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EXAMPLES OF CAPACITY FOR SOME ELLIPTIC OPERATORS

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ABSTRACT. We study L-capacities for uniformly elliptic operators of nondivergence form

$$L = \sum_{i,j} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_j a_j(x)\frac{\partial}{\partial x_j};$$

and construct examples of large sets having zero L-capacity for some L, and small sets having positive L-capacity. The relations between ellipticity constants of the coefficients and the sizes of these sets are also considered.

A compact set $S \subseteq \{|x| < 1\} \subseteq \mathbb{R}^n$, $n \geq 2$, has zero capacity for the Laplacian if and only if it is a removable set for the class of bounded subharmonic functions on $\{|x| < 1\}$; equivalently, there exists a positive superharmonic function ($\not\equiv +\infty$) on $\{|x| < 1\}\backslash S$ which approaches $+\infty$ continuously on S.

In this note, we study L-capacities for uniformly elliptic operators of nondivergence form

$$L = \sum_{i,j} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_j a_j(x)\frac{\partial}{\partial x_j};$$

and construct examples of large sets having zero L-capacity for some L, and small sets having positive L-capacity. We also study the relations between ellipticity constants of the coefficients and the sizes of these sets.

When (a_{ij}) are Dini continuous and (a_j) are bounded, sets of L-capacity zero are precisely those of capacity zero for the Laplacian; this follows from the growth of the Green function for these operators. (See [1] and [7].) In general, there are elliptic operators L with continuous coefficients for $n = 2$, bounded coefficients for $n \geq 3$, for which a single point has positive L-capacity; again this reflects the behavior of the Green function. (See [2, 4, 5 and 12].)

For uniformly elliptic operators of divergence form, the growth of Green function near its pole is comparable to that for the Laplacian [10]. Therefore sets of capacity zero are exactly those for the Laplacian.

Let L be the above operator and coefficients of L be continuous in a domain $\Omega \subseteq \mathbb{R}^n$. We consider strong solutions of $L = 0$ in $W^{2,n}_{\text{loc}}(\Omega)$ and call them L-solutions. The maximum principle, the existence and uniqueness of the solution to the Dirichlet problem and the Harnack principle are well known. A lower semicontinuous function v is called an L-supersolution on Ω, if for any closed ball $B \subseteq \Omega$ and any L-solution u in B with u continuous on $\overline{B}$, the

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inequality $v \geq u$ on ∂B implies that $v \geq u$ in B. A function $v \in C^2(\Omega)$ is an L-supersolution if and only if $Lv \leq 0$. A function v is called an L-subsolution if $-v$ is an L-supersolution. (See [6, Chapter 9].)

Let $D \equiv \{|x| < 1\}$, $\overline{x} = (\frac{1}{2}, 0, 0, \ldots 0)$ and S be a compact set in $\{|x| \leq \frac{1}{4}\}$. We define the L-capacity of S as

$$L\text{-cap } S = \inf\{v(\overline{x}) \colon v \text{ is a positive } L\text{-supersolution on } D \text{ and } v \geq 1 \text{ on } S\},$$

when the coefficients of L are bounded continuous in D. When the coefficients of L are only known to be bounded continuous on $D\backslash S$, we say L-cap $S = 0$ provided that

$$\tag{0.1} \inf\left\{v(\overline{x}) \colon v \text{ is a positive } L\text{-supersolution on } D\backslash S, \right. \\ \left. \text{and } \liminf_{x \to x_0} v(x) \geq 1 \text{ at each } x_0 \in S\right\} = 0;$$

otherwise we say L-cap $S > 0$. Both definitions of L-capacity zero agree when the coefficients of L are continuous on D. We note that if there exists a positive L-supersolution on $D\backslash S$ which approaches $+\infty$ continuously on S, then L-cap $S = 0$; and that if there exists a bounded positive L-subsolution on $D\backslash S$ which approaches 0 continuously on ∂D, then L-cap $S > 0$.

We recall that for the Laplacian, a set has positive capacity if it has positive h-Hausdorff measure for some $h > 0$ satisfying $\int_0^1 h(r)/r^{n-1}\,dr < \infty$; and a set has zero capacity if it has finite $(n-2)$-dimensional Hausdorff measure when $n \geq 3$, or finite logarithmic measure when $n = 2$. Therefore $n - 2$ is the critical dimension for studying sets of capacity zero.

We shall prove the following:

Theorem 1. *Let $n \geq 2$ and $n - 2 < \alpha < n$. Then there exist a constant $\Lambda_{n,\alpha} > 1$, a compact set $S \subseteq D$ of Hausdorff dimension α, an operator $L = \sum a_{ij}\partial^2/\partial x_i \partial x_j$ with coefficients bounded smooth in $D\backslash S$, satisfying*

$$\tag{0.2} |\xi|^2 \leq \sum a_{ij}(x)\xi_i\xi_j \leq \Lambda_{n,\alpha}|\xi|^2, \qquad x, \xi \in \mathbb{R}^n,$$

so that S has zero L-capacity in the sense (0.1). In fact, there is a positive L-supersolution v ($\not\equiv +\infty$) in $D\backslash S$ approaching $+\infty$ continuously on S. Moreover,

$$\tag{0.3} \Lambda_{n,\alpha} = 1 + O(1)(\alpha - n + 2) \quad \text{as } \alpha \to n - 2,$$

and

$$\tag{0.4} \Lambda_{n,\alpha} = O(1)(n - \alpha)^{-1} \quad \text{as } \alpha \to n,$$

with the $O(1)$ terms positive and independent of n and α.

We believe that (0.3) is sharp, and do not know whether (0.4) can be improved.

Theorem 2. *Let $n \geq 2$, $a > 0$, and $h(r) = r^{n-2}(\log\frac{1}{r})^{-1-a}$. Then there exist a constant $\beta > 0$, a compact set $S \subseteq D$ of dimension $n - 2$, positive Hausdorff h-measure, an operator $L = \sum a_{ij}\partial^2/\partial x_i \partial x_j$ with coefficients continuous in $\mathbb{R}^n$, smooth off S satisfying*

$$\tag{0.5} |\xi|^2 \leq \sum a_{ij}(x)\xi_i\xi_j \leq \left\{1 + \beta\left(\log\frac{1}{\operatorname{dist}(x, S)}\right)^{-1}\right\}|\xi|^2$$

for all $x, \xi \in \mathbb{R}^n$, *and a positive* L*-supersolution* v *on* D, *which is an* L*-solution on* $D\backslash S$ *and approaches* $+\infty$ *continuously on* S. *In particular,* S *has zero* L*-capacity, and positive capacity for the Laplacian.*

Theorem 3. *Let* $n \geq 3$, $a > 0$, *and* $h(r) = r^{n-2}(\log \frac{1}{r})^a$. *Then there exist a compact set* $S \subseteq D$ *of dimension* $n-2$, *vanishing* h*-measure, a constant* $\beta > 0$, *an operator* $L = \sum a_{ij}\partial^2/\partial x_i \partial x_j$ *with coefficients continuous in* $\mathbb{R}^n$, *smooth off* S *and satisfying*

$$\left\{1 - \beta\left(\log\frac{1}{\operatorname{dist}(x, S)}\right)^{-1}\right\}|\xi|^2 \leq \sum a_{ij}(x)\xi_i\xi_j \leq |\xi|^2$$

for $x, \xi \in \mathbb{R}^n$, *and a bounded positive* L*-supersolution* w *on* D, *which is an* L*-solution on* $D\backslash S$. *Thus* S *has zero capacity for the Laplacian, and positive capacity for the operator* L.

