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LANNES' T FUNCTOR ON SUMMANDS OF $H^*(B(\mathbf{Z}/p)^s)$

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ABSTRACT. Let H be the mod-p cohomology of the classifying space $B(\mathbf{Z}/p)$ thought of as an object in the category, $\mathcal{U}$, of unstable modules over the Steenrod algebra. Lannes constructed a functor $T: \mathcal{U} \to \mathcal{U}$ which is left adjoint to the functor $A \mapsto A \otimes H$. In this paper we evaluate T on the indecomposable $\mathcal{U}$-summands of $H^{\otimes s}$, the tensor product of s copies of H. Our formula involves the composition factors of certain tensor products of irreducible representations of the semigroup ring $\mathbf{F}_p[\mathrm{M}_{s,s}(\mathbf{Z}/p)]$. The main application is to determine the homotopy type of the space of maps from $B(\mathbf{Z}/p)$ to X when X is a wedge summand of the space $\Sigma(B(\mathbf{Z}/p)^s)$.

INTRODUCTION

Let H be the mod-p cohomology of the classifying space $B(\mathbf{Z}/p)$ thought of as an object in the category, $\mathcal{U}$, of unstable modules over the Steenrod algebra. This object is of fundamental importance: the fact that it is injective in $\mathcal{U}$ was proved and used in Carlsson's proof of the Segal conjecture and in Miller's proof of the Sullivan conjecture [C, M]. The functor $A \mapsto A \otimes H$ from $\mathcal{U}$ to $\mathcal{U}$ has a left adjoint $T: \mathcal{U} \to \mathcal{U}$ constructed by Lannes [L]. He used T to reformulate the proofs of the above conjectures for elementary abelian p-groups and to prove the generalized Sullivan conjecture.

In this paper we evaluate the T functor on the $\mathcal{U}$-summands of $H^{\otimes s}$, the tensor product of s copies of H. These summands were described by the first author and Kuhn using the modular representation theory of the semigroup ring $R = \mathbf{F}_p[\mathrm{M}_{s,s}(\mathbf{Z}/p)]$ [HK]. Our main application is to determine the homotopy type of the space of maps from $B(\mathbf{Z}/p)$ to X when X is a wedge summand of the space $\Sigma(B(\mathbf{Z}/p)^s)$.

Our formula for T involves the composition factors of certain tensor products of irreducible representations of R. Because these representations are not completely understood (even their dimensions are not known in general), these composition factors cannot always be determined. A number of examples are given in cases where the representation theory is sufficiently well known.

Lannes and Zarati have shown that if I is injective in $\mathcal{U}$, then so is $I \otimes H$ [LZ]. In particular, $H^{\otimes s}$ is injective, as are its $\mathcal{U}$-summands. In [LS], Lannes and Schwartz classified all of the $\mathcal{U}$-injectives. In particular, they proved that the indecomposable injectives are of the form $L \otimes J(n)$, where L is an

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indecomposable summand of some $H^{\otimes s}$ and $J(n)$ is one of the dual Brown-Gitler modules studied by Miller [M]. Since $J(n)$ is a finite module $T(J(n)) = J(n)$. The functor T preserves tensor products, so our formula can be used to evaluate T on any $\mathcal{U}$-injective. Since T is exact, a possible application of our results might be to evaluate T on an object A using the first two terms of an injective resolution. We do not pursue this here.

We now outline the paper. In §1, we recall some results from the representation theory of finite-dimensional algebras and apply these to the semigroup ring R. In §2, we give Lannes' calculation of $T(H^{\otimes s})$ and describe the $\mathcal{U}$-decompositions of $H^{\otimes s}$ and $T(H^{\otimes s})$.

The evaluation of T on the indecomposable summands is stated in §3 as Theorem 3.2. Using an explicit description of the ring homomorphism τ appearing in this theorem, we give a more computational formula in Theorem 3.8. A number of examples complete the section.

In §4, we first describe the analogs of our previous results implied by the representation theory of the group ring $R' = \mathbf{F}_p[\mathrm{GL}_s(\mathbf{Z}/p)]$. Because of the close relationship between the representations of R and R', these results are essentially equivalent to the earlier ones (see Theorem 4.11). However, in this context we can more easily describe the techniques that were used to evaluate our formulas for small values of p and s (tabulated in the Appendix). At the end of the section we find the value of T on the 'Steinberg' summand, $L(s)$ (for $p = 2$). We note the intriguing isomorphism $T(L(s)) \cong L(s) \oplus (H \otimes L(s-1))$.

In §5, we give a new determination of $T(M_s(j))$, where $M_s(j)$ is a certain summand of $H^{\otimes s}$ studied by Campbell and Selick [CS]. The original result is due to the second author [S2].

Section 6 contains our application to mapping spaces. The $\mathcal{U}$-decompositions of $H^{\otimes s}$ are topologically realized by wedge sum decompositions of the space $\Sigma(B(\mathbf{Z}/p)^s_+)$. We apply a result of Lannes to describe the homotopy type of the space of maps from $B(\mathbf{Z}/p)$ to X, where X is a wedge summand of $\Sigma(B(\mathbf{Z}/p)^s)$ (see Theorem 6.11). When X is indecomposable, we describe $\mathrm{map}(B(\mathbf{Z}/p), X)$ using our earlier evaluation of T (Theorem 6.12).

Throughout the paper, $\mathrm{M}_{s,s}(\mathbf{Z}/p)$ denotes the multiplicative *semigroup* of $s \times s$ matrices over $\mathbf{Z}/p$, while $\mathrm{M}_{n,n}(A)$ denotes the matrix *ring* for any A other than $\mathbf{Z}/p$. We denote the semigroup ring $\mathbf{F}_p[\mathrm{M}_{s,s}(\mathbf{Z}/p)]$ by R and the tensor product $\mathrm{M}_{p^s,p^s}(\mathbf{F}_p) \otimes R$ by S. Similarly, R' and S' denote the group ring $\mathbf{F}_p[\mathrm{GL}_s(\mathbf{Z}/p)]$ and tensor product $M_{p^s,p^s}(\mathbf{F}_p) \otimes R'$. We write $\langle v_1, \dots, v_n \rangle$ for the $\mathbf{F}_p$-vector space with basis $\{v_1 \dots, v_n\}$. A positive integer n, in front of a module or space will denote the coproduct of n copies of that module or space (i.e., direct sum for modules, wedge sum for spaces). All cohomology groups have $\mathbf{F}_p$-coefficients.

Some of the results in this paper first appeared in the second author's thesis written under the direction of P. S. Selick.

1. General representation theory

This section recalls some results from the general representation theory of finite-dimensional algebras over $\mathbf{F}_p$ (see, e.g., [CR1]). Included are those examples needed for §§2 and 3. We will recall further representation results in §4.

Let A be a finite-dimensional algebra over $\mathbf{F}_p$. An element e in A is called *idempotent* if $e^2 = e$. Two idempotents, e_1 and e_2, are called *orthogonal* if $e_1e_2 = e_2e_1 = 0$. An idempotent e which is not the sum of nonzero orthogonal idempotents is called *primitive*. There exist primitive orthogonal idempotents $e_1, \dots, e_m$ summing to the identity with $A \cong \bigoplus_{i=1}^m Ae_i$ as left A-modules. This decomposition is unique up to isomorphism: if $f_1, \dots, f_n$ is another collection of primitive orthogonal idempotents summing to the identity, then $m = n$ and there is a unit u in A and a permutation σ of the f_j with $u^{-1}e_iu = \sigma(f_i)$ for $i = 1, \dots, m$.

