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DIFFERENTIALS OF COMPLEX INTERPOLATION PROCESSES FOR KÖTHE FUNCTION SPACES

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ABSTRACT. We continue the study of centralizers on Köthe function spaces and the commutator estimates they generate (see [29]). Our main result is that if X is a super-reflexive Köthe function space then for every real centralizer Ω on X there is a complex interpolation scale of Köthe function spaces through X inducing Ω as a derivative, up to equivalence and a scalar multiple. Thus, in a loose sense, all real centralizers can be identified with derivatives of complex interpolation processes. We apply our ideas in an appendix to show, for example, that there is a twisted sum of two Hilbert spaces which fails to be a (UMD)-space.

1. INTRODUCTION

In this paper we develop ideas first suggested by work of Rochberg and Weiss [45] and continued in [29]. Our basic program is to study "twisted sums" of Banach spaces (see [24, 25, 26, 32, 35]) as differentials of interpolation processes, and the corresponding commutator estimates obtained for linear operators. In [29] we gave several applications of these ideas, in particular applying them to obtain results of Davis [17] and Ceretelli [10] on the distributions of functions in the Hardy class H^1. (See also [27, 30].) Our results in this paper complement and improve the results we obtained in [29]. In this paper we study general Köthe function spaces in place of rearrangement-invariant spaces and use a different approach. The main idea here is to characterize those twisted sums of Köthe function spaces which can be obtained by differentiating a complex interpolation scale of Köthe function spaces.

Let us now give some definitions and an informal discussion of our main results.

If $0 \to Y \to Z \to X \to 0$ is an exact sequence of (quasi-)Banach spaces (over the field $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$) we refer to Z as a *twisted sum* of X and Y (or an *extension* of X by Y). If $\dim Y = 1$, we say that Z is a *minimal extension* of X.

Suppose X is a quasi-Banach space with a dense linear subspace X_0 and that Y is a quasi-Banach space contained in a linear space $\widetilde{Y}$ (which does not necessarily carry any topology). Consider a map $\Omega\colon X_0 \to \widetilde{Y}$ satisfying, for

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suitable $\rho = \rho(\Omega)$,

$$\Omega(\alpha x) = \alpha\Omega(x), \qquad \alpha \in \mathbf{K},\ x \in X, \tag{1.1}$$

$$\|\Delta_\Omega(x_1, x_2)\|_Y \le \rho(\|x_1\|_X + \|x_2\|_X), \qquad x_1, x_2 \in X_0, \tag{1.2}$$

where we introduce the notation

$$\Delta_\Omega(x_1, \dots, x_n) = \sum_{k=1}^{n} \Omega(x_k) - \Omega\left(\sum_{k=1}^{n} x_k\right).$$

Then we can create a twisted sum $X \oplus_\Omega Y$ as the completion of the space Z_0 of all $(x, y) \in X_0 \oplus \widetilde{Y}$ such that

$$\|(x, y)\|_Z = \|x\|_X + \|y - \Omega(x)\|_Y < \infty.$$

Conversely every twisted sum of X and Y can be described in this way taking $X = X_0$ and $Y = \widetilde{Y}$ (see [24, 33, 44]).

We refer to [2, 3] for a general discussion of interpolation theory. In [23, 45], Jawerth, Rochberg, and Weiss have shown that twisted sums arise very naturally in interpolation theory, as the 'differential' of an interpolation process. The complex method is treated in [45] (see also some similar ideas in [47]) and real methods are considered in [23]. The results are surveyed in [15]. See also related work in [40]. We would also like to mention the work of Coifman and Semmes [14, 48] which seems somewhat related to our ideas; also see the work of Slodkowski [49].

In the discussion that follows we will try to give the flavor of the ideas of the paper, without perhaps being too precise about technical definitions.

Suppose X_0, X_1 is a pair of Banach spaces and $X = X_\theta = [X_0, X_1]_\theta$ is an intermediate space obtained by the complex method of interpolation. Then Rochberg and Weiss associate to the scale $[X_0, X_1]$ and X a map $\Omega\colon X \to X_0 + X_1$ satisfying (1.1) and (1.2) in such a way that if T is a linear operator bounded on both X_0 and X_1 then $T \oplus T$ is also bounded on $X \oplus_\Omega X$ (which we will call $d_\Omega X$ later in the paper). This implies a commutator relationship

$$\|[T, \Omega](x)\|_X \le C\|x\|_X, \tag{1.3}$$

where $[T, \Omega] = T\Omega - \Omega T$.

Now consider a Köthe function space X; for the sake of definiteness let us consider spaces over [0,1]. In [29], the author defined a map $\Omega\colon X \to L_0$ to be a (homogeneous) *centralizer* if it satisfies (1.1) and (1.3) uniformly for all multiplication operators, i.e.,

$$\|\Omega(ux) - u\Omega(x)\|_X \le \rho\|u\|_\infty\|x\|_X \tag{1.4}$$

for $f \in X$, $u \in L_\infty$ where $\rho(\Omega)$ is a constant independent of u, f. It turns out that (1.2) then also holds. Ω is called *symmetric* if for some ρ'

$$\|\Omega(x \circ \sigma) - \Omega(x) \circ \sigma\|_X \le \rho'\|x\|_X$$

for all measure-preserving rearrangements σ of [0,1]. The work of Jawerth, Rochberg, and Weiss then shows that if $p_0 < p < p_1$ and T is an operator of strong types (p_0, p_0) and (p_1, p_1) then (1.3) holds for certain symmetric centralizers on L_p such as

$$\Omega(x) = x \log \frac{|x|}{\|x\|_p} \quad \text{and} \quad \Omega(x) = x \log r_x,$$

where $r_x(t) = \mu\{x: |f(s)| > |f(t)| \text{ or } |f(s)| = |f(t)| \text{ and } s \leq t\}$. The aim in [29] was to show that the same hypotheses in fact ensure that (1.3) holds for every symmetric centralizer.

In this paper our objective is to show at least for super-reflexive Köthe function spaces X that every centralizer with the additional property that $\Omega(x)$ is a real function whenever x is real (Ω is *real*) can be obtained by differentiating a suitable complex interpolation scale of Köthe function spaces. Precisely, we show that if X is a super-reflexive Köthe function space and $\Omega: X \to L_0$ is a real centralizer on X then there are constants $a > 0$ and $B < \infty$ and Köthe function spaces X_0, X_1 so that $X = [X_0, X_1]_{1/2}$ (isomorphically) and if Ω_0 is the induced centralizer on X obtained by differentiation then

$$\|\Omega(x) - a\Omega_0(x)\|_X \leq B\|x\|_X$$

for all $x \in X$. The spaces X_0, X_1 are determined up to isomorphism by Ω and a; if X is rearrangement-invariant and Ω is symmetric then both X_0 and X_1 are rearrangement-invariant. Thus the main interpolation theorem of [29] can be recaptured as a consequence of the Boyd interpolation theorem [5].

The key idea here is to exploit the intimate relationship between centralizers on X and minimal extensions of L_1. To every Köthe function space we may associate a map $\Phi_X: I \to \mathbf{C}$ defined on a dense order ideal in L_1 and satisfying (1.1) and (1.2). For nonnegative functions f

$$\Phi_X(f) = \sup_{\|x\|_X \leq 1} \int f \log|x|\, d\mu .$$

We then show that to every centralizer Ω defined on X corresponds a similar densely defined functional Φ^Ω; if Ω is obtained by differentiating a scale $[X_0, X_1]$ then $\Phi^\Omega = \Phi_{X_1} - \Phi_{X_0}$. The main result is then obtained by showing that if Φ is real for real functions and satisfies (1.1) and (1.2) with $\rho < \log 2$ then it is 'equivalent' to $\Phi_X - \Phi_{X^*}$ for some suitable Köthe function space X.

To illustrate this let us state precisely a finite-dimensional version.

Theorem 1.1. *Given $\varepsilon > 0$ there exists a constant $B = B(\varepsilon)$ so that if $n \in \mathbf{N}$, P is the nonnegative cone in l_1^n and $\Phi: P \to \mathbf{R}$ is a functional satisfying $\Phi(\alpha x) = \alpha(\Phi(x))$ and*

$$|\Delta_\Phi(x_1, x_2)| \leq (1 - \varepsilon)\log 2(\|x_1\| + \|x_2\|),$$

whenever $x, x_1, x_2 \in P$ and $\alpha > 0$, then there exists a lattice norm $\| \ \|_X$ on $\mathbf{R}^n$ so that

$$|\Phi(x) - (\Phi_X(x) - \Phi_{X^*}(x))| \leq B\|x\|$$

for $x \in P$.

One could also state this in terms of approximating nearly affine functions on the unit simplex.

We now briefly outline the contents of the paper. In §2 we set up a formal framework for complex interpolation, including the case of families of spaces studied in [11, 12]. Our framework is a slight variation on the usual approach as outlined in [3], for example, but seems appropriate to our setting. We introduce the notion of a derivation obtained by differentiating an interpolation scale. In §3 we study how these ideas adapt to Köthe function spaces. We study in §4 and §5 the functional Φ_X which we associate to any Köthe function space. Our

main approximation theorem (specialized to Theorem 1.1) above is proved in §6 as Theorem 6.6. Our main result on centralizers is Theorem 7.6. In §8 we generalize our ideas to Schatten ideals.

Finally in §9, we apply the methods of the paper to the study of the Riesz projection and its vector-valued analogue. For $f \in L_1(\mathbf{T})$ we define Rf by $Rf \sim \sum_{n\geq 0} \hat{f}(n)e^{int}$. We then show that if X_0, X_1 are Köthe function spaces on $\mathbf{T}$ so that R is bounded on $X = X_{\theta_0} = [X_0, X_1]_{\theta_0}$ for some $0 < \theta_0 < 1$ where X is super-reflexive then the boundedness of the associated commutator $[R, \Omega]$ on X is equivalent to the boundedness of R on X_θ in a neighborhood of θ_0. We show how this relates to and extends several well-known results on A_p-weights (cf. [13, 41, 50]). In the vector-valued case we can apply the methods to the study of the class of (UMD)-spaces introduced by Burkholder [7, 8]. As an application we show that there is a twisted sum of two Hilbert spaces which fails to be (UMD) thus resolving, negatively, the question of whether the (UMD)-property is a three-space property.

2. Admissible spaces and derivations

Let S be a Polish space and let μ be a σ-finite Borel measure on S. Let $L_0 = L_0(\mu)$ be the space of all measurable complex-valued functions on S endowed with the topology of convergence in measure relative to each Borel set of finite measure. We define an *admissible norm* to be a map $f \to \|f\|_X$ $(L_0 \to [0, \infty])$ such that if $X = \{x \in L_0 : \|x\|_X < \infty\}$ then:

(2.1) X is a vector subspace of L_0 and $\| \ \|_X$ is a norm on X.

(2.2) $B_X = \{x : \|x\|_X \leq 1\}$ is closed in L_0.

(2.3) There exist strictly positive $h, k \in L_0$ so that

$$\|xh\|_1 \leq \|x\|_X \leq \|xk\|_\infty$$

for every $x \in L_0$.

The corresponding space X is then easily proved to be a Banach space which is continuously embedded in L_0. We refer to X as an admissible space. If X is an admissible space then a map $\Omega: X \to L_0$ will be called a derivation if (2.4)–(2.6) hold, where

(2.4) Ω is homogeneous, i.e., $\Omega(\alpha x) = \alpha\Omega(x)$, $\alpha \in \mathbf{C}$, $x \in X$.

(2.5) Ω is quasi-additive, i.e., $\|\Delta_\Omega(x_1, x_2)\|_X \leq \rho(\|x_1\|_X + \|x_2\|_X)$, $x_1, x_2 \in X$, where ρ is a constant independent of x_1, x_2. We refer to the least such constant as $\rho(\Omega)$.

(2.6) The set $\Omega(B_X)$ is bounded in L_0.

For any derivation we may introduce the *derived space* $d_\Omega X$ as the subset of $L_0 \times L_0$ of all pairs (x, y) such that

$$\|(x, y)\|_{X,\Omega} = \|x\|_X + \|y - \Omega(x)\|_X < \infty.$$

Lemma 2.1. *$\| \ \|_{X,\Omega}$ is a quasinorm on $d_\Omega X$ and $d_\Omega X$ is a quasi-Banach space which is continuously embedded in $L_0 \times L_0$. If X is a B-convex Banach space, or X^* is isomorphic to a subspace of an L_1-space then $d_\Omega X$ is a Banach space, and there is a constant C depending only on X so that for any $x_1, \ldots, x_n \in X$,*

$$\|\Delta_\Omega(x_1, \ldots, x_n)\|_X \leq C \sum_{k=1}^{n} \|x_k\|_X.$$

Proof. It is easy to check that if $a, b \in d_\Omega X$ then

$$\|a+b\|_{X,\Omega} \le (1+\rho)(\|a\|_{X,\Omega} + \|b\|_{X,\Omega}).$$

Clearly, if $a_n = (x_n, y_n)$ and $\|a_n\|_{X,\Omega} \to 0$ then $\|x_n\|_X \to 0$ and so, by using (2.4) and (2.6), $\Omega(x_n) \to 0$ in L_0. From this it follows easily that $y_n \to 0$ in L_0. Clearly the subspace $\{(0, y): y \in X\}$ is a closed subspace F of $d_\Omega X$ such that F is isometric to X and $d_\Omega X/F$ is also isometric to X. This is easily seen to imply completeness of $d_\Omega X$ (cf. [24]).

For the second half of the lemma, we quote the results of [24] and [34] that any Banach space X satisfying either criterion is a $\mathscr{K}$-space, i.e., every twisted sum with a Banach is locally convex. Then if we consider any sequence Ω_n of derivations with $\rho(\Omega_n) \le 1$ we can construct a derivation on either $l_2(X)$ or $l_\infty(X)$ according to a case given by $\Omega((x_n)) = (\Omega_n(x_n))$. (We regard both spaces as admissible spaces for a suitable measure space.) Then $\rho(\Omega) \le 1$. Thus the derived space is locally convex from which it easily follows that there is a constant C independent of m, so that

$$\|\Delta_{\Omega_m}(x_1, \dots, x_n)\|_X \le C \sum_{k=1}^{n} \|x_k\|$$

for all $x_1, \dots, x_n$. The lemma then follows. □

We remark that two derivations Ω_1 and Ω_2 give rise to equivalent derived spaces if for some constant B we have $\|\Omega_1(x) - \Omega_2(x)\|_X \le B\|x\|_X$. In this case we term Ω_1, Ω_2, *equivalent*.

We denote by D the open unit disk in the complex plane; its boundary, the unit circle, is denoted by $\mathbf{T}$ and λ will denote the Haar measure $(2\pi)^{-1}\, d\theta$ on $\mathbf{T}$. Now consider a family $\mathscr{X} = \{X_w\}$ of admissible spaces indexed by $w \in \mathbf{T}$; we write $\|x\|_w = \|x\|_{X_w}$. We say that $\mathscr{X}$ is an *admissible family* if:

(2.7) The map $(w, x) \to \|x\|_w$ $(\mathbf{T} \times L_0 \to [0, \infty])$ is a Borel map.

(2.8) There exist strictly positive $h, k \in L_0$ so that for all $x \in L_0$ we have λ-a.e. on $\mathbf{T}$,

$$\|xh\|_1 \le \|x\|_w \le \|xk\|_\infty.$$

We may extend this definition to the case when X_w is defined only on a Borel subset of λ-measure one by simply setting $\|x\|_w = \|xh\|_1$ otherwise.

Let $\mathscr{N}^+$ be the collection of functions $F: D \to L_0$ which can be written in the form $F(z)(s) = F(z, s)$ where μ-almost everywhere we have $F_s \in N^+$ where $F_s(z) = F(z, s)$ and N^+ denotes the Smirnov class (cf. [18]). If $F \in \mathscr{N}^+$ it may easily be shown by an application of Fubini's theorem that relative to the L_0-topology F has radial limits almost everywhere on $\mathbf{T}$. Thus we can define $F(e^{i\theta}) = \lim_{r\to 1} F(re^{i\theta})$ almost everywhere.

If $\mathscr{X}$ is an admissible family then we define for $F \in \mathscr{N}^+$

$$\|F\|_{\mathscr{X}} = \text{ess sup}\, \|F(e^{i\theta})\|_{e^{i\theta}}.$$

Then for $z \in D$ we define

$$\|x\|_z = \inf\{\|F\|_{\mathscr{X}}: F(z) = x\}$$

and let $X_z = \{x: \|x\|_z < \infty\}$.

Lemma 2.2. *Let $\mathscr{X}$ be an admissible family and let Y be a separable admissible space so that $\|x\|_Y \le \|x\|_w$ for all $w \in \mathbf{T}$. Then, if $\|F\|_{\mathscr{X}} < \infty$, we have $\|F(z)\|_Y \le \|F\|_{\mathscr{X}}$ for all $z \in D$. Furthermore $F(re^{i\theta}) \to F(e^{i\theta})$ a.e. in Y.*

For any $z \in D$, $x \in L_0$ we have $\|x\|_Y \le \|x\|_z$.

Proof. We may assume the existence of a strictly positive h satisfying (2.3) for Y and (2.8). Then

$$\begin{aligned}\int_{-\pi}^{\pi}\int_S |F(e^{i\theta}, s)|h(s)\,d\mu(s)\,\frac{d\theta}{2\pi} &\le \int_S\int_{-\pi}^{\pi} |F(e^{i\theta}, s)|h(s)\,\frac{d\theta}{2\pi}\,d\mu(s)\\ &= \int_{-\pi}^{\pi} \|F(e^{i\theta})\|_{L_1(hd\mu)}\frac{d\theta}{2\pi}\\ &\le \|F\|_{\mathscr{X}}.\end{aligned}$$

Hence

$$\int_{-\pi}^{\pi} \|F(re^{i\theta})\|_{L_1(hd\mu)}\frac{d\theta}{2\pi} \le \|F\|_{\mathscr{X}}$$

for $0 < r < 1$ so that $F \in H_1(L_1(hd\mu))$. Since L_1 has the analytic Radon-Nikodym Property [6], this implies that $F(re^{i\theta}) \to F(e^{i\theta})$ a.e. in $L_1(hd\mu)$ and, further, that

$$F(re^{i\theta}) = \int_{-\pi}^{\pi} P(r, \theta - t)F(e^{it})\frac{dt}{2\pi}$$

as a Bochner integral in $L_1(hd\mu)$, where P is the Poisson kernel. Since Y is separable, $F(e^{it})$ is also measurable in Y, and, since it is also bounded in Y, the same equation holds as a Bochner integral in Y. Hence for $z \in D$ we have $\|F(z)\|_Y \le \|F\|_{\mathscr{X}}$. It is also easy to verify that $F(re^{i\theta}) \to F(e^{i\theta})$ in Y almost everywhere.

It now follows that $\|x\|_z \ge \|x\|_Y$ for any $x \in L_0$. □

Lemma 2.3. *Suppose $F_n \in \mathscr{N}^+$ and $\|F_n\|_{\mathscr{X}} \le 1$. Then there is a sequence G_n of convex combinations of $\{F_k : k \ge n\}$ and a $G \in \mathscr{N}^+$ so that $\|G\|_{\mathscr{X}} \le 1$ and*

$$\lim_{n\to\infty} G_n^{(k)}(z) = G^{(k)}(z), \qquad z \in D,\ k = 0, 1, 2, \dots, \tag{2.9}$$

$$\lim_{n\to\infty} G_n(e^{i\theta}) = G(e^{i\theta}) \quad \lambda\text{-a.e.}, \tag{2.10}$$

where both limits are computed in L_0.

Proof. Since

$$\int_{-\pi}^{\pi}\int_S |F_n(e^{i\theta}, s)|h(s)\,d\mu(s)\,\frac{d\theta}{2\pi} \le 1,$$

we can apply Komlos's theorem [36] to extract a sequence G_n of convex combinations as specified so that we have both that $G_n(e^{i\theta}, s)$ converges $\mu \times \lambda$-a.e. and $\int_{-\pi}^{\pi} |G_n(e^{i\theta}, s)|\,d\theta/2\pi$ is bounded μ-a.e. (The sequence of convex combinations $\sum_{k=n}^{N(n)} a_k^n F_k$ is chosen so that μ-a.e.

$$\int_{-\pi}^{\pi} \sum_{k=n}^{N(n)} a_k^n |F_k(e^{i\theta}, s)|\,\frac{d\theta}{2\pi}$$

converges.)

If we write $G_n(z, s) = \sum_{j=0}^{\infty} g_{n,j}(s) z^j$ then we have $G_{n,s} \in N^+$ μ-a.e. Let S_0 be the set of $s \in S$ such that we have, simultaneously, $G_{n,s} \in N^+$, $G_n(e^{i\theta}, s)$ converges λ-a.e. and $\int_{-\pi}^{\pi} |G_n(e^{i\theta}, s)|\, d\theta/2\pi$ is bounded. Then $\mu(S \backslash S_0) = 0$.

For $s \in S_0$ the sequence $G_{n,s}$ is uniformly bounded in $L_1(\mathbf{T}) \cap N^+ = H_1(\mathbf{T})(\mathbf{T})$ and converges a.e. Hence it converges in $L_p(\mathbf{T})$ for every $p < 1$ to a function $G_s \in H_p(\mathbf{T})$. Further if $z \in D$, $k \geq 0$, $\lim_{n\to\infty} G_{n,s}^{(k)}(z) = G_s^{(k)}(z)$ while on $\mathbf{T}$, $\lim_{n\to\infty} G_{n,s}(e^{i\theta}) = G_s(e^{i\theta})$ a.e. Thus for each j, $g_{n,j}$ converges in L_0 (in fact a.e.) to some $g_j \in L_0$ and, if $G(z) = \sum g_j z^j$, then for a.e. $s \in S$ $G(z, s) = G_s(z)$. Hence $G \in \mathcal{N}^+$ and $\|G\|_{\mathscr{X}} \leq 1$. □

Proposition 2.4. *Each space X_z is admissible and if $x \in X_z$ there is an extremal $F \in \mathcal{N}^+$ such that $F(z) = x$ and $\|F\|_{\mathscr{X}} = \|x\|_z$.*

Proof. Conditions (2.1) and (2.3) are elementary. We prove only (2.2), and simultaneously prove the existence of an extremal. To do this we need only consider a sequence x_n with $\|x_n\|_z < 1$ so that $x_n \to x$ a.e. and prove that this implies the existence of $F \in \mathcal{N}^+$ with $F(z) = x$ and $\|F\|_{\mathscr{X}} \leq 1$. However, there exist $F_n \in \mathcal{N}^+$ with $\|F_n\|_{\mathscr{X}} \leq 1$ and $F_n(z) = x_n$ and so the conclusion follows easily from Lemma 2.3. □

These considerations can now be lifted to any open subset U of $\mathbf{C}$ conformally equivalent to D. We define an admissible family of spaces $\mathscr{X} = (X_w : w \in \partial U)$ to be a family of admissible spaces so that

(2.11) The map $(w, x) \to \|x\|_w$ is Borel on $\partial U \times L_0$.

(2.12) There exist strictly positive $h, k \in L_0$ so that for $x \in L_0$ we have for every $w \in \partial U$, $\|xh\|_1 \leq \|x\|_w \leq \|xk\|_\infty$.

Let $\varphi : D \to U$ be any conformal equivalence. Then $\varphi \in H_p$ for any $p < \frac{1}{2}$ [18] and so has radial limits $\varphi(e^{i\theta}) \in \partial U$ a.e. The family $\varphi(\mathscr{X}) = (X_{\varphi(e^{i\theta})})$ is then admissible for D. Define $\mathcal{N}^+(U)$ to be the space of all functions $F : U \to L_0$ for which $F \circ \varphi \in \mathcal{N}^+$ and set $\|F\|_{\mathscr{X}} = \|F \circ \varphi\|_{\varphi(\mathscr{X})}$. It is easy to show that these definitions are independent of φ. For $z \in U$ we define

$$\|x\|_z = \inf\{\|F\|_{\mathscr{X}} : F(z) = x\}$$

and set $X_z = \{x : \|x\|_z < \infty\}$. We then automatically obtain

Proposition 2.5. *Each X_z is admissible and if $x \in X_z$ then there is an extremal $F \in \mathcal{N}^+(U)$ with $F(z) = x$ and $\|F\|_{\mathscr{X}} = \|x\|_z$.*

In particular if X_0, X_1 is a pair of admissible spaces we let $U = \mathscr{S}$ be the strip $\{z : 0 < \Re z < 1\}$ and define $X_{k+it} = X_k$ for $k = 0, 1$ and $t \in \mathbf{R}$. Then for $0 < \theta < 1$ we obtain the complex interpolation space $X_\theta = [X_0, X_1]_\theta$.