Theorem 4. *Let* $n \geq 3$ *and* $0 < \alpha < n-2$. *Then there exist a positive constant* $\lambda_{n,\alpha} < 1$, *a compact set* $S \subseteq D$ *of dimension* α, *an operator* $L = \sum a_{ij}\partial^2/\partial x_i\partial x_j$ *with* a_{ij} *bounded smooth off* S, *satisfying*

$$\lambda_{n,\alpha}|\xi|^2 \leq \sum a_{ij}(x)\xi_i\xi_j \leq |\xi|^2, \qquad x, \xi \in \mathbb{R}^n,$$

so that S *has positive* L*-capacity. In fact, there exists a bounded positive* L*-subsolution* w *on* $D\backslash S$ *which vanishes continuously on* ∂D. *Moreover*

$$\lambda_{n,\alpha} = (1-2\alpha)(n-1)^{-1}, \qquad 0 < \alpha < 1/4, \tag{0.6}$$

and

$$\lambda_{n,\alpha} = 1 - O(1)(n-2-\alpha) \quad \text{as } \alpha \to n-2, \tag{0.7}$$

with the $O(1)$ *term positive and independent of* n *and* α.

Since it is known that a point can have positive L-capacity, the only new part of Theorem 4 is the relation between the ellipticity constants and the dimension.

In the proofs of all four theorems, we start with the Laplace operator, then modify the coefficients on a sequence of rings, accumulating on a Cantor set S, so that on the rings all eigenvalues are greater than 1 (or less than 1). When all are chosen properly, it will produce an L-supersolution which grows faster than (or slower than) the fundamental solution of the Laplacian near each point in S. This explains the relation between the normalization of the ellipticity constants and the size of the set S.

A related subject, the boundary regularity problem for the operator L, has been studied by many. A partial list includes [4, 7, 8, 9. 10, 11, 12, and 13].

1. Preliminary lemmas

Let Δ be the Laplace's operator and $r = |x|$ for $x \in \mathbb{R}^n$.

Lemma 1. *Let* $B(x)$ *be positive continuous in a domain* Ω, *and let*

$$L = \frac{B(x)}{n-1}\sum\frac{\partial^2}{\partial x_i^2} - \left(\frac{B(x)}{n-1} - 1\right)\sum\frac{x_i x_j}{|x|^2}\frac{\partial^2}{\partial x_i \partial x_j}$$

on $\Omega\backslash\{0\}$. Then the coefficients a_{ij} of L are continuous, symmetric on $\Omega\backslash\{0\}$, satisfying

$$|\xi|^2 \le \sum a_{ij}(x)\xi_i\xi_j \le \frac{B(x)}{n-1}|\xi|^2 \quad \textit{when } B(x) \ge n-1, \tag{1.1}$$

and

$$\frac{B(x)}{n-1}|\xi|^2 \le \sum a_{ij}(x)\xi_i\xi_j \le |\xi|^2 \quad \textit{when } B(x) \le n-1. \tag{1.2}$$

The coefficients of L can be extended to be continuous on Ω if $B(0) = n-1$. Moreover, $|x|^{-B+1}$ is a solution of $L = O(x \neq 0)$, when $B(x) \equiv$ a constant B.

The characteristic values of $(a_{ij}(x))$ are 1, $B(x)/(n-1)$, $B(x)/(n-1), \dots,$ $B(x)/(n-1)$; and for $x \neq 0$,

$$L = \frac{B(x)}{n-1}\Delta - \left(\frac{B(x)}{n-1} - 1\right)\frac{\partial^2}{\partial r^2} = \frac{\partial^2}{\partial r^2} + \frac{B(x)}{r}\frac{\partial}{\partial r} + \frac{B(x)}{n-1}r^{-2}\delta,$$

where δ is the Beltrami operator in the spherical coordinates. Whence the lemma follows.

Denote by $D(x, a)$ the closed ball centered at x of radius a, and recall that $D = D(0, 1)$. When U is a ball, denote by cU the ball concentric to U of radius c times that of U.

Lemma 2. *Let $0 < \delta < \frac{1}{16}$ and $D(a, r) \subseteq D(0, \delta)$. Then there exists a diffeomorphism $y = Tx$ from $\mathbb{R}^n$ onto $\mathbb{R}^n$, which fixes every point in $\mathbb{R}^n\backslash D(0, \frac{9}{16})$, maps each point x in $D(a, r)$ to $x - a$, and satisfies on $D(0, \frac{9}{16})$:*

$$\frac{\partial y_i}{\partial x_j} - \delta_{ij} = c(x)a_i(x_j - a_j), \tag{1.3}$$

$$\left|\sum_i \frac{\partial^2 y_i}{\partial x_l \partial x_m}\xi_i\right| \le 272\delta|\xi|, \tag{1.4}$$

where $0 < c(x) < 32$, $\delta_{ij} = 1$ when $i = j$ and $\delta_{ij} = 0$ when $i \neq j$. Moreover if $(a_{ij}(x))$ is symmetric positive definite with all its eigenvalues bounded above by Λ, and

$$b_{ij}(x) = \sum_{l,m} a_{lm}(x)\frac{\partial y_i}{\partial x_l}\frac{\partial y_j}{\partial x_m},$$

then

$$\left|\sum_{i,j} b_{ij}\xi_i\xi_j - \sum_{i,j} a_{ij}\xi_i\xi_j\right| \le 128\delta\Lambda|\xi|^2, \tag{1.5}$$

and

$$\left|\sum b_{ii} - \sum a_{ii}\right| \le 128\delta\Lambda. \tag{1.6}$$

Proof. Let

$$\psi(s) = \begin{cases} 10s^3 - 15s^4 + 6s^5, & 0 \le s \le 1, \\ 0, & s < 0, \\ 1, & s > 1; \end{cases}$$

and note that ψ is C^2, $0 \le \psi \le 1$, $0 \le \psi' \le 15/8$ and $|\psi''| \le 10/\sqrt{3}$. Let

$$\varphi(t) = 1 - \psi\left(\frac{t-\delta^2}{\frac{1}{4}-\delta^2}\right).$$

Thus $\varphi = 0$ for $t \ge \frac{1}{4}$, $\varphi = 1$ for $t \le \delta^2$, $0 \le \varphi \le 1$, $-8 \le \varphi' \le 0$ and $|\varphi''| < 160$. Then $Tx = x - \varphi(|x-a|^2)a$ is a diffeomorphism on $\mathbb{R}^m$ that fixes every point in $\mathbb{R}^m \backslash D(0, \frac{9}{16})$ and maps $x \in D(a, r)$ to $x - a$. Moreover, on $D(0, \frac{9}{16})$, T satisfies (1.3) and (1.4) with $c(x) = -2\varphi'(|x-a|^2)$.