The isomorphism classes of the modules Ae_i appearing in the above decomposition are called the *principal indecomposables* for A. Each of these modules has a unique maximal submodule (given by Je_i, where J is the radical of A), so the top quotient in any composition series is well defined. The modules Ae_i and Ae_j are isomorphic if and only if their top quotients are isomorphic. Also, if V is any irreducible A-module, then V occurs as the top quotient of some principal indecomposable, called the *projective cover* of V. The number of times that the projective cover of V occurs in an indecomposable decomposition of A is equal to the dimension of V over its endomorphism ring, $\hom_A(V, V)$, which is a finite field of characteristic p. We summarize some of these results in the following.

Proposition 1.1. *Let A be a finite-dimensional $\mathbf{F}_p$-algebra and let $\{V_\lambda\}$ be the collection of irreducible A-modules. Let $e_1, \dots, e_m$ be a set of primitive orthogonal idempotents summing to the identity, and, for each λ, choose a primitive idempotent e_λ in A so that $P_\lambda = Ae_\lambda$ is the projective cover for V_λ. Then as left A-modules, $A \cong \bigoplus_{i=1}^m Ae_i \cong \bigoplus_\lambda \dim_{E_\lambda}(V_\lambda)P_\lambda$, where $E_\lambda = \hom_A(V_\lambda, V_\lambda)$.*

Example 1.2. The ring $A = \mathrm{M}_{n,n}(\mathbf{F}_p)$ of $n \times n$ matrices over $\mathbf{F}_p$ has a unique irreducible module V^n with $\mathbf{F}_p$-dimension n. If we let e_i be the $n \times n$ matrix having (i, i)-entry 1 and all other entries 0, then $\{e_1, \dots, e_n\}$ is a set of primitive orthogonal idempotents summing to the identity. So $A \cong \bigoplus_{i=1}^n Ae_i \cong nV^n$. (Here V^n is its own projective cover and the endomorphism ring is $\mathbf{F}_p$.)

Example 1.3. [HK §6]. Let R be the semigroup ring $\mathbf{F}_p[\mathrm{M}_{s,s}(\mathbf{Z}/p)]$. The irreducible R-modules can be described using Young diagrams (see §4). They are $\{V_\lambda | \lambda \in \Lambda\}$, where $\Lambda = \{(\lambda_1, \dots, \lambda_s) | 0 \le \lambda_k \le p-1\}$. For each $\lambda \in \Lambda$, the endomorphism ring $\hom_R(V_\lambda, V_\lambda)$ is $\mathbf{F}_p$. Let $f_1, \dots, f_N$ be a set of primitive orthogonal idempotents summing to the identity and choose f_λ so that $P_\lambda = Rf_\lambda$ is the projective cover of V_λ. Then $R \cong \bigoplus_{j=1}^N Rf_j \cong \bigoplus_{\lambda\in\Lambda} \dim(V_\lambda)P_\lambda$.

Remark 1.4. The above indexing set Λ will be used throughout the paper.

If K is an algebraic extension field of $\mathbf{F}_p$ and M is an A-module, then we can consider the $(A \otimes K)$-module $M \otimes K$. If K is an algebraic closure of $\mathbf{F}_p$ and $M \otimes K$ is irreducible, then M is called *absolutely irreducible*. If every irreducible for A is absolutely irreducible, then $\mathbf{F}_p$ is called a *splitting field* for A.

Proposition 1.5 [CR2, 3.56(iii)]. *Let A and B be finite-dimensional $\mathbf{F}_p$-algebras each having $\mathbf{F}_p$ as a splitting field. If M and N are irreducible modules for A and B respectively, then $M \otimes N$ is an irreducible $(A \otimes B)$-module.*

For the rings in the above two examples, $\mathbf{F}_p$ is indeed a splitting field.

Applying the proposition gives the following.

Example 1.6. Let S be the ring

$$\mathbf{M}_{n,n}(\mathbf{F}_p) \otimes \mathbf{F}_p[\mathbf{M}_{s,s}(\mathbf{Z}/p)].$$

Then $\{V^n \otimes V_\lambda\}_{\lambda \in \Lambda}$ is a complete set of irreducibles, and $\{e_i \otimes f_j | i = 1, \ldots, n; j = 1, \ldots, N\}$ is a set of primitive orthogonal idempotents summing to the identity. (These idempotents are clearly orthogonal and sum to the identity. They are primitive since there are $n \cdot N$ of them, which is the sum of the dimensions of the irreducibles.) We can take $e_1 \otimes f_\lambda$ as a primitive idempotent for the projective cover of $V^n \otimes V_\lambda$.

Remarks 1.7. The rings $\mathbf{M}_{n,n}(A)$ and $\mathbf{M}_{n,n}(\mathbf{F}_p)\otimes A$ are isomorphic in the usual way. Example 1.6 can also be thought of as a special case of Morita equivalence.

Now let B be another finite-dimensional algebra over $\mathbf{F}_p$ with irreducibles $\{W_\mu\}$, primitive idempotents f_μ, projective covers $Q_\mu = Bf_\mu$, and endomorphism rings $E_\mu = \hom_B(W_\mu, W_\mu)$. Given an algebra homomorphism $\tau: A \to B$ and a primitive idempotent e_λ in A, the following proposition describes the idempotent decomposition of $\tau(e_\lambda)$ in B. Given a B-module W, $\tau_*(W)$ denotes the A-module obtained by restriction of scalars.

Proposition 1.8. *The following numbers are equal*:

(i) *the number of copies of* Q_μ *in a principal indecomposable decomposition of* $B\tau(e_\lambda)$,

(ii) *the* $\mathbf{F}_p$*-dimension of* $\hom_B(B\tau(e_\lambda), W_\mu)$ *divided by the* $\mathbf{F}_p$*-dimension of* E_μ,

(iii) *the* $\mathbf{F}_p$*-dimension of* $\hom_A(Ae_\lambda, \tau_*(W_\mu))$ *divided by the* $\mathbf{F}_p$*-dimension of* E_μ, *and*

(iv) *the multiplicity of* V_λ *in* $\tau_*(W_\mu)$ *times the* $\mathbf{F}_p$*-dimension of* $\hom_A(V_\lambda, V_\lambda)$ *divided by the* $\mathbf{F}_p$*-dimension of* E_μ.

(*The multiplicity of an irreducible module* V *in a finite-dimensional module* W *is the number of times that* V *occurs as a quotient in a composition series for* W.)

Proof. The equality of (i) and (ii) follows from [CR1, 54.11, 54.14], the equality of (ii) and (iii) follows from [CR2, 2.19, 2.6] and the fact that $B\tau(e_\lambda) \cong B \otimes_A Ae_\lambda$, and the equality of (iii) and (iv) follows from [CR1, 54.14, 54.19]. □

2. Decompositions of $H^{\otimes s}$ and $T(H^{\otimes s})$

This section describes the $\mathscr{U}$-decompositions of the objects $H^{\otimes s}$ and $T(H^{\otimes s})$. We begin with Lannes' calculation of $T(H^{\otimes s})$. The complete proof is shown to set up notation to be used later. Next we show that the endomorphism rings for $H^{\otimes s}$ and $T(H^{\otimes s})$ are the rings R and S from above. By applying results of §1 to these rings, the complete decompositions of $H^{\otimes s}$ and $T(H^{\otimes s})$ are stated.

Let $\mathscr{K}$ denote the category of unstable algebras over the Steenrod algebra. There is a free functor, denoted $F_{\mathscr{K}}$ from the category of graded vector spaces to the category $\mathscr{K}$. It is well known that $H^{\otimes s}$ is the free unstable algebra, $F_{\mathscr{K}}(\langle t_1, \ldots, t_s\rangle)$, on the graded vector space $\langle t_1, \ldots, t_s\rangle$, where the t_j have degree one, and that

$$F_{\mathscr{K}}(\langle t_1, \dots, t_s\rangle) = \begin{cases} \mathbf{F}_2[t_1, \dots, t_s], \quad \text{with } Sq_1(t_j) = t_j^2, q^1 \text{ for } p = 2, \\ \mathrm{E}[t_1, \dots, t_s] \otimes \mathbf{F}_p[\beta t_1, \dots, \beta t_s], \\ \qquad\qquad \text{with } P^1(\beta t_j) = (\beta t_j)^p, \text{ for } p \text{ odd}, \end{cases}$$

where E denotes the exterior algebra over $\mathbf{F}_p$.