Now let U be any open subset of $\mathbf{C}$ which is conformally equivalent to D and let $\mathscr{X}$ be an admissible family. For $z \in U$ and $x, y \in L_0$ we define

$$\|(x, y)\|_{dX_z} = \inf\{\|F\|_{\mathscr{X}} : F(z) = x,\ F'(z) = y\}.$$

For each $x \in X_z$ we may pick an extremal F_x by Proposition 2.5 with $\|F_x\|_{\mathscr{X}} = \|x\|_z$ and $F_x(z) = x$. We may further suppose that $F_{\alpha x} = \alpha F_x$ for $\alpha \in \mathbf{C}$. Then we define $\Omega = \Omega_{\mathscr{X}, z}$ by $\Omega(x) = F_x'(z)$.

Theorem 2.6. *Ω is a derivation on X_z and $\| \ \|_{dX_z}$ is equivalent to $\| \ \|_{X_z,\Omega}$.*

Remark. Theorem 2.6 is essentially due to Rochberg and Weiss [45]. If each X_w is strictly convex, then, as they observe, the extremal F_x is unique and hence Ω is uniquely determined. See also Corollary 4.9 below for more discussion of the uniqueness question.

Proof. Property (2.4) of derivations is built into our construction. Let $\varphi: U \to D$ be a conformal equivalence, with $\varphi(z) = 0$, and let $|\varphi'(z)| = \delta$. Then

$$\begin{aligned}\|\Omega(x)\|_{L_1(hd\mu)} &= \|F_x'(z)\|_{L_1(hd\mu)} \\ &\le \delta \operatorname{ess\ sup} \|F_x(e^{i\theta})\|_{L_1(hd\mu)} \\ &\le \delta \|x\|_{X_z},\end{aligned}$$

so that property (2.6) holds. The remaining statements are essentially contained in [45]. It is readily verified that $\|(0, x)\|_{dX_z} = \delta\|x\|_{X_z}$. It then follows from the triangle law for the norm on dX_z that $\|\Delta_\Omega(x, y)\|_{X_z} \le 2\delta(\|x\|_{X_z} + \|y\|_{X_z})$ so that (2.5) hold with $\rho = 2\delta$. It can also be checked that

$$\min(1, \delta^{-1})\|(x, y)\|_{dX_z} \le \|(x, y)\|_{X_z,\Omega} \le (1 + 2\delta)\|(x, y)\|_{dX_z}.$$

Finally, let us observe, in our setting, the standard commutator interpolation theorem for linear operators.

Theorem 2.7. *Suppose $\mathscr{X}$ is admissible family for U, where U is conformally equivalent to D. Let Y be a separable admissible space such that $\| \ \|_Y \le \| \ \|_w$ for $w \in \partial U$. Let $T: Y \to L_0$ be a continuous linear map such that for some $M < \infty$ and every $w \in U$, $x \in X_w$ $\|Tx\|_w \le M\|x\|_w$.*

Then for $z \in U$ we have $T(X_z) \subset X_z$ and $\|T\|_{X_z \to X_z} \le M$. Further there is a constant $C = C(z)$ so that for every $x \in X_z$

$$\|[T, \Omega]x\|_{X_z} \le CM\|x\|_z,$$

where $\Omega = \Omega_{\mathscr{X},z}$ and $[T, \Omega] = T\Omega - \Omega T$.

Proof. From Nikishin's theorem [39, 42] for any fixed $0 < p < 1$ there exists a strictly positive $h_0 \in L_0$ so that for any $y \in Y$ we have $\|h_0 Ty\|_p \le \|y\|_Y$. Let $\varphi: D \to U$ be a conformal equivalence with $\varphi(0) = z$.

Suppose $\|x\|_{X_z} = 1$. Then $F = F_x \circ \varphi$ is a bounded analytic function with values in Y and has radial limits a.e. in Y by Lemma 2.2. Let $F_x \circ \varphi(\zeta) = \sum_{k=0}^\infty u_k \zeta^k$. Then let $G(\zeta) = \sum_{k=0}^\infty Tu_k \zeta^k$. Then $\|h_0 G(\zeta)\|_p \le 1$ for all $\zeta \in D$. It follows easily that $G(\zeta, s) \in H_p$ for almost every $s \in S$ so that $G \in \mathscr{N}^+$. On $\mathbf{T}$ we have, almost everywhere that $G(e^{i\theta}) = \lim_{r\to 1} TF(re^{i\theta})$ in L_0 so that, again by Lemma 2.2 $G(e^{i\theta}) = TF(e^{i\theta})$. Hence $\|G \circ \varphi^{-1}\|_{\mathscr{X}} \le M$. Thus $\|Tx\|_z \le M\|x\|_z$.

If we let $G_0 = G \circ \varphi^{-1}$ we obtain $\|(Tx, G_0'(z))\|_{dX_z} \le M$ or

$$\|(Tx, TF_x'(z))\|_{dX_z} \le M.$$

This implies that $\|[T, \Omega]x\|_{X_z} \le C(z)M$ as required. □

3. Function spaces

We define a Köthe function space X on S to be an admissible space for which $\| \ \|_X$ is a lattice norm i.e., $\|x\|_X \le \|y\|_X$ whenever $|x| \le |y|$. Since, by

assumption B_X is closed in L_0 it is not difficult to see that we restrict attention to those function spaces which are maximal in the sense of Lindenstrauss-Tzafriri [37, p. 118]. We define the dual X^* to be the function space defined by

$$\|y\|_{X^*} = \sup_{x \in B_X} \int_S |xy| \, d\mu .$$

Our assumptions force X^* to be a norming subspace of the full dual and X^* is a function space.

If X is a function space, then a (homogeneous) *centralizer* is a map $\Omega\colon X \to L_0$ satisfying (2.4), (2.6), and

$$\|\Omega(ux) - u\Omega(x)\|_X \le \rho \|x\|_X \tag{3.1}$$

whenever $\|u\|_\infty \le 1$ and $x \in X$. Here ρ is a constant independent of x and we define $\rho(\Omega)$ to be the least such constant. It is proved in [29] Lemma 4.2, that a centralizer is automatically a derivation. Conversely we have from Theorem 2.7:

Proposition 3.1. *Suppose U is conformally equivalent to D and $\mathscr{X}$ is an admissible family of Köthe function spaces on S. Then for $z \in U$, X_z is a Köthe function space and the corresponding derivation Ω is a centralizer on X_z.*

If we interpolate between two Köthe function spaces X_0 and X_1 then it is well known (due to Calderon [9]) that X_θ is given by $X_\theta = X_0^{1-\theta} X_1^\theta$ i.e.,

$$\|x\|_{X_\theta} = \inf\{\max(\|u\|_{X_0}, \|v\|_{X_1}) \colon |x| = u^{1-\theta} v^\theta, \ u, v \ge 0\}.$$

Furthermore the extremal F_x is given by

$$F_x(z) = (\operatorname{sgn} x) u^{1-z} v^z ,$$

where uv is the optimal factorization; the fact that an optimal factorization exists follows easily from the existence of an extremal. Thus the corresponding derivation is given by

$$\Omega(x) = F_x'(\theta) = x(\log v - \log u). \tag{3.2}$$

Let us note at this point, that there is some ambiguity in this definition since there may not be a unique optimal factorization. At such x we must make a selection, and thus Ω is only fixed up to equivalence. Fortunately, this does not present a significant complication and many situations (such as when X_0 and X_1 have strictly convex norms) uniqueness is guaranteed.

We remark that the identity $X^{1/2} X^{*1/2} = L_2$ is an important result of Lozanovskii [38], (see also [20]) which asserts in our setting that if $f \in L_1$ then f has a unique factorization $f = xy$ where $\|x\|_X = 1$, $\|y\|_{X^*} = \|f\|_1$, $x \ge 0$, and $\operatorname{supp} x = \operatorname{supp} y = \operatorname{supp} f$. We refer to this factorization as the *Lozanovskii factorization of f* for X. Note here the connection with the idea of the duality map on X, whose relationship to differential estimates was observed in [15].

Let us now extend these ideas to the case of interpolation of families. We consider only the case $U = D$. To avoid certain measurability difficulties, we say that an admissible family $\mathscr{X}$ is *strongly admissible* if it is admissible and there is a countable dimensional subspace V of L_0 so that $V \cap B_{X_w}$ is L_0-dense in B_{X_w} for almost every $w \in \mathbf{T}$. It is easy to see that this holds in most reasonable situations.

We introduce a Köthe function space $\mathscr{E}$ on $S \times \mathbf{T}$ by setting

$$\|\phi\|_{\mathscr{E}} = \operatorname{ess\,sup} \|\phi(\cdot, e^{i\theta})\|_{X_{e^{i\theta}}}.$$

Lemma 3.2. *Suppose $\mathscr{X}$ is a strongly admissible family of Köthe function spaces. Then $\mathscr{X}^* = \{X_w^*, w \in \mathbf{T}\}$ is an admissible family and*

$$\|\psi\|_{\mathscr{E}^*} = \int_{-\pi}^{\pi} \|\psi(\cdot, e^{it})\|_{X_{e^{it}}^*} \frac{dt}{2\pi}.$$

We omit the proof of this lemma which is routine. The following theorem is essentially due to Hernandez [22].

Theorem 3.3. *Let $\mathscr{X}$ be a strongly admissible family of Köthe function spaces for D. For $z \in D$, $z = re^{i\theta}$, we have $\|x\|_z = \inf \|\phi\|_{\mathscr{E}}$ where the infimum is taken over all $\phi \in \mathscr{E}$ such that $\phi \geq 0$ and*

$$|x(s)| = \exp\left(\int_{-\pi}^{\pi} P(r, \theta - t) \log \phi(s, e^{it}) \frac{dt}{2\pi}\right).$$

Furthermore there is an extremal choice of ϕ for which the corresponding centralizer $\Omega = \Omega_{\mathscr{X}, z}$ is given by

$$\Omega(x)(s) = x(s)\left(\int_{-\pi}^{\pi} \frac{e^{it}}{(e^{it} - z)^2} \log \phi(s, e^{it}) \frac{dt}{\pi}\right).$$

Proof. We first remark that if $\phi \in \mathscr{E}_+$, then

$$\int_S \int_{-\pi}^{\pi} \phi(s, e^{it}) h(s) \frac{dt}{2\pi} \, d\mu(s) < \infty$$

so that for a.e. $s \log_+ \phi(s, e^{it})$ is integrable. Thus the integrals in the statement of the theorem are unambiguously defined, but may be $-\infty$.

For $\phi \in \mathscr{E}_+$ define

$$F(z, s) = \exp\left(\int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} \log \phi(s, e^{it}) \frac{dt}{2\pi}\right),$$

where $F(z, s) = 0$ at all s such that $\log \phi$ is not integrable. Then $F \in \mathscr{N}^+$, $|F(e^{it}, s)| \leq \phi(s, e^{it})$ a.e. and

$$|F(re^{i\theta, s})| = \exp\left(\int_{-\pi}^{\pi} P(r, \theta - t) \log \phi(s, e^{it}) \frac{dt}{2\pi}\right).$$

We conclude that $\|x\|_z \leq \inf \|\phi\|_{\mathscr{E}}$ in the statement of the theorem. However conversely there exists an extremal $F_x \in \mathscr{N}^+$ and

$$|x(s)| = |F(z, s)| \leq \exp\left(\int_{-\pi}^{\pi} P(r, \theta - t) \log |F(e^{it}, s)| \frac{dt}{2\pi}\right).$$

Now there exists u with $0 \leq u \leq 1$ so that if $\phi(s, e^{it}) = u(s)|F(e^{it}, s)|$ then $\|\phi\|_{\mathscr{E}} = \|x\|_z$ and

$$|x(s)| = \exp\left(\int_{-\pi}^{\pi} P(r, \theta - t) \log |\phi(s, e^{it})| \frac{dt}{2\pi}\right).$$

This choice of ϕ then is optimal and further gives rise to an extremal $F \in \mathscr{N}^+$. The last part of the theorem now follows by evaluating its derivative. □

We conclude this section by introducing some further definitions which will be useful in the next section. We will need to consider certain quasi-Banach function spaces. Let us say that a g-convex quasinorm on L_0 is a functional $x \to \|x\|_X$ $(L_0 \to [0, \infty])$ such that if $X = \{x\colon \|x\|_X < \infty\}$, then (2.2) holds, B_X contains some strictly positive element (this is equivalent to half of (2.3)) and

(3.4) $\|\alpha x\|_X = |\alpha|\|x\|_X$, $\alpha \in \mathbf{C}$, $x \in X$.

(3.5) For every $x, u, v \in L_0$ and $0 \le \theta \le 1$ we have that if $|x| \le |u|^\theta |v|^{1-\theta}$ then $\|x\|_X \le \|u\|_X^\theta \|v\|_X^{1-\theta}$.

(3.6) Suppose for $0 < \tau < \infty$ we define $\eta_X(\tau)$ to be the least η such that if $x, y \in X$ have disjoint supports and $\|x\|_X \le 1$, $\|y\|_X \le \tau$ then $\|x+y\|_X \le 1 + \eta_X(\tau)$. Then $\eta(\tau) < \infty$ for all τ and $\lim_{\tau\to 0} \eta_X(\tau) = 0$.

Here condition (3.6) forces $\| \ \|_X$ to be a quasinorm on X and hence X becomes a quasi-Banach lattice using (3.5). In addition (3.6) imposes a mild continuity property on the quasinorm, while (3.5) (geometrical convexity) implies that X is **A**-convex in the sense of [28] (see Theorem 4.4 of [28]). In fact if X is **A**-convex it is automatically renormable to be p-convex for some $p > 0$. We recall that X is p-convex if for every $x_1, \dots, x_n \in X$ we have

$$\left\| \left(\sum_{i=1}^n |x_i|^q \right)^{1/q} \right\|_X \ge \left(\sum_{i=1}^n \|x_i\|_X^q \right)^{1/q}.$$

We remark that if X is p-convex then X^p is a Köthe function space where X^p is defined by $\|x\|_{X^p} = \||x|^{1/p}\|_X^p$.

4. Indicators of Köthe function spaces

We shall say that a subset $\mathscr{I}$ of the positive cone L_1^+ of $L_1(\mu)$ is a semi-ideal if $\mathscr{I}$ is a cone and if $0 \le f \le g \in \mathscr{I}$ implies $f \in \mathscr{I}$. I is called a strict semi-ideal if it contains a strictly positive member. Let X be a g-convex function space on S. We define $\mathscr{I}_X$ to be the semi-ideal of all $f \in L_1^+$ such that

$$\sup_{x \in B_X} \int_S f \log_+ |x| \, d\mu < \infty \tag{4.1}$$

and

$$\exists x \in B_X, \qquad \int_S f |\log |x|| \, d\mu < \infty. \tag{4.2}$$

For $f \in \mathscr{I}_X$ we define the *indicator* of X, Φ_X, by

$$\Phi_X(f) = \sup_{x \in B_X} \int_S f \log |x| \, d\mu \tag{4.3}$$

and then $-\infty < \Phi_X(f) < \infty$. It is appropriate to note that this idea is suggested by Gillespie's proof of the Lozanovskii theorem [20].

Let us start by identifying Φ_X for some simple spaces. We shall henceforward write $\|f\|$ without subscript for $\|f\|_1$. We say that $f \in L \log L$ if $f(1 + |(\log |f|)|) \in L_1$.

Lemma 4.1. *Suppose* $X = L_1$. *Then* $\mathscr{I}_X = (L\log L)_+$ *and*

$$\Phi_X(f) = \int f \log \frac{f}{\|f\|}\, d\mu .$$

Proof. If $f \in (L\log L)_+$ and $\|f\| = 1$ then for $\|x\| \le 1$,

$$\begin{aligned}\int f \log_+ |x|\, d\mu &= \int_{|x|>1} f \log |x|\, d\mu \\ &= \int_{|x|>1} f \log f\, d\mu + \int_{|x|>1} f \log \frac{|x|}{f}\, d\mu \\ &\le \int_{|x|>1} f \log |f|\, d\mu + \frac{1}{e}\int_{|x|>1} |x|\, d\mu \\ &\le \int f |\log f|\, d\mu + e^{-1} .\end{aligned}$$

Since $f \log f \in L_1$, (4.1) and (4.2) hold so that $(L\log L)_+ \subset \mathscr{I}_X$. Conversely if $f \in \mathscr{I}_X$ then for some $x \in B_X$ $f \log |x|$ is integrable and by the Geometric Mean Inequality,

$$\int f \log \frac{|x|}{f}\, d\mu \le \log \int |x|\, d\mu \le 0 ,$$

so that

$$\int f \log |x|\, d\mu \le \int f \log f\, d\mu .$$

Since $f \log_+ f$ is integrable we obtain both that $\mathscr{I}_X = (L\log L)_+$ and that $\Phi_X(f) = \int f \log f\, d\mu$ and this in turn implies the lemma for general f by homogeneity of Φ_X. □

We now define $\Lambda = \Phi_{L_1}$ on $(L\log L)_+$. It is trivial to show that for $1 < p < \infty$ we have $\mathscr{I}_{L_p} = (L\log L)_+$ and $\Phi_{L_p} = \frac{1}{p}\Lambda$. For $p = \infty$, $\mathscr{I}_{L_\infty} = L_1^+$, and $\Phi_{L_\infty} = 0$.

In order to describe the properties of indicators we introduce some further definitions. Let $\mathscr{I}$ be any semi-ideal in L_1^+ and let $\Phi\colon \mathscr{I} \to \mathbf{C}$ be a map. We say that Φ is *semilinear* if

(4.4) $\Phi(\alpha f) = \alpha \Phi(f)$ for $f \in \mathscr{I}$, $\alpha \ge 0$.

(4.5) There exists a constant $\delta < \infty$ so that for $f, g \in \mathscr{I}$,

$$|\Delta_\Phi(f, g)| \le \delta(\|f\| + \|g\|) .$$

The least such constant δ is denoted by $\delta(\Phi)$.

(4.6) If $f \in \mathscr{I}$ and $0 \le f_n \le f$ with $\lim_{n\to\infty} \|f_n\| = 0$ then $\lim_{n\to\infty} \Phi(f_n) = 0$.

We call Φ real if it takes only real values. We further say that Φ is *continuous* if

$$\lim_{\varepsilon\to 0} \sup_{\|f\|\le 1} \sup_{\|g\|\le\varepsilon} |\Delta_\Phi(f, g)| = 0 , \tag{4.7}$$

and *convex* if

$$\Delta_\Phi(f, g) \ge 0 , \qquad f, g \in \mathscr{I} . \tag{4.8}$$

Proposition 4.2. *Let X be a g-convex function space. Then $\mathcal{J}_X$ is a strict semi-ideal and Φ_X is a continuous, convex real semilinear map. If X is a Köthe function space then $\delta(\Phi_X) \leq \log 2$.*

Proof. First of all, we observe that for some $\alpha > 0$, X^α is isomorphic to a Köthe function space. Thus any $f \in (X^\alpha)^* \cap L_1$ satisfies

$$\sup_{x \in B_X} \int f \log_+ |x| \, d\mu < \infty .$$

If we then pick some fixed strictly positive $x_0 \in B_X$, then there is a strictly positive $f \in (X^\alpha)^* \cap L_1$ for which $f|\log|x_0||$ is integrable. Thus $\mathcal{J}_X$ is a strict semi-ideal.

Now conditions (4.4) and (4.8) are obvious. Let us prove (4.5) and (4.7). Recall the definition of $\eta_X(\tau)$ from §3. Then if $f, g \in \mathcal{J}_X$ with $\|f\| \leq 1$, $\|g\| \leq \tau$, we have for any $\varepsilon > 0$, $x, y \in B_X$ with

$$\int f \log|x| \, d\mu \geq \Phi_X(f) - \frac{\varepsilon}{2}, \qquad \int g \log|y| \, d\mu \geq \Phi_X(g) - \frac{\varepsilon}{2}.$$

Let $v = \eta_X(\tau)^{-1}(\max(|x|, \tau|y|))$. Then $v \in B_X$ and

$$\int (f+g) \log v \, d\mu \geq \Phi_X(f) + \Phi_X(g) - \varepsilon - \log \eta_X(\tau)\|f\| - \log \frac{\eta_X(\tau)}{\tau}\|g\|$$

so that

$$\Delta_{\Phi_X}(f, g) \leq (1+\tau) \log \eta_X(\tau) + \tau \log \frac{1}{\tau}.$$

As $\lim_{\tau \to 1} \eta_X(\tau) = 1$ this implies (4.5) and (4.7). If X is a Köthe function space then $\eta_X(\tau) \leq 1 + \tau$ so that $\Delta_{\Phi_X}(f, g) \leq (1+\tau)\log(1+\tau) - \tau \log \tau \leq \log 2(1+\tau)$ provided $0 \leq \tau \leq 1$ which establishes that $\delta(\Phi_X) \leq \log 2$.

It remains to establish (4.6). Assume $f \in \mathcal{J}_X$ and $0 \leq f_n \leq f$ with $\|f_n\| \to 0$. If $x \in B_X$ is chosen so that $f \log|x|$ is integrable, then clearly by an application of the Dominated Convergence Theorem we have $\lim \int f_n \log|x| \, d\mu = 0$ and hence $\liminf \Phi_X(f_n) \geq 0$.

To complete the proof we introduce a Köthe function space F defined by

$$\|g\|_F = \sup_{x \in B_X} \int |g| \log_+ |x| \, d\mu + \|g\|.$$

We claim that the norm on F is order continuous. To prove this it suffices to take $g \in F_+$ and a disjoint sequence $A_n \in \Sigma$ and show that $\|g 1_{A_n}\|_F \to 0$. In fact, if not, passing to a subsequence if necessary, we can find $x_n \in B_X$ with

$$\int_{A_n} g \log_+ |x_n| \, d\mu \geq \varepsilon > 0,$$

for suitable ε. Let $y_n = \max(|x_1|, \ldots, |x_n|)$. For suitable $p > 0$ and $\beta < \infty$ we have $\|y_n\|_X \leq \beta n^{1/p}$ and hence $\beta^{-1} n^{-1/p} y_n \in B_X$. However,

$$\int g \log_+ y_n \, d\mu \geq n\varepsilon - \left(\log \beta + \frac{1}{p} \log n\right) \|g\|,$$

which leads to a contradiction.

Returning to the proof we obtain $\|f_n\|_F \to 0$ and so $\limsup_{n \to \infty} \Phi_X(f_n) \leq 0$. □

Lemma 4.3. *Let $\mathscr{I}$ be a semi-ideal and let Φ be a semilinear functional on $\mathscr{I}$. Then*

(i) *For every $f \in \mathscr{I}$, Φ is bounded on $[0, f]$.*
(ii) *If $f_n \in [0, f]$ and $\lim \|f_n - g\| = 0$ then $\limsup_{n\to\infty} |\Phi(f_n) - \Phi(g)| \leq 2\delta(\Phi)\|f\|$.*
(iii) *If Φ is continuous then Φ is continuous on $[0, f]$ for the L_1-norm.*

Proof. (i) For $f \in \mathscr{I}$, let $\Gamma(f) = \sup\{|\Phi(g)|: 0 \leq g \leq f\}$. Then $\Gamma(f+g) \leq \Gamma(f)+\Gamma(g)+\delta\|f+g\|$ where $\delta = \delta(\Phi)$. Suppose $\Gamma(f) = \infty$. Then there exists $g \in [0, f]$ with $|\Phi(g)| > 3|\Phi(f)| + \delta\|f\| + 1$. Thus $|\Phi(f-g)| > 2|\Phi(f)| + 1$. Arguing by induction, there exists a descending sequence f_n with $f_0 = f$, $\Gamma(f_n) = \infty$, and $|\Phi(f_n)| \to \infty$. Let $g = \lim f_n$. Then

$$|\Phi(f_n)| \leq |\Phi(g)| + |\Phi(f_n - g)| + \delta\|f\|.$$

As $\lim \Phi(f_n - g) = 0$ this yields a contradiction.