To show (1.5), we let $x \in D(0, \frac{9}{16})$, and note that

$$\begin{aligned} b_{ij} - a_{ij} &= \sum_{l,m} a_{lm}[\delta_{il} + c(x)a_i(x_l - a_l)][\delta_{jm} + c(x)a_j(x_m - a_m)] - a_{ij} \\ &= c(x)\sum_m a_{im}a_j(x_m - a_m) + c(x)\sum_l a_{lj}a_i(x_l - a_l) \\ &\quad + c(x)^2 \sum_{l,m} a_{lm}a_ia_j(x_l - a_l)(x_m - a_m). \end{aligned}$$

Thus

$$\begin{aligned} &\left|\sum_{i,j} b_{ij}\xi_i\xi_j - \sum_{i,j} a_{ij}\xi_i\xi_j\right| \\ &\quad \le c(x)\left|\sum_j \sum_{i,m} a_{im}\xi_i(x_m - a_m)a_j\xi_j\right| + c(x)\left|\sum_i \sum_{j,l} a_{lj}\xi_j(x_l - a_l)a_i\xi_i\right| \\ &\quad + c(x)^2\left|\sum_{i,j}\sum_{l,m} a_{lm}(x_l - a_l)(x_m - a_m)a_ia_j\xi_i\xi_j\right|. \end{aligned}$$

Since $|a| < \delta$, $|x - a| < 1$ and eigenvalues of (a_{ij}) are bounded above by Λ, we conclude that

$$\left|\sum b_{ij}\xi_i\xi_j - \sum a_{ij}\xi_i\xi_j\right| \le 2c(x)\Lambda\delta + c(x)^2\Lambda\delta^2 \le 128\Lambda\delta.$$

Similarly,

$$\begin{aligned} \left|\sum_i b_{ii} - \sum a_{ii}\right| &\le c(x)\left|\sum_{i,m} a_{im}a_i(x_m - a_m)\right| + c(x)\left|\sum_{i,l} a_{li}a_i(x_l - a_l)\right| \\ &\quad + c(x)^2\left|\sum_i a_i^2 \sum_{l,m} a_{lm}(x_l - a_l)(x_m - a_m)\right| \le 128\Lambda\delta. \end{aligned}$$

2. The construction

Given $B^* \ge n-1$, integer $k_0 > 0$, let $\{\delta_k\}$, $\{r_k\}$, and $\{N_k\}$ be sequences satisfying $0 < \delta_k < (2400B^*)^{-1}$, $0 < r_k < r_{k-1} < r_1 \le \frac{1}{2}$ and $16\sqrt{n}/\delta_k < N_k < r_{k-1}/r_k$, for $k \ge k_0$. Then $r_{k+1} < \delta_{k+1}N_{k+1}r_{k+1} < N_{k+1}r_{k+1} < r_k$ for $k \ge k_0$. Let [ ] be the greatest integer function, $I_{k_0} = 1$ and $I_k = \prod_{j=k_0}^{k}[\delta_j N_j/16\sqrt{n}]^n$ for $k > k_0$.

Denote by $D_{k_0,1} = D(0, r_{k_0})$. After $\{D_{k,l}: 1 \le l \le I_k\}$ are selected for some $k \ge k_0$ we let $\mathscr{D}_{k,l}$ be the ball $\delta_{k+1}N_{k+1}r_{k+1}r_k^{-1}D_{k,l}$ of radius $\delta_{k+1}N_{k+1}r_{k+1}$; and choose from each $\mathscr{D}_{k,l}$ a number of $[\delta_{k+1}N_{k+1}/16\sqrt{n}]^n$ balls of radius r_{k+1} to form the collection $\{D_{k+1,l}: 1 \le l \le I_{k+1}\}$. Moreover, we require their doublings $\{2D_{k+1,l}\}$ to be mutually disjoint and contained in $\bigcup_l \mathscr{D}_{k,l}$. Let S be the Cantor set defined by $S = \bigcap_{k=k_0}^{\infty}(\bigcup_{l=1}^{I_k} D_{k,l})$. And let μ be the continuous measure on S, defined by $\mu(D_{k,l}) = I_k^{-1}$ for all $k \ge k_0$ and $1 \le l \le I_k$. For $k \ge k_0$, denote by $P_{k,l}$ the center of $D_{k,l}$,

$$R_{k,l} = \{N_{k+1}r_{k+1} \le |x - P_{k,l}| \le r_k\},$$

and

$$R'_{k,l} = \{\tfrac{3}{4}N_{k+1}r_{k+1} \le |x - P_{k,l}| \le \tfrac{5}{4}r_k\},$$

and note that $\{R'_{k,l}: k \ge k_0,\ 1 \le l \le I_k\}$ are mutually disjoint.

Let $B(r)$ be a smooth function for $r > 0$, satisfying $n - 1 \le B(r) \le B^*$, with

$$B(r) \equiv n - 1 \quad \text{on } \{r > \tfrac{5}{4}r_{k_0}\} \cup \bigcup_{k=k_0}^{\infty} [\tfrac{5}{4}r_k,\ \tfrac{3}{4}N_k r_k], \tag{2.1}$$

$$B(r) > n - 1 \quad \text{on } \bigcup_{k \ge k_0} [N_k r_k,\ r_{k-1}],$$

and $B(r)$ monotone in each of the remaining intervals. Define on $\mathbb{R}^n$ an elliptic operator

$$L = \begin{cases} \Delta, & \text{on } \mathbb{R}^n \setminus \bigcup_{k,l} R'_{k,l}, \\[2ex] \dfrac{B(r)}{n-1}\Delta - \left(\dfrac{B(r)}{n-1} - 1\right)\dfrac{\partial^2}{\partial r^2}, & \text{at } x + P_{k,l} \in R'_{k,l}, \end{cases} \tag{2.2}$$

where $r = |x|$. Rewrite L in the standard form $\sum a_{ij}\partial^2/\partial x_i \partial x_j$. We note from Lemma 1 and properties of $B(r)$ that the coefficients a_{ij} are symmetric and are smooth off S; and that a_{ij} are continuous on $\mathbb{R}^n$ if $\lim_{r \to 0} B(r) = n - 1$. Let

$$B_k = \sup\{B(r): 0 < r \le N_k r_k\}, \tag{2.3}$$

from (1.1) it follows that

$$|\xi|^2 \le \sum a_{ij}\xi_i\xi_j \le \frac{B_k}{n-1}|\xi|^2 \quad \text{on } R'_{k,l}. \tag{2.4}$$

Next, we construct positive L-supersolutions.