In addition to $T\colon \mathscr{U} \to \mathscr{U}$, Lannes constructed a functor $T_{\mathscr{K}}\colon \mathscr{K} \to \mathscr{K}$ which is left adjoint to $K \mapsto K \otimes H$. The following proposition gives a relationship between T and $T_{\mathscr{K}}$.

Proposition 2.1 [L, 3.4]. *There is a natural equivalence between $T_{\mathscr{K}}$ and the restriction of T to $\mathscr{K}$. In particular, $T(H^{\otimes s}) \cong T_{\mathscr{K}}(H^{\otimes s})$.*

Since $H^{\otimes s}$ is a free $\mathscr{K}$-object, it is easy to compute $T_{\mathscr{K}}(H^{\otimes s})$ and hence $T(H^{\otimes s})$. Let $B(s)$ denote the free unstable algebra on the graded vector space $\langle b_1, \dots, b_s\rangle$, where the b_i have degree zero. Then $B(s)$ is the free p-Boolean algebra (an $\mathbf{F}_p$-algebra with $x^p = x$ for each element x) with generators $b_1, \dots, b_s$ and with the Steenrod algebra acting trivially.

Proposition 2.2. [L, 3.4]. *As objects in $\mathscr{K}$, $T(H^{\otimes s}) \cong B(s) \otimes H^{\otimes s} \cong p^s H^{\otimes s}$.*

Proof. It follows from the construction of $T_{\mathscr{K}}$ that

$$\begin{aligned} T_{\mathscr{K}}(H^{\otimes s}) &\cong T_{\mathscr{K}}(F_{\mathscr{K}}(\langle t_1, \dots, t_s\rangle)) \\ &\cong F_{\mathscr{K}}(\langle t_1 \otimes t^*, \dots, t_s \otimes t^*, t_1 \otimes 1^*, \dots, t_s \otimes 1^*\rangle), \\ &\cong F_{\mathscr{K}}(\langle t_1 \otimes t^*, \dots, t_s \otimes t^*\rangle \oplus \langle t_1 \otimes 1^*, \dots, t_s \otimes 1^*\rangle) \\ &\cong B(s) \otimes H^{\otimes s}. \end{aligned}$$

Here $B(s)$ is the free p-Boolean algebra on $b_1, \dots, b_s$, where $b_j = t_j \otimes t^*$, and the last $H^{\otimes s}$ is the free unstable algebra on $\bar{t}_1, \dots \bar{t}_s$, where $\bar{t}_j = t_j \otimes 1^*$.

As an algebra, $B(s)$ is semisimple and commutative, so must be isomorphic to a direct sum of fields. Since $\mathbf{F}_p$ is the only field of characteristic p which is p-Boolean, it follows that $B(s) \cong \bigoplus_{i=1}^{p^s} \mathbf{F}_p$. The second isomorphism of the proposition follows. □

Now we turn to the direct sum decompositions in $\mathscr{U}$. For any module M in $\mathscr{U}$, a (finite) orthogonal idempotent decomposition of the identity, $1 = \Sigma_i e_i$, in the endomorphism ring, $\hom_{\mathscr{U}}(M, M)$, induces a $\mathscr{U}$-decomposition of M: $M \cong \bigoplus_i e_i M$. If the endomorphism ring is a finite-dimensional $\mathbf{F}_p$-algebra, then, as in §1, there is a *primitive* orthogonal idempotent decomposition of the identity giving a splitting of M into *indecomposables.* The next proposition shows that the endomorphism rings of the modules $H^{\otimes s}$ and $T(H^{\otimes s})$ are indeed finite-dimensional $\mathbf{F}_p$-algebras.

Proposition 2.3. (i) $\hom_{\mathscr{U}}(H^{\otimes s}, H^{\otimes s}) \cong \mathbf{F}_p[\mathrm{M}_{s.s}(\mathbf{Z}/p)]$,

(ii) $\hom_{\mathscr{U}}(B(s), B(s)) \cong \mathrm{M}_{p^s, p^s}(\mathbf{F}_p)$, *and*

(iii) $\hom_{\mathscr{U}}(T(H^{\otimes s}), T(H^{\otimes s})) \cong \mathrm{M}_{p^s, p^s}(\mathbf{F}_p) \otimes \mathbf{F}_p[\mathrm{M}_{s,s}(\mathbf{Z}/p)]$.

Proof. Part (i) is a result of Adams, Gunawardena, and Miller [AGM, p. 438; L, W]. Since the Steenrod algebra action on $B(s)$ is trivial, part (ii) is clear. Part (iii) follows from (i), (ii), and Proposition 2.2. □

Note that we are denoting the ring in (i) by R and the ring in (iii) by S. The representation theory of these rings was described in §1. Each of the idempotents f_λ in R (Example 1.3) gives an indecomposable summand $f_\lambda H^{\otimes s}$ of $H^{\otimes s}$. Throughout the paper, this summand will be denoted $H_\lambda^{\otimes s}$. Part (i) of the next proposition is from [HK].

Proposition 2.4. *As objects in* $\mathscr{U}$,

(i) $H^{\otimes s} \cong \bigoplus_{\lambda\in\Lambda} \dim_{\mathbf{F}_p}(V_\lambda) H_\lambda^{\otimes s}$, *and*

(ii) $T(H^{\otimes s}) \cong \bigoplus_{\lambda\in\Lambda} \dim_{\mathbf{F}_p}(V^{p^s} \otimes V_\lambda) H_\lambda^{\otimes s} \cong \bigoplus_{\lambda\in\Lambda} p^s \dim_{\mathbf{F}_p}(V_\lambda) H_\lambda^{\otimes s}$.

Proof. Part (i) follows from Example 1.3 and Proposition 2.3. Part (ii) follows similarly using Example 1.6, except for the fact that the summand corresponding to $(V^{p^s} \otimes V_\lambda)$ is $H_\lambda^{\otimes s}$. This follows from

$$(e_1 \otimes f_\lambda)(B(s) \otimes H^{\otimes s}) \cong \mathbf{F}_p \otimes f_\lambda H^{\otimes s} \cong f_\lambda H^{\otimes s}.$$

(Of course, (ii) follows easily from (i) and Proposition 2.2, but we wish to emphasize the representation theory here. The fact that $H_\lambda^{\otimes s}$ corresponds to $V^{p^s} \otimes V_\lambda$ is used in the proof of Theorem 3.2 below.) □

Remark 2.5. The above decomposition of $H^{\otimes s}$ can be realized by stable wedge decompositions of the classifying space $B(\mathbf{Z}/p)_+^s$. The summand corresponding to $\lambda \in \Lambda$ is denoted by $e_\lambda B(\mathbf{Z}/p)_+^s$ and can be defined by a certain telescope construction (see §6).

We end this section with a result which relates the summands of $H^{\otimes(s-1)}$ to those of $H^{\otimes s}$. When $\lambda_s = 0$, the $\mathrm{M}_{s,s}(\mathbf{Z}/p)$-representation $V_{(\lambda_1,\dots,\lambda_{s-1},0)}$ is 'induced' from the representation $V_{(\lambda_1,\dots,\lambda_{s-1})}$ of $\mathrm{M}_{s-1,s-1}(\mathbf{Z}/p)$. This induction procedure is quite complicated (see [HK, §4] or [CP]); however, it is possible to prove the following.

Proposition 2.6 [HK, 6.2(2)]. $H^{\otimes s}_{(\lambda_1,\dots,\lambda_{s-1},0)} \cong H^{\otimes(s-1)}_{(\lambda_1,\dots,\lambda_{s-1})}$.