(ii) Let $h_n = \max(f_n, g)$. Then

$$|\Phi(h, n) - \Phi(f_n) - \Phi(h_n - f_n)| \leq \delta\|f\|, \qquad |\Phi(h_n) - \Phi(g) - \Phi(h_n - g)| \leq \delta\|f\|.$$

Thus

$$|\Phi(f_n) - \Phi(g)| \leq |\Phi(h_n - f_n)| + |\Phi(h_n - g)| + 2\delta\|f\|$$

and since $\lim \Phi(h_n - f_n) = \lim \Phi(h_n - g) = 0$ (ii) follows.

(iii) In this case, arguing as in case (ii)

$$\lim_{n\to\infty} |\Delta_\Phi(f_n, h_n - f_n)| = \lim_{n\to\infty} |\Delta_\Phi(g, h_n - g)| = 0$$

and so $\lim |\Phi(f_n) - \Phi(g)| = 0$. □

Theorem 4.4. *Let X be a g-convex function space and suppose $\mathscr{I}$ be a strict semi-ideal contained in $\mathscr{I}_X$. Then $x \in B_X$ if and only if $f \log_+ |x|$ is integrable for every $f \in \mathscr{I}$ and*

$$\int f \log|x|\, d\mu \leq \Phi_X(f), \qquad f \in \mathscr{I}.$$

Proof. One direction is trivial. For the other, we first prove that we may suppose $\mathscr{I} = \mathscr{I}_X$. Indeed suppose $f \in \mathscr{I}_X$ and that $f_n \in \mathscr{I}_X$ with $0 \leq f_n \uparrow f$ a.e. Let $A = \{s: |x(s)| > 1\}$. Then $\int f_n 1_A \log|x|\, d\mu \leq \Phi_X(f_n 1_A)$. Now by the Monotone Convergence Theorem, Proposition 4.2, and Lemma 4.3 we have $\int f 1_A \log|x|\, d\mu \leq \Phi_X(f 1_A)$ so that $f \log_+ |x| \in L_1$. Now if $f \log|x|$ fails to be integrable then $\int f \log|x|\, d\mu = -\infty < \Phi_X(f)$. Otherwise, we have

$$\int f \log|x|\, d\mu = \lim_{n\to\infty} \int f_n \log|x|\, d\mu \leq \lim_{n\to\infty} \Phi_X(f_n) = \Phi_X(f)$$

again by Lemma 4.3. Thus we will assume $\mathscr{I} = \mathscr{I}_X$.

To continue the proof assume $x \notin B_X$. Let u be a strictly positive member of B_X and let w be a strictly positive function such that

$$\int w(1 + |\log|x|| + |\log u|)^2\, d\mu < \infty.$$

Let $L_2(w\, d\mu)$ be the corresponding weighted Hilbert space of all $\phi \in L_0$ such that $\|\phi\|_{w,2}^2 = \int |\phi|^2 w\, d\mu < \infty$. Let V be the subset of $L_2(w\, d\mu)$ of all functions of the form $\log|y|$ where $y \in B_X$. We observe that V is convex

(since X is g-convex) and closed in the relative L_0-topology on $L_2(w\,d\mu)$ and hence closed in $L_2(w\,d\mu)$. V is nonempty since $\log u \in V$. Further if $v \in V$ and $v_1 \le v$ a.e. with $v_1 \in L_2(w\,d\mu)$ then $v_1 \in V$.

By assumption, $\log|x| \notin V$. Thus by the Hahn-Banach theorem there exists $f \in L_2(w^{-1}\,d\mu)$ and $\alpha \in \mathbf{R}$ so that $\int f \log|x|\,d\mu > \alpha$ but $\int f\phi\,d\mu \le \alpha$ for $\phi \in V$. From the properties of V, we must have $f \ge 0$ a.e. Since $\int |f|^2 w^{-1}\,d\mu < \infty$ and w is integrable we have $f \in L_1$ by Hölder's inequality.

We show $f \in \mathscr{I}_X$. Assume $y \in B_X$, and let $y_{m,\tau} = \max(\min(|y|, m|x|), \tau u)$ for $m \in \mathbf{N}$, $\tau > 0$. Then $\eta_X(\tau)^{-1} y_{m,\tau} \in B_X$ and $\log y_{m,\tau} - \log \eta_X(\tau) \in L_2(w\,d\mu)$. Hence

$$\int f \log y_{m,\tau}\,d\mu \le \alpha + \|f\| \log \eta_X(\tau). \tag{4.9}$$

If we set $A = \{s\colon |y(s)| > 1\}$ then

$$\int_A f \log y_{m,1}\,d\mu \le \alpha + \|f\| \log \eta_X(1) + \int_{S\setminus A} f \log \frac{1}{|u|}\,d\mu,$$

and letting $m \to \infty$

$$\int f \log_+ |y|\,d\mu = \int_A f \log|y|\,d\mu \le \alpha + \|f\| \log \eta_X(1) + \int f|\log u|\,d\mu.$$

Again

$$\int f|\log u|\,d\mu \le \|f\|_{w^{-1},2} \left(\int |\log u|^2 w\,d\mu\right)^{1/2}$$

so that we may conclude that $\sup_{y\in B_X} \int f \log_+ |y|\,d\mu < \infty$. We also observe that (4.9) implies that for arbitrary $y \in B_X$

$$\int f \log(\max(|y|, \tau u)\,d\mu \le \alpha + \|f\| \log \eta_X(\tau)$$

for any $\tau > 0$. Hence

$$\int f \log|y|\,d\mu \le \alpha + \|f\| \log \eta_X(\tau).$$

Letting $\tau \to 0$ we have $\Phi_X(f) \le \alpha$. Since $\int f \log|x|\,d\mu > \alpha$, this contradiction proves the theorem. □

Our next proposition summarizes a few simple properties of the indicator.

Proposition 4.5. (i) *Let X, Y be g-convex function spaces and $\mathscr{I}$ be a strict semi-ideal contained in $\mathscr{I}_X \cap \mathscr{I}_Y$. Then $\|x\|_X \le M\|x\|_Y$ for all $x \in L_0$ if and only if $\Phi_X(f) \ge \Phi_Y(f) - (\log M)\|f\|$ for all $f \in \mathscr{I}$. Further, X and Y have equivalent quasinorms if and only if*

$$d(\Phi_X, \Phi_Y) = \sup_{\|f\|\le 1} |\Phi_X(f) - \Phi_Y(f)| < \infty.$$

(ii) *If $Z = X^\alpha Y^\beta$ (i.e., $\|x\|_Z = \inf\{\|u\|_X^\alpha \|v\|_Y^\beta\colon |x| \le |u|^\alpha |v|^\beta\}$), then $\mathscr{I}_Z \supset \mathscr{I}_X \cap \mathscr{I}_Y$ and*

$$\Phi_Z(f) = \alpha\Phi_X(f) + \beta\Phi_Y(f)$$

for $f \in \mathscr{I}_X \cap \mathscr{I}_Y$.

We omit the proofs. Part (i) is almost immediate from the definitions and Theorem 4.4, while (ii) is left as an easy exercise. We shall call two semilinear

functionals Φ, Ψ defined on some strict semi-ideal $\mathscr{I}$ *equivalent* if $d(\Phi, \Psi) < \infty$ where $d(\Phi, \Psi)$ is defined above.

Example. If we take $X = L(p, \infty)$ then it is relatively easy to compute

$$\Phi_X(f) = \sup_{x \in B_X} \int f \log|x| \, d\mu = \frac{1}{p} \int_0^\infty f^*(t) \log t \, dt$$

for $f \in L \log L$ where f^* is the decreasing rearrangement of f. However X is not g-convex, even though Φ_X is continuous and convex. In fact $\Phi_X = \Phi_{\widetilde{X}}$ where $\widetilde{X}$ is the "g-convexification" of X i.e., $x \in B_X$ if and only if

$$\int_0^\tau \log x^*(t) \, dt \le \frac{1}{p} \int_0^\tau \log \frac{1}{t} \, dt .$$

Clearly $\| \ \|_{\widetilde{X}}$ is an equivalent quasinorm on $L(p, \infty)$ since $L(p, \infty)$ is isomorphic to a g-convex space.

Proposition 4.6. *Let X be a Köthe function space. Then $\mathscr{I}_X \cap \mathscr{I}_{X^*} \subset (L \log L)_+$ and for $f \in \mathscr{I}_X \cap \mathscr{I}_{X^*}$, we have $\Phi_X(f) + \Phi_{X^*}(f) = \Lambda(f)$.*

Furthermore if $f \in \mathscr{I}_X \cap \mathscr{I}_{X^*}$, and $\|f\| = 1$, then $\Phi_X(f) = \int f \log x \, d\mu$ where $f = xy$ is the Lozanovskii factorization for (X, X^*) i.e., $x, y \ge 0$, $\|x\|_X = \|y\|_{X^*} = 1$, and $\operatorname{supp} x = \operatorname{supp} y = \operatorname{supp} f$.

Proof. The first part is essentially Lozanovskii's theorem that $B_X . B_{X^*} = B_{L_1}$. For the second suppose $\xi \in B_X$. Then

$$\begin{aligned} \int f \log|\xi| \, d\mu &= \int f \log|\xi y| \, d\mu - \int f \log|y| \, d\mu \\ &\le \int f \log f \, d\mu - \int f \log|y| \, d\mu \\ &= \int f \log x \, d\mu . \quad \square \end{aligned}$$

We conclude this section by generalizing the fact which we have already noticed that $\Phi_{X^\theta Y^{1-\theta}}(f) = \theta \Phi_X(f) + (1-\theta)\Phi_Y(f)$ for all $f \in \mathscr{I}_X \cap \mathscr{I}_Y$.

Theorem 4.7. *Let $\mathscr{X} = (X_w : w \in \mathbf{T})$ be a strongly admissible family of Köthe function spaces. Then there is a strict semi-ideal $\mathscr{I}$ so that if $f \in \mathscr{I}$, $\Phi_{X_{e^{it}}}(f)$ is a bounded measurable function of t and for $z = re^{i\theta} \in D$ we have $f \in \mathscr{I}_{X_z}$ with*

$$\Phi_{X_z}(f) = \int_{-\pi}^{\pi} P(r, \theta - t) \Phi_{X_{e^{it}}}(f) \frac{dt}{2\pi} .$$

Proof. There exist strictly positive h, k so that for a.e. $w \in \mathbf{T}$, $\|xh\|_1 \le \|x\|_{X_w} \le \|xk\|_\infty$. Let $\mathscr{I} = \mathscr{I}_{L_1(h d\mu)} \cap \mathscr{I}_{L_1(k^{-1} d\mu)}$. Then $\mathscr{I} \subset \mathscr{I}_{X_w}$ for a.e. $w \in \mathbf{T}$. We also observe that it suffices to establish the formula for the case $z = 0$ by a conformal mapping argument.

Consider, as in §3, the Köthe function space $\mathscr{E}$ on $(S \times \mathbf{T}, \mu \times \lambda)$ given by $\|\phi\|_{\mathscr{E}} = \operatorname{ess\,sup} \|\phi(\cdot, e^{i\theta})\|_{X_{e^{i\theta}}}$. Now if $f \in \mathscr{I}$ with $\|f\| = 1$, then by Lozanovskii's theorem there is a factorization $f(s) = \phi_1(s, e^{it})\phi_2(s, e^{it})$ with $\phi_1, \phi_2 \ge 0$, $\operatorname{supp} \phi_1 = \operatorname{supp} \phi_2 = \operatorname{supp} f \times \mathbf{T}$, and $\|\phi_1\|_{\mathscr{E}} = \|\phi_2\|_{\mathscr{E}^*} = 1$.

Define $x \in X_0$ by

$$x(s) = \exp\left(\int_{-\pi}^{\pi} \log \phi_1(s, e^{it}) \frac{dt}{2\pi} \right) .$$

As discussed in §3, this is well defined since $\log_+ \phi_1(s, e^{it})$ is integrable for a.e. $s \in S$. Similarly

$$\int_{-\pi}^{\pi} \int_S \phi_2(s, e^{it}) k(s)^{-1} \, d\mu(s) \frac{dt}{2\pi} \leq 1$$

so that $\log_+ \phi_2(s, e^{it})$ is integrable for a.e. $s \in S$ and we may define

$$y(s) = \exp\left(\int_{-\pi}^{\pi} \log \phi_2(s, e^{it}) \frac{dt}{2\pi} \right).$$

Now suppose $\xi \in B_{X_0}$ and $\xi \geq 0$. Then there exists $\psi \in B_{\mathscr{E}}$ with

$$\xi(s) = \exp\left(\int_{-\pi}^{\pi} \log \psi(s, e^{it}) \frac{dt}{2\pi} \right).$$

Thus

$$\begin{aligned} \int_S y\xi \, d\mu &= \int_S \exp\left(\int_{-\pi}^{\pi} \log(\phi_2 \psi) \frac{dt}{2\pi} \right) d\mu \\ &\leq \int_S \int_{-\pi}^{\pi} \phi_2 \psi \frac{dt}{2\pi} \, d\mu \\ &\leq \|\phi_2\|_{\mathscr{E}^*} \|\psi\|_{\mathscr{E}} \leq 1. \end{aligned}$$

Thus $\|y\|_{X_0^*} \leq 1$. However $xy = f$, so this is the Lozanovskii factorization of f and

$$\Phi_{X_0}(f) = \int f \log x \, d\mu = \int_S \int_{-\pi}^{\pi} f \log \phi_1(s, e^{it}) \frac{dt}{2\pi} \, d\mu(s).$$

As in §3, we have

$$\|\phi_2\|_{\mathscr{E}^*} = \int_{-\pi}^{\pi} \|\phi_2(\cdot, e^{it})\|_{X_{e^{it}}^*} \frac{dt}{2\pi},$$

and as $\phi_1 \phi_2 = f \otimes 1_{\mathbf{T}}$ we must have $\|\phi_2(\cdot, e^{it})\|_{X_{e^{it}}^*} \geq 1$ a.e. So we have $\|\phi_2(\cdot, e^{it})\|_{X_{e^{it}}^*} = 1$ a.e. Thus λ-a.e. $f(s) = \phi_1(s, e^{it}) \phi_2(s, e^{it})$ is the Lozanovskii factorization of f for the pair $X_{e^{it}}$, $X_{e^{it}}^*$. Hence

$$\Phi_{X_{e^{it}}}(f) = \int f \log \phi_1(s, e^{it}) \, d\mu(s).$$

and the theorem follows. □

Corollary 4.8. *We have the duality theorem that the space X_z^* obtained by interpolating $\mathscr{X}^* = \{X_w^* : W \in \mathbf{T}\}$ is isometric to $(X_z)^*$.*

This is immediate from 4.6 and 4.7.

Corollary 4.9. *If $f \in (L_1)_+$ with $\|f\| = 1$ has a Lozanovskii factorization $f = xx^*$ for X_z^* where $|z| < 1$, and if $|u| = 1$ a.e., then there is a unique $F \in \mathscr{N}^+$, with $F(z) = ux$, F vanishes identically off the support of x and $\|F\|_{\mathscr{X}} = 1$. Hence the induced centralizer Ω is uniquely defined at ux.*

Proof. Let G be an extremal for $u^{-1}x^*$ in $\mathscr{X}^*$, i.e., suppose $G \in \mathscr{N}^+$, $G(z) = u^{-1}x^*$, and $\|G\|_{\mathscr{X}} = 1$. If F is an extremal for ux in $\mathscr{X}$, then it follows from the maximum modulus principle that since $\int F(z)G(z) \, d\mu = 1$ then $\int F(\zeta)G(\zeta) \, d\mu = 1$ for all $|\zeta| < 1$. Hence $F(\zeta)G(\zeta)$ is real and nonnegative a.e. Thus for a.e. s $F(\zeta, s)G(\zeta, s)$ is real on a dense subset of the disk

and hence everywhere. Thus $F(\zeta)G(\zeta) = f$ for all ζ and it follows easily that F is uniquely determined on the support of x. This implies that $\Omega(ux)$ is also uniquely determined. □

5. Characterizing indicators

Theorem 5.1. *Let $\mathscr{I}$ be a strict semi-ideal, and let Φ be a continuous convex real semilinear map defined on $\mathscr{I}$. Then there is a (unique) g-convex function space X for which $\mathscr{I} \subset \mathscr{I}_X$ and $\Phi(f) = \Phi_X(f)$ for $f \in \mathscr{I}$.*

Proof. Let B_X be the subset of L_0 of all x such that $f \log_+ |x| \in L_1$ for any $f \in \mathscr{I}$ and $\int f \log|x|\, d\mu \le \Phi(f)$ for $f \in \mathscr{I}$. We define $\|x\|_X = \inf\{\alpha > 0 : \alpha^{-1}|x| \in B_X\}$ and $\|x\|_X = \infty$ if $x \notin \alpha B_X$ for every $\alpha > 0$. Clearly $B_X = \{x : \|x\|_X \le 1\}$.

We now show that $\| \ \|_X$ defines a g-convex quasinorm. Suppose $x, y \in B_X$ with disjoint supports contained in A_x, A_y where $A_x \cup A_y = S$ and $A_x \cap A_y = \varnothing$. For $f \in \mathscr{I}$,

$$\begin{aligned}\int f \log(\max(|x|, \tau|y|))\, d\mu &\le \Phi(f1_{A_x}) + \Phi(f1_{A_y}) + \log\tau \|f1_{A_y}\| \\ &\le \Phi(f) + \Delta_\Phi(f1_{A_x}, f1_{A_y}) + \log\tau\|f1_{A_y}\|.\end{aligned}$$

Let

$$\sigma(\varepsilon) = \sup_{\|f\|\le 1} \sup_{\|g\|\le\varepsilon} |\Delta_\Phi(f, g)|.$$

Then

$$\int f \log(\max(|x|, \tau|y|))\, d\mu \le \Phi(f) + \|f\|(\sigma(\alpha) + \alpha\log\tau),$$

where $\alpha = \|f1_{A_y}\|/\|f\|$. If we let

$$\sigma^*(\tau) = \sup_{0\le\alpha\le 1} (\sigma(\alpha) + \alpha\log\tau),$$

then $\lim_{\tau\to 0}\sigma^*(\tau) = 0$ and

$$\int f \log(\max(|x|, \tau|y|))\, d\mu \le \Phi(f) + \|f\|\sigma^*(\tau).$$

Thus $\|\max(|x|, \tau|y|)\|_X \le e^{\sigma^*(\tau)}$ or $\eta_X(\tau) \le e^{\sigma^*(\tau)}$.

To complete the proof that X is a g-convex function space we need only show that B_X contains a strictly positive function and that B_X is closed in L_0, since condition (3.5) is trivial. Let us postpone the former statement and simply show that B_X is L_0-closed. Suppose $x_n \in B_X^+$ and $x_n \to x$ a.e. We may suppose, without loss of generality that $\operatorname{supp} x_n \subset \operatorname{supp} x = A$ for every n. Let A_k be an increasing sequence of measurable subsets of A so that $\mu(A \setminus \bigcup A_k) = 0$ and $x_n x^{-1} \to 1$ uniformly on each A_k. Then, given $\varepsilon > 0$, $k \in \mathbf{N}$, there exists $N \in \mathbf{N}$ so that if $n \ge N$, $x_n 1_{A_k} \ge (1-\varepsilon)x1_{A_k}$. Hence if $f \in \mathscr{I}$, $f \log_+ x1_{A_k} \in L_1$ and

$$\begin{aligned}\int f \log x1_{A_k}\, d\mu &\le -\log(1-\varepsilon)\|f\| + \int f \log x_n 1_{A_k}\, d\mu \\ &\le -\log(1-\varepsilon)\|f\| + \Phi(f).\end{aligned}$$

If we let $\varepsilon \to 0$ we obtain that $x 1_{A_k} \in B_X$ for every $k \in \mathbf{N}$. Now let $G = \{s: x(s) > 1\}$. If $f \in \mathscr{I}$,

$$\int f \log_+ x \, d\mu = \int_G f \log x \, d\mu = \lim_{k\to\infty} \int_{G\cap A_k} f \log x \, d\mu$$
$$\leq \liminf_{k\to\infty} \Phi(f 1_{G\cap A_k}) = \Phi(f 1_G),$$

using Lemma 4.3 and the fact that Φ is continuous. Hence $f \log_+ x \in L_1$. If $f \log x \in L_1$ then the same argument yields that $\int f \log x \, d\mu \leq \Phi(f)$ and this also holds in the case when $f \log x$ fails to be integrable. Thus $x \in B_X$ and B_X is closed in L_0.

To conclude we will check that B_X contains a strictly positive element and that $\Phi = \Phi_X$ on $\mathscr{I}$. Notice first that if $f \in \mathscr{I}$, $x \in B_X$, then

$$\int f \log_+ |x| \, d\mu \leq \Phi(f 1_{|x|>1}) \leq \sup_{g\in[0,f]} \Phi(g) < \infty$$

by Lemma 4.3. Hence

$$\sup_{x\in B_X} \int f \log_+ |x| \, d\mu < \infty.$$

Now suppose $f \in \mathscr{I}$ and $\Phi(f) > \alpha$. Then we show the existence of $x \in B_X$ so that $\int f \log |x| \, d\mu < \alpha$. This will imply, using some strictly positive f that B_X contains a strictly positive function and hence X is a g-convex function space. Further, from the above remarks we will have that $\mathscr{I} \subset \mathscr{I}_X$ and we will be able to conclude that $\Phi_X = \Phi$ on $\mathscr{I}$.

Let Y be the AM-space $\bigcup n[-f, f]$ normed by $\|g\|_Y = \operatorname{ess\,sup} |g(s)| f(s)^{-1}$. Let $K \subset Y^+$ be the convex set of all $g \geq 0$ such that $\Phi(g) \leq \alpha$. For $m \in \mathbf{N}$, $K \cap m[0, f]$ is closed for convergence in L_1-norm by Lemma 4.3 and the continuity of Φ. Let Y_* be the predual of Y, i.e. the space of all measurable functions h with $\operatorname{supp} h \subset \operatorname{supp} f$ such that $\|h\|_{Y_*} = \int |h| f \, d\mu < \infty$. Then $K \cap m[-f, f]$ is closed for the weak* topology $\sigma(Y, Y_*)$ which is weaker on $m[-f, f]$ than the L_1-norm. Now by the Banach-Dieudonné theorem, K is $\sigma(Y, Y_*)$-closed in Y. Thus there exists $h \in Y_*$, so that $\int h\phi \leq \alpha$ for $\phi \in K$ but $\int hf \, d\mu > \alpha$. Clearly if $g \in Y$, $g \geq 0$, $\int hg \, d\mu \leq \Phi(g)$.