Fix a point $x_0 \in S$ and rearrange the indices if necessary, we may assume that $x_0 \in \bigcap_k D_{k,1}$. Let

$$D'_{k,1} = \{|x - P_{k,1}| \le \tfrac{5}{4}r_k\},$$
$$D''_{k,1} = \{|x - P_{k,1}| \le \tfrac{3}{4}N_{k+1}r_{k+1}\},$$
$$S_{k,1} = D''_{k-1,1} \setminus D'_{k,1};$$

and note that $D''_{k,1} \subseteq D_{k,1} \subseteq D'_{k,1} \subseteq D''_{k-1,1}$ and $D'_{k,1} \subseteq D(P_{k-1,1}, \delta_k N_k r_k)$. Observe also that

$$\mathbb{R}^n \backslash \{x_0\} = \bigcup_{k \geq k_0} R'_{k,1} \cup \bigcup_{k \geq k_0+1} S_{k,1} \cup \{|x| \geq \tfrac{5}{4} r_{k_0}\};$$

and that $\bigcup_k R'_{k,1}$ and $\bigcup_k S_{k,1}$ meet on the boundaries only. Denote by

$$a^k = (a_1^k, a_2^k, \dots, a_n^k) = P_{k,1} - P_{k-1,1},$$
$$x^k = (x_1^k, x_2^k, \dots, x_n^k) = x - P_{k-1,1},$$

then $|a^k| < \delta_k N_k r_k$ and $|x^k - a^k| \leq N_k r_k$ if $x^k \in S_{k,1}$.

Applying Lemma 2 to $D''_{k-1,1}$ and $D'_{k,1}$ instead of $D(0, \frac{3}{4})$ and $D(a, r)$ for each $k \geq 1 + k_0$ in succession, we obtain, after a scale change, a diffeomorphism T from $\mathbb{R}^n \backslash \{x_0\}$ onto $\mathbb{R}^n \backslash \{0\}$ so that T fixes every point in $\{|x| > \frac{3}{4} N_{k_0+1} r_{k_0+1}\}$, and is a translation on $R'_{k,1}$ for each $k \geq 1 + k_0$ with

$$T(R'_{k,1}) = \{\tfrac{3}{4} N_{k+1} r_{k+1} \leq |y| \leq \tfrac{5}{4} r_k\};$$

and that for $x \in S_{k,1}$,

$$\left| \frac{\partial y_i}{\partial x_j} - \delta_{ij} \right| \leq 32 |a_i^k| \, |x_j^k - a_j^k| (N_k r_k)^{-2} \leq 32 \delta_k, \tag{2.5}$$

$$\left| \sum_j \frac{\partial^2 y_j}{\partial x_l \partial x_m} \xi_j \right| \leq 272 \delta_k |\xi| (N_k r_k)^{-1}, \tag{2.6}$$

and

$$T(S_{k,1}) = \{\tfrac{5}{4} r_k < |y| \leq \tfrac{3}{4} N_k r_k\}.$$

Let $T(x_0) = 0$ and note that T is homeomorphic on $\mathbb{R}^n$.

Let M be the operator on $\mathbb{R}^n \backslash \{0\}$ defined by $Mv(y) \equiv L(v \circ T)(x)$ when $y = Tx$; that is,

$$M = \sum_{i,j} b_{ij} \frac{\partial^2}{\partial y_i \partial y_j} + \sum_j b_j \frac{\partial}{\partial y_j},$$

with

$$b_{ij} = \sum_{l,m} a_{lm} \frac{\partial y_i}{\partial x_l} \frac{\partial y_j}{\partial x_m} \quad \text{and} \quad b_j = \sum_{l,m} a_{lm} \frac{\partial^2 y_j}{\partial x_l \partial x_m}.$$

Thus M is the Laplacian on $\{|y| \geq \frac{5}{4} r_{k_0}\}$. Since T is a translation on $R'_{k,1}$,

$$M = \frac{B(\rho)}{n-1} \Delta - \left(\frac{B(\rho)}{n-1} - 1 \right) \frac{\partial^2}{\partial \rho^2} \quad \text{on } T(R'_{k,1}) \tag{2.7}$$

where $\rho = |y|$. In view of (1.5) and (2.5), we obtain after a scale change that for $x \in S_{k,1}$,

$$\begin{aligned} &\left| \sum b_{ij}(Tx) \xi_i \xi_j - \sum a_{ij}(x) \xi_i \xi_j \right| \\ &\quad \leq 128 \delta_k \sup_{S_{k,1}} \sum a_{ij}(x) \xi_i \xi_j \leq \frac{128}{n-1} B_k \delta_k |\xi|^2. \end{aligned} \tag{2.8}$$

The last inequality follows from (2.3) and the fact that $S_{k,1}$ contains rings from $\{R'_{k,l}\}_l$ but none from the larger ones $\{R'_{k-1,l}\}_l$. Similarly it follows from (1.6) and (2.5) that

$$\left|\sum b_{ii}(Tx) - \sum a_{ii}(x)\right| \le \frac{128}{n-1} B_k \delta_k \quad \text{on } S_{k,1}. \tag{2.9}$$

Let f be a smooth function on $r > 0$, bounded above by $B(r)$, with values $f(r) \equiv 0$ for $r \ge \frac{5}{4} r_{k_0}$, $f(r) = B(r)$ on $\bigcup_{k \ge k_0} [N_{k+1} r_{k+1}, r_k]$,

$$f(r) = \left(n - 2 + \frac{n-1}{B_k}\right)(1 - 1200 B^* \delta_k) \quad \text{on } [\tfrac{5}{4} r_k, \tfrac{3}{4} N_k r_k], \tag{2.10}$$

for each $k \ge 1 + k_0$, and monotone in each of the remaining intervals. Define for $\rho = |y| < 1$,

$$u(y) \equiv u(\rho) \equiv \int_\rho^1 \exp \int_t^1 \frac{f(s)}{s}\, ds\, dt, \tag{2.11}$$

and claim that

$$Mu \le 0 \quad \text{in } D\backslash\{0\}. \tag{2.12}$$

The idea of defining a radial M-supersolution in the form (2.11) comes from Gilbarg and Serrin [5] and Bauman [4]. It follows from (2.7) and the fact that $f(r) \le B(r)$ that $Mu \le 0$ on $\{\frac{5}{4} r_{k_0} < |y| < 1\} \cup \bigcup_k T(R'_{k,1})$. On $T(S_{k,1})$, we note that

$$Mu(y) = \frac{u'(\rho)}{\rho}\left[-\sum_{i,j} b_{ij} \frac{y_i y_j}{\rho^2} f(\rho) + \sum_i b_{ii} - \sum_{i,j} b_{ij} \frac{y_i y_j}{\rho^2} + \sum_j b_j y_j\right].$$