3. The ring homomorphism τ

Since T is an adjoint, it is additive and therefore induces a ring homomorphism

$$T\colon \hom_{\mathscr{U}}(H^{\otimes s}, H^{\otimes s}) \to \hom_{\mathscr{U}}(T(H^{\otimes s}), T(H^{\otimes s})).$$

We denote the associated homomorphism from R to S by τ. The first result of this section gives a formula for the evaluation of the T functor on the indecomposable summand $H_\lambda^{\otimes s}$ of $H^{\otimes s}$ (Theorem 3.2). The coefficients $a_{\lambda\mu}$ appearing in this formula involve the composition factors of the restrictions of certain S modules to R. To simplify the evaluation of these coefficients we give an explicit algebraic description of the homomorphism τ, and we determine the composition series of $B(s)$ regarded as an R-module (Theorem 3.8). The end of the section lists a number of conditions on the $a_{\lambda\mu}$ resulting from the representation theory.

Lemma 3.1. *If* $e \in \hom_{\mathscr{U}}(H^{\otimes s}, H^{\otimes s})$ *is an idempotent associated to the summand* A *of* $H^{\otimes s}$, *then* $T(e)$ *is an idempotent associated to the summand* $T(A)$ *of* $T(H^{\otimes s})$.

Proof. The composition $A \xrightarrow{i} H^{\otimes s} \xrightarrow{e} A$ is the identity, so $T(A) \xrightarrow{T(i)} T(H^{\otimes s}) \xrightarrow{T(e)} T(A)$ is the identity, and $T(A) = T(e)T(H^{\otimes s})$.

Recall that $\tau_*(W)$ denotes the R-module obtained from the S-module W via restriction along τ.

Theorem 3.2. *For $\lambda \in \Lambda$, $T(H_\lambda^{\otimes s}) = \bigoplus_{\mu\in\Lambda} a_{\lambda\mu} H_\mu^{\otimes s}$, where $a_{\lambda\mu}$ is the multiplicity of V_λ in $\tau_*(V^{p^s} \otimes V_\mu)$.*

Proof. Let e_λ in R be an idempotent defining the summand $H_\lambda^{\otimes s}$. By Lemma 3.1, $\tau(e_\lambda)$ is an idempotent for $T(H_\lambda^{\otimes s})$. By Proposition 1.8, the number of copies of the summand corresponding to $e_1 \otimes e_\mu$ in $\tau(e_\lambda)$ is the multiplicity of V_λ in $\tau_*(V^{p^s} \otimes V_\mu)$. By the proof of Proposition 2.4, the summand corresponding to $e_1 \otimes e_\mu$ is $H_\mu^{\otimes s}$. □

Remark 3.3. For a spectrum Y, let $k_t(Y)$ be the $K(t)^*$-dimension of $K(t)^*(Y)$, where $K(t)$ denotes the tth mod-p Morava K-theory. Kuhn has shown that $k_t(e_\lambda B(\mathbf{Z}/p)_+^s)$ is the $\mathbf{F}_p$-dimension of $\hom_{\mathscr{U}}(H_\lambda^{\otimes s}, H^{\otimes t})$. Using the adjointness of the T functor, this equals the $\mathbf{F}_p$-dimension of $\hom_{\mathscr{U}}(T^t(H_\lambda^{\otimes s}), \mathbf{F}_p)$ which in turn equals the number of copies of $H_{(0,\dots,0)}^{\otimes s}$ in $T^t(H_\lambda^{\otimes s})$. The $a_{\lambda\mu}$ can therefore be used to determine these Morava K-theory dimensions (compare [K2, 1.6]).

We now describe the homomorphism $\beta\colon R \to M_{p^s,p^s}(\mathbf{F}_p)$ given by the action of $M_{s,s}(\mathbf{Z}/p)$ on $B(s)$. Recall that $B(s)$ is the free p-Boolean algebra on $b_1, \dots, b_s$, so has as basis the monomials $\{b_1^{i_1} b_2^{i_2} \cdots b_s^{i_s} \mid 0 \le i_j \le (p-1)\}$. Let $m \in \mathrm{M}_{s,s}(\mathbf{Z}/p)$ act on the vector space $\langle b_1, \dots, b_s\rangle$ in the usual way and extend this action multiplicatively to the basis elements. Let $\beta(m)$ be the associated linear mapping of $B(s)$. Since $\beta(m_1 m_2) = \beta(m_1)\beta(m_2)$, β extends to give a ring homomorphism from $\mathbf{F}_p[\mathrm{M}_{s,s}(\mathbf{Z}/p)]$ to $\mathrm{M}_{p^s,p^s}(\mathbf{F}_p)$.

Now let $\Delta\colon R \to R \otimes R$ be the linear extension of the map $m \mapsto m \otimes m$ for $m \in \mathrm{M}_{s,s}(\mathbf{Z}/p)$. This is the 'diagonal' map used to define the internal tensor product of two R-modules. The next theorem first appeared in the second author's thesis (for $p = 2$).

Theorem 3.4. *The ring homomorphism $\tau\colon R \to S$ is equal to $(\beta \otimes 1)\circ\Delta$.*

Proof. Since both maps are ring homomorphisms it suffices to show that $\tau(m) = (\beta \otimes 1)(\Delta(m))$ for each matrix $m \in \mathrm{M}_{s,s}(\mathbf{Z}/p)$. Furthermore, both $\tau(m)$ and $(\beta\otimes 1)(\Delta(m))$ represent algebra maps in $\hom_{\mathscr{U}}(T(H^{\otimes s}), T(H^{\otimes s}))$, so it suffices to show that $\tau(m)$ and $(\beta \otimes 1)(\Delta(m))$ are the same on the generators $b_j \otimes 1$ and $1 \otimes \bar{t}_j$ defined in the proof of Proposition 2.2.

As before, $H^{\otimes s} = F_{\mathscr{K}}(\langle t_1, \dots, t_s\rangle)$, where the t_j have degree 1, and $T(H^{\otimes s}) \cong F_{\mathscr{K}}(\langle t_j \otimes t^*, t_j \otimes 1^*\rangle)$. It follows from the construction of T [L] that the adjoint of the identity on $T(H^{\otimes s})$ is a map

$$\mathrm{ad}(1)\colon H^{\otimes s} \to H \otimes T(H^{\otimes s})$$

which takes t_j to $t\otimes(t_j\otimes t^*) + 1\otimes(t_j\otimes 1^*)$. The matrix m, acting on V, induces an algebra map on $H^{\otimes s}$. The adjoint of $\tau(m)$ is the composition of this map with $\mathrm{ad}(1)$. This composite takes t_j to $t\otimes(m(t_j)\otimes t^*) + 1\otimes(m(t_j)\otimes 1^*)$. Thus $\tau(m)(t_j\otimes t^*) = m(t_j)\otimes t^*$ and $\tau(m)(t_j\otimes 1^*) = m(t_j)\otimes 1^*$. From the definitions of b_j and $\bar{t}_j$, we have

$$\tau(m)(b_j \otimes 1) = \tau(m)(t_j \otimes t^*) = m(t_j) \otimes t^* = m(b_j) \otimes 1$$

and

$$\tau(m)(1 \otimes \bar{t}_j) = \tau(m)(t_j \otimes 1^*) = m(t_j) \otimes 1^* = 1 \otimes m(\bar{t}_j).$$

On the other hand,

$$(\beta(m)\otimes m)(b_j\otimes 1)=m(b_j)\otimes 1 \quad\text{and}\quad (\beta(m)\otimes m)(1\otimes \bar{t}_j)=1\otimes m(\bar{t}_j). \quad \square$$

Corollary 3.5. *As R-modules, $\tau_*(V^{p^s}\otimes V_\mu)$ and $\beta_*(V^{p^s})\otimes V_\mu$ are isomorphic. So the coefficient $a_{\lambda\mu}$ in Theorem* 3.2 *is the multiplicity of V_λ in $\beta_*(V^{p^s})\otimes V_\mu$.*

We now determine the composition factors of $\beta_*(V^{p^s})\cong\beta_*(B(s))$. For $i=0,\ldots,s(p-1)$, let B_i be the subspace of $B(s)$ spanned by the monomials of polynomial degree less than or equal to i. For each i, B_i is an R-submodule of $B(s)$, and $B_0\subseteq B_1\subseteq\cdots\subseteq B_{s(p-1)}=B(s)$.