Let $x(s) = e^{h(s)}$ for $s \in \operatorname{supp} f$ and $x(s) = 0$ otherwise. Then if $g \in \mathscr{I}$

$$\int g \log_+ x \, d\mu = \int gh \, d\mu = \lim_{n\to\infty} \int_{h>0} \min(g, nf) h \, d\mu$$
$$\leq \liminf_{n\to\infty} \Phi(\min(g 1_{(h>0)}, nf))$$
$$= \Phi(g 1_{(h>0)})$$

by Lemma 4.3. If $\operatorname{supp} g$ is not a subset of $\operatorname{supp} f$ then $\int g \log x \, d\mu = -\infty$. Otherwise, if $g \log x$ is integrable

$$\int g \log x \, d\mu = \int gh \, d\mu = \lim_{n\to\inf} \int \max(g, nf) h \, d\mu \leq \Phi(g)$$

by the same reasoning. Hence we can conclude that $x \in B_X$ and the proof is complete. □

Theorem 5.2. *Let $\mathscr{I}$ be a strict semi-ideal and let Φ be a real semilinear functional on $\mathscr{I}$. Then, in order that there exist a Köthe function space X with*

$\mathscr{I} \subset \mathscr{I}_X$, and $\Phi(f) = \Phi_X(f)$ for $f \in \mathscr{I}$, it is necessary and sufficient that for any $f, g \in \mathscr{I}$,

$$0 \le \Delta_\Phi(f, g) \le \Delta_\Lambda(f, g). \tag{5.1}$$

Proof. Necessity is clear since $\Phi_{X^*} = \Lambda - \Phi_X$. Conversely, assume (5.1). Since Λ is continuous, both Φ and $\Lambda - \Phi$ are convex and continuous. Thus there exist g-convex function spaces X, Y with $\mathscr{I} \subset \mathscr{I}_X \cap \mathscr{I}_Y$ and so that $\Phi = \Phi_X$ and $\Lambda - \Phi_X = \Phi_Y$. It suffices to show that X is a Köthe function space. Clearly $B_X.B_Y \subset B_{L_1}$. Let $B_{\widetilde{X}}$ be the L_0-closed convex hull of B_X. Then $B_{\widetilde{X}}.B_Y \subset B_{L_1}$, and so on $\mathscr{I}_{\widetilde{X}} \cap \mathscr{I}$ we have $\Phi_X \le \Phi_{\widetilde{X}} \le \Lambda - \Phi_Y = \Phi_X$. Hence by Theorem 4.4 $\widetilde{X} = X$ and the proof is completed. □

Theorem 5.3. *Let X be a g-convex function space and let $\mathscr{I} \subset \mathscr{I}_X \cap (L \log L)_+$ be a strict semi-ideal. Then*

(i) *X is p-convex $(0 < p < \infty)$ if and only if for every $f, g \in \mathscr{I}$,*

$$\Delta_{\Phi_X}(f, g) \le \frac{1}{p}\Delta_\Lambda(f, g).$$

(ii) *If X is p-convex for some $p > 0$ then X is also q-concave where $0 < q < \infty$ if and only if for every $f, g \in \mathscr{I}$,*

$$\Delta_{\Phi_X}(f, g) \le \frac{1}{q}\Delta_\Lambda(f, g).$$

Proof. Part (i) is obvious. For (ii), suppose $Y = X^p$. Then Y is a Köthe function space and $\Phi_{Y^*} = \Lambda - p\Phi_X$. Thus $\Delta_{\Phi_{Y^*}}(f, g) = \Delta_\Lambda(f, g) - p\Delta_{\Phi_X}(f, g)$.

Hence Y^* is r-convex if and only if

$$\Delta_{\Phi_X}(f, g) \ge \left(1 - \frac{1}{r}\right)\frac{1}{p}\Delta_\Lambda(f, g)$$

for $f, g \in \mathscr{I}$. Now X is q-concave if and only if Y is q/p-concave if and only if Y^* is $(1 - p/q)^{-1}$-concave. This yields the theorem. □

Theorem 5.3 has a simple illustration in a theorem of Pisier [43].

Corollary 5.4 (Pisier). *Suppose X is a Köthe function space and $0 < \theta < 1$. In order that there is a Köthe function space Y with $X = L_2^\theta Y^{1-\theta}$ it is necessary and sufficient that X is p-convex and q-concave where $\frac{1}{q} = 1 - \frac{1}{p} = \frac{\theta}{2}$.*

Proof. This is a simple calculation based on the fact that Y can be determined from the equation $\Phi_X = \frac{\theta}{2}\Lambda + (1-\theta)\Phi_Y$ on a suitable semi-ideal together with the preceding two theorems. □

We remark that some recent results of Cwikel and Nilsson [16] on complex interpolation between Banach lattices can also be proved by this method.

Lemma 5.5. *For any Köthe function space X, if $f_1, \dots, f_n \in \mathscr{I}_X \cap (L \log L)_+$ and $\sum_{i=1}^n \|f_i\| = 1$ then*

$$\Delta_{\Phi_X}(f_1, \dots, f_n) \le \sum_{i=1}^n \|f_i\| \log \frac{1}{\|f_i\|}.$$

In particular $\delta_{\Phi_X} \le \delta_\Lambda = \log 2$.

Proof. This follows immediately from the observation:

$$\begin{aligned}\Delta_{\Phi_X}(f_1, \dots, f_n) &\le \Delta_\Lambda(f_1, \dots, f_n)\\ &= \sum_{i=1}^n \int f_i \log \frac{f_i}{F}\, d\mu + \sum_{i=1}^n \|f_i\| \log \frac{1}{\|f_i\|},\end{aligned}$$

where $F = f_1 + \cdots + f_n$. □

In the above proof that if $f_1, \dots, f_n$ have disjoint supports then

$$\Delta_\Lambda(f_1, \dots, f_n) = -\sum \|f_i\| \log \|f_i\|.$$

Lemma 5.6. *Suppose X is a Köthe function space and that $\mathscr{J}$ is the ideal generated by $\mathscr{I}_X \cap (L\log L)_+$ (thus $\mathscr{J}$ consists of all complex $f \in L_0$ so that $|f| \in \mathscr{I} \cap (L\log L)_+$). Then extend Φ_X to $\mathscr{J}$ by $\Phi_X(f) = f\log x$ where $f = xy$ is the Lozanovskii factorization of $|f|$. Then for $f, g \in \mathscr{J}$,*

$$\Delta_{\Phi_X}(f, g) \le \tfrac{4}{e}(\|f\| + \|g\|).$$

Proof. First suppose $0 \le g \le f \in \mathscr{I}_X \cap (L\log L)_+$. Then if we let $f = x_1y_1$ and $g = x_2y_2$ be the corresponding Lozanovskii factorizations, we have

$$\begin{aligned}\Phi_X(g) &= \Lambda(g) - \int g \log \frac{|y_2|}{\|g\|}\, d\mu\\ &\le \Lambda(g) - \int g \log \frac{|y_1|}{\|f\|}\, d\mu\\ &= \int g \log x_1\, d\mu + \|g\| \log \frac{\|f\|}{\|g\|}\\ &\le \int g \log x_1\, d\mu + \frac{1}{e}\|f\|.\end{aligned}$$

If we then set $A = \{s\colon x_1(s) > x_2(s)\}$ we have

$$\int_A g \log x_1 d\mu \le \Phi_X(g 1_A) \le \int_A g \log x_2 d\mu + \frac{1}{e}\|g\|,$$

and

$$\int_{S\setminus A} g \log x_2\, d\mu \le \Phi_X(g 1_{s\setminus A}) \le \int_{S\setminus A} g \log x_1 d\mu + \frac{1}{e}\|f\|.$$

We conclude that

$$\int g|\log x_1 - \log x_2|\, d\mu \le \frac{2}{e}\|f\|$$

whenever $0 \le g \le f$.

Now suppose f, g are in $\mathscr{J}$ and let $h = |f| + |g|$. Let h have the Lozanovskii factorization $h = \xi\eta$. Then by the above argument

$$\left|\Phi_X(f) - \int f \log \xi\, d\mu\right| \le \frac{2}{e}\|h\|$$

with a similar inequality for g. Thus $\Delta_{\Phi_X}(f, g) \le \frac{4}{e}\|h\|$ and the lemma follows. □

We will refer to Φ_X extended in this way as the (*extended*) *indicator* of X.

We conclude this section with a result on the construction of Köthe function spaces which we will need later in §9.

Theorem 5.7. *Suppose μ is a nonatomic probability measure on S and that X is a Köthe function space such that $L_\infty \subset X \subset L_p$ for some $p > 0$. Let $T: L_1 \to L_1$ be a doubly stochastic operator, i.e., $T \geq 0$, $T1_S = T^*1_S = 1_S$. Then there is a unique Köthe function space Y such that $\Phi_Y(f) = \Phi_X(Tf)$ whenever $f \in (L\log L)_+$.*

Proof. We remark that $\mathscr{I}_X \supset L\log L$ and that if $f \in L\log L$ then $Tf \in L\log L$. We need the following lemma.

Lemma 5.8. *Let $(a_{ij})_{1\leq i,j\leq n}$ be a doubly stochastic matrix. For any nonnegative $(x_i)_{i=1}^n$, $(y_i)_{i=1}^n$, let $\xi_i = \sum_{j=1}^n a_{ij}x_j$, $\eta_i = \sum_{j=1}^n a_{ij}y_j$. We then have*

$$\sum_{i=1}^n \left(\xi_i \log\frac{\xi_i+\eta_i}{\xi_i} + \eta_i \log\frac{\xi_i+\eta_i}{\eta_i}\right) \leq \sum_{i=1}^n \left(x_i \log\frac{x_i+y_i}{x_i} + y_i \log\frac{x_i+y_i}{y_i}\right).$$

Proof of Lemma 5.8. We define for $(s, t) \in \mathbf{R}^2$ with $s, t \geq 0$,

$$u(s, t) = s\log\frac{s+t}{s} + t\log\frac{s+t}{t}.$$

Then by differentiation it can be seen that u is a concave function on its cone of definition, since the second derivative is negative semidefinite in the interior. Now consider the function v defined on the convex set of doubly stochastic matrices, given by

$$v(A) = \sum_{i=1}^n u\left(\sum_{j=1}^n a_{ij}x_j, \sum_{j=1}^n a_{ij}y_j\right).$$

Then v is also concave and assumes its maximum at an extreme point, i.e., a permutation matrix $S_\sigma = (\delta_{i,\sigma(j)})$ where σ is a permutation of $[1, 2, \dots, n]$. Then the lemma follows immediately. □

Proof of Theorem 5.7. Let $\Phi(f) = \Phi_X(Tf)$ for $f \in L\log L$. Let $D(n, k)$; $1 \leq k \leq 2^n$, $1 \leq n$, be a family of measurable sets so that $D(0, 1) = S$, $D(n, k) \cap D(n, k') = \varnothing$ for $k \neq k'$, $D(n, k) = D(n+1, 2k-1) \cup D(n+1, 2k)$, $\mu(D(n, k)) = 2^{-n}$ and the sets $(D(n, k))$ generate Σ. Let E_n be the conditional expectation operator of L_1 onto $L_1(\Sigma_n)$ where Σ_n is generated by $D(n, k)$ for $1 \leq k \leq 2^n$. Clearly the lemma implies that for any nonnegative $f, g \in L_1$, if $f_n = E_nf$, $g_n = E_ng$, and $\phi_n = E_nTE_nf$, $\psi_n = E_nTE_ng$,

$$\int \phi_n \log\frac{\phi_n+\psi_n}{\phi_n}\,d\mu + \int \psi_n \log\frac{\phi_n+\psi_n}{\psi_n}\,d\mu$$
$$\leq \int f_n \log\frac{f_n+g_n}{f_n}\,d\mu + \int g_n \log\frac{f_n+g_n}{g_n}\,d\mu.$$

Since $\|f_n\| = \|\phi_n\| = \|f\|$ and $\|g_n\| = \|\psi_n\| = \|g\|$ this translates to $\Delta_\Lambda(\phi_n, \psi_n) \leq \Delta_\Lambda(f_n, g_n)$.

If we now assume $f, g \in L\log L$ the continuity properties of Λ on the Köthe function space $L\log L$ imply that $\Delta_\Lambda(Tf, Tg) \leq \Delta_\Lambda(f, g)$.

Now it follows that

$$0 \leq \Delta_\Phi(f, g) = \Delta_{\Phi_X}(Tf, Tg) \leq \Delta_\Lambda(Tf, Tg) \leq \Delta_\Lambda(f, g).$$

Hence Φ satisfies the conditions of (4.4)–(4.6) and the theorem follows by Theorem 5.2. □

It now follows easily from Theorem 5.7 that

Corollary 5.9. *Under the hypotheses of Theorem* 5.7 *we have for any* $f \in L\log L$,

$$|\Phi_Y(f) - \Phi_X(Tf)| \leq \frac{24}{e}\|f\|.$$

Proof. Simply split each f into real and imaginary and then positive and negative parts and use Theorem 5.7. □

6. Approximation theorems for semilinear functionals

Proposition 6.1. *Let* $\mathscr{I}$ *be a strict semi-ideal contained in* $(L\log L)_+$ *and let* Φ *be a semilinear functional on* $\mathscr{I}$. *Suppose that for some* M *and every* $f_1, \dots, f_n \in \mathscr{I}$,

$$-M\sum_{j=1}^{n}\|f_j\| \leq \Delta_\Phi(f_1, \dots, f_n) \leq \Delta_\Lambda(f_1, \dots, f_n) + M\sum_{j=1}^{n}\|f_j\|. \tag{6.1}$$

Then, there exists a Köthe function space X *so that* $\mathscr{I} \subset \mathscr{I}_X$ *and*

$$d(\Phi, \Phi_X) = \sup_{\|f\|\leq 1}|\Phi(f) - \Phi_X(f)| \leq M + 4\delta,$$

where $\delta = \delta(\Phi)$.

Proof. We first prove this in the special case when S is a finite set and $\mathscr{I} = L_1^+(S)$ is thus a closed cone in a finite dimensional space. Define $\Phi_0(f) = \inf\sum_{j=1}^n \Phi(f_j)$ where the infimum is taken over all $f_1, \dots, f_n \geq 0$ with $\sum_{j=1}^n f_j = f$. Then $\Phi(f) - M\|f\| \leq \Phi_0(f) \leq \Phi(f)$ and Φ_0 is a convex semilinear functional.

Let P be the positive cone $\{x\colon x(s) > 0\ \forall s \in S\}$. We argue first that Φ_0 is necessarily continuous on P. First notice that for some K we have $|\Phi_0(f)| \leq K$ whenever $\|f\|_\infty \leq 1$. Now suppose $f \in P$ and $f_n \to f$. Then, given $\varepsilon > 0$ there exists N so that if $n \geq N$, $(1-\varepsilon)f \leq f_n \leq (1+\varepsilon)f$. Then $|\Phi_0((1+\varepsilon)f - f_n)|$, $|\Phi_0(f_n - (1-\varepsilon)f)| \leq 2\varepsilon\|f\|_\infty K$ provided $n \geq N$. Thus

$$(1+\varepsilon)\Phi_0(x) \leq \Phi_0(f_n) + 2\varepsilon\|f\|_\infty K,$$
$$\Phi_0(f_n) \leq (1-\varepsilon)\Phi_0(f) + 2\varepsilon\|f\|_\infty K.$$

We conclude that for $n \geq N$, $|\Phi_0(f) - \Phi_0(f_n)| \leq \varepsilon|\Phi_0(f)| + 2\varepsilon\|f\|_\infty K$, so that Φ_0 is continuous on P.

Now let B_0 be the set of x so that $\int f\log|x|\,d\mu \leq \Phi_0(f)$ whenever $f \in P$. Now a standard Hahn-Banach separation argument shows that for $f \in P$,

$$\Phi_0(f) = \sup_{x\in B_0}\int f\log|x|d\mu.$$

Arguing similarly, let $\Phi_1(f) = \inf\sum_{j=1}^n(\Lambda(f_j) - \Phi(f_j))$ where the infimum is taken over all $f_1, \dots, f_n \geq 0$ so that $\sum f_i = f$. Then by the same reasoning

$$\Lambda(f) - \Phi(f) - M\|f\| \leq \Phi_1(f) \leq \Lambda(f) - \Phi(f),$$

and if $B_1 = \{x\colon \int f \log|x|\, d\mu \le \Phi_1(f)\ \forall f \in P\}$ then for $f \in P$,

$$\Phi_1(f) = \sup_{x \in B_1} \int f \log|x|\, d\mu .$$

Since $\Phi_0 + \Phi_1 \le \Lambda$ we conclude that $B_0.B_1 \subset B_{L_1}$. Let B_X be the closed convex hull of B_0. Then $B_X.B_1 \subset B_{L_1}$ so that $\Phi_X(f) + \Phi_1(f) \le \Lambda(f)$ for $f \in P$. Thus $\Phi(f) - M\|f\| \le \Phi_X(f) \le \Phi(f) + M\|f\|$ for every $f \in P$. In general if $f \ge 0$ then there is a sequence $f_n \downarrow f$ with $f_n \in P$. By Lemma 4.3 $\limsup |\Phi(f_n) - \Phi(f)| \le 2\delta\|f\|$ and it follows that

$$|\Phi(f) - \Phi_X(f_n)| \le (M + 2\delta)\|f\|$$

whenever $f \ge 0$.

We now turn to the general case. Let $\mathscr{A} = (A_1, \dots, A_n)$ be any finite collection of disjoint measurable sets so that each $1_{A_k} \in \mathscr{I}$. Let $E_{\mathscr{A}}$ be the linear span of 1_{A_k} for $1 \le k \le n$. By the above there exists a semilinear map $\Psi_{\mathscr{A}}$ on $E_{\mathscr{A}}$ so that

$$0 \le \Delta_{\Psi_{\mathscr{A}}}(f, g) \le \Delta_\Lambda(f, g), \qquad f, g \in \mathscr{I},$$

and

$$|\Phi(f) - \Psi_{\mathscr{A}}(f)| \le (M + 2\delta)\|f\|$$

for $f \in \mathscr{I}$. By a simple compactness argument we can then determine a functional Ψ on the cone $\mathscr{V}$ of simple functions in $\mathscr{I}$ so that

$$0 \le \Delta_\Psi(f, g) \le \Delta_\Lambda(f, g), \qquad |\Phi(f) - \Psi(f)| \le (M + 2\delta)\|f\|$$

for $f \in \mathscr{V}$.

Suppose $f \in \mathscr{I}$. Then $\mathscr{V} \cap [0, f]$ is dense in $[0, f]$ for the L_1-norm. We now use the reasoning of Lemma 4.3 to argue that Ψ is uniformly continuous on $\mathscr{V} \cap [0, f]$. First notice that since Φ is semilinear, it follows that $|\Psi(g)| \le \eta_0(\|g\|)$ for $g \in \mathscr{V} \cap [0, f]$ where $\lim_{\varepsilon \to 0} \eta_0(\varepsilon) = 0$. Suppose $g, h \in [0, f]$ and let $k = \max(g, h)$. Then if $\|g - h\| \le \varepsilon$,

$$|\Psi(g) - \Psi(h)| \le \Delta_\Psi(g, k - g)| + \Delta_\Psi(h, k - h)| + |\Psi(k - g)| + |\Psi(k - h)|.$$

Now

$$|\Delta_\Psi(g, k - g)| \le |\Delta_\Lambda(g, k - g)| \le \|f\|\phi(\varepsilon),$$

where $\lim_{\varepsilon \to 0} \phi(\varepsilon) = 0$ since Λ is a continuous semilinear functional. Similarly estimating the other terms we obtain that $|\Psi(g) - \Psi(h)| \le \eta_1(\varepsilon)$, where $\lim_{\varepsilon \to 0} \eta_1(\varepsilon) = 0$.

Thus Ψ extends continuously and uniquely to each $[0, f]$, and hence to $\mathscr{I}$ in such a way that it is continuous on each order-interval. In particular, we must have $0 \le \Delta_\Psi(f, g) \le \Delta_\Lambda(f, g)$ for every $f, g \in \mathscr{I}$. Further, if $f \in \mathscr{I}$, we may pick $f_n \in \mathscr{V}$ so that $f_n \uparrow f$. Then as $\limsup |\Phi(f_n) - \Phi(f)| \le 2\delta\|f\|$ by Lemma 4.3 we obtain $d(\Phi, \Psi) \le M + 4\delta$. This in turn implies that Ψ is semilinear and we can apply Theorem 5.2 to show that $\Psi = \Phi_X$ for some Köthe function space X. □

We remark that if Φ is continuous then we can improve the estimate to give $d(\Phi, \Phi_X) \le M$.

We now prove a result on set functions which is the key to our main results.

Theorem 6.2. *Let S be a finite set and let μ be a positive measure on S. Let $\phi\colon 2^S \to \mathbf{R}$ be a set function, continuous with respect to μ in the sense that $\mu(A) = 0$ implies $\phi(A) = 0$, and such that whenever A, B are disjoint sets then*

$$\phi(A) + \phi(B) - \mu(A \cup B) \le \phi(A \cup B) \le \phi(A) + \phi(B).$$

Then there is a signed measure ν on S such that $\nu(A) \le \phi(A)$ for $A \subset S$ and $\nu(S) \ge \phi(S) - 4\mu(S)$.

Proof. For each $A \subset S$ let Γ_A be the set of signed measures ρ defined on A so that $\rho \le \phi$. Let

$$h(A) = \mu(A)^{-1}\left(\phi(A) - \sup_{\rho \in \Gamma_A} \rho(A)\right).$$

Here we define $h(A) = 0$ if $\mu(A) = 0$. Then $h(S) \le \max_{A \subset S} h(A)$. We shall show that $\max_{A \subset S} h(A) \le 4$. Once this is done it will follow that there is a sequence $\nu_n \in \Gamma_S$ with $\nu_n(S) \ge \phi(S) - 4\mu(S) - 1/n$; it is then easy to show that $(\nu_n(A))$ is bounded for every A and so a compactness argument will complete the proof. Thus, by restricting to the set where h attains its maximum, it suffices to consider the case when $h(A) \le h(S) = \gamma$, say for every A.

We may clearly suppose $\mu(S) > 0$. For $0 < \varepsilon < 1$, pick a measure $\lambda \in \Gamma_S$ so that $\lambda(S) \ge \phi(S) - \varepsilon - \gamma\mu(S)$. Let E be a maximal subset of S so that $\lambda(E) \ge \phi(E) - 2\mu(E)$. Let $F = S \backslash E$. If F is empty we are done. Otherwise, if $A \subset F$ then $\lambda(A \cup E) < \phi(A \cup E) - 2\mu(A \cup E)$, so that $\lambda(A) < \phi(A \cup E) - \phi(E) - 2\mu(A) \le \phi(A) - 2\mu(A)$. Now consider the signed measure λ_0 where $\lambda_0(B) = \lambda(B) + \mu(B \cap F) - \mu(B \cap E)$. Then

$$\begin{aligned}\lambda_0(B)&\lambda(B \cap E) - \mu(B \cap E) + \lambda(B \cap F) + \mu(B \cap F)\\ &< \phi(B \cap E) - \mu(B \cap E) + \phi(B \cap F) - \mu(B \cap F)\\ &\le \phi(B)\end{aligned}$$

so that $\lambda_0 \in \Gamma_S$. Thus $\lambda_0(S) \le \phi(S) - \gamma\mu(S)$ or $\lambda(S) + \mu(F) - \mu(E) \le \lambda(S) + \varepsilon$. Thus $\mu(E) \ge \frac{1}{2}(1 - \varepsilon)\mu(S)$.

Now we utilize the fact that $h(F) \le \gamma$. There exists a signed measure ν on F with $\nu \in \Gamma_F$ and $\nu(F) \ge \phi(f) - \gamma\mu(F) - \varepsilon$. Define ν_0 on S by

$$\nu_0(B) = \lambda(B \cap E) + \nu(B \cap F) - \mu(B).$$

Then

$$\nu_0(B) \le \phi(B \cap E) + \phi(B \cap F) - \mu(B) \le \phi(B).$$

Thus $\nu_0 \in \Gamma_S$, and

$$\nu_0(S) = \lambda(E) + \nu(F) - \mu(S) \le \phi(S) - \lambda\mu(S).$$

Hence

$$\phi(E) + \phi(F) - 2\mu(E) - \gamma\mu(F) - \varepsilon - \mu(S) \le \phi(S) - \gamma\mu(S).$$

Since $\phi(E) + \phi(F) \ge \phi(S)$ this simplifies to $(\gamma - 2)\mu(E) \le \mu(S) + \varepsilon$. Thus

$$\gamma - 2 \le \frac{2}{1 - \varepsilon} + \frac{2\varepsilon}{1 - \varepsilon}\frac{1}{\mu(S)}.$$

Letting $\varepsilon \to 0$ we obtain $\gamma \le 4$. □

In order to apply Theorem 6.2 we will need some preliminary estimates for semilinear functionals. Essentially, the next theorem is a translation of a result in [25] but we give a more precise proof.