Eigenvalues of $(a_{ij}(x))$ are in the form $1, \Lambda(x), \Lambda(x), \dots, \Lambda(x)$, with

$$1 \le \Lambda(x) \le \frac{B_k}{n-1} \quad \text{on } S_{k,1}. \tag{2.13}$$

We obtain from (2.6) that

$$\left|\sum b_j y_j\right| = \left|\sum_{l,m} a_{lm} \sum_j \frac{\partial^2 y_j}{\partial x_l \partial x_m} y_j\right| \le 272 \delta_k B_k \frac{n}{n-1}$$

on $T(S_{k,1})$. For $x \in S_{k,1}$, $y = Tx$, and $\rho = |y|$, we obtain from (2.8), (2.9), (2.10), (2.13) and the assumptions $B(r) \le B^*$ and $\delta_k < (2400B^*)^{-1}$ that

$$\begin{aligned} \frac{\sum b_{ii} + \sum b_j y_j}{\sum b_{ij} \frac{y_i y_j}{\rho^2}} - 1 &\ge \frac{\sum a_{ii} - \frac{128}{n-1} B_k \delta_k - 272 \frac{n}{n-1} B_k \delta_k}{\sum a_{ij} \frac{y_i y_j}{\rho^2} + \frac{128}{n-1} B_k \delta_k} - 1 \\ &\ge \frac{1 + (n-1)\Lambda(x) - \left(\frac{128}{n-1} + 272 \frac{n}{n-1}\right) B_k \delta_k}{\Lambda(x) + \frac{128}{n-1} B_k \delta_k} - 1 \\ &\ge \left(n - 2 + \frac{n-1}{B_k}\right)(1 - 1200 B^* \delta_k) \ge f(|y|). \end{aligned} \tag{2.14}$$

Hence $Mu \le 0$ on $T(S_{k,1})$, and (2.12) is proved.

Let $H_{x_0}(x_0) = +\infty$ and

$$H_{x_0}(x) = u(Tx) \quad \text{on } D\backslash\{x_0\}. \tag{2.15}$$

Since $LH_{x_0} \leq 0$ on $D\backslash\{x_0\}$ and the coefficients of L are smooth off S, H_{x_0} is an L-supersolution in $D\backslash S$. We shall estimate the growth of $H_{x_0}(x)$ near x_0.

In the rest of the paper, C denotes positive constants depending at most on n, α and a in the theorems; its value may vary from line to line.

3. Proof of Theorem 1

Let

$$B = \frac{2(n-1) + 2n(\alpha + 2 - n)}{n - \alpha},$$

and note that $B > \alpha + 1 > n - 1$, $B \to n - 1$ as $\alpha \to n - 2$, and

$$\frac{\alpha}{n} < \frac{B - (1+\alpha)}{B - (n-2) - (n-1)/B} < 1. \tag{3.1}$$

Choose α', $\alpha < \alpha' < n$, so that

$$\frac{\alpha}{n} < \frac{B - (1+\alpha')}{B - (n-2) - (n-1)/B} < 1,$$

and denote by

$$A = \frac{B - (1+\alpha')}{B - (n-2) - (n-1)/B} \quad \text{and} \quad E = A - \frac{\alpha}{n}.$$

Let

$$\delta_k = \frac{16\sqrt{n}}{k}, \quad N_k = k^{A/E} \quad \text{and} \quad r_k = (k!)^{-1/E} \tag{3.2}$$

and note that $N_k \leq r_{k-1}/r_k$ for $k \geq 1$. To specify $B(r)$ in (2.1), we let

$$B(r) \equiv B \quad \text{on} \bigcup_{k \geq k_0} [N_{k+1} r_{k+1}, r_k]; \tag{3.3}$$

and note that $n - 1 \leq B(r) \leq B$. Choose and fix integer $k_0 \geq 10^5 B\sqrt{n}$.

It is clear that

$$\lim_{k\to\infty} (k!)^{\alpha/E}(\delta_k N_k r_k)^{\gamma} = 0 \quad \text{if } \gamma > \alpha; \tag{3.4}$$

and that

$$\lim_{k\to\infty} (2^{-k}(k-1)!)^{-\alpha/E}/(\delta_k N_k r_k)^{\eta} = 0, \quad \text{if } \eta < \alpha. \tag{3.5}$$

Note that there are $I_k = \prod_{j=k_0}^{k} [j^{A/E-1}]^n$ balls in $\{D_{k,l}\}_l$ and that

$$\left(\frac{k!}{2^k k_0!}\right)^{\alpha/E} \leq I_k \leq k!^{\alpha/E} \tag{3.6}$$

when k is large. And recall that μ is the continuous measure on S defined by

$$\mu(D_{k,l}) = I_k^{-1} \quad \text{for each } l \text{ and } k \geq k_0. \tag{3.7}$$

The smallest balls that carry a μ-measure I_{k-1}^{-1} have radii proportional to $\delta_k N_k r_k$. Using (3.5) and (3.6) one may check that for each $\eta < \alpha$, $\mu(D(x, r)) \leq C_\eta r^\eta$ for all $x \in \mathbb{R}^n$ and $r > 0$. Hence S has positive η-dimensional measure for all $\eta < \alpha$. In view of (3.4), S has Hausdorff dimension α.

For $0 < t < r_{k_0}$, let $K \equiv K(t)$ be the largest integer so that $r_K \geq t$. We deduce from (2.10), (2.11), and (3.3) that, for $0 < \rho < r_{k_0}$,

$$u(\rho) \geq \int_\rho^{r_{K_0}} \exp\left\{\int_t^{r_{K_0}} \frac{B}{S}\,ds - \sum_{k=k_0}^{K} \int_{r_k}^{N_k r_k} B - \left(n-2+\frac{n-1}{B}\right)(1-1200B\delta_k)\frac{ds}{s}\right\} dt.$$

And note from the choices of A and δ_k that

$$\begin{aligned}
&\sum_{k=k_0}^{K} \int_{r_k}^{N_k r_k} B - \left(n-2+\frac{n-1}{B}\right)(1-1200B\delta_k)\frac{ds}{s} \\
&\quad \leq \sum_{k=k_0}^{K} \left[\left(B-n+2-\frac{n-1}{B}\right) + 1200B\delta_k(n-1)\right]\frac{A}{E}\log k \\
&\quad \leq \sum_{k=k_0}^{K} \frac{(B-\alpha'-1)}{E}\log k + C\frac{\log k}{k} \\
&\quad = C + (B-\alpha'-1)\log\frac{1}{t} + C\left(\log\log\frac{1}{t}\right)^2.
\end{aligned}$$

Therefore, for $0 < \rho < r_{k_0}$,

$$u(\rho) \geq \int_\rho^{r_{k_0}} \exp\left\{-C + (\alpha'+1)\log\frac{1}{t} - C\left(\log\log\frac{1}{t}\right)^2\right\} dt \geq C\rho^{-\gamma}$$

for some γ satisfying $\alpha < \gamma < \alpha'$. From the property (2.5) of the transformation T, it follows that

$$H_{x_0}(x) \geq C|x-x_0|^{-\gamma} \quad \text{when } |x| < r_{k_0}. \tag{3.8}$$

Let $v(x) = \int_S H_z(x)\,d\mu(z)$, where μ is the measure defined in (3.7). Clearly v is an L-supersolution on $D\backslash S$. In view of (3.4), (3.6), and (3.8), v approaches $+\infty$ as $x \to x_0$ for every $x_0 \in S$. Since $f(r)$ is bounded, $v < +\infty$ on $D\backslash S$.