Let $P=\mathbf{F}_p[x_1,\ldots,x_s]/(x_1^p,\ldots,x_s^p)$ be regarded as an R-module in the usual way, and let P^d denote the span of the monomials in P having polynomial degree d. It is easy to see that $B_d/B_{d-1}\cong P^d$.

Notation 3.6. For $0\le d\le s(p-1)$, write $d=a(p-1)+b$ with $0\le b<p-1$. Let $\lambda[d]\in\Lambda$ be the sequence $(\lambda_1,\ldots,\lambda_s)$ with $\lambda_a=p-1-b$, $\lambda_{a+1}=b$, and $\lambda_i=0$ for $i\ne a, a+1$.

Lemma 3.7. *For $d=0,\ldots,s(p-1)$, the modules P^d are distinct and irreducible, and $P^d\cong V_{\lambda[d]}$*

Proof. The first statement is [K2, 4.1], and the second is [CK1, 6.1]. □

So the $V_{\lambda[d]}$, $d=0,\ldots,s(p-1)$, are the composition factors of $B(s)$. Combining this with Corollary 3.5 gives the following formula for $a_{\lambda\mu}$.

Theorem 3.8. *In Theorem* 3.2, *we have*

$$a_{\lambda\mu}=\sum_{d=0}^{s(p-1)} \textit{the multiplicity of } V_\lambda \textit{ in } (V_{\lambda[d]}\otimes V_\mu).$$

To state the next proposition, we need a few definitions. A finite sequence of nonnegative integers $\lambda=(\lambda_1,\lambda_2,\ldots)$ is *p-regular* if $0\le\lambda_i\le p-1$ for all i, and is *bounded by* s if $\lambda_i=0$ for $i>s$. (Note that the indexing set Λ consists of the p-regular sequences that are bounded by s.) Given another finite sequence $\mu=(\mu_1,\mu_2,\ldots)$, we say that $\lambda\le\mu$ if, for each k, $\sum_{j=1}^k(\sum_{i=j}^\infty\lambda_i)\le\sum_{j=1}^k(\sum_{i=j}^\infty\mu_i)$. The sequence $\lambda+\mu$ equals $(\lambda_1+\mu_1,\lambda_2+\mu_2,\ldots)$.

Proposition 3.9 [CK1, 3.7]. *Let λ, μ and ν be p-regular sequences.*

(i) *If V_λ is a composition factor of $V_\mu\otimes V_\nu$, then $\lambda\le\mu+\nu$.*
(ii) *If $\mu+\nu$ is again p-regular, then $V_{\mu+\nu}$ occurs precisely once in $V_\mu\otimes V_\nu$.*

Corollary 3.10. (i) $a_{\lambda\mu}=0$ *if* $\lambda>\mu+\lambda[s(p-1)]$.
(ii) $a_{\lambda\mu}=1$ *if* $\lambda=(\lambda_1,\ldots,\lambda_{s-1},p-1)$ *and* $\mu=(\lambda_1,\ldots,\lambda_{s-1},0)$.

Proof. It is easy to see that $\lambda\le\mu$ if and only if $\lambda+\nu\le\mu+\nu$. The sequences $\{\lambda[d]\}$ are ordered by $\lambda[0]<\lambda[1]<\cdots<\lambda[s(p-1)]$. By taking $\nu=\lambda[s(p-1)]$ in Proposition 3.9, the corollary follows from Theorem 3.8. □

We end this section with some simple results for the $a_{\lambda\mu}$ implied by the above.

Example 3.11. $H^{\otimes s}_{(0,\dots,0)} = \mathbf{F}_p$ and $T(\mathbf{F}_p) = \mathbf{F}_p$, so

$$a_{(0,\dots,0)(\mu_1,\dots,\mu_s)} = \begin{cases} 1 & \text{if } (\mu_1, \dots, \mu_s) = (0, \dots, 0), \\ 0 & \text{otherwise}. \end{cases}$$

Example 3.12. $V_{(0,\dots,0)}$ is the trivial R-module (sending each $m \in \mathrm{M}_{s,s}(\mathbf{Z}/p)$ to 1), so

$$\mu = (0, \dots, 0) \Rightarrow a_{\lambda\mu} = \begin{cases} 1 & \text{if } \lambda = \lambda[d], \\ 0 & \text{otherwise}. \end{cases}$$

Example 3.13. By Proposition 2.6,

$$a_{(\lambda_1,\dots,\lambda_{s-1},0)(\mu_1,\dots,\mu_{s-1},0)} = a_{(\lambda_1,\dots,\lambda_{s-1})(\mu_1,\dots,\mu_{s-1})},$$

and

$$a_{(\lambda_1,\dots,\lambda_{s-1},0)(\mu_1,\dots,\mu_{s-1},\mu_s)} = 0 \quad \text{when } \mu_s \neq 0.$$

Example 3.14. $V_{(0,\dots,0,1)}$ is the determinant representation, Det, and, for $\lambda_s < p-1$, $V_{(\lambda_1,\dots,\lambda_s)} \otimes \mathrm{Det} \cong V_{\lambda_1,\dots,\lambda_{s-1},\lambda_s+1}$. For $1 \leq k \leq p-1$, Det^k sends the singular matrices to zero and is invertible when restricted to the nonsingular matrices. It follows that

$$\text{if } 0 < \lambda_s, \mu_s < p-1, \quad \text{then } a_{(\lambda_1,\dots,\lambda_s)(\mu_1,\dots,\mu_s)} = a_{(\lambda_1,\dots,\lambda_s+1)(\mu_1,\dots,\mu_s+1)}.$$

Example 3.15. For $s = 1$, each of the V_λ is a power of Det (where Det^0 is the trivial representation). The $a_{(\lambda_1)(\mu_1)}$ for $\lambda_1 = 0$ or $\mu_1 = 0$ are given above.

$$\text{For } \lambda_1, \mu_1 \neq 0, \quad a_{(\lambda_1)(\mu_1)} = \begin{cases} 2 & \text{if } \lambda_1 = \mu_1, \\ 1 & \text{otherwise}. \end{cases}$$

Example 3.16. For $s = 2$, the tensor products of all of the irreducible representations of $\mathbf{F}_p[\mathrm{M}_{2,2}(\mathbf{Z}/p)]$ have been determined by Glover [G]. It is therefore possible to determine all of the $a_{\lambda\mu}$ when $s = 2$. (It is also possible to use the techniques from Carlisle's thesis described in the next section to compute these $a_{\lambda\mu}$.)

4. Representations of $\mathbf{F}_p[\mathrm{GL}_s(\mathbf{Z}/p)]$

This section has two parts. In the first part we describe the analogs of our previous results induced by the representation theory of the group ring $R' = \mathbf{F}_p[\mathrm{GL}_s(\mathbf{Z}/p)]$. We state in Theorem 4.11 how the coefficients $a'_{\lambda\mu}$ appearing in this section relate to the $a_{\lambda\mu}$ appearing above. In the second part of this section we sketch how the irreducible modules for R' are constructed using Young diagrams. We also describe those results which were used to tabulate the $a_{\lambda\mu}$ for small values of p and s (see the Appendix). The final result in this section gives a closed formula for $T(H^{\otimes s}_{(1,\dots,1)})$ (the 'Steinberg' summand) when $p = 2$. We were somewhat surprised to see that this module is almost isomorphic to the tensor product $H \otimes H^{\otimes(s-1)}_{(1,\dots,1)}$.