Theorem 6.3. *Suppose $\sigma > 1$. Then there are constants $C(\sigma)$ and $C'(\sigma)$ so that whenever $\mathscr{I} \subset (L\log L)_+$ is a strict semi-ideal and $\Phi\colon \mathscr{I} \to \mathbf{R}$ is a map which satisfies*

$$\Delta_\Phi(f, g) \le \alpha \log 2(\|f\| + \|g\|)$$

for $f, g \in \mathscr{I}$ (respectively, for disjoint $f, g \in \mathscr{I}$) then for any $f_1, \dots, f_n \in \mathscr{I}$ (respectively, any disjoint $f_1, \dots, f_n \in \mathscr{I}$) we have, if $f = \sum_{k=1}^n f_k$

$$\Delta_\Phi(f_1, \dots, f_n) \le \alpha \left(\sigma \sum_{k=1}^n \|f_k\| \log \frac{\|f\|}{\|f_k\|} + C\|f\| \right),$$

$$\Delta_\Phi(f_1, \dots, f_n) \le \alpha \left(\sigma \sum_{k=1}^n (\log k)\|f_k\| + C'\|f\| \right).$$

Proof. By a simple continuity argument there exists $0 < \theta_0 < 1$ and $\tau > 0$ so that if $\frac{1}{2} \le \alpha \le \frac{1}{2}(1 + \theta_0)$ then

$$\sigma \left(a \log \frac{1}{a} + (1-a) \log \frac{1}{1-a} \right) \ge \log 2 + \tau.$$

Let c_n be the least constant so that if $f_1, \dots, f_n \in \mathscr{I}$ satisfy $\sum_{k=1}^n \|f_k\| = 1$ (and are disjoint) then

$$\Delta_\Phi(f_1, \dots, f_n) \le \alpha \left(\sigma \sum_{k=1}^n \|f_k\| \log \frac{1}{\|f_k\|} + c_n \right).$$

It is clear that for every n, c_n is finite.

For any such $f_1, \dots, f_n$ we let $\theta = \max_{1 \le k \le n} \|f_k\|$. Let us first assume that $\theta \le \theta_0$. Then we may choose signs $\varepsilon_k = \pm 1$ so that $0 \le \sum_{k=1}^n \varepsilon_k \|f_k\| \le \theta$. Let $A = \{k\colon \varepsilon_k = 1\}$ and $B = \{k\colon \varepsilon_k = -1\}$. Let $a = \sum_{k \in A} \|f_k\|$ so that $\frac{1}{2} \le a \le \frac{1}{2}(1 + \theta)$. Let $f_A = \sum_{k \in A} f_k$ and let $f_B = \sum_{k \in B} f_k$. Then

$$\Delta_\Phi(f_A, f_B) \le \alpha \log 2,$$

$$\Delta_\Phi(f_k\colon k \in A) \le \alpha \left(\sum_{k \in A} \|f_k\| \log \frac{a}{\|f_k\|} + c_n a \right),$$

$$\Delta_\Phi(f_k\colon k \in B) \le \alpha \left(\sum_{k \in B} \|f_k\| \log \frac{1-a}{\|f_k\|} + c_n(1-a) \right),$$

$$\Delta_\Phi(f_1, \dots, f_n) = \Delta_\Phi(f_k\colon k \in A) + \Delta_\Phi(f_k\colon k \in B) + \Delta_\Phi(f_A, f_B).$$

Combining these equations gives us

$$\begin{aligned} &\Delta_\Phi(f_1, \dots, f_n) - \alpha\sigma \sum_{k=1}^n \|f_k\| \log \frac{1}{\|f_k\|} \\ &\qquad \le \alpha(c_n + \sigma(a \log a + (1-a)\log(1-a)) + \log 2). \end{aligned}$$

By the hypothesis on θ we conclude that

$$\Delta_\Phi(f_1, \dots, f_n) - \alpha\sigma \sum_{k=1}^n \|f_k\| \log \frac{1}{\|f_k\|} \le \alpha(c_n - \tau).$$

On the other hand, if $\theta > \theta_0$ we may assume $\|f_n\| = \theta$. Then

$$\begin{aligned}\Delta_\Phi(f_1, \dots, f_n) &= \Delta_\Phi\left(\sum_{k=1}^n f_k, f_n\right) + \Delta_\Phi(f_1, \dots, f_{n-1}) \\ &\le \alpha\left(\log 2 + \sigma \sum_{k=1}^{n-1} \|f_k\| \log \frac{1-\theta}{\|f_k\|} + c_n(1-\theta)\right) \\ &\le \alpha\left(\log 2 + \sigma \sum_{k=1}^{n} \|f_k\| \log \frac{1}{\|f_k\|} + c_n(1-\theta)\right)\end{aligned}$$

so that

$$\Delta_\Phi(f_1, \dots, f_n) - \sigma\alpha \sum_{k=1}^n \|f_k\| \log \frac{1}{\|f_k\|} \le \alpha(\log 2 + c_n(1-\theta_0)).$$

Combining the two cases we must have that $c_n \le \log 2 + c_n(1-\theta_0)$ or $c_n \le \log 2/\theta_0$ independent of n. This leads to the first equation. The second is obtained by using Lemma 3.5 of [25], observing that it first suffices to consider the case $\|f_1\| \ge \|f_2\| \ge \cdots \ge \|f_n\|$. □

Lemma 6.4. *Given $\varepsilon > 0$ there exists a constant $\beta = \beta(\varepsilon) < \infty$ so that whenever $\mathcal{J}$ is a semi-ideal and Φ is a real semilinear functional on $\mathcal{J}$ and $f_1, \dots, f_n \in \mathcal{J}$*

$$|\Delta_\Phi(f_1, \dots, f_n)| \le \delta(\Phi)\left(\varepsilon \sum_{k=1}^n k\|f_k\| + \beta \left\|\sum_{k=1}^n f_k\right\|\right).$$

Proof. We have

$$|\Delta_\Phi(f_1, \dots, f_n)| \le 2\delta \sum_{k=1}^n (\log k)\|f_k\| + C \sum_{k=1}^n \|f_k\|$$

for a suitable constant C, by applying Theorem 6.3 with $\sigma = 2\log 2 > 1$. Since for arbitrary ε there exists $\gamma = \gamma(\varepsilon)$ so that $\log k \le \varepsilon k + \gamma$ for all k the lemma follows easily. □

Lemma 6.5. *Given $\varepsilon > 0$ there is a constant $C(\varepsilon)$ so that if $\mathcal{J}$ is a strict semi-ideal and Φ is a real semilinear functional on $\mathcal{J}$ such that for any disjoint f, g we have $\Delta_\Phi(f, g) \ge 0$, then for any $f_1, \dots, f_n \in \mathcal{J}$,*

$$\Delta_\Phi(f_1, \dots, f_n) + \delta\varepsilon\Delta_\Lambda(f_1, \dots, f_n) \ge -C\delta \sum_{k=1}^n \|f_k\|,$$

where $\delta = \delta(\Phi)$.

Proof. We first note that we need only establish the lemma for simple functions. Indeed, if we establish the lemma for simple functions then it follows for arbitrary functions by using Lemma 4.3 with C replaced by $C + 4$. Assume then that $f_1, \dots, f_n$ are simple. We may further assume that if $f = f_1 + \cdots + f_n$ then $\|f\| = 1$. Let Σ_0 be the finite algebra generated by $f_1, \dots, f_n$.

If we set $\phi(A) = \Phi(f 1_A)$ then ϕ is a set function on Σ_0 which satisfies for disjoint sets A, B

$$\phi(A) + \phi(B) - \delta \int_{A\cup B} f\, d\mu \le \phi(A \cup B) \le \phi(A) + \phi(B).$$

Thus Theorem 6.3 can be applied to produce a signed measure ν on Σ_0 with $\nu(A) \le \phi(A)$ for all $A \in \Sigma_0$ and $\nu(S) \ge \phi(S) - 4\delta$.

Now suppose g is any Σ_0-measurable function whose support is contained in that of f. Suppose $\|g\| = 1$ and let $h = g/f$ ($h = 0$ when $f = 0$). Let $A_0 = \{h < 1\}$ and then for $k > 0$ set $A_k = \{2^{k-1} \le h < 2^k\}$. There exists N so that $A_k = \varnothing$ for $k > N$. We first notice that

$$\Delta_\Phi(g1_{A_0}, g1_{A_1}, \dots, g1_{A_N}) \le \delta\left(\varepsilon \log 2 \sum_{k=0}^{N}(k+1)\|g1_{A_k}\| + C_0\right)$$

by Lemma 6.4, where C_0 depends only on ε. Now

$$\log 2 \sum_{k=0}^{N}(k+1)\int_{A_k} g\,d\mu \le \int g(\log_+ h + 2\log 2)\,d\mu.$$

Further

$$\begin{aligned}\int g \log_+ h\,d\mu &\le \int g \log h\,d\mu + \int_{A_0} g \log\frac{f}{g}\,d\mu \\ &\le \int g\log h\,d\mu + \frac{1}{e}\int_{A_0} f\,d\mu \\ &\le \int g \log h\,d\mu + \frac{1}{e}.\end{aligned}$$

Combining these equations gives us

$$\Delta_\Phi(g1_{A_0}, \dots, g1_{A_N}) \le \delta\left(\varepsilon\int g\log h\,d\mu + C_1\right),$$

where C_1 depends only on ε. Now we can write each $h1_{A_k}$ in the form

$$h1_{A_k} = 2^k \sum_{j=1}^{\infty} 2^{-j} 1_{B_{kj}},$$

where $B_{kj} \in \Sigma_0$. Now for arbitrary m we have

$$\begin{aligned}\Delta_{km} &= \Delta_\Phi(2^{-1}f1_{B_{k1}}, \dots, 2^{-m}f1_{B_{km}}) \\ &\le \delta\left(\sum_{j=1}^{m} j2^{-j}\|f1_{B_{kj}}\| + C_2\sum_{j=1}^{m}2^{-j}\|f1_{B_{kj}}\|\right),\end{aligned}$$

where C_2 is a constant. Thus $\Delta_{km} \le C_3\delta\|f1_{A_k}\|$ for some constant C_3. Now this means that

$$\Phi\left(\sum_{j=1}^{m} 2^{k-j} f1_{B_{kj}}\right) \ge \sum_{j=1}^{m} 2^{k-j}\phi(B_{kj}) - C_3\delta 2^k\|f1_{A_k}\|.$$

Thus for $k \ge 1$,

$$\Phi\left(\sum_{j=1}^{m} 2^{k-j} f1_{B_{kj}}\right) \ge \sum_{j=1}^{m} 2^{k-j}\nu(B_{kj}) - 2C_3\delta\|g1_{A_k}\|.$$

Letting $m \to \infty$ and invoking Lemma 4.3 we have

$$\Phi(g 1_{A_k}) \geq \int_{A_k} h \, d\nu - C_4 \delta \int_{A_k} g \, d\mu ,$$

where C_4 is a constant. In the case $k = 0$ the same reasoning leads to

$$\Phi(g 1_{A_0}) \geq \int_{A_0} h \, d\nu - C_4 \delta \int_{A_0} f \, d\mu .$$

Combining we obtain $\sum_{k=0}^{N} \Phi(g 1_{A_k}) \geq \int h \, d\nu - 2C_4 \delta$. Thus

$$\Phi(g) \geq \int h \, d\nu - \varepsilon\delta \int g \log h \, d\mu - C_5 \delta ,$$

where $C_5 = C_5(\varepsilon)$. Now we apply this to $g_k = f_k / \|f_k\|$. The result is that

$$\Phi(f_k) \geq \int \frac{f_k}{f} \, d\nu - \varepsilon\delta \int f_k \log \frac{f_k}{f \|f_k\|} \, d\mu - C_5 \delta \|f_k\| .$$

Summing over k

$$\sum_{k=1}^{n} \Phi(f_k) \geq \nu(S) - \varepsilon\delta \Delta_\Lambda(f_1, \dots, f_n) - C_5 \delta .$$

Recalling the choice of ν this implies

$$\Delta_\Phi(f_1, \dots, f_n) + \varepsilon\delta \Delta_\Lambda(f_1, \dots, f_n) \geq -(C_5 + 4)\delta . \quad \square$$

We finally come to the main theorem of the section.

Theorem 6.6. *Given $\varepsilon > 0$ there is a constant $C = C(\varepsilon)$ so that whenever $\mathcal{J}$ is a strict semi-ideal contained in $(L \log L)_+$ and Φ is a real semilinear functional with $\delta(\Phi) \leq (1 - \varepsilon) \log 2$ then there is a Köthe function space X with $d(\Phi, \Phi_X - \Phi_{X^*}) \leq C$.*

Proof. By Theorem 6.3 we have that if $f_1, \dots, f_n$ are disjoint and $f = f_1 + \cdots + f_n$ then

$$|\Delta_\Phi(f_1, \dots, f_n)| \leq \left(1 - \frac{1}{2}\varepsilon\right) \sum_{k=1}^{n} \|f_k\| \log \frac{\|f\|}{\|f_k\|} + C_0 \|f\| ,$$

where C_0 depends only on ε. But this can be reworded as

$$|\Delta_\Phi(f_1, \dots, f_n)| \leq \left(1 - \frac{1}{2}\varepsilon\right) \Delta_\Lambda(f_1, \dots, f_n) + C_0 \|f\| .$$

Let us define $\Psi_0 = \Phi + (1 - \frac{1}{2}\varepsilon)\Lambda$. We clearly then have for disjoint $f_1, \dots, f_n$

$$\Delta_{\Psi_0}(f_1, \dots, f_n) \geq -C_0 \sum_{k=1}^{n} \|f_k\| ,$$

and $\delta(\Psi_0) \leq 2 \log 2$. Let $\Psi(f) = \inf \sum_{k=1}^{n} \Psi_0(f_k)$ where the infimum is taken over all disjoint $f_1, \dots, f_n$ with $f = \sum f_k$. Clearly $\Psi_0(f) - C_0 \|f\| \leq \Psi(f) \leq \Psi_0(f)$ and hence for any $f_1, \dots, f_n$

$$|\Delta_\Psi(f_1, \dots, f_n) - \Delta_{\Psi_0}(f_1, \dots, f_n)| \leq C_0 \sum_{k=1}^{n} \|f_k\| .$$

Also $\delta(\Psi) \leq 2\log 2 + C_0$. Now $\Delta_\Psi(f, g) \geq 0$ for disjoint f, g so that there is a constant $C_1 = C_1(\varepsilon)$ from Lemma 6.5 with

$$\Delta_\Psi(f_1, \dots, f_n) \geq -\frac{\varepsilon}{2}\Delta_\Lambda(f_1, \dots, f_n) - C_1 \sum_{k=1}^{n} \|f_k\|$$

for arbitrary $f_1, \dots, f_n$. But this means that

$$\Delta_{\Psi_0}(f_1, \dots, f_n) \geq \frac{\varepsilon}{2}\Delta_\Lambda(f_1, \dots, f_n) - C_2 \sum_{k=1}^{n} \|f_k\|.$$

Hence

$$\Delta_\Phi(f_1, \dots, f_n) \geq -\Delta_\Lambda(f_1, \dots, f_n) - C_2 \sum_{k=1}^{n} \|f_k\|.$$

Clearly this reasoning may also be applied to $-\Phi$ and so we conclude that

$$|\Delta_\Phi(f_1, \dots, f_n)| \leq \Delta_\Lambda(f_1, \dots, f_n) + C_2 \sum_{k=1}^{n} \|f_k\|.$$

Now we can apply Proposition 6.1 to $\frac{1}{2}(\Phi + \Lambda)$ to deduce that there is a Köthe function space X with $d(\frac{1}{2}(\Phi+\Lambda), \Phi_X) \leq C_3$ where C_3 depends only on ε. Hence $d(\Phi, \Phi_X - \Phi_{X^*}) \leq 2C_3$ $\square$.

We now state an extension to maps defined on ideals rather than semi-ideals. Let $\mathcal{J}$ be an ideal contained in $L\log L$. Then a homogeneous map $\Phi\colon \mathcal{J} \to \mathbf{C}$ will be called semilinear if $\Phi|_{\mathcal{J}^+}$ is semilinear and (4.5) holds for all $f, g \in \mathcal{J}$. Φ is called real if its restriction to $\mathcal{J}_+$ is real.

Corollary 6.7. *Suppose $\varepsilon > 0$; then there exists a constant $C = C(\varepsilon)$ so that if $\mathcal{J}$ be a strict ideal contained in L_1 and if $\Phi\colon \mathcal{J} \to \mathbf{C}$ be a real semilinear map with $\delta(\Phi) < (1-\varepsilon)\log 2$ then there exists a Köthe function space X so that*

$$|\Phi(f) - (\Phi_X(f) - \Phi_{X^*}(f))| \leq C\|f\|$$

for all $f \in \mathcal{J}$. If Φ is not necessarily real, then we can conclude that there exist Köthe function spaces X, Y so that

$$|\Phi(f) - (\Phi_X(f) + i\Phi_Y(f) - \Phi_{X^*}(f) - i\Phi_{Y^*}(f))| \leq C\|f\|.$$

Proof. We apply Theorem 6.6, when Φ is real, to approximate $\Phi|_{\mathcal{J}_+}$ and then notice that for arbitrary f we can write $f = (\Re f)_+ - (\Re f)_- + i(\Im f)_+ - i(\Im f)_- = f_1 - f_2 + if_3 - if_4$, say. Thus

$$|\Phi(f) - \Phi(f_1) + \Phi(f_2) - i\Phi(f_3) + i\Phi(f_4)| \leq C_0\|f\|$$

for some constant C. A similar inequality holds for Φ_X and Φ_{X^*} by Lemma 5.6 and so the result follows. If Φ is complex, one uses the same argument on its real and imaginary parts when restricted to $\mathcal{J}_+$. $\square$

Let us give an application to minimal extensions (see §1). Compare the original constructions of minimal extensions of l_1 in [23, 44].

Theorem 6.8. *Let Z be a minimal extension of the real space $L_{1,\mathbf{R}}(S, \Sigma, \mu)$. Then there exists a Köthe function space X so that Z is linearly homeomorphic to the completion of $L_2(\mu) \oplus \mathbf{R}$ under the quasinorm*

$$\|(f, \alpha)\| = \|f\| + |\alpha - \Phi_X(f) + \Phi_{X^*}(f)|.$$

Proof. In fact there is a quasilinear map $F: L_1 \to \mathbf{R}$ so that Z can be identified with the space $L_1 \oplus_F \mathbf{R}$ of all pairs (f, α) quasinormed by $\|(f, \alpha)\| = \|f\| + |\alpha - F(f)|$. Now consider F restricted to E, where E is a suitable weighted L_2-space so that $\|x\|_E \geq \|x\|$ for every $x \in L_0$. Since L_2 is a $\mathscr{K}$-space [23], there exists a linear map (not necessarily continuous!) $\phi: L_2 \to \mathbf{R}$ with $|F(f) - \phi(f)| \leq C\|f\|_E$ for some constant C. Now $F - \phi$ will be semilinear according to our definition on the ideal E. Thus there exists a Köthe function space X and $\alpha > 0$ so that

$$|F(f) - \phi(f) - \alpha(\Phi_X(f) - \Phi_{X^*}(f))| \leq C_1\|f\|$$

for $f \in E$ by the real version of Corollary 6.7. Now this implies [23, Theorem 2.5] the isomorphism of the space Z with the minimal extension generated by $\Phi_X - \Phi_{X^*}$. □

7. Applications to centralizers

Suppose X is a Köthe function space and $\Omega: X \to L_0$ is a centralizer on X. We say Ω is real if $\Omega(f)$ is a real function whenever f is a real function.

Lemma 7.1. *Let Ω be a centralizer on X with $\rho(\Omega) = \rho$. Then there exist real centralizers Ω_1, Ω_2 so that $\rho(\Omega_j) \leq 2\rho$ for $j = 1, 2$ and*

$$\|\Omega(f) - \Omega_1(f) - i\Omega_2(f)\|_X \leq 2\rho\|f\|_X \quad \textit{for every } f \in X. \quad \square$$

Proof. Set $\Omega_1(f) = \Re\Omega(\Re f) + i\Re\Omega(\Im f)$ and $\Omega_2(f) = \Im\Omega(\Re f) - i\Im(\Omega(\Im f))$. The lemma then follows quickly. □

We now use [29, Theorem 5.1]. If $\Omega: X \to L_0$ is a centralizer then there is a centralizer $\Omega^{[1]}$ on L_1 with $\rho(\Omega^{[1]}) \leq 36\rho(\Omega)$ and so that for $x \in X$ and $y \in X^*$ we have

$$\|\Omega^{[1]}(xy) - \Omega(x)y\| \leq 18\rho(\Omega)\|x\|_X\|y\|_{X^*}.$$

Furthermore $\Omega^{[1]}$ is unique up to equivalence, i.e., if Ω' is any other centralizer on L_1 satisfying

$$\|\Omega'(xy) - \Omega(x)y\| \leq C\|x\|_X\|y\|_{X^*},$$

then for some constant C_1 we have $\|\Omega^{[1]}(f) - \Omega'(f)\| \leq C_1\|f\|$ for all $f \in L_1$. $\Omega^{[1]}$ is defined by $\Omega^{[1]}(f) = \Omega(x)y$, where $f = xy$ is the Lozanovskii factorization of f for X. Clearly if Ω is real the so is $\Omega^{[1]}$.

Theorem 7.2. (i) *Let X_0, X_1 be Köthe function spaces and let $X = [X_0, X_1]_\theta$. Let Ω be the induced centralizer on X. Then for $f \in L_1$,*

$$\Omega^{[1]}(f) = f(\log x_1 - \log x_0),$$

where $|f| = x_0y_0 = x_1y_1$ are the Lozanovskii factorizations for $|f|$ with respect to X_0, X_1.

(ii) *Let $\mathscr{X} = \{X_w : w \in \mathbf{T}\}$ be a strongly admissible family of Köthe function spaces, and $X = X_0$ and Ω is the centralizer induced on X. Then for $f \in L_1$*

$$\Omega^{[1]}(f)(s) = f(s)\left(\int_{-\pi}^{\pi} e^{-it}\log g(s, t)\,\frac{dt}{\pi}\right),$$

where $f(s) = g(s, t)h(s, t)$ is the Lozanovskii factorization for the space $\mathscr{E}$ on $S \times \mathbf{T}$ defined by

$$\|\phi\|_{\mathscr{E}} = \operatorname{ess\,sup}\|\phi(\cdot, w)\|_{X_w}.$$

Proof. (i) Assume f has norm one and is factored as stated. On $\mathscr{S} = \{z : 0 < \Re z < 1\}$ consider the functions $G(z) = x_0^{1-z} x_1^z$ and $H(z) = |y_0|^{1-z}|y_1|^z$. Then both $G, H \in \mathscr{N}^+(\mathscr{S})$. Also $\|G(\theta)\|_X \leq 1$ and $\|H(\theta)\|_{X^*} \leq 1$ since $X^* = [X_0^*, X_1^*]_\theta$. However $G(z)H(z) = |f|$ from which it follows that $|f| = G(\theta)H(\theta)$ is the Lozanovskii factorization for X. Now consider any optimal factorization $x = u_0^{1-\theta} u_1^\theta$ with $u_0, u_1 \geq 0$ and $\|u_0\|_{X_0} = \|u_1\|_{X_1} = 1$. Let $K(z) = u_0^{1-z} u_1^z$; then $\|K(z)H(z)\| \leq 1$ but $\int K(\theta)H(\theta)\,d\mu = 1$. Hence $\int K(z)H(z)\,d\mu = 1$ for all z and so $K(z, s)H(z, s)$ is real and positive for all z for almost every $s \in S$. (An argument with a dense countable set in $\mathscr{S}$ is required here.) Thus $K(z, s)H(z, s) = f(s)$ a.e. and so $u_0(s) = x_0(s)$, $u_1(s) = x_1(s)$ for $s \in \operatorname{supp} f = \operatorname{supp} x$. Now by (3.2) this implies that $\Omega(x)$ can only take the value $x(\log x_1 - \log x_0)$.