Clearly (0.2) holds with $\Lambda_{n,\alpha} = B/(n-1)$.

The number B was chosen so that among other properties, (3.1) holds. As a consequence, (0.3) and (0.4) follow.

4. Proof of Theorem 2 ($n > 3$)

Let

$$\delta_k = \frac{16\sqrt{n}}{k^{3/2}}, \qquad N_k = k^{2n-5/2} \tag{4.1}$$

and

$$r_k = (k!)^{-2n+(4+(2+a)/(n-2))/k}$$

for $k \geq 1$. Choose an integer $k_0 \geq 10^5(1+a)n$, so that $N_k \leq r_{k-1}/r_k$ for $k \geq k_0$. It is easy to check that

$$\lim_{k\to\infty} (k!)^{2n(n-2)} r_k^\gamma = 0 \quad \text{if } \gamma > n-2, \tag{4.2}$$

$$\lim_{k\to\infty} (k-1)!^{-2n(n-2)}/(\delta_k N_k r_k)^{n-2} \left(\log \frac{1}{\delta_k N_k r_k}\right)^{-1-a} = 0, \tag{4.3}$$

$$\lim_{k\to\infty} (k-1)!^{-2n(n-2)}/(\delta_k N_k r_k)^{n-2} \left(\log \frac{1}{\delta_k N_k r_k}\right)^{-3-a} = \infty. \tag{4.4}$$

There are $(k!/k_0!)^{2n(n-2)}$ balls in $\{D_{k,l}\}_l$, and that

$$\mu(D_{k,l}) = (k_0!/k!)^{2n(n-2)} \tag{4.5}$$

for $k \geq k_0$. In view of (4.2) and (4.3), S has Hausdorff dimension $n-2$ and positive h-measure, where $h(r) = r^{n-2}(\log \frac{1}{r})^{-1-a}$.

Let

$$\beta = 12n(1+a), \tag{4.6}$$

and $B(r)$ be the function described in §2, with

$$B(r) = n - 1 + \frac{\beta}{\log \frac{1}{r}} \quad \text{on } \bigcup_{k\geq k_0} [N_{k+1}r_{k+1}, r_k].$$

Clearly $n-1 \leq B(r) \leq n-1+\beta/k_0 \equiv B^*$, and $\delta_k < (2400B^*)^{-1}$ for $k \geq k_0$.

Stirling's formula shows that

$$\left(\log \frac{1}{r_{k-1}}\right)^{-1} \leq \frac{1}{2nk\log k}\left(1 + \frac{C}{\log k}\right), \tag{4.7}$$

when k is sufficiently large. Since $(\log \frac{1}{s})^{-1}$ is an increasing function of s $(0 < s < 1)$, and $r_{k-1} \geq N_k r_k$, we obtain from (4.7) that

$$\begin{aligned} 1 - \frac{n-1}{B_k} = \frac{B_k - (n-1)}{B_k} &\leq \frac{\beta}{(n-1)\log 1/r_{k-1}} \\ &\leq \frac{\beta}{2n(n-1)k\log k}\left(1 + \frac{C}{\log k}\right) \end{aligned} \tag{4.8}$$

for large k, here B_k is the number defined in (2.3).

Thus, for $0 < \rho < r_{k_0}$,

(4.9)

$$\begin{aligned} u(\rho) \geq \int_\rho^{r_{k_0}} \exp\Bigg\{ &\int_t^{r_{k_0}} \frac{n-1}{S} + \frac{\beta}{s\log\frac{1}{s}}\,ds - \sum_{k=k_0}^{K} \int_{r_k}^{N_k r_k} \frac{\beta}{s\log\frac{1}{s}}\,ds \\ &- \sum_{k=k_0}^{K} \int_{r_k}^{N_k r_k} (n-1) - \left(n-2+\frac{n-1}{B_k}\right)(1-1200B^*\delta_k)\frac{ds}{s} \Bigg\}\,dt, \end{aligned}$$

where $K = k(t)$ is the largest integer satisfying $r_K \geq t$. We deduce from (4.8) that

$$
\begin{aligned}
&\sum_{k=k_0}^{K}\int_{r_k}^{N_k r_k}(n-1)-\left(n-2+\frac{n-1}{B_k}\right)(1-1200B^*\delta_k)\frac{ds}{s}\\
&\le\sum_{k=k_0}^{K}\left[\frac{\beta}{2n(n-1)k\log k}\left(1+\frac{C}{\log k}\right)+Ck^{-3/2}\right]\log N_k\\
&\le C+\frac{\beta(2n-\frac{5}{2})}{2n(n-1)}\log K+C\log\log K\\
&\le C+\frac{\beta(2n-\frac{5}{2})}{2n(n-1)}\log\log\frac{1}{t}+C\log\log\log\frac{1}{t}.
\end{aligned}
\tag{4.10}
$$

Again, from (4.7) and monotonicity of $(\log\frac{1}{s})^{-1}$, it follows that

$$
\begin{aligned}
\sum_{k=k_0}^{K}\int_{r_k}^{N_k r_k}\frac{\beta}{s\log\frac{1}{s}}\,ds&\le\frac{\beta(2n-\frac{5}{2})}{2n}\sum_{k=k_0}^{K}\left(1+\frac{C}{\log k}\right)\Big/k\\
&\le C+\frac{\beta(2n-\frac{5}{2})}{2n}\log\log\frac{1}{t}+C\log\log\log\frac{1}{t}.
\end{aligned}
\tag{4.11}
$$

We conclude from (4.6), (4.9), (4.10) and (4.11) that

$$
\begin{aligned}
u(\rho)&\ge\int_\rho^{r_{k_0}}\exp\left\{-C+(n-1)\log\frac{1}{t}\right.\\
&\qquad+\beta\left[1-\frac{2n-\frac{5}{2}}{2n}\left(1+\frac{1}{n-1}\right)\right]\log\log\frac{1}{t}\\
&\qquad\left.-C\log\log\log\frac{1}{t}\right\}dt\\
&\ge C\int_\rho^{r_{k_0}}t^{-n+1}\left(\log\frac{1}{t}\right)^{\beta/4(n-1)}\left(\log\log\frac{1}{t}\right)^{-C}dt\\
&\ge C\rho^{-n+2}\left(\log\frac{1}{\rho}\right)^{3+a}
\end{aligned}
\tag{4.12}
$$

for $0<\rho<r_{k_0}$. Therefore,

$$
H_{x_0}(x)\ge C|x-x_0|^{-n+2}\left(\log\frac{1}{|x-x_0|}\right)^{3+a}\quad\text{for } |x|<r_{k_0}.
\tag{4.13}
$$

The relation

$$
\frac{2n-\frac{5}{2}}{2n}\left(1+\frac{1}{n-1}\right)<1
$$

used in (4.12) is prepared in the choices of r_k and N_k.

The ellipticity of a_{ij}, (0.5), follows from the choice of $B(r)$; and the continuity of a_{ij} in $\mathbb{R}^n$ follows from $\lim_{r\to0}B(r)=n-1$. Recall that $LH_{x_0}\le0$ on $D\backslash\{x_0\}$; it follows from the maximum principle and the solvability of the Dirichlet problem for operators with continuous coefficients [5, pp. 220 and 252] and H_{x_0} is L-supersolution in D.

