 2$, the maps α_{i-1} and β_{i-1} are surjective by the computation of the intersection cohomology of a cone. It follows that Ψ_{i-1} is also surjective, hence Φ_i is injective. Similarly, since α_i and β_i are injective for $i \geq q(m) + q(n) + 3$, Φ_i is the zero map in these dimensions.

Using these facts, it is an exercise in homological algebra to verify that

$$0 \to \ker(\alpha_i) \to IH^i(X * Y) \to \ker(\beta_i) \to 0$$

is split exact for $i \leq q(m) + q(n) + 2$, and that

$$0 \to \operatorname{coker}(\alpha_{i-1}) \to IH^i(X * Y) \to \operatorname{coker}(\beta_{i-1}) \to 0$$

is split exact for $i \geq q(m) + q(n) + 3$. Direct computations involving these sequences finish the proof of the proposition. □

4. Proof of Proposition 2

We will use induction on $k = \dim(X) + \dim(Y)$ with the case $k = 0$ trivial. We shall refer to Theorem 1_k and Proposition 2_k when the additional hypothesis $\dim(X) + \dim(Y) \leq k$ is satisfied. Similarly, we refer to Proposition 3_k when the additional hypothesis $m + n - 1 \leq k$ is satisfied. Then Proposition $2_{k-1} \Rightarrow$ Theorem $1_{k-1} \Rightarrow$ Proposition 3_{k-1} and we now show that Proposition $3_{k-1} \Rightarrow$ Proposition 2_k.

The multiplication map $\pi_1^*(\mathbf{R}_{X^\circ}) \otimes \pi_2^*(\mathbf{R}_{Y^\circ}) \to \mathbf{R}_{X^\circ \times Y^\circ}$ is a quasi-isomorphism (in fact it is an isomorphism). Let us assume (by induction on t) that this extends uniquely to a quasi-isomorphism

$$\pi_1^*(\mathbf{I}^{\bar{\mathbf{p}}}\mathbf{C}^\bullet(X; R)) \otimes \pi_2^*(\mathbf{I}^{\bar{\mathbf{p}}}\mathbf{C}^\bullet(Y; R))|_{(X \times Y - V)} \cong \mathbf{I}^{\bar{\mathbf{p}}}\mathbf{C}^\bullet(X \times Y - V; R)$$

over the complement of the union V of all strata with codimension greater than $t - 1$. We must show that the quasi-isomorphism extends over the complement of the set of strata with codimension greater than t. Let $(x, y) \in X \times Y$ be a point in the complement of the union of all strata of codimension greater than t. By [GM2] it suffices to check that $\mathbf{IC}^\bullet(X) \boxtimes \mathbf{IC}^\bullet(Y)$ and $\mathbf{IC}^\bullet(X \times Y)$

have the same stalk cohomology at $p = (x, y)$. Let U_x and U_y denote basic open neighborhoods in X and Y respectively, of the points x and y. The local cohomology at p of the sheaf $\mathbf{IC}^\bullet(X) \boxtimes \mathbf{IC}^\bullet(Y)$ is given by the Künneth formula discussed in §2, namely:

$$H^i_p(\mathbf{IC}^\bullet(X) \boxtimes \mathbf{IC}^\bullet(Y)) \cong \bigoplus_{a+b=i} IH^a(U_x) \otimes IH^b(U_y) \oplus \bigoplus_{a+b=i-1} \operatorname{Tor}^R(IH^a(U_x), IH^b(U_y)).$$

There are homeomorphisms $U_x \cong E \times cL_x$ and $U_y \cong E \times cL_y$, where L_x and L_y are the links of x and y in X and Y respectively, and E is a disk of suitable dimension in each case. Since the Künneth formula in intersection cohomology holds for a product with a disk [GM2], we have

$$H^i_p(\mathbf{IC}^\bullet(X) \boxtimes \mathbf{IC}^\bullet(Y)) \cong \bigoplus_{a+b=i} IH^a(cL_x) \otimes IH^b(cL_y) \oplus \bigoplus_{a+b=i-1} \operatorname{Tor}^R(IH^a(cL_x), IH^b(cL_y)).$$

Now the stalk cohomology of $\mathbf{IC}^\bullet(X \times Y)$ is given by $H^i_p(\mathbf{IC}^\bullet(X \times Y)) = IH^i(U)$, where U is a basic open neighborhood in $X \times Y$ of $p = (x, y)$. Here U is homeomorphic to $E \times cL$, where again E is a disk of suitable dimension. In this case $cL \cong cL_x \times cL_y$, so we find the local cohomology of $\mathbf{IC}^\bullet(X \times Y)$ by computing the intersection cohomology groups of $cL_x \times cL_y$. The computation of $IH^i(cL_x \times cL_y)$ hinges on the following straightforward result:

Lemma. *For topological spaces X and Y, the product $cX \times cY$ is homeomorphic to $c(X * Y)$.*

Hence we find $H^i_p(\mathbf{IC}^\bullet(X \times Y))$ using the computation of the intersection cohomology of a cone. Explicitly,

$$H^i_p(\mathbf{IC}^\bullet(X \times Y)) = \begin{cases} IH^i(L_x * L_y), & i \le m + n - p(m) - p(n) - 2, \\ 0, & i \ge m + n - p(m) - p(n) - 1, \end{cases}$$

where $m = \dim(cL_x)$ and $n = \dim(cL_y)$.