Now

$$\Omega^{[1]}(f) = \Omega(x)H(\theta) \operatorname{sgn} f,$$

where $\operatorname{sgn} f = f/|f|$ as long as $f \neq 0$ and is zero otherwise. Hence

$$\Omega^{[1]}(f) = f(\log x_1 - \log x_0). \quad \square$$

(ii) The proof is very similar. Suppose as above that $f \in L_1$ and that $|f(s)| = g(s, t)h(s, t)$ is the Lozanovskii factorization of $|f|$ for $\mathscr{E}$. Define $G, H \in \mathscr{N}^+$ by

$$G(z) = \exp\left(\int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} \log g(s, t) \frac{dt}{2\pi}\right),$$

$$H(z) = \exp\left(\int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} \log h(s, t) \frac{dt}{2\pi}\right).$$

It follows from the duality theorem Corollary 4.9 that $|f| = G(0)H(0)$ is the Lozanovskii factorization for X. Further, arguing as in the case (i) it can be shown that if $x = G(0)$ and if $K \in \mathscr{N}^+$ satisfies $K(0) = x$ and $\|K(w)\|_{X_w} \leq 1$ for $w \in \mathbf{T}$ then $K(z, s) = G(z, s)$ on $\operatorname{supp} f$. Thus applying Theorem 3.3 we obtain the theorem. $\square$

Notice in the above theorem that in case (i) the centralizer $\Omega^{[1]}$ is independent of θ. We now connect our results with those of the preceding sections. If Ω is a centralizer on X we defined a strict ideal $\mathscr{J} = \mathscr{J}_\Omega \subset L_1$ by $f \in \mathscr{J}$ if and only if $\Omega^{[1]}(f) \in L_1$. We omit the simple proof that $\mathscr{J}$ is a strict ideal. On $\mathscr{J}$ we define the functional Φ^Ω by $\Phi^\Omega(f) = \int \Omega^{[1]}(f)\,d\mu$.

Lemma 7.3. *Φ^Ω is a semilinear functional on $\mathscr{J}$ with $\delta(\Phi^\Omega) \leq 108\rho(\Omega)$. Φ^Ω is real whenever Ω is real.*

Proof. Condition (4.3) is obvious and (4.4) follows from Lemma 4.2 of [29]. (4.5) follows from Lemma 4.3 of [29].

Proposition 7.4. (i) *Let X_0, X_1 be a pair of Köthe function spaces and let $X = [X_0, X_1]_\theta$ and suppose Ω is the corresponding centralizer on X. Then $\mathscr{J}_{X_0} \cap \mathscr{J}_{X_1} \subset \mathscr{J}_\Omega$ and*

$$\Phi^\Omega(f) = \Phi_{X_1}(f) - \Phi_{X_0}(f), \qquad f \in \mathscr{J}_{X_0} \cap \mathscr{J}_{X_1}.$$

(ii) *If $\mathscr{X} = \{X_w : w \in \mathbf{T}\}$ is a strongly admissible family of Köthe function spaces, and $X = X_0$ with Ω the corresponding centralizer, then there is strict*

semi-ideal $\mathscr{I} \subset \mathscr{J}_\Omega$ *so that*

$$\Phi^\Omega(f) = \int_{-\pi}^{\pi} e^{-it}\Phi_{X_{e^{it}}}(f)\,\frac{dt}{2\pi}, \qquad f \in \mathscr{I}.$$

The proofs of these statements are immediate.

We now prove our main theorem. We recall that two centralizers Ω_1 and Ω_2 on X are called equivalent if there exists a constant B so that $\|\Omega_1(f) - \Omega_2(f)\|_X \le B\|f\|_X$ for all $f \in X$.

Theorem 7.6. *There is a constant* $\gamma > 0$ *and a constant* $C < \infty$ *so that whenever* $1 < p \le 2$ *and* X *is a* p*-convex and* q*-concave Köthe function space with* $\frac{1}{p} + \frac{1}{q} = 1$ *and* Ω *is a real centralizer on* X *with* $\rho(\Omega) \le \gamma/q$ *then there is a pair of Köthe function spaces* X_0, X_1 *so that* $X = [X_0, X_1]_{1/2}$ *and if* Ω_0 *is the induced centralizer then* $\|\Omega(f) - \Omega_0(f)\|_X \le C\|f\|_X$ *for* $f \in X$. *In particular* Ω *is equivalent to* Ω_0.

Furthermore X_0 *and* X_1 *are unique up to equivalent norming. If* Y_0, Y_1 *are Köthe function spaces such that* $[Y_0, Y_1]_{1/2} = X$ *with an equivalent norm and the induced centralizer* Ω_1 *is equivalent to* Ω *then* $Y_0 = X_0$ *with an equivalent norm and* $Y_1 = X_1$ *with an equivalent norm.*

Proof. Let us dispose of the uniqueness question. If Ω_1 is equivalent to Ω then $\Omega_1^{[1]}$ and $\Omega^{[1]}$ are equivalent. Hence on a suitable strict semi-ideal, Φ^Ω is equivalent to $\Phi_{Y_1} - \Phi_{Y_0}$, while $\Phi_{Y_1} + \Phi_{Y_0}$ is equivalent to $2\Phi_X$. Thus, up to equivalence Φ_{Y_1} and Φ_{Y_0} are uniquely determined. Proposition 4.5 shows then that the spaces Y_0, Y_1, if they exist, are unique up to equivalence of norm.

The above argument quickly modifies to establish existence of X_0, X_1. Indeed we will take $\gamma = \frac{1}{200}$. Then $\rho(\Omega^{[1]}) \le 0.18q^{-1}$ and hence by Lemma 4.3 of [29], $\delta(\Phi^\Omega) \le 0.54q^{-1}$. Thus we can find a Köthe function space W so that if $f \in \mathscr{I} = L\log L \cap \mathscr{J}_\Omega$ then

$$|q\Phi^\Omega(f) - (\Phi_W(f) - \Phi_{W^*}(f))| \le C_0\|f\|,$$

where C_0 is a universal constant (Corollary 6.7). Now on $\mathscr{J}_+$ we consider $\Phi_0 = \Phi_X - \frac{1}{q}(\Phi_W - \Phi_{W^*})$ and $\Phi_1 = \Phi_X + \frac{1}{q}(\Phi_W - \Phi_{W^*})$. If $f, g \in \mathscr{J}_+$ then

$$\begin{aligned}\Delta_{\Phi_0}(f, g) &\ge \Delta_{\Phi_X}(f, g) - \frac{1}{q}\Delta_{\Phi_W}(f, g)\\ &\ge \Delta_{\Phi_X}(f, g) - \frac{1}{q}\Delta_\Lambda(f, g) \ge 0\end{aligned}$$

since X is q-concave (Theorem 5.3). Similarly,

$$\begin{aligned}\Delta_{\Phi_0}(f, g) &\le \Delta_{\Phi_X}(f, g) + \frac{1}{q}\Delta_{\Phi_{W^*}}(f, g)\\ &\le \Delta_{\Phi_X}(f, g) + \left(1 - \frac{1}{p}\right)\Delta_\Lambda(f, g)\\ &\le \Delta_\Lambda(f, g)\end{aligned}$$

since X is p-convex, again utilizing Theorem 5.3. Thus by Theorem 5.2 there is a Köthe function space X_0 so that on $\mathscr{J}_+$ we have $\Phi_0 = \Phi_{X_0}$. A similar calculation shows that $\Phi_1 = \Phi_{X_1}$ for some Köthe function space X_1. Since $\Phi_{X_0} + \Phi_{X_1} = 2\Phi_X$ on $\mathscr{J}_+$ we have $[X_0, X_1]_{1/2} = X$, by Proposition 4.5. Let

Ω_0 be the corresponding induced centralizer. Then $\Phi^{\Omega_0} = \frac{1}{q}(\Phi_W - \Phi_{W^*})$ on $\mathscr{J}_+$. Thus for $f \in \mathscr{J}_+$, $|\Phi^{\Omega}(f) - \Phi^{\Omega_0}(f)| \le C_0 q^{-1}\|f\|$.

Now $\delta(\Phi^{\Omega}) \le 0.54q^{-1}$ and $\delta(\Phi^{\Omega_0}) \le q^{-1}\log 2$. Hence, by breaking up an arbitrary $f \in \mathscr{J}$ into its real and imaginary and then positive and negative parts as in Theorem 6.7, it follows that for all $f \in \mathscr{J}$ we have $|\Phi^{\Omega}(f) - \Phi^{\Omega_0}(f)| \le C_1 q^{-1}\|f\|$ for some universal constant C_1.

Now suppose $x \in X$ and let $y \in X^*$ be such that $xy \in \mathscr{J}$. Then $\Omega^{[1]}(xy)$, $\Omega_0^{[1]}(xy) \in L_1$ and hence both $\Omega(x)y$ and $\Omega_0(x)y$ are integrable. Further, it follows from Theorem 2.6 that $\rho(\Omega_0)$ is bounded by a universal constant C_2. Thus,

$$\begin{aligned}\left|\int (\Omega(x) - \Omega_0(x))y\,d\mu\right| &\le \left|\int \Omega^{[1]}(xy) - \Omega_0^{[1]}(xy)\,d\mu\right| \\ &\quad + 18(\rho(\Omega) + \rho(\Omega_0))\|x\|_X\|y\|_{X^*} \\ &\le |\Phi^{\Omega}(xy) - \Phi^{\Omega_0}(xy)| + C_3\|x\|_X\|y\|_{X^*} \\ &\le C_4\|x\|_X\|y\|_{X^*},\end{aligned}$$

where C_3, C_4 are universal constants. By considering all such y we obtain

$$\|\Omega(x) - \Omega_0(x)\|_X \le C_4\|x\|_X$$

and the proof is complete. □

Remark. In order that X can be renormed to be p-convex and q-concave where $q < \infty$, it is necessary and sufficient that X is super-reflexive. Thus Theorem 7.6 holds for super-reflexive X but not necessarily isometrically.

We recall that if X is a rearrangement-invariant Banach function space on S then a centralizer Ω is called *symmetric* if for some constant C we have $\|\Omega(x) - \Omega(x \circ \sigma)\|_X \le C\|x\|_X$ for all $x \in X$ and all measure-preserving automorphisms σ of S.

Corollary 7.7. *If μ is nonatomic or $S = \mathbf{N}$ with μ counting measure, and the space X in Theorem 7.6 is rearrangement invariant and Ω is a symmetric centralizer then the spaces X_0, X_1 can be chosen to be rearrangement invariant.*

Proof. In the proof of Theorem 7.6 observe that if follows from the equivalence of $q^{-1}\Phi^{\Omega}$ and $2\Phi_W - \Phi_\Lambda$ on $\mathscr{J}_+$ that for some constant C_1 one has $|\Phi_W(f) - \Phi_W(f \circ \sigma)| \le C_1\|f\|$ for every measure-preserving automorphism. From this it is easy to deduce that if $x \in B_W$ then $e^{-C_1} \le \|x \circ \sigma\|_W \le e^{C_1}$ so that W can be renormed to be rearrangement invariant. Continuing with the proof then yields that X_0, X_1 are also rearrangement invariant. □

We now use Corollary 7.7 to prove [29, Theorem 6.10] under the additional hypothesis of super-reflexivity.

Corollary 7.8. *Suppose either μ is nonatomic or $S = \mathbf{N}$ with counting measure. Suppose $1 < p_0 < p_1 < \infty$, and that T is an operator of strong types (p_0, p_0) and (p_1, p_1). Let X be any super-reflexive rearrangement-invariant Banach function space whose Boyd indices satisfy $p_0 < p_X < q_X < p_1$. Then for any symmetric centralizer Ω on X there is a constant C so that $\|[T, \Omega](x)\|_X \le C\|x\|_X$.*

Proof. We may suppose X is p-convex and q-concave where $\frac{1}{p} + \frac{1}{q} = 1$. We may further simplify by noting that it is only necessary by Lemma 7.1 to

consider the case when Ω is real and $\rho(\Omega)$ is sufficiently small. But then there are r.i. spaces X_0, X_1 so that $X = [X_0, X_1]_{1/2}$ and the induced centralizer is equivalent to Ω. Now by interpolation there exists $\varepsilon > 0$ so that if $\theta = \frac{1}{2} \pm \varepsilon$, then $p_0 < p_{X_\theta} < q_{X_\theta} < p_1$. Hence T is bounded on $X_{1/2\pm\varepsilon}$ and Theorem 2.6 will yield the result.

The above theorem works for real centralizers. For the general case it is possible to realize an arbitrary centralizer as a derivation induced by a family of three spaces. We have:

Theorem 7.9. *There is a constant $\gamma > 0$ so that if X is a Köthe function space which is p-convex and q-concave where $\frac{1}{p} + \frac{1}{q} = 1$ and $1 < p \leq 2$ and Ω is a centralizer on X with $\rho(\Omega) \leq \gamma q^{-1}$ then there are three Köthe function spaces $X(j)$: $j = 1, 2, 3$ so that if we consider the family $\mathscr{X} = \{X_w : w \in \mathbf{T}\}$ where $X_{e^{i\theta}} = X(j)$ when $2(j-1)\pi/3 \leq 2j\pi/3$ then $X_0 = X$ and the induced centralizer Ω_0 is equivalent to Ω.*

We shall not prove this in detail; the ideas are similar to those of Theorem 7.6. The key observation is that

$$\Phi^{\Omega_0} = \frac{i(1-\omega)}{\pi}(\omega^2 \Phi_{X(1)} + \omega \Phi_{X(2)} + \Phi_{X(3)})$$

on some strict suitable semi-ideal, where $\omega = e^{2\pi i/3}$. By splitting Φ^Ω into its real and imaginary parts and utilizing the equation

$$\Phi_X = \tfrac{1}{3}(\Phi_{X(1)} + \Phi_{X(2)} + \Phi_{X(3)})$$

one obtains equations to determine $X(j)$ for $j = 1, 2, 3$.

8. Applications to Schatten classes

We now extend our ideas to Schatten ideals (see [1] for a discussion of interpolation in this setting). Suppose $\mathscr{H}$ is a separable Hilbert space and $\mathscr{B}(\mathscr{H})$ is the algebra of all bounded operators on $\mathscr{H}$. Let E be a Köthe sequence space (i.e., a Köthe function space on $\mathbf{N}$) which is symmetric (or rearrangement invariant) then we define the corresponding Schatten ideal $\mathscr{C}_E$ to be the algebra of all operators $A \in \mathscr{B}(\mathscr{H})$ whose singular values $s_n(A)$ satisfy $(s_n) \in E$. We define $\|A\|_E = \|(s_n(A))\|_E$ and then $\mathscr{C}_E$ is a Banach space which is an ideal in $\mathscr{B}(\mathscr{H})$. Further if we regard each $A \in \mathscr{B}(\mathscr{H})$ as an infinite matrix then the spaces $\mathscr{C}_E$ are all admissible spaces regarded as spaces of functions on $\mathbf{N} \times \mathbf{N}$ with the usual counting measure. Thus our approach to complex interpolation derived in §2 applies to this setting.

Following [31] we define a map $\Omega: \mathscr{C}_E \to \mathscr{B}(\mathscr{H})$ to be a *bicentralizer* provided for some constant $\rho = \rho(\Omega)$ we have

$$\|\Omega(VAW) - V\Omega(A)W\|_E \leq \rho\|V\|\,\|A\|_E\|W\| \tag{8.1}$$

for $A \in \mathscr{C}_E$ and $V, W \in \mathscr{B}(\mathscr{H})$. By [31, Proposition 4.1] (where it is proved only for $\mathscr{C}_p$) every bicentralizer is a derivation on $\mathscr{C}_E$ with $\rho(\Omega) \leq 8\rho$. Let us say that a bicentralizer is hermitian if $\Omega(H)$ is hermitian for every hermitian H. The argument of Lemma 7.1 yields

Lemma 8.1. *Let Ω be a bicentralizer on $\mathscr{C}_E$. Then there exist hermitian bicentralizers Ω_1, Ω_2 and a constant C so that if $A \in \mathscr{C}_E$ then $\|\Omega(A) - \Omega_1(A) - i\Omega_2(A)\| \leq C\|A\|_E$.*

Let us regard $\mathscr{C}_E$ as a space of infinite matrices by distinguishing a particular orthonormal basis of $\mathscr{H}$. For any $A = (a_{jk}) \in \mathscr{B}(\mathscr{H})$ we define $\mathscr{D}(A) = \operatorname{diag}(a_{jj})$ and $\mathscr{D}_n(A) = \operatorname{diag}(a_{11}, \dots, a_{nn}, 0, \dots)$.

Lemma 8.2. *If $\mathscr{C}_E$ is B-convex, then there is a constant $C = C(E)$ so that if Ω is a bicentralizer on $\mathscr{C}_E$ then for any $A \in \mathscr{C}_E$ we have $\|[\mathscr{D}, \Omega](A)\|_E \leq C\rho(\Omega)\|A\|_E$.*

Proof. By Lemma 2.1, there is a constant $C_0 = C_0(E)$ so that if $A_1, \dots, A_n \in \mathscr{C}_E$ then

$$\|\Delta_\Omega(A_1, \dots, A_n)\|_E \leq C_0\rho \sum_{k=1}^{n} \|A_k\|_E .$$

Hence if $U_1, \dots, U_n \in \mathscr{B}(\mathscr{H})$ with $\|U_k\| \leq 1$ and if $\mathscr{T}(A) = \frac{1}{n}\sum_{k=1}^{n} U_k A U_k$ then

$$\|[\mathscr{T}, \Omega]\|_E = \sup_{\|A\|_E \leq 1} \|[\mathscr{T}, \Omega](A)\|_E \leq C_1\rho ,$$

where $C_1 = C_1(E)$.

For fixed $m \in \mathbf{N}$ let $U_{\varepsilon_1, \dots, \varepsilon_n} = \operatorname{diag}(\varepsilon_1, \dots, \varepsilon_n, 0, \dots)$ where $\varepsilon_k = \pm 1$. By averaging over all choices of signs we obtain $\|[\mathscr{D}_n, \Omega]\|_E \leq C_1\rho$. Let P_n be the orthogonal projection onto the first n basis vectors. Then

$$\|\Omega(\mathscr{D}_n A) - \mathscr{D}_n\Omega(A)\|_E \leq C_1\rho\|A\|_E .$$

Thus

$$\|\Omega(P_n\mathscr{D}AP_n) - \mathscr{D}_n\Omega(A)\|_E \leq C_1\rho\|A\|_E .$$

Hence

$$\|P_n(\Omega(\mathscr{D}A) - \mathscr{D}\Omega(A))P_n\|_E \leq (C_1 + 1)\rho\|A\|_E$$

for every n and the result follows. □

Theorem 8.3. *Let $\mathscr{C}_E$ be a super-reflexive Schatten ideal. Then for every symmetric centralizer Ω_E on E we can define a bicentralizer Ω on $\mathscr{C}_E$ by*

$$\Omega(A) = U^* \operatorname{diag}(\Omega_E(s_n))V^* , \tag{8.2}$$

where (s_n) are the singular values of A and U, V are norm operators chosen so that $U^*U = VV^* = I$ and $UAV = \operatorname{diag}(s_n)$.

Conversely, if Ω' is a bicentralizer on $\mathscr{C}_E$ then there is a symmetric centralizer Ω_E on E so that if Ω is given by (8.2) then Ω is equivalent to Ω'. Further there is a constant $\gamma = \gamma(E) > 0$ so that if Ω' is hermitian, and $\rho(\Omega') < \gamma$, then there are symmetric Köthe sequence spaces E_0, E_1 so that $[\mathscr{C}_{E_0}, \mathscr{C}_{E_1}]_{1/2} = \mathscr{C}_{\widetilde{E}}$ where $\widetilde{E}$ is equivalent to E (i.e., has an equivalent norm), and the induced derivation is equivalent to Ω'.

Proof. We break the proof up into several steps. First consider the situation when E_0, E_1 are symmetric Köthe sequence spaces and $[E_0, E_1]_{1/2} = E$, with the induced centralizer Ω_E. We claim that $[\mathscr{C}_{E_0}, \mathscr{C}_{E_1}]_{1/2} = \mathscr{C}_E$ and that the induced derivation the bicentralizer Ω given by (8.2). To see this suppose $\|A\|_E = 1$ so that $\|s\|_E = 1$. If $F(z)$ is an extremal for A by considering

$\mathscr{D}(UF(z)V)$ we must have $\|A\|_{1/2} \geq 1$. Conversely if $u \in B_{E_0}$ and $v \in B_{E_1}$ satisfying $s^2 = uv$ then $F(z) = U^* \operatorname{diag}(u_n^{1-z} v_n^z) V^*$ is an extremal for A. Thus by differentiating, we may suppose that the induced derivation is given by $\Omega(A) = U^* \operatorname{diag}(\Omega_E(s)) V^*$, and this must be a bicentralizer by interpolation.

Now if $\mathscr{C}_E$ is super reflexive so that E is super reflexive (and hence can be renormed to be p-convex and q-concave where $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$) then Theorem 7.6 and Lemma 7.1 can be used to prove the first statement of the theorem.

Now suppose Ω' is an arbitrary bicentralizer on $\mathscr{C}_E$. Define Ω_E on E by $\Omega_E(x) = y$ where $\operatorname{diag}(y_n) = \mathscr{D}(\operatorname{diag}(x_n))$. It is fairly easy to check that Ω_E is a symmetric centralizer on E by considering the effect of multiplying $\operatorname{diag}(x_n)$ by $\operatorname{diag}(u_n)$ or replacing it by $S_\pi \operatorname{diag}(x_n) S_{\pi^{-1}}$ where π is a permutation of $\mathbf{N}$ and S_π is the corresponding operator obtained by permuting the basis elements. If $\|A\|_E = 1$ and A has singular values (s_n) then we must have

$$\|\Omega'(A) - U^* \Omega'(\operatorname{diag}(s_n)) V^*\|_E \leq \rho \|A\|_E$$

where $\rho = \rho(\Omega')$. Now

$$\|\Omega'(\operatorname{diag}(s_n)) - \mathscr{D}\Omega'(\operatorname{diag}(s_n))\| \leq C\rho,$$

where C depends only on E. Thus

$$\|\Omega'(A) - U^* \operatorname{diag}(\Omega_E(s_n)) V^*\|_E \leq (C+1)\rho$$

and the second part follows.

For the last statement, observe that if $\rho(\Omega')$ is small enough then so is $\rho(\Omega_E)$ and we can apply Theorem 7.6 and the first part of the proof. □

We remark that the proof of Theorem 8.3 essentially classifies bicentralizers on $\mathscr{C}_E$ as being, up to equivalence, in bijective correspondence with symmetric centralizers on E. The correspondence is given by the relationship $\Omega_0(A) = U^* \operatorname{diag}(\Omega_{0,E}(s_n)) V^*$ established in the proof.

Theorem 8.4. *Suppose $1 \leq p_0 < p < p_1 < \infty$. Suppose $\mathscr{T}$ is a bounded operator on both $\mathscr{C}_{p_0}$ and $\mathscr{C}_{p_1}$. Then for every bicentralizer Ω on $\mathscr{C}_p$ there is a constant C so that for $A \in \mathscr{C}_p$ we have $\|[\mathscr{T}, \Omega](A)\|_p \leq C\|A\|_p$.*

The proof of Theorem 8.4 is the same as that of Corollary 7.8, using Arazy's generalization of the Boyd interpolation theorem [1] and the above Theorem 8.3. Of course, we could state Theorem 8.4 in somewhat more generality.

We conclude by studying the operator versions of indicators maps Φ_X. Let us first notice that if E is a symmetric Köthe sequence space then $l_1 \subset E \subset l_\infty$ so that $\mathscr{I}_E \supset l \log l$. We also recall that the sequence space h_1^{sym} [19, 31] is defined to be the space of all sequences $s \in l_1$ such that

$$\|s\|_h = \sum_{k=1}^{\infty} |s_k| + \sum_{k=1}^{\infty} \frac{|\tilde{s}_1 + \cdots + \tilde{s}_k|}{k} < \infty,$$

where $\tilde{s}$ is any rearrangement of s so that $|\tilde{s}_k|$ is monotone decreasing. (Strictly, we require only that $\tilde{s}$ is a rearrangement of s in the sense that if $\alpha \in \mathbf{C}\backslash\{0\}$ then the sets $\{k : s_k = \alpha\}$ and $\{k : \tilde{s}_k = \alpha\}$ have the same cardinalities; also the quasinorm $\| \ \|_h$ is well defined only up to equivalence since it depends on the choice of the rearrangement.)