Example 4.1 [JK, Exercise 8.4]. The irreducibles for R' are $\{V'_\lambda \mid \lambda \in \Lambda'\}$, where $\Lambda' = \{\lambda \in \Lambda \mid 1 \le \lambda_s \le p-1\}$. For each $\lambda \in \Lambda'$, the finite field $\hom_{R'}(V'_\lambda, V'_\lambda)$ is $\mathbf{F}_p$; also, $\mathbf{F}_p$ is a splitting field for R'. Let $f'_1, \dots, f'_{N'}$, be a set of primitive orthogonal idempotents summing to the identity, and for each $\lambda \in \Lambda'$, choose f'_λ so that $P'_\lambda = R'f'_\lambda$ is the projective cover of V'_λ. Then $R' \cong \bigoplus_{j=1}^{N'} R'f'_j \cong \bigoplus_{\lambda\in\Lambda'} \dim_{\mathbf{F}_p}(V'_\lambda)P'_\lambda$.

Example 4.2. Using Proposition 1.5 and the notation from Example 1.2, we see that the set of irreducible representations for the ring $S' = \mathrm{M}_{n,n}(\mathbf{F}_p) \otimes \mathbf{F}_p[\mathrm{GL}_s(\mathbf{Z}/p)]$ is $\{V^n \otimes V'_\lambda\}_{\lambda\in\Lambda'}$. The set $\{e_i \otimes f'_j \mid i = 1, \dots, n;\ j = 1, \dots, N'\}$ is a collection of primitive orthogonal idempotents summing to the identity, and we can take $e_1 \otimes f'_\lambda$ as an idempotent defining the projective cover of $(V^n \otimes V'_\lambda)$.

As in §2, each idempotent f'_λ in R' gives a summand $f'_\lambda H^{\otimes s}$ of $H^{\otimes s}$. We denote this summand by $(H^{\otimes s})'_\lambda$. (Note that the subscripts for the $(H^{\otimes s})'_\lambda$ are taken from Λ', not Λ.)

Proposition 4.3. *As objects in* $\mathscr{U}$,

(i) $H^{\otimes s} \cong \bigoplus_{\lambda\in\Lambda'} \dim_{\mathbf{F}_p}(V'_\lambda)(H^{\otimes s})'_\lambda$, *and*

(ii) $T(H^{\otimes s}) \cong \bigoplus_{\lambda\in\Lambda} \dim_{\mathbf{F}_p}(V^{p^s} \otimes V'_\lambda)(H^{\otimes s})'_\lambda$.

It is clear from the description of τ in Theorem 3.4 that $\tau(R') \subseteq S'$. Let $\beta' \colon R' \to \mathrm{M}_{p^s,p^s}(\mathbf{F}_p)$ and $\tau' \colon R' \to S'$ denote the restrictions of β and τ to R'. The same proofs as before give the following.

Theorem 4.4. $T((H^{\otimes s})'_\lambda) = \bigoplus_{\mu\in\Lambda'} a'_{\lambda\mu}(H^{\otimes s})'_\mu$, *where*

$$\begin{aligned} a'_{\lambda\mu} &= \textit{the multiplicity of } V'_\lambda \textit{ in } \tau'_*(V^{p^s} \otimes V'_\mu) \\ &= \textit{the multiplicity of } V'_\lambda \textit{ in } \beta'_*(V^{p^s}) \otimes V'_\mu \\ &= \sum_{d=0}^{s(p-1)} \textit{the multiplicity of } V'_\lambda \textit{ in } (V'_{\lambda'[d]} \otimes V'_\mu). \end{aligned}$$

Here $\lambda'[d]$ *is the same as* $\lambda[d]$, *but with* λ_s *changed to* $p-1$ *whenever it is zero.* (*Note that* $\lambda'[0] = \lambda'[s(p-1)] = (0, \dots, 0, p-1)$.)

Definition 4.5. An object in $\mathscr{U}$ which can be written as a sum of terms of the form $(H^{\otimes s})'_\lambda$ will be called a *group summand*.

The following corollary of Theorem 4.4 is immediate.

Corollary 4.6. *T takes group summands to group summands.*

Here are a few simple examples (which should be compared to §3).

Example 4.7. For $\mu = (0, \dots, 0, p-1)$,

$$a'_{\lambda\mu} = \begin{cases} 2 & \text{if } \lambda = (0, \dots, 0, p-1), \\ 1 & \text{if } \lambda = \lambda'[d],\ d \ne 0, s(p-1), \\ 0 & \text{otherwise}. \end{cases}$$

Example 4.8. $V'_{(0,\dots,0,k)}$ is the kth power of the determinant, so

$$a'_{(\lambda_1,\dots,\lambda_s)(\mu_1,\dots,\mu_s)} = a'_{(\lambda_1,\dots,\lambda_s+1)(\mu_1,\dots,\mu_s+1)},$$

where, when $\lambda_s + 1$ or $\mu_s + 1$ equals p it is changed to 1.

Example 4.9. $s = 1$. Here $1 \le \lambda_1, \mu_1 \le p-1$, and

$$a'_{(\lambda_1)(\mu_1)} = \begin{cases} 2 & \text{if } \lambda_1 = \mu_1, \\ 1 & \text{otherwise}. \end{cases}$$

We now relate these results (using R') to those in the previous sections (which used R). For $\lambda \in \Lambda$, let $\lambda^0 = (\lambda_1, \dots, \lambda_{s-1}, 0)$ and let $\lambda^{p-1} = (\lambda_1, \dots, \lambda_{s-1}, p-1)$.

Proposition 4.10 [HK, 6.2]. *For* $\lambda \in \Lambda'$,

(i) *if* $\lambda_s \ne p-1$, *then* $(H^{\otimes s})'_\lambda \cong H_\lambda^{\otimes s}$; *and*
(ii) *if* $\lambda_s = p-1$, *then* $(H^{\otimes s})'_\lambda \cong H_{\lambda^0}^{\otimes s} \oplus H_\lambda^{\otimes s}$.

Proof. This follows from Examples 4.1 and 1.3, and Proposition 1.8 applied to the inclusion of R' in R. □

Theorem 4.11. *Let* λ *and* μ *be elements of* Λ. *Then*

$$a_{\lambda\mu} = \begin{cases} a'_{\lambda\mu^{p-1}} & \text{if } \lambda_s \ne 0, p-1, \ \mu_s = 0, \\ a'_{\lambda\mu} & \text{if } \lambda_s \ne 0, p-1, \ \mu_s \ne 0, \\ a'_{\lambda\mu^{p-1}} - a_{\lambda^0\mu} & \text{if } \lambda_s = p-1, \ \mu_s = 0, \\ a'_{\lambda\mu} & \text{if } \lambda_s = p-1, \ \mu_s \ne 0. \end{cases}$$

(*The values for* $\lambda_s = 0$ *were given in Example* 3.13.)