Again, because a_{ij} are continuous, Green functions $G(x, x_0)$ exist in D (see Bauman [3, 4]). In fact, for each $x_0\in D$, $G(\cdot, x_0)$ is a positive L-solution in $D\backslash\{x_0\}$ with boundary value vanishing continuously on $|x| = 1$. Let $\overline{x} = (\frac{1}{2}, 0, 0, 0, \ldots 0)$ and assume that G is normalized so that $G(\overline{x}, x_0) = 1$. We claim that for each $x_0 \in S$,

$$G(x, x_0) \geq C|x - x_0|^{-n+2}\left(\log \frac{1}{|x - x_0|}\right)^{3+a} \tag{4.14}$$

whenever $0 < |x - x_0| < r_{k_0}$.

Let $g(r) = \sup\{G(x, x_0)\colon |x - x_0| = r\}$ for $0 < r < r_{k_0}$. Applying (4.13) and the maximum principle to the region $D\backslash\{|x - x_0| \leq r\}$, we obtain

$$1 = G(\overline{x}, x_0) \leq Cg(r)r^{n-2}(\log \tfrac{1}{r})^{-3-a}H_{x_0}(\overline{x}).$$

Because $f(r)$ is bounded and $|x_0 - \overline{x}| > \frac{1}{4}$, $H_{x_0}(\overline{x}) < C < \infty$ for all $x_0 \in S$. Hence $g(r) \geq Cr^{-n+2}(\log \frac{1}{r})^{3+a}$ for $0 < r < r_{k_0}$. Thus (4.14) follows from the Harnack principle.

In view of (4.14), the maximum principle and the solvability of the Dirichlet problem, $G(\cdot, x_0)$ is actually an L-supersolution on D. The function $v(x) = \int_S G(x, z)\, d\mu(z)$ approaches $+\infty$ on S in view of (4.4) and (4.14), and it is the function desired.

5. Proof of Theorem 2 ($n = 2$)

Let $\delta_k \equiv \delta \equiv [50000(1 + a)^2]^{-1}$, $N_k \equiv N \equiv 1600000\sqrt{2}(1 + a)^2$, $r_k \equiv e^{-4^{k/(1+a)}}/32\sqrt{2}$ for $k \geq 1$. Choose integers $k_0 \geq 1 + a$ so that

$$r_{k-1}/r_k \geq N^4 \quad \text{when } k \geq k_0. \tag{5.1}$$

Note that

$$4^{-k}\left(\log \frac{1}{N\delta r_k}\right)^{1+a} = 1. \tag{5.2}$$

We note that there are 4^{k-k_0} disks in $\{D_{k,l}\}_l$, and that $\mu(D_{k,l}) = 4^{-k+k_0}$. In view of (5.2), S has positive finite h-measure for $h(r) = (\log \frac{1}{r})^{-1-a}$.

Choose

$$B(r) = 1 + 20(1 + a)^2\left(\log \frac{1}{r}\right)^{-1} \quad \text{on } \bigcup_{k \geq k_0} [N_{k+1}r_{k+1}, r_k]. \tag{5.3}$$

Clearly $1 \leq B(r) \leq 6(1 + a)^2$. Let $f(r)$ be the function in §2, satisfying all the properties there except (2.10); instead, let $f(r) \equiv 0$ on $\bigcup[\frac{5}{4}r_k, \frac{3}{4}N_k r_k]$. The fact that $Mu \leq 0$ in $D\backslash\{0\}$ is not affected by the change of $f(r)$ due to the estimate (2.14).

For $0 < \rho < r_{k_0}$,

$$u(\rho) \geq \int_\rho^{r_{k_0}} \exp\left\{\int_t^{r_{k_0}} \frac{1}{s} + \frac{20(1 + a)^2}{s \log \frac{1}{s}}\, ds - \sum_{k=k_0}^{K} \int_{r_k}^{N_k r_k} \frac{1}{s} + \frac{20(1 + a)^2}{s \log \frac{1}{s}}\, ds\right\} dt,$$

where $K = K(t)$ is the largest integer so that $r_K \geq t$. We deduce from (5.1) that

$$\sum_{k=k_0}^{K} \int_{r_k}^{Nr_k} \frac{1}{s\log\frac{1}{s}}\,ds \leq C + \frac{1}{4}\int_{t}^{r_{k_0}} \frac{1}{s\log\frac{1}{s}}\,ds \leq C + \frac{1}{4}\log\log\frac{1}{t},$$

and that

$$\sum_{k=k_0}^{K} \int_{r_k}^{Nr_k} \frac{1}{s}\,ds = (K-K_0)\log N \leq C + 14(1+a)^2\log\log\frac{1}{t}.$$

Combining the above estimates, we obtain

$$u(\rho) \geq \int_{\rho}^{1} \exp\left\{C + \log\frac{1}{t} + (1+a)^2\log\log\frac{1}{t}\right\} dt \geq C\left(\log\frac{1}{\rho}\right)^{1+(1+a)^2}$$

for $0 < \rho < r_{k_0}$.

In view of (5.3), a_{ij} are continuous in D. Thus the normalized Green function exists on D and satisfies

$$G(x, x_0) \geq C\left(\log\frac{1}{|x-x_0|}\right)^{1+(1+a)^2} \quad \text{for } |x-x_0| < r_{k_0}.$$

The function $v(x) = \int_S G(x, y)\,d\mu(y)$ has all the properties in the theorem.

6. Proofs of Theorems 3 and 4

We follow the constructions in §2 and indicate the necessary changes.

Given $B^* = n-1$, k_0, $\{\delta_k\}$, $\{r_k\}$ and $\{N_k\}$, let S be the Cantor set and μ be the measure on S defined in §2.

Let $B(r)$ be a new function, smooth on $r > 0$, with values $\frac{1}{2} < B(r) \leq n-1$, satisfying (2.1),

$$B(r) < n-1 \quad \text{on } \bigcup_{k\geq k_0} [N_{k+1}r_{k+1}, r_k], \tag{6.1}$$

and monotone in each of the remaining intervals. Define an operator L associated with this $B(r)$ as in (2.2). Let

$$\beta_k = \inf\{B(r) : 0 < r \leq N_k r_k\},$$

then

$$\frac{\beta_k}{n-1}|\xi|^2 \leq \sum a_{ij}\xi_i\xi_j \leq |\xi|^2 \quad \text{on } R'_{k,l}.$$

Fix $x_0 \in S$, let $y = Tx$ be the diffeomorphism and M be the operator defined before. Clearly (2.5) $\sim$ (2.7) are retained; and (2.8) and (2.9) can be replaced respectively by

$$\left|\sum b_{ij}(Tx)\xi_i\xi_j - \sum a_{ij}(x)\xi_i\xi_j\right| \leq 128\delta_k|\xi|^2 \quad \text{on } S_{k,1},$$

and

$$\left|\sum b_{ii}(Tx)\xi_i\xi_j - \sum a_{ii}(x)\xi_i\xi_j\right| \leq 128\delta_k \quad \text{on } S_{k,1}.$$