Lemma 8.5. *There is a constant C so that if E is a symmetric Köthe sequence space, then for any $s \in h_1^{\mathrm{sym}} \cap l \log l$ we have $|\Phi_E(s)| \le C\|s\|_h$.*

Proof. For any sequence $\beta = (\beta_n)_{n=0}^{\infty}$ of complex numbers let $D(\beta)$ denote the sequence $(\beta_0, \beta_1, \beta_1, \dots)$ where each β_n is repeated 2^n times. Thus $\|D(\beta)\| = \sum 2^n|\beta_n|$. Next let $\beta_n' = \frac{1}{2}\beta_{n-1}$ for $n \ge 1$ and $\beta_0' = 0$. Then it is clear that if $D(\beta) \in l \log l$ then $|\Phi_E(D(\beta)) - \Phi_E(D(\beta'))| \le \frac{4}{e}\|D(\beta)\|$, by Lemma 5.6. Thus if $\alpha_0 = \beta_0$ and $\alpha_n = \beta_n - \frac{1}{2}\beta_{n-1}$ for $n \ge 1$ then $\|\Phi_E(D(\alpha))\| \le \frac{12}{e}\|D(\beta)\|$. Now if $\alpha = (\alpha_n)_{n=0}^{\infty}$ is any finitely nonzero sequence which satisfies $\sum 2^n\alpha_n = 0$ then we can set $\beta_0 = \alpha_0$ and subsequently $\beta_n = \sum_{k=0}^{n} 2^{k-n}\alpha_k$. Thus β is finitely nonzero and we conclude that

$$|\Phi_E(D(\alpha))| \le \frac{12}{e}\sum_{n=0}^{\infty}\left|\sum_{k=1}^{n} 2^k\alpha_k\right|.$$

Now suppose $s = (s_k)_{k=1}^{\infty}$ is any finitely nonzero real sequence arranged in decreasing order of absolute value. Let $\alpha_n = 2^{-n}\sum_{k=2^n}^{2^{n+1}-1} s_k$. Also let $\theta_n = |s_{2^n}|$ for $n \ge 0$. Then we obviously have that $\sum_{n=1}^{\infty} 2^{n-1}\theta_n \le \|s\|$. If $2^n \le r \le 2^{n+1} - 1$ note that

$$\left|\sum_{k=1}^{2^{n+1}-1} s_k\right| \le \left|\sum_{k=1}^{r} s_k\right| + 2^n\theta_n$$

and hence for $n \ge 1$,

$$\left|\sum_{k=0}^{n} 2^k\alpha_k\right| \le \sum_{r=2^n}^{2^{n+1}-1}\frac{1}{r}\left|\sum_{k=1}^{r} s_k\right| + 2^n\theta_n.$$

Summing, including the obvious term when $n = 0$, we obtain

$$|\Phi_E(D(\alpha))| \le 6\|s\|_h.$$

Now consider $t = s - D(\alpha)$. We have $|t_k| \le 2\theta_n$ and $\sum_{k=2^n}^{2^{n+1}-1} t_k = 0$ for $n \ge 1$ and $2^n \le k \le 2^{n+1} - 1$; also $t_1 = 0$. We may thus permute the elements of t in the blocks $(2^n, 2^n + 1, \dots, 2^{n+1} - 1)$ to construct a sequence $\tilde{t}$ with $|\sum_{k=1}^{r}\tilde{t}_k| \le 2\theta_n$ for $n \ge 1$ and $2^n \le r \le 2^{n+1} - 1$. Thus if $u_r = \sum_{k=1}^{r}\tilde{t}_k$ then $\|u\| \le \sum_{n=1}^{\infty} 2^{n+1}\theta_n \le 4\|s\|$. Since $\tilde{t}_k = u_k - u_{k-1}$ and $u_0 = 0$ we have $\Phi_E(t) \le \frac{32}{e}\|s\|$. Combining we have that

$$\Phi_E(s) \le 22\|s\|_h + \tfrac{4}{e}(\|t\| + \|D(\alpha)\|) \le C_0\|s\|_h$$

for a suitable absolute constant C_0.

Next if s is real but not necessarily finitely nonzero, consider for each N the sequence $s^{(N)}$ defined by $s_k^{(N)} = s_k$ for $k \le N$ and $s_k = 0$ otherwise. Then, denoting the basis vectors by e_k,

$$\left|\Phi_E\left(s^{(N)} - \sum_{k=1}^{N} s_k e_{N+1}\right)\right| \le C_0\|s\|_h$$

by the above reasoning. If $s \in h_1^{\mathrm{sym}}$ then $\sum_{k=1}^{N} s_k \to 0$ as $N \to \infty$ and so we conclude $\limsup_{N\to\infty}|\Phi_E(s^{(N)})| \le C_1\|s\|_h$ for some absolute constant C_1. Let $s = s_+ - s_-$ be the splitting of s into positive and negative parts.

Then $\Phi_E(s_+) = \lim_{N\to\infty} \Phi_E(s_+^{(N)})$ and $\Phi_E(s_-) = \lim_{N\to\infty} \Phi_E(s_-^{(N)})$. It quickly follows that $|\Phi_E(s)| \le C_3\|s\|_h$ for some absolute constant C_3.

Finally, if s is a complex sequence the proof is completed by taking real and imaginary parts. □

Let $\mathscr{C}_{l\log l}$ denote the ideal of operators in $\mathscr{C}_1$ whose singular value $s = (s_n) \in l\log l$. It is easy to show that if $T \in \mathscr{C}_{l\log l}$ then the eigenvalues of T repeated according to multiplicity belong to $l\log l$. (Here we adjoin zeroes if the set of eigenvalues is finite.) To see this observe that if (λ_n) are the eigenvalues of T, arranged in decreasing order of absolute value then $|\lambda_1\cdots\lambda_n| < s_1\cdots s_n$ for every n [21]. Hence since the function xe^x is concave for $x < -2$, if we suppose, without loss of generality that $s_1 \le e^{-2}$ then for every N,

$$\sum_{n=1}^{N} |\lambda_n| \log|\lambda_n| \ge \sum_{n=1}^{N} s_n \log s_n .$$

Now if E is a symmetric Köthe sequence space for $T \in \mathscr{C}_{l\log l}$ we define $\Phi_E(T) = \Phi_E((\lambda_k)_{k=1}^{\infty})$.

Theorem 8.6. *There is a universal constant C so that, whenever E is a symmetric Köthe sequence space, then Φ_E satisfies*

$$|\Delta_{\Phi_E}(T_1, T_2)| \le C(\|T_1\|_1 + \|T_2\|_1)$$

whenever $T_1, T_2 \in \mathscr{C}_{l\log l}$.

Proof. Let $\lambda^{(1)}$ and $\lambda^{(2)}$ denote the eigenvalues of T_1, T_2 respectively both arranged in decreasing order of absolute value. Let ν denote the similar sequence for $T_1 + T_2$. Then, arguing as in [19, Lemma 5.2] it may be shown that the operator $V = T_1 \oplus T_2 \oplus -(T_1+T_2)$ on $\mathscr{H}\oplus\mathscr{H}\oplus\mathscr{H}$ can be written as a commutator $V = [A, B]$ where $\|A\| \le 2$ and $\|B\|_1 \le \|T_1\|_1 + \|T_2\|_1$. Hence its eigenvalues $\alpha = \lambda^{(1)} \oplus \lambda^{(2)} \oplus (-\nu)$ are in h_1^{sym}. Further for some universal constant C_0 we have $\|\alpha\|_h \le C_0(\|T_1\|_1 + \|T_2\|_1)$. Thus $|\Phi_E(\alpha)| \le C_1(\|T_1\|_1 + \|T_2\|_1)$ for some universal C_1. It then follows quickly that $|\Phi_E(\nu) - \Phi_E(\lambda^{(1)}) - \Phi_E(\lambda^{(2)})| \le C_2(\|T_1\|_1 + \|T_2\|_1)$ from which the theorem is immediate. □

9. Appendix: an application to harmonic analysis

In this section we consider some applications to harmonic analysis. We shall suppose that S is a compact metric space and that μ is a probability measure on S. We consider Köthe function spaces modelled on $S\times\mathbf{T}$ with the product measure $\tilde{\mu} = \mu\times\lambda$ where $d\lambda = (2\pi)^{-1}\,d\theta$ is normalized Haar measure on the circle.

We will be specifically interested in the (vector-valued) Riesz transform R. If $x \in L_0(S\times\mathbf{T})$ satisfies $x_s \in L_1(\mathbf{T})$ μ-a.e. where $x_s(e^{it}) = x(s, e^{it})$ then we define

$$Rx(s, e^{it}) = \lim_{r\to 1} \int_0^{2\pi} \frac{x(s, e^{i\tau})}{1 - re^{i(t-\tau)}} \frac{d\tau}{2\pi}$$

where the limit exists $\tilde{\mu}$-a.e.

Suppose X is a Köthe function space containing L_∞. We shall say that R is bounded on X if there exists a constant C so that $\|Rx\|_X \le C\|x\|_X$ for all

$x \in L_\infty$. If R is bounded on X then so are $\overline{R}$ and R_0 where $\overline{R}(f) = \overline{R(\overline{f})}$ and $R_0(x) = e^{it}R(e^{it}f)$.

If $h \in L_1(S)$ satisfies $h > 0$ a.e. and $\int h\,d\mu = 1$ then we say that X is *h-acceptable* if X contains L_∞ and for some constant C we have

$$\int_{S\times\mathbf{T}} h(s)|x(s, e^{it})|d\tilde{\mu} \le C\|x\|_X$$

for all $x \in X$. If X is 1_S-acceptable we simply say that X is *acceptable.* If X is h-acceptable then $R\colon X \to L_0$ is well defined.

In this section we adopt the convention that C is a constant independent of x, y, ξ, etc. but possibly depending on $X, X_0, \ldots$ which may vary from line to line.

Lemma 9.1. *Suppose X is a Köthe function space for which R is bounded. Then there exists h so that X is h-acceptable and $\|Rx\|_X \le C\|x\|_X$ for all $x \in X$.*

Proof. Let $Jx = Rx - R_0x$. For $x \in L_\infty$

$$Jx(s, e^{it}) = \int_0^{2\pi} x(s, e^{i\theta})\frac{d\theta}{2\pi}.$$

Thus $\|Jx\|_X \le C\|x\|_X$ for $x \in L_\infty$. Let X_S be the restriction of X to S, so that if $x \in L_0(S)$ then $x \in X_S$ if and only if $\tilde{x} \in X$ where $\tilde{x}(s, e^{it}) = x(s)$ and then $\|x\|_{X_S} = \|\tilde{x}\|_X$. Pick any $h \in X_S^*$ with $h > 0$ a.e. and $\int h\,d\mu = 1$. Then

$$\int h(s)|x(s, e^{it})|\,d\tilde{\mu} \le C\|x\|_X$$

for $x \in L_\infty$. This clearly extends to X and so X is h-acceptable. It follows quickly that $\|Rx\|_X \le C\|x\|_X$ for all $x \in X$, since now R is well defined on X. □

If X is acceptable (i.e., $L_\infty \subset X \subset L_1$) then X^* is also acceptable. We shall say that X is *weakly acceptable* if $L_\infty \subset X \subset L_p$ for some $p > 0$; thus X is weakly acceptable if X^p is acceptable for some $p > 0$. In this case we at least have $X^* \subset L_1$ so it follows that $L\log L \subset \mathscr{I}_X \cap \mathscr{I}_{X^*}$. It will be convenient to restrict attention largely to weakly acceptable spaces X so that the indicators Φ_X have common domain $L\log L$. We shall also need the corresponding centralizer on L_1, $\Omega_X^{[1]}(f) = f\log|x|$ where $|f| = xx^*$ is the Lozanovskii factorization of $|f|$ for X.

Lemma 9.2. *Suppose X is a weakly acceptable Köthe function space on $S \times \mathbf{T}$. Then there is a constant C so that if $f \in L\log L$, $\|\Omega^{[1]}(f)\| \le C\|f\|_{L\log L}$.*

Proof. The assumptions imply the existence of C so that

$$-C\|f\| \le \Phi_X(f) \le C\|f\| + \Lambda(f)$$

for $f \in (L\log L)_+$. Thus $|\Phi_X(f)| \le C\|f\|_{L\log L}$ for all $f \in L\log L$ by the quasiadditivity of Φ_X. Hence if $\|\eta\|_\infty \le 1$ and A is any Borel set over which $\Omega_X^{[1]}(f)$ is integrable, we have

$$\|\Omega^{[1]}(f)\eta 1_A - \Omega_X^{[1]}(f\eta 1_A)\| \le C\|f\|$$

and so

$$\left|\int_A \Omega_X^{[1]}(f)\eta\,d\tilde{\mu} - \Phi_X(f\eta 1_A)\right| \le C\|f\|.$$

Hence $\|\Omega^{[1]}(f)1_A\| \le C\|f\|_{L\log L}$ and the lemma follows easily. □

We denote by H^1 the subspace of $L_1(S \times \mathbf{T})$ of all f such that

$$\int_0^{2\pi} f(s, e^{it})e^{int}\,dt = 0, \qquad \mu\text{-a.e.}$$

for $n > 0$. H_0^1 is the subspace of all $f \in H^1$ such that

$$\int_0^{2\pi} f(s, e^{it})\,dt = 0, \qquad \mu\text{-a.e.}$$

We shall need the concept of a *generalized outer function.* If $x \in L_1$ is such that for μ-a.e. s either $x_s = 0$ λ-a.e. or $\log|x_s| \in L_1(\mathbf{T})$ we define the corresponding generalized outer function $\xi \in H^1$ by

$$\xi(s, e^{it}) = \lim_{r\to 1} \exp\left(\int_0^{2\pi} \frac{e^{i\tau} + re^{it}}{e^{i\tau} - re^{it}} \log|x(s, e^{i\tau})| \frac{d\tau}{2\pi}\right)$$

when $\log|x_s| \in L_1$ and $\xi(s, e^{it}) = 0$ otherwise. Then ξ is defined a.e. and $|\xi| = |x|$ a.e.

Proposition 9.3. *Suppose X_0, X_1 are acceptable Köthe function spaces and that $X = [X_0, X_1]_\theta$ where $0 < \theta < 1$. Suppose that R is bounded on X and that $X \subset L\log L$. Let Ω: $X \to L_0$ be the induced centralizer. Then $\Omega(X) \subset L_1$.*

If, further, R is bounded on X, and X is separable, then

$$\|[R, \Omega]\|_X = \sup_{\|x\|_X \le 1} \|R\Omega(x) - \Omega R(x)\|_X < \infty$$

if and only if there is a constant C so that

$$|\Phi(f)| \le C\|f\|$$

for $f \in H_0^1 \cap L\log L$, where $\Phi = \Phi^\Omega = \Phi_{X_1} - \Phi_{X_0}$.

Proof. We first observe that if $f \in X$ then $\Omega(f) - \Omega^{[1]}(f) \in L_1$, where $\Omega^{[1]}$ is the corresponding centralizer on L_1, by Theorem 5.1 of [29] since $1 \in X^*$. However $\Omega^{[1]} = \Omega^{[1]}_{X_1} - \Omega^{[1]}_{X_0}$ and so Lemma 9.2 gives the first part of our conclusion.

We shall need the following observation. If $\xi \in L_\infty$ then by considering the constant function $F(z) = \xi$ for $0 \le \Re z \le 1$ we see that $(\xi, 0) \in d_\Omega X$ and hence $\Omega(\xi) \in X$.

Now assume $\|[R, \Omega]\|_X < \infty$. Suppose $f \in H_0^1 \cap L\log L$ and that $\|f\| = 1$. Let $|f| = x_j y_j$ be the Lozanovskii factorization of $|f|$ for X_j for $j = 0, 1$. Let $x = x_0^{1-\theta}x_1^\theta$ and $y = y_0^{1-\theta}y_1^\theta$; then $\|x\|_X = \|y\|_{X^*} = 1$. Since X is acceptable, $\log_+ x$, $\log_+ x^* \in L_1$. Let S_0 be the set of $s \in S$ such that $f(s, e^{it})$ is nonzero on a set of positive λ-measure. For μ-a.e. $s \in S_0$ we have $f_s \in H^1(\mathbf{T})$ so that

$$\int_0^{2\pi} |\log|f_s(e^{it})|| \frac{dt}{2\pi} < \infty$$

and hence for almost every such s, $\log x_s$ is integrable. For $s \notin S_0$, $x_s = 0$, a.e.

Now let ξ be the generalized outer function corresponding to x. Clearly $\xi \in B_X \cap H^1$ and $f\xi^{-1} \in B_{X^*} \cap H_0^1$ (here we define $f\xi^{-1}$ to be zero if $s \notin S_0$.) It follows from the uniqueness principle of Corollary 4.9 that

$$\Omega(\xi) = \xi(\log x_1 - \log x_0).$$

As $f \in L\log L$, we have $\Omega^{[1]}(f) \in L_1$ and hence by Theorem 5.1 of [29] we have $f\xi^{-1}\Omega(\xi) \in L_1$. Since $R\xi = \xi$ we have an estimate

$$\|R\Omega(\xi) - \Omega(\xi)\|_X \le C.$$

Thus also $f\xi^{-1}R\Omega(\xi) \in L_1$. But $f\xi^{-1} \in H_0^1$ and $(R\Omega(\xi))_s$ is μ-a.e. in $H_{1/2}$. Thus $f\xi^{-1}R\Omega(\xi) \in H_0^1$, and in particular $\int f\xi^{-1}R\Omega(\xi)\,d\tilde{\mu} = 0$ from which it follows that $\left|\int f\xi^{-1}\Omega(\xi)\,d\tilde{\mu}\right| \le C$, i.e.

$$\left|\int f(\log x_1 - \log x_0)\,d\tilde{\mu}\right| = |\Phi(f)| \le C.$$

Conversely, suppose we have the estimate $|\Phi(f)| \le C\|f\|$ for $f \in L\log L \cap H_0^1$. Suppose $\xi, \eta \in L_\infty$. Then both $\eta\Omega(R\xi)$ and $\eta R\Omega(\xi)$ are in L_1, and

$$\left|\int \eta\Omega(R\xi)d\tilde{\mu} - \Phi(\eta R\xi)\right| \le C\|\xi\|_X\|\eta\|_{X^*},$$
$$\left|\int R^*\eta\Omega(\xi)d\tilde{\mu} - \Phi(\xi R^*\eta)\right| \le C\|\xi\|_X\|\eta\|_{X^*},$$

by Theorem 5.1 of [29]. Now observe that $\eta R\xi - \xi R^*\eta = R_0\eta.R\xi - \overline{R}\eta.\overline{R}_0\xi$. Since $R_0\eta.R\xi \in L\log L$ we have

$$|\Phi(R_0\eta.R\xi)| \le C\|R_0\eta.R\xi\| \le C\|\xi\|_X\|\eta\|_{X^*}.$$

Since Φ is real, we also have $|\Phi(\overline{R}\eta.\overline{R}_0\xi)| \le C\|\xi\|_X\|\eta\|_{X^*}$.

Now using the quasi-addivity of Φ this implies

$$|\Phi(\eta R\xi - \xi R^*\eta)| \le C\|\xi\|_X\eta\|_{X^*}.$$

Hence

$$\left|\int(\eta\Omega(R\xi) - R^*\eta\Omega(\xi))\,d\tilde{\mu}\right| \le C\|\xi\|_X\|\eta\|_{X^*}$$

and thus $\|[R, \Omega]\xi\|_X \le C\|\xi\|_X$ for bounded ξ.

This inequality can now be extended to the whole of X, when X is separable using the argument of [29, p. 82]. Notice first that if $\xi \in X$ then $R\Omega(\xi)$ is well defined. If $x \in X$ we may write $x = \sum_{n=1}^\infty \xi_n$ where $\xi_n \in L_\infty$, and $\sum_{n=1}^\infty \|\xi_n\|_X \le 2\|x\|_X$. Let $v = \sum|\xi_n|$ and $w = \sum|R\xi_n|$. If $x_n = \sum_{j=1}^n \xi_j$, then

$$\|\Omega(x_n) - x_n v^{-1}\Omega(v)\|_X \le C\|x\|_X,$$
$$\|R\Omega(x_n) - R(x_n v^{-1}\Omega(v))\|_X \le C\|x\|_X,$$
$$\|\Omega(Rx_n) - (Rx_n)w^{-1}\Omega(w)\|_X \le C\|x\|_X.$$

Combining these statements we have

$$\|R(x_n v^{-1}\Omega(v)) - (Rx_n)w^{-1}\Omega(w)\|_X \le C\|x\|_X.$$

As $n \to \infty$ we have $x_n \to x$ and $Rx_n \to Rx$ in both X and L_0. By Lemma 4.3 of [29],

$$\|R(xv^{-1}\Omega(v)) - (Rx)w^{-1}\Omega(w)\|_X \le C\|x\|_x$$

which implies

$$\|R\Omega(x) - \Omega(Rx)\|_X \le C\|x\|_X . \quad \square$$

We now come to the crucial calculation in our argument which essentially allows us to integrate a differential estimate for the Riesz transform.

Proposition 9.4. *Given $0 < \theta_0 < 1$ and $1 \le M < \infty$ there is a constant $\delta = \delta(\theta_0, M) > 0$ so that whenever X_0, X_1 are weakly acceptable Köthe function spaces on $S \times \mathbf{T}$ so that R is bounded on both X_0 and X_1, and such that if $X_\theta = [X_0, X_1]_\theta$ then $\|R\|_{X_{\theta_0}} \le M$, and $|\Phi(f)| \le M\|f\|$, for $f \in H_0^1 \cap L\log L$, (where $\Phi = \Phi_{X_1} - \Phi_{X_0}$) then for $|\theta - \theta_0| \le \delta$ we have $\|R\|_{X_\theta} \le 2M$.*

Remark. It is important to observe that here δ is independent of the original spaces X_0, X_1. In the argument that follows we shall use the letter K to denote a constant depending only on θ_0 and M but not on the spaces X_0, X_1.

Proof. Denote by $\Omega_\theta\colon X_\theta \to L_0$ the centralizer induced on X_θ, $0 < \theta < 1$, by the scale $[X_0, X_1]$. Similarly let Ω_θ^* be the centralizer on X_θ^* induced by $[X_0^*, X_1^*]$. We first observe that it is possible to choose $\delta_0 = \delta_0(\theta_0, M) > 0$ and $\gamma = \gamma(\theta_0, M) < \infty$ (both independent of X_0, X_1) so that if $|\theta - \theta_0| \le \delta_0$ then $\rho(\Omega_\theta)$, $\rho(\Omega_\theta^*) \le \gamma$ and we have the inequalities

$$\|x\|_{X_\theta} + \|y - \Omega_\theta\|_{X_\theta} \le \gamma\|(x, y)\|_{dX_\theta},$$
$$\|x^*\|_{X_\theta^*} + \|y^* - \Omega_\theta^* x^*\|_{X_\theta^*} \le \gamma\|(x^*, y^*)\|_{dX_\theta^*}.$$

Let $N(\theta)$, $0 < \theta \le 1$, denote the norm of R on X_θ. By our assumptions $N(\theta)$ is a bounded function and further $\log N(\theta)$ is a convex function of $\log\theta$. It follows that $N(\theta)$ is absolutely continuous on closed subintervals of $(0, 1)$. Since $N(\theta) \ge 1$ for all θ the function $J(\theta) = N(\theta)^{-1}$ is similarly absolutely continuous on closed subintervals of $(0, 1)$.