Proof. By Proposition 4.10, $(H^{\otimes s})'_\lambda = H_\lambda^{\otimes s}$ for $\lambda_s \ne 0, p-1$, and $(H^{\otimes s})'_\lambda = H_{\lambda^0}^{\otimes s} \oplus H_\lambda^{\otimes s}$ for $\lambda_s = p-1$. Theorems 3.2 and 4.4 give

$$\begin{aligned} T(H_{\lambda^0}^{\otimes s}) &= \bigoplus_{\mu \in \Lambda} a_{\lambda^0\mu} H_\mu^{\otimes s} \\ &= \bigoplus_{\mu_s = 0} a_{\lambda^0\mu} H_\mu^{\otimes s}, \\ T(H_\lambda^{\otimes s}) &= \bigoplus_{\mu \in \Lambda} a_{\lambda\mu} H_\mu^{\otimes s} \\ &= \bigoplus_{\mu_s = 0} a_{\lambda\mu} H_\mu^{\otimes s} \oplus \bigoplus_{\mu_s \ne 0, p-1} a_{\lambda\mu} H_\mu^{\otimes s} \oplus \bigoplus_{\mu_s = p-1} a_{\lambda\mu} H_\mu^{\otimes s}, \quad \text{and} \\ T((H^{\otimes s})'_\lambda) &= \bigoplus_{\mu \in \Lambda'} a'_{\lambda\mu} (H^{\otimes s})'_\mu = \bigoplus_{\mu_s = p-1} a'_{\lambda\mu} (H^{\otimes s})'_\mu \oplus \bigoplus_{\mu_s \ne 0, p-1} a'_{\lambda\mu} (H^{\otimes s})'_\mu \\ &= \bigoplus_{\mu_s = p-1} a'_{\lambda\mu} H_\mu^{\otimes s} \oplus \bigoplus_{\mu_s \ne 0, p-1} a'_{\lambda\mu} H_{\mu^0}^{\otimes s} \oplus \bigoplus_{\mu_s = p-1} a'_{\lambda\mu} H_\mu^{\otimes s}. \end{aligned}$$

The result then follows by comparing coefficients. □

Using Theorem 4.11, we now determine the values of the $a_{\lambda\lambda}$.

Example 4.12. $a_{\lambda\lambda}$ equals one more than the number of nonzero λ_i's in λ.

Proof. $a_{(0,\ldots,0)(0,\ldots,0)} = 1$ by Example 3.11. The result follows by induction from the following. If $\lambda_s \neq 0$, then

$$\begin{aligned} a_{\lambda\lambda} = a'_{\lambda\lambda} &= a'_{(\lambda_1,\ldots,\lambda_{s-1},p-1)(\lambda_1,\ldots,\lambda_{s-1},p-1)} \\ &= a_{(\lambda_1,\ldots,\lambda_{s-1},0)(\lambda_1,\ldots,\lambda_{s-1},0)} + a_{(\lambda_1,\ldots,\lambda_{s-1},p-1)(\lambda_1,\ldots,\lambda_{s-1},0)} \\ &= a_{(\lambda_1,\ldots,\lambda_{s-1})(\lambda_1,\ldots,\lambda_{s-1})} + 1 . \end{aligned}$$

The first and third equalities are from Theorem 4.11, the second is from Example 4.8, and the forth is from Theorem 4.11 and Corollary 3.1(ii). □

For the second part of this section we recall some further results about the representations of $R' = \mathbf{F}_p[\mathrm{GL}_s(\mathbf{Z}/p)]$ (see [JK, Chapter 8]). Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ be a finite sequence of nonnegative integers and, for $j = 1, 2, \ldots$, let $\alpha_j = \sum_{i=j}^{\infty} \lambda_i$. The Weyl module, W'_λ is defined using the Young diagram having a_j nodes in the jth row. For example, $W'_{(1,2,2)}$ has diagram:

$$\begin{matrix} X & X & X & X & X \\ X & X & X & X & \\ X & X & & & \end{matrix}$$

There is a certain bilinear form $\Phi_\lambda\colon W'_\lambda \otimes W'_\lambda \to \mathbf{F}_p$ satisfying $\Phi_\lambda(mx, y) = \Phi_\lambda(x, m^t y)$ for $m \in \mathrm{GL}_s(\mathbf{Z}/p)$ and $x, y \in W'_\lambda$, where m^t denotes the transpose of m. It follows that $W'^{\perp}_\lambda = \{w \in W'_\lambda \mid \Phi_\lambda(w, v) = 0,\ \forall v \in W'_\lambda\}$ is an $\mathbf{F}_p[\mathrm{GL}_s(\mathbf{Z}/p)]$-module. Let $V'_\lambda = W'_\lambda / W'^{\perp}_\lambda$.

Recall that λ is p-regular if $0 \le \lambda_i \le p-1$ for each i, and is bounded by s if $\lambda_i = 0$ for $i > s$. In the following proposition we collect some of the properties of the Weyl modules.

Proposition 4.13 [JK, Chapter 8]. (i) *If λ is not bounded by s, then $W'_\lambda = 0$.*

(ii) *For λ bounded by s, V'_λ is irreducible and occurs exactly once as the top composition factor in any composition series for W'_λ.*

(iii) *For λ p- regular and bounded by s, the V'_λ for $0 \le \lambda_s \le p-2$ are distinct and give a complete set of irreducibles for R'.*

(iv) $W'_{(0,\ldots,0)} \cong V'_{(0,\ldots,0)}$ *is the trivial module.*

(v) $W'_{(0,\ldots,0,1)} \cong V'_{(0,\ldots,1)}$, *the determinant representation,* Det, *and* Det^{p-1} *is trivial.*

(vi) $W'_\lambda \otimes \mathrm{Det} \cong W'_{(\lambda_1,\ldots,\lambda_{s-1},\lambda_s+1)}$, *so* $V'_\lambda \otimes \mathrm{Det} \cong V'_{(\lambda_1,\ldots,\lambda_{s-1},\lambda_s+1)}$.

Remark 4.14. For $R = \mathbf{F}_p[\mathrm{M}_{s,s}(\mathbf{Z}/p)]$, there are Weyl modules W_λ and irreducibles V_λ defined in a similar fashion [HK, §6].

Notation 4.15. Recall that $\Lambda = \{\lambda \mid \lambda\ p\text{-regular, bounded by } s\}$ and $\Lambda' = \{\lambda \in \Lambda \mid 1 \le \lambda_s \le p-1\}$. Now let $\Lambda'' = \{\lambda \in \Lambda \mid 0 \le \lambda_s \le p-2\}$. By (iii) above, the set $\{V'_\lambda \mid \lambda \in \Lambda''\}$ gives a complete set of irreducibles for R'. By (v) and (vi), the set $\{V'_\lambda \mid \lambda \in \Lambda'\}$ gives the same set of irreducibles. We hope that the reader will not be confused by the appearance of two indexing sets for these modules. When comparing the R'- and R-modules (e.g., Theorem 4.11, Appendix) we prefer the set Λ'. But when dealing only with R' (e.g., Proposition 4.16) the set Λ'' is preferred.

Part (ii) of Proposition 4.13 says that V'_λ occurs exactly once as a composition factor of W'_λ. The next result is concerned with the other composition factors of W'_λ. Recall the definition of $\lambda \le \mu$ given before Proposition 3.9.

Proposition 4.16 [CK1, 3.6]. *If V'_λ, with $\lambda \in \Lambda''$, occurs as a composition factor of W'_μ, then $\lambda \le \mu$.*

Since we are only interested in composition factors of various modules, we may as well work in the representation ring of R', where $[M] = [M'] + [M'']$ when $0 \to M' \to M \to M'' \to 0$ is exact. There is a well-known formula called the Littlewood-Richardson rule which writes the tensor product $W'_\lambda \otimes W'_\mu$ as a sum of Weyl modules [JK, 2.8.14]. In fact this formula is combinatorial depending only on λ and μ (and independent of p and s). We remark that each W'_ν occurring in $W'_\lambda \otimes W'_\mu$ has $n(\nu) = n(\lambda) + n(\mu)$, where $n(\lambda)$ is the number of nodes in the Young diagram corresponding to λ.

It follows from Propositions 4.13(ii) and 4.16 that if we knew the composition factors of all of the Weyl modules, then using the Littlewood-Richardson rule we could inductively determine the tensor products of all of the irreducibles.

For small values of s and p, Carlisle determined the composition factors of many of the Weyl modules in his thesis [Car]. For p-regular λ, he proved the following.