The link $L \cong L_x * L_y$ is contained in the complement of the union of all strata of codimension greater than $t - 1$. Also notice that $\dim(L_x) < \dim(X)$ and $\dim(L_y) < \dim(Y)$, so all of the product spaces $L_x \times L_y$, $L_x \times cL_y$, and $cL_x \times L_y$ are of dimension less than k. Furthermore, if, say, X is locally $\bar{p}$-torsion free (over R), then so are L_x and cL_x. Thus, since L is compact, we find $IH^i(L_x * L_y)$ using Proposition 3_{k-1}, which holds by induction. The conditions in either case (A) or (B) then insure that

$$H^i_p(\mathbf{IC}^\bullet(X \times Y)) \cong \bigoplus_{a+b=i} IH^a(cL_x) \otimes IH^b(cL_y) \oplus \bigoplus_{a+b=i-1} \operatorname{Tor}^R(IH^a(cL_x), IH^b(cL_y))$$

which completes the (t-)inductive step of the proof of Proposition 2_k. □

5. Counterexamples

1. In this example, we show that the Künneth formula (with field coefficients) need not hold in the case where the perversity fails to satisfy the inequalities in condition (B).

Let $X = Y = cT^2$, and let $\bar{p}$ be a perversity such that $p(3) = 0$. Check that the betti numbers for $I^{\bar{p}}H^*(cT^2; \mathbb{Q})$ are $(1, 2, 0, 0)$, and that $I^{\bar{p}}H^*(X; \mathbb{Q}) \otimes I^{\bar{p}}H^*(Y; \mathbb{Q}) = (1, 4, 4, 0, 0, 0, 0)$. Using a Mayer-Vietoris sequence as in §3, we obtain $(1, 4, 4, 0, 0, 1)$ as the betti numbers for $I^{\bar{p}}H^*(T^2 * T^2; \mathbb{Q})$.

Now $X \times Y = cT^2 \times cT^2 \cong c(T^2 * T^2)$, so we find $I^{\bar{p}}H^*(X \times Y; \mathbb{Q})$ using the computation of the intersection cohomology of a cone. For a perversity $\bar{p}$ with $p(3) = 0$, there are four possible values for $p(6)$, namely $p(6) = 0, 1, 2,$ or 3. If $0 \leq p(6) \leq 2$, then the perversity satisfies the inequalities in condition (B) and we have $I^{\bar{p}}H^*(X \times Y; \mathbb{Q}) = (1, 4, 4, 0, 0, 0, 0) = I^{\bar{p}}H^*(X; \mathbb{Q}) \otimes I^{\bar{p}}H^*(Y; \mathbb{Q})$. However if $p(6) = 3$, then $\bar{p}$ does not satisfy condition (B) and $I^{\bar{p}}H^*(X \times Y; \mathbb{Q}) = (1, 4, 0, 0, 0, 0, 0)$.

2. This example demonstrates that the Künneth formula need not hold in the case where the perversity satisfies the inequalities in condition (B), but neither X nor Y is locally $\bar{p}$-torsion free.

Let M denote the lens space $S^3/(\mathbb{Z}/n\mathbb{Z})$, and let $X = Y = cM$. If $\bar{p}$ is a perversity such that $p(4) = 0$, then X and Y are not locally $\bar{p}$-torsion free (over $\mathbb{Z}$) since the link of the singular stratum (the cone point) in cM is just M, and $I^{\bar{p}}H^2(M; \mathbb{Z}) = H^2(M; \mathbb{Z}) = \mathbb{Z}/n\mathbb{Z}$.

Now $X \times Y = cM \times cM \cong c(M * M)$, so we find $I^{\bar{p}}H^*(X \times Y; \mathbb{Z})$ by again using the computation of the intersection cohomology of a cone. In particular if $p(8) = 2$, then $I^{\bar{p}}H^5(X \times Y; \mathbb{Z}) = 0$. On the other hand,

$$\bigoplus_{a+b=5} IH^a(X) \otimes IH^b(Y) \oplus \bigoplus_{a+b=4} \mathrm{Tor}(IH^a(X), IH^b(Y))$$
$$= \mathrm{Tor}(H^2(M), H^2(M)) = \mathbb{Z}/n\mathbb{Z}.$$

Notice that any perversity with $p(4) = 0$ and $p(8) = 2$ must satisfy condition (B).

Analogous examples involving pseudomanifolds without boundary may be obtained by considering suspensions rather than cones in each of the above.

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