Suppose that F is a smooth function on $r > 0$, with values $F(r) \geq B(r)$, $F(r) \equiv n-1$ for $r \geq \frac{5}{4}r_{k_0}$, $F(r) = B(r)$ on $\bigcup_{k\geq k_0}[N_{k+1}r_{k+1}, r_k]$,

$$F(r) = \left(n-2+\frac{n-1}{\beta_k}\right)(1+5000\delta_k) \quad \text{on } [\tfrac{5}{4}r_k, \tfrac{3}{4}N_k r_k],$$

for each $k \geq 1+k_0$, and that F is monotone in each of the remaining intervals. Define for $\rho = |y| < 1$,

$$U(y) = U(\rho) = \int_\rho^1 \exp \int_t^1 \frac{F(s)}{s}\, ds\, dt.$$

Arguing as in §2, we conclude that for $x \in S_{k,1}$, and $y = Tx$,

$$\frac{\sum b_{ii} + \sum b_j y_j}{\sum b_{ij} \frac{y_i y_j}{\rho^2}} - 1 \leq f(|y|).$$

From this, we may deduce that $MU(y) \geq 0$ on $\{|y| < 1\}\backslash\{0\}$. Thus,

$$Q_{x_0}(x) \equiv U(Tx) \quad \text{on } D\backslash\{x_0\}$$

is an L-subsolution in $D\backslash S$.

To complete the proof of Theorem 3, we let δ_k and N_k be the numbers defined in (4.1), let $\tau > a/(n-2)$ and $r_k = (k!)^{-2n-\tau/k}$. Fix an integer $k_0 \geq 20(n^2+\tau^2)$, so that $N_k \leq r_{k-1}/r_k$ and $\delta_k \leq (2400n)^{-1}$ for $k \geq k_0$. It is ready to check that

$$\lim_{k\to\infty} (k!)^{2n(n-2)} r_k^{n-2} \left(\log \frac{1}{r_k}\right)^a = 0, \tag{6.2}$$

$$\lim_{k\to\infty} ((k-1)!)^{-2n(n-2)} (\delta_k N_k r_k)^{-\eta} = 0 \quad \text{if } \eta < n-2, \tag{6.3}$$

and

$$\sum_{k\geq k_0} ((k-1)!)^{-2n(n-2)} r_k^{-n+2} \left(\log \frac{1}{r_k}\right)^{-2n(n+\tau)} < \infty. \tag{6.4}$$

There are $(k!/k_0!)^{2n(n-2)}$ balls in $\{D_{k,l}\}_l$ for each $k \geq k_0$, and $\mu(D_{k,l}) = (k_0!/k!)^{2n(n-2)}$. From (6.2) and (6.3) it follows that S has Hausdorff dimension $n-2$, and zero h-measure for $h(r) = r^{n-2}(\log\frac{1}{r})^a$.

Let

$$\beta = 16n^2(n+\tau), \tag{6.5}$$

and

$$B(r) = n - 1 - \frac{\beta}{\log\frac{1}{r}} \quad \text{on } \bigcup_{k\geq k_0} [N_{k+1}r_{k+1}, r_k].$$

Thus, for $0 < \rho < r_{k_0}$,

$$\begin{aligned} U(\rho) \leq \int_\rho^1 \exp\Bigg\{ &\int_t^1 \frac{n-1}{s} - \frac{\beta}{s\log\frac{1}{s}}\, ds + C + \sum_{k=k_0}^{K} \int_{r_k}^{N_k r_k} \frac{\beta}{s\log\frac{1}{s}}\, ds \\ &+ \sum_{k=k_0}^{K} \int_{r_k}^{N_k r_k} \left[-(n-1) + \left(n-2+\frac{n-1}{\beta_k}\right)(1+5000\delta_k)\right]\frac{ds}{s} \Bigg\}\, dt, \end{aligned}$$

where $K = K(t)$ is the largest integer satisfying $r_K \geq t$. Note that for large k, inequality (4.7) still holds and

$$\frac{n-1}{\beta_k} - 1 \leq \frac{9\beta}{16n(n-1)k\log k}.$$

Thus $U(\rho) \leq C\rho^{-n+2}(\log\frac{1}{\rho})^{-\beta/8n}$.

Let $G(\ ,x_0)$ be the normalized Green function on D with

$$G((\tfrac{1}{2},0,0,\dots 0),x_0)=1.$$

Arguing as in §4, we obtain

$$G(x,x_0)\le C|x-x_0|^{-n+2}\left(\log\frac{1}{|x-x_0|}\right)^{-\beta/8n} \tag{6.6}$$

for all $x_0\in S$ and $|x-x_0|<r_{k_0}$. We may also prove that

$$G(x,x_0)\to+\infty \quad \text{as } x\to x_0$$

for each $x_0\in S$, by constructing a positive L-supersolution approaching $+\infty$ at x_0. Thus $G(\cdot,x_0)$ is an L-supersolution on D. In view of (6.4), (6.5), and (6.6),

$$w(x)=\int_S G(x,y)\,d\mu(y)$$

has all the properties stated in Theorem 3.

To prove Theorem 4, we need only to verify that the coefficients of L can be chosen so that (0.6) and (0.7) are fulfilled. Let

$$b=\begin{cases}1-2\alpha, & \text{if } 0<\alpha<\dfrac{1}{4},\\[2ex] (n-1)\left(1-\dfrac{n-2-\alpha}{n-\frac{3}{2}-\alpha}\right), & \text{if } n-\dfrac{9}{4}<\alpha<n-2,\end{cases}$$

and note that $\frac{1}{2}<b<1+\alpha<n-1$ and that

$$\frac{\alpha}{n}<\frac{1+\alpha-b}{n-2+(n-1)/b-b}<1.$$

Choose α', $0<\alpha'<\alpha$ so that

$$\frac{\alpha}{n}<\frac{1+\alpha'-b}{n-2+(n-1)/b-b}<1,$$

and denote by

$$A=\frac{1+\alpha'-b}{n-2+(n-1)/b-b},\qquad E=A-\frac{\alpha}{n}.$$

Let δ_k, N_k and r_k be defined according to (3.2), associated with the current choices of A and E; and let the function $B(r)$ in (6.1) be chosen so that

$$B(r)\equiv b \quad \text{on } \bigcup_{k\ge k_0}[N_{k+1}r_{k+1},r_k].$$

It is ready to check that S has dimension α and that the L-subsolution $Q_{x_0}(x)$ satisfies

$$Q_{x_0}(x)\le C|x-x_0|^{-\gamma},\quad \text{when } |x-x_0|<r_{k_0}$$

for some γ with $\alpha'<\gamma<\alpha$. The rest of the proof is routine and follows from the observation

$$\sum_{k\ge k_0}(k-1)!^{-\alpha/E}(\delta_kN_kr_k)^{-\gamma}<\infty \quad \text{if } \gamma<\alpha.$$

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