Now suppose that $\xi_0, \xi_1, \eta_0, \eta_1$ are bounded nonnegative functions with $\|\xi_0\|_{X_0} = \|\xi_1\|_{X_1} = \|\eta_0\|_{X_0^*} = \|\eta_1\|X_1^* = 1$. Suppose also $u, v \in L_\infty$ with $\|u\|_\infty = \|v\|_\infty \le 1$. Consider the function $F(z) = u\xi_0^{1-z}\xi_1^z$ defined in the strip $\mathscr{S} = \{z\colon 0 < \Re z < 1\}$. Then F is analytic into L_∞ (this can be checked by computing local Taylor series). Further if $0 < \theta < 1$ then $\|(F(\theta), F'(\theta))\|_{dX_\theta} \le 1$ and so if $|\theta - \theta_0| \le \delta_0$ $\|F'(\theta) - \Omega_\theta(F(\theta))\|_{X_\theta} \le \gamma$.

Similarly if we set $G(z) = v\eta_0^{1-z}\eta_1^z$, then for $|\theta - \theta_0| \le \delta_0$,

$$\|G'(\theta) - \Omega_\theta^*(G(\theta))\|_{X_\theta^*} \le \gamma .$$

Now the map $z \mapsto RF(z)$ is analytic into L_2 and so we may define a scalar-valued analytic function h on $\mathscr{S}$ by $h(z) = \int RF(z)G(z)\,d\tilde{\mu}$. Then

$$h'(\theta) = \int R^*G(\theta)F'(\theta)\,d\tilde{\mu} + \int RF(\theta)G'(\theta)\,d\tilde{\mu}.$$

Estimating the first term we have

$$\left|\int R^*G(\theta)F'(\theta)\,d\tilde{\mu} - \int R^*G(\theta)\Omega_\theta(F(\theta))\,d\tilde{\mu}\right|$$
$$\le \gamma\|R^*G(\theta)\|_{X_\theta^*} \le \gamma N(\theta)\|G(\theta)\|_{X_\theta^*} \le \gamma N(\theta).$$

We also have, using Theorem 5.1 of [29],

$$\begin{aligned}&\|R^*G(\theta)\Omega_\theta(F(\theta)) - \Omega^{[1]}(R^*G(\theta).F(\theta))\|\\ &\quad\le 36\gamma\|R^*G(\theta)\|^*_{X_\theta}\|F(\theta)\|_{X_\theta} < KN(\theta).\end{aligned}$$

Here $\Omega^{[1]}$ is the induced centralizer on L_1 which is independent of θ, and $\Omega^{[1]} = \Omega^{[1]}_{X_1} - \Omega^{[1]}_{X_0}$. For $f \in L\log L$ we have by Lemma 8.2 $\int \Omega^{[1]}(f)\,d\tilde{\mu} = \Phi(f)$. Thus

$$\left|\int R^*G(\theta)\Omega_\theta(F(\theta))\,d\tilde{\mu} - \Phi(R^*G(\theta).F(\theta))\right| \le KN(\theta).$$

Combining the estimates we have

$$\left|\int R^*G(\theta).F'(\theta)d\tilde{\mu} - \Phi(R^*G(\theta).F(\theta))\right| \le KN(\theta).$$

If we let $\Phi^* = \Phi^*_{X_1} - \Phi^*_{X_0}$ then a precisely similar calculation shows that

$$\left|\int RF(\theta).G'(\theta)d\tilde{\mu} - \Phi^*(RF(\theta).G(\theta))\right| \le KN(\theta).$$

However $\Phi^* = -\Phi$ and so by the quasi-additivity of Φ

$$|h'(\theta) - \Phi(R^*G(\theta).F(\theta) - RF(\theta).G(\theta))| \le KN(\theta).$$

Now,

$$R^*G(\theta).F(\theta) - RF(\theta).G(\theta) = \overline{R}G(\theta).\overline{R}_0F(\theta) - RF(\theta).R_0G(\theta)$$

and so the estimate on Φ and the fact that Φ is real together give, as in the preceding proposition,

$$|\Phi(R^*G(\theta).F(\theta) - RF(\theta).G(\theta))| \le KN(\theta)^2$$

and so $|h'(\theta)| \le KN(\theta)^2$ for $|\theta - \theta_0| \le \delta_0$. Thus if $\theta_0 - \delta_0 \le \theta_1 < \theta_2 \le \theta_0 + \delta_0$ we have

$$|h(\theta_2) - h(\theta_1)| \le K\int_{\theta_1}^{\theta_2} N(t)^2\,dt.$$

Now $|h(\theta) \le N(\theta)$. Thus, it follows by taking suprema over all possible choices of ξ_0, ξ_1, η_0, η_1, u, v that

$$|N(\theta_2) - N(\theta_1)| \le K\int_{\theta_1}^{\theta_2} N(t)^2\,dt$$

and at points of differentiability we obtain $|N'(\theta)| \le KN(\theta)^2$. Thus $|J'(\theta)| \le K$ at such points from which we have the estimate $J(\theta) \ge M^{-1} - K|\theta - \theta_0|$ for $|\theta - \theta_0| \le \delta_0$. The result follows. □

Lemma 9.5. *Suppose $p > 1$ and $1 \le M < \infty$. Then there is a constant $\delta = \delta(M, p)$ with $0 < \delta < 1 - \frac{1}{p}$ so that whenever X and Y are weakly acceptable Köthe function spaces, X is p-convex and q-concave (where $\frac{1}{p} + \frac{1}{q} = 1$), $\|R\|_X \le M$, and $|\Phi_Y(f)| \le M\|f\|$ for $f \in H^1_0 \cap L\log L$, then $\|R\|_{XY^\alpha} \le 2M$ for $0 \le \alpha \le \delta$.*

Proof. We start by remarking that XY^α is a Köthe function space for $0 \le \alpha \le \frac{1}{q}$.

Let Σ_N be an increasing sequence of finite subalgebras of the Borel sets of S whose union is μ-dense. If Σ_N is generated by atoms $A_1, \dots, A_m$ we define E_N be the conditional expectation operator of $L_1(S \times \mathbf{T})$ onto $L_1(\Sigma'_N)$ where Σ'_N is generated by $A_1 \times \mathbf{T}, \dots, A_m \times \mathbf{T}$. Let T_N be the doubly stochastic operator on $L_1(S \times \mathbf{T})$ defined by

$$T_N f(s, e^{it}) = \int_0^{2\pi} k_N(t-\tau) E_N f(s, e^{it}) \frac{d\tau}{2\pi},$$

where k_N is the Fejer kernel

$$k_N(t) = \frac{\sin^2 (N+1)t/2}{(N+1)\sin^2 t/2}.$$

Now, by Theorem 5.7, there is a Köthe function space Y_N so that $\Phi_{Y_N}(f) = \Phi_Y(T_N f)$ for $f \in (L \log L)_+$. We also have that if $f \in L \log L \cap H_0^1$ then, using Corollary 5.9,

$$\begin{aligned} |\Phi_{Y_N}(f)| &\le |\Phi_Y(T_N f)| + \frac{24}{e}\|f\| \\ &\le \left(\frac{qM}{2} + \frac{24}{e}\right)\|f\| = K\|f\|, \end{aligned}$$

where K is a constant depending only on M and p, and independent of N. Furthermore T_N has finite-dimensional range in L_∞ so that for each N there is a constant B_N so that $|\Phi_{Y_N}(f)| \le B_N \|f\|$ for all $f \in L \log L$.

From the convexity and concavity conditions on X it follows that we may define Köthe function spaces Z_0^N, Z_1^N by the equations $\Phi_{Z_1^N} = \Phi_X + \frac{1}{q}\Phi_{Y_N}$ and $\Phi_{Z_0^N} = \Phi_X - \frac{1}{q}\Phi_{Y_N}$. Then each Z_0^N, Z_1^N is simply a renorming of X so that R is bounded on both spaces. Furthermore if $f \in L \log L \cap H_0^1$,

$$|\Phi_{Z_1^N}(f) - \Phi_{Z_0^N}(f)| = \frac{2}{q}|\Phi_{Y_N}(f)| \le K\|f\|,$$

where K again depends only on M and p. We are thus in a position to apply Lemma 8.4. There exists $0 < \delta_0 < \frac{1}{2}$ depending only on M and p, so that if $|\theta - \frac{1}{2}| \le \delta_0$ then $\|R\|_{Z_\theta} \le 2M$, where $Z_\theta = [Z_0, Z_1]_\theta$.

Now suppose $\delta = \frac{2}{q}\delta_0$ and that $0 \le \alpha \le \delta$. To prove the lemma, it suffices to consider $\xi \in XY^\alpha \cap L_\infty$ of the form $\xi = u|x||y|^\alpha$ where $u, x, y \in L_\infty$, $\|u\|_\infty$, $\|x\|_X$, $\|y\|_X \le 1$, and we additionally suppose that $\log|y| \in L_\infty$.

Let $y_N = \exp(T_N \log|y|)$. For any $f \in (L \log L)_+$ we have

$$\int f \log y_N \, d\tilde{\mu} = \int T_N f \log|y| \, d\tilde{\mu} \le \Phi_Y(T_N f) = \Phi_{Y_N}(f)$$

so that $y_N \in B_{Y_N}$. If we let $\xi_N = u|x|y_N^\alpha$ then $\|\xi_N\|_{Z_\theta^N} \le 1$, where $\theta - \frac{1}{2} = \frac{q}{2}\alpha \le \delta_0$. Hence $\|R\xi_N\|_{Z_\theta^N} \le 2M$.

Now, as $N \to \infty$ we have $\|R\xi_N - R\xi\|_2 \to 0$. Let A be the measurable subset of $S \times \mathbf{T}$ where $\log|R\xi| > -\infty$. For any $\eta > 0$ we may find a subset A_η of A with $\tilde{\mu}(A \backslash A_\eta) < \eta$ and a subsequence such that $\log|R\xi_{N_i}|$ converges

uniformly on A_η to $\log|R\eta|$. Now suppose $f \in (L_\infty)_+$.

$$\begin{aligned}\int_{A_\eta} f \log|R\eta|\, d\tilde{\mu} &= \lim_{l\to\infty} \int_{A_\eta} f \log|R\xi_{N_l}|\, d\tilde{\mu} \\ &\le \limsup_{l\to\infty} (\log(2M) + \Phi_{Z_\theta^{N_l}}(f 1_{A_\eta})) \\ &\le \log(2M) + \limsup_{l\to\infty} (\Phi_X(f 1_{A_\eta}) + \alpha \Phi_Y(T_{N_l}(f 1_{A_\eta})) \\ &= \log(2M) + \Phi_X(f 1_{A_\eta}) + \alpha \Phi_Y(f 1_{A_\eta}),\end{aligned}$$

where the last step uses the continuity of Φ_Y on order intervals. Using this same continuity we obtain

$$\int_A f \log|R\xi|\, d\tilde{\mu} \le \log(2M) + \Phi_X(f 1_A) + \alpha \Phi_Y(f 1_A).$$

Now it follows that we have

$$\int f \log|R\xi|\, d\tilde{\mu} \le \log(2M) + \Phi_X(f) + \alpha \Phi_Y(f),$$

since if $f - f 1_A \neq 0$ then the integral is $-\infty$. Hence $\|R\xi\|_{XY^\alpha} \le 2M$ as required. □

Lemma 9.6. *Suppose X is an acceptable Köthe function space and that R is bounded on X. Then there is a constant M so that $|\Phi_X(f)| \le M\|f\|$ for $f \in H_0^1 \cap L\log L$.*

Proof. Clearly R is bounded on both X and X^*. By interpolation we obtain that $[R, \Omega]$ is bounded on $L_2 = [X, X^*]_{1/2}$ where Ω is the induced centralizer. By Proposition 8.3 there exists B so that

$$|\Phi_X(f) - \Phi_{X^*}(f)| = |2\Phi_X(f) - \Lambda(f)| \le B\|f\|$$

for $f \in H_0^1 \cap L\log L$. Applying the same reasoning to $X = L_p$ where $1 < p < 2$ shows that $|\Lambda(f)| \le B'\|f\|$ for $f \in H_0^1 \cap L\log L$. The lemma follows. □

We now state our main theorem of the section.

Theorem 9.7. *Let X_0, X_1 be acceptable Köthe function spaces on $S \times \mathbf{T}$. Suppose $0 < \theta_0 < 1$ and that $X = [X_0, X_1]_{\theta_0}$ is super reflexive and satisfies $L_r \subset X \subset L_s$ for some $1 < s < r < \infty$. Suppose also that R is bounded on X. Let Ω be the induced centralizer on X. Then the following conditions are equivalent.*

(1) $\|[R, \Omega]\|_X < \infty$.
(2) *There exists $\delta > 0$ so that R is bounded on X_θ for $|\theta - \theta_0| \le \delta$.*
(3) *For some constant M we have $|\Phi_{X_j}(f)| \le M\|f\|$ for $j = 0, 1$ whenever $f \in H_0^1 \cap L\log L$.*

Proof. (1) ⇒ (3). This follows by combining Proposition 9.3 which yields an estimate for $\Phi_{X_1} - \Phi_{X_0}$ and Lemma 9.6 which yields an estimate for $\theta_0 \Phi_{X_1} + (1-\theta_0)\Phi_{X_0}$.

(3) ⇒ (2). Let Y be a renorming of X which is p-convex and q-concave where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Assume $\|R\|_Y = M$. Then (by Lemma 9.5) there exists $\delta_0 = \delta_0(r, p, M) > 0$ so that if $0 \le \alpha \le \delta_0$ then YX_1^α is acceptable, $\|R\|_{YX_1^\alpha} \le 2M$, and YX_1^α is both $2q$-concave and p'-convex where $\frac{1}{p'} + \frac{1}{2q} = 1$.

Using Lemma 9.5 on $(YX_1^\alpha)^*$ there exists $\delta_1 = \delta_1(r, p, M) > 0$ so that if $0 \le \beta \le \delta_1$ then $\|R\|_Z \le 4M$ where $Z = (YX_1^\alpha)^*(X_0)^\beta$. In particular if $\alpha \le \delta = \min(\delta_0, \delta_1)$ then R is bounded on $Z_\alpha = (YX_1^\alpha)^*(X_0)^\alpha$. Now

$$\begin{aligned}\Phi_{Z_\alpha} &= \Phi^*_{(YX_1^\alpha)} + \alpha\Phi_{X_0} = \Lambda - (\Phi_Y + \alpha\Phi_{X_1}) + \alpha\Phi_{X_0}\\ &= \Lambda - (\Phi_Y + \alpha(\Phi_{X_1} - \Phi_{X_0}))\end{aligned}$$

and hence Z_α^* is a renorming of $X_{\theta_0+\alpha}$ whence R is bounded on this space for $0 \le \alpha \le \delta$. A similar argument gives the case $-\delta \le \alpha \le 0$.

(2) $\Rightarrow$ (1). This is simply Theorem 2.7. □

Let us now specialize to the case when S reduces to one point so that $S \times \mathbf{T}$ is simply $\mathbf{T}$. In this case R cannot be bounded unless X is acceptable. Then it follows from Lemma 9.5 that XL_1^α is also acceptable for some $\alpha > 0$ so that $X \subset L_s$ for some $s > 1$. By duality $X^* \supset L_r$ for some finite r. It then follows that in the hypotheses of Theorem 9.7 we need only assume X_0, X_1 weakly acceptable, since then for some $\alpha < \theta_0 < \beta$ we will have X_α, X_β acceptable. Thus we restate our result, adding one more equivalence.

Theorem 9.8. *Let X_0, X_1 be weakly acceptable Köthe function spaces on $\mathbf{T}$. Suppose $0 < \theta_0 < 1$ and that $X = [X_0, X_1]_{\theta_0}$ is super reflexive and R is bounded on X. Then the following are equivalent.*

(1) $\|[R, \Omega]\|_X < \infty$.
(2) *There exists $\delta > 0$ so that R is bounded on X_θ for $|\theta - \theta_0| < \delta$.*
(3) *For some constant C we have $|\Phi_{X_j}(f)| \le C\|f\|$ for $f \in H_0^1 \cap L\log L$ and $j = 0, 1$.*
(4) *The set $\{\theta : \|R\|_{X_\theta} < \infty\}$ is open.*

Let us now illustrate this result. Suppose w is an A_p-weight where $1 < p < \infty$, i.e. R is bounded on $L_p(w)$. Then R is bounded on $L_p(w^\theta)$ for $0 \le \theta \le 1$. An application of (3) of Theorem 9.8 gives

$$\left|\int_{\mathbf{T}} f \log w \, d\lambda\right| \le C\|f\|$$

for $f \in H_0^1 \cap L\log L$. The point is that if $X = L_p(w)$ then $\Phi_X(f) = \frac{1}{p}\Lambda(f) - \int f \log w \, d\lambda$. Thus $\log w \in \mathrm{BMO}$.

Conversely if $\log w \in \mathrm{BMO}$ then taking $X_0 = L_p(w^{-1})$ and $X_1 = L_p(w)$ we see that R is bounded on $L_p(w^\alpha)$ for some $\alpha > 0$, i.e. w^α is an A_p-weight. Thus we have the following result of Coifman-Rochberg [13] as a special case.

Corollary 9.9. *Suppose $1 < p < \infty$. Then $u \in \mathrm{BMO}(\mathbf{T})$ if and only if $e^{\alpha u}$ is an A_p-weight for some $\alpha > 0$.*

Further well-known facts can also be recovered this way. If $1 < p < 2$ and w is an A_p-weight, let $X_0 = L_2$, and $X_1 = L_1(w^\alpha)$ where $\alpha^{-1} = 2 - p$. Then $L_p(w) = X_\theta$ where $\theta = \alpha^{-1}$. Since R is bounded at both X_0 and X_θ it is clear that (3) and hence (4) of Theorem 9.8 hold. Thus R is bounded on $L_{p-\varepsilon}(w^\beta)$ for some $\varepsilon > 0$ and $\beta > 1$. By further interpolation w is an $A_{p-\varepsilon}$-weight. (See Muckenhoupt [41].)

Let us now turn to the vector valued case. Suppose X is a super reflexive Köthe function space on S. Then X is a (UMD)-space (Burkholder [7]) if

and only if R is bounded on the space $L_2(X)$ ([4, 8]) where $L_2(X)$ may be identified with the Köthe function space on $S \times \mathbf{T}$ of all x such that

$$\|x\|_{L_2(X)} = \left(\int_0^{2\pi} \|x_t\|_X^2 \frac{dt}{2\pi} \right) < \infty,$$

where $x_t(s) = x(s, e^{it})$. We now state our main theorem.

Theorem 9.10. *Let X_0, X_1 be Köthe function spaces on S and that $X = [X_0, X_1]_{\theta_0}$ is a* (UMD)*-space. Let Ω be the induced centralizer at X. Then $d_\Omega X$ is a* (UMD)*-space if and only if there exists $\delta > 0$ so that X_θ is a* (UMD)*-space for $|\theta - \theta_0| \le \delta$.*

Proof. Let us first prove the assertion under the additional hypotheses that X_0, X_1 are acceptable and $L_r \subset X \subset L_s$ where $1 < s < r < \infty$. Under these hypotheses, Theorem 9.7 applies to the interpolation scale

$$L_2(X_\theta) = [L_2(X_0), L_2(X_1)]_\theta.$$

The proof will be complete when we show that $d_\Omega X$ is (UMD) if and only if the induced centralizer $\widetilde{\Omega}$ at $L_2(X)$ satisfies $[R, \widetilde{\Omega}]_{L_2(X)} < \infty$. It will be convenient to suppose X_0, X_1 are renormed to be strictly convex Köthe function spaces; this involves no loss of generality, and results in uniqueness of the choice of extremal (see Theorem 2.6 and remarks).

If $x \in L_2(X)$ and $\|x\|_{L_2(X)} = 1$ then we write $x = u|x|$ where $|u| = 1$ a.e. Let $|x| = |x_0|^{1-\theta_0}|x_1|^{\theta_0}$ where $\|x_0\|_{L_2(X_0)} = \|x_1\|_{L_2(X_1)} = 1$. By Hölder's inequality $\|x_t\|_X = \|x_{0,t}\|_{X_0} = \|x_{1,t}\|_{X_1} = 1$ a.e. Thus, utilizing the uniqueness of both Ω and $\widetilde{\Omega}$

$$\widetilde{\Omega}(x) = x(\log|x_1| - \log|x_0|),$$
$$(\widetilde{\Omega}(x))_t = x_t(\log|x_{1,t}| - \log|x_{0,t}|) = (\Omega(x))_t \quad \text{a.e.}$$

Now if $\phi, \psi \in L_0(S \times \mathbf{T})$, then $\|(\phi, \psi)\|_{L_2(d_\Omega X)}$ is equivalent to

$$\left(\int \|\phi_t\|_X^2 \, d\lambda \right)^{1/2} + \left(\int \|\psi_t - \Omega(\phi_t)\|_X^2 \, d\lambda \right)^{1/2}.$$

Here to avoid cumbersome measurability problems we may interpret the integrals as upper integrals; however, it can be verified that all functions are suitably measurable. Thus $\|(\phi, \psi)\|_{d_\Omega X}$ is equivalent to $\|\phi\|_{L_2(X)} + \|\psi - \widetilde{\Omega}(\psi)\|_{L_2(X)}$ and $L_2(d_\Omega X)$ can be identified with $d_{\widetilde{\Omega}}(L_2(X))$. Thus the vector-valued Riesz transform R is bounded on $L_2(d_\Omega X)$ if and only if the map $(\phi, \psi) \mapsto (R\phi, R\psi)$ is bounded on $d_{\widetilde{\Omega}}(L_2(X))$ i.e., if and only if $\|[R, \widetilde{\Omega}]\|_{L_2(X)} < \infty$. The result will then reduce to Theorem 9.7.

It remains to justify our hypotheses. Since X is super reflexive it is equivalent to a space which is p-convex and q-concave where $\frac{1}{p} + \frac{1}{q} = 1$ and $p > 1$. By Pisier's theorem [43], X is equivalent to $[L_2, Y]_\tau$ where Y is a Köthe function space and $\tau > 0$. Now, from Lozanovksii's theorem, there is a weight function $w_2 > 0$ a.e. such that $L_\infty \subset w_2 Y \subset L_1$. Here $\|x\|_{wY} = \|xw^{-1}\|_Y$. Similarly there exist w_0, w_1 so that $L_\infty \subset w_0 X_0, w_1 X_1 \subset L_1$. Now $L_{2/(1-\tau)} \subset w_2^\tau X \subset L_{2/(1+\tau)}$, so that for some $\eta > 0$ we can find weights, w_3, w_4 with

$L_\alpha \subset w_3 X_{\theta_{0_{-\eta}}}$, $w_4 X_{\theta_0+\eta} \subset L_\beta$ with $1 < \beta < \alpha < \infty$. Thus the problem may be reduced by a change of weight to the type discussed above. □

Theorem 9.11. *There exists a centralizer Ω on $L_2(0, 1)$ so that the twisted sum $d_\Omega L_2$ is not* (UMD). *Thus the* (UMD)*-property is not a three-space property.*

Proof. Bourgain [4] shows that for any $p < 2$ there is a Köthe function space X_p on $(0, 1)$ which is p-convex and q-concave but not (UMD). By Pisier's theorem [43] if $0 < \theta < 1$ there exists a Köthe space $X^{[\theta]}$ so that $[L_2, X^{[\theta]}]_\theta$ fails to be (UMD). It follows by combining isomorphic copies of the spaces $X^{[1/n]}$ as bands in a single space X that there is a Köthe function space X so that $[X^*, X]_\theta$ is (UMD) if and only if $\theta = \frac{1}{2}$ when $X_\theta = L_2$. If Ω is the induced centralizer, $d_\Omega L_2$ cannot be (UMD). □

Theorem 9.12. *If X_0, X_1 be Köthe function spaces on S. Then the set $\{\theta : 0 < \theta < 1$, and X_θ is* (UMD)$\}$ *is either open or a single point.*

We omit the proof of Theorem 9.12 which is similar to Theorem 9.8; this theorem extends a result of Rubio de Francia [46].

Added in proof. See also [51] where the similar results to Theorem 9.8 and 9.12 are conjectured.

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