Proposition 4.17 [Car, §4]. *Let λ be in Λ''.*

(i) *For $s = 1$ or 2: $W'_\lambda = V'_\lambda$.*

(ii) *For $s = 3$: if $\lambda_1 + \lambda_2 > p - 2$ and $0 < \lambda_1, \lambda_2 < p - 1$, then*

$$W'_{(\lambda_1, \lambda_2, \lambda_3)} = V'_{(\lambda_1, \lambda_2, \lambda_3)} + V'_{(p-2-\lambda_2, p-2-\lambda_1, \lambda_1+\lambda_2+\lambda_3-p+2)};$$

otherwise

$$W'_{(\lambda_1, \lambda_2, \lambda_3)} = V'_{(\lambda_1, \lambda_2, \lambda_3)}.$$

(iii) *For $s = 4$ and $p = 2$: if $\lambda \ne (1, 0, 1, 0)$, then $W'_\lambda = V'_\lambda$.*

$$W'_{(1,0,1,0)} = V'_{(1,0,1,0)} + V'_{(0,0,0,0)}.$$

For the smallest non-p-regular sequence, there is the following formula.

Proposition 4.18 [Car, 2.21]. $W_{(p,0,\dots,0)} = W_{(1,0,\dots,0)} + \sum_{a=0}^{p-2}(-1)^a W_{(p-a-2,0^a,1)}$ *where 0^a denotes the sequence with a zeros.*

To determine further decompositions, Carlisle tensors the above formula with a fixed Weyl module and applies the Littlewood-Richardson rule. The resulting equations for the W'_λ can often be used to determine the composition factors inductively.

With these techniques and Proposition 4.17, we have determined all of the $a'_{\lambda\mu}$ for $p = 2$, $s = 4$; $p = 3$, $s = 3$; and $p = 5$, $s = 2$. (Many of the formulas we needed were already in Carlisle's thesis.) The corresponding $a_{\lambda\mu}$ can then be determined from Theorem 4.11. The results are tabulated in the Appendix.

The module $V'_{(p-1,\dots,p-1)}$ is known as the Steinberg representation and has a number of nice properties. For instance, it is both irreducible and projective. Also, it equals $W'_{(p-1,\dots,p-1)}$. The corresponding R-module $V_{(p-1,\dots,p-1)}$ is associated to the summand $H^{\otimes s}_{(p-1,\dots,p-1)}$ which is isomorphic to $\widetilde{H}^*(L(s))$, where $L(s)$ is a stable wedge summand of the classifying space $B(\mathbf{Z}/p)^s_+$ [MP]. For $p = 2$, we give the complete decomposition of $T(H^{\otimes s}_{(p-1,\dots,p-1)})$.

Theorem 4.19. *For $p = 2$,*

$$T(H^{\otimes s}_{(1,\dots,1)}) \cong H^{\otimes s}_{(1,\dots,1,0)} \oplus H^{\otimes s}_{(1,\dots,1,0,1)} \oplus \cdots \oplus H^{\otimes s}_{(0,1,\dots,1)} \oplus (s+1)H^{\otimes s}_{(1,\dots,1)}.$$

Proof. The zero coefficients follow from Corollary 3.10(i): $a_{(1,\dots,1)\mu}=0$ when $\sum_{i=1}^{s}\mu_i<s-1$. The first nonzero coefficient follows from Corollary 3.10(ii), $a_{(1,\dots,1)(1,\dots,1,0)}=1$.

Recall that λ^0 denotes the sequence $(\lambda_1,\dots,\lambda_{s-1},0)$ when $\lambda=(\lambda_1,\dots,\lambda_s)$. Fix $\lambda=(1,\dots,1)$ and $\mu=(1,\dots,1,0,1,\dots,1)$, where $\mu_s\neq 0$. Again by Corollary 3.10(i), $a_{\lambda\mu^0}=0$. By induction on s (using Example 3.13), $a_{\lambda^0\mu^0}=1$. Two applications of Theorem 4.11 give $a_{\lambda\mu}=a'_{\lambda\mu}=a'_{\lambda\mu^0}+a_{\lambda^0\mu^0}=1$.

For the final coefficient, $a_{(1,\dots,1)(1,\dots,1)}=s+1$, use Example 4.12. □

This result is surprisingly similar to the following tensor product.

Proposition 4.20 [CK2, 6.1]. *For* $p=2$,

$$H\otimes H^{\otimes(s-1)}_{(1,\dots,1)}\cong H^{\otimes s}_{(1,\dots,1,0)}\oplus H^{\otimes s}_{(1,\dots,1,0,1)}\oplus\cdots\oplus H^{\otimes s}_{(0,1,\dots,1)}\oplus sH^{\otimes s}_{(1,\dots,1)}.$$

It follows that $\widetilde{T}(H^{\otimes s}_{(1,\dots,1)})\cong H\otimes H^{\otimes(s-1)}_{(1,\dots,1)}$, where $\widetilde{T}$ is the functor left adjoint to $A\mapsto A\otimes\widetilde{H}$ and $H=\mathbf{F}_2\oplus\widetilde{H}$.

For a graded vector space $N=\bigoplus_{i\geq 0}N_i$, denote the Poincaré series by $f_N(t)=\sum_{i\geq 0}\dim_{\mathbf{F}_p}(N_i)t^i$. We then have the following result for the Poincaré series of $T(H^{\otimes s}_{(1,\dots,1)})$.

Corollary 4.21. *For* $p=2$ *and* $N=T(H^{\otimes s}_{(1,\dots,1)})$,

$$f_N(t)=t^{(1+3+\cdots+2^{s-1}-1)}\cdot\frac{(1+t)(1+t^2)\dots(1+t^{2^{s-1}})}{(1-t)(1-t^3)\dots(1-t^{2^s-1})}.$$

Proof. This follows easily from Proposition 4.20 and the known Poincaré series for the Steinberg summands [MP]:

$$f_{H^{\otimes s}_{(1,\dots,1)}}(t)=\frac{t^{(1+3+\cdots+2^s-1)}}{(1-t)(1-t^3)\dots(1-t^{2^s-1})}.$$

This result also follows from results of Carlisle and Walker [CW] who determined the Poincaré series of the non-Steinberg summands occurring on the right in Theorem 4.19.

For odd primes we suggest the following analog to Theorem 4.19.

Conjecture 4.22. *Let* $\lambda[St]=(p-1,\dots,p-1)$ *and let* $\alpha_1(\mu)=\mu_1+\cdots+\mu_s$, *then*

(i) $a_{\lambda[St]\mu}=0$ *if* $\alpha_1(\mu)<(s-1)(p-1)$;
(ii) $a_{\lambda[St]\mu}=1$ *if* $(s-1)(p-1)\leq\alpha_1(\mu)<s(p-1)$; *and*
(iii) $a_{\lambda[St]\mu}=s+1$ *if* $\mu=\lambda[St]$.

Remark 4.23. Parts (i) and (iii) follow from Corollary 3.10(i) and Example 4.12, respectively. The above inductive proof for (ii) fails since μ_s may be something other than 0 or $p-1$.

5. The Campbell and Selick summands

Campbell and Selick [CS] have studied a certain collection of unstable modules $M_s(j)$ for $j\in\mathbf{Z}/(p^s-1)$, with $H^{\otimes s}\cong\bigoplus_{j\in\mathbf{Z}/(p^s-1)}M_s(j)$. In [S2], the second author determined the value of the T functor on these summands. In this section, we give a representation theoretic proof of his result.

We first recall the description of the $M_s(j)$ given by the first author using the representation theory of the cyclic group $C = \langle c \rangle$ of order $p^s - 1$ (see [H]). Fix a primitive $(p^s - 1)$st root of unity ω in $\mathbf{F}_{p^s}$. The irreducible representations of C over $\mathbf{F}_{p^s}$ are one-dimensional; we denote them by $\overline{W}_j$, $j \in \mathbf{Z}/(p^s - 1)$, with c acting by multiplication by ω^j.

To obtain $\mathbf{F}_p$-representations of C we apply the Frobenius automorphism $\phi\colon x \mapsto x^p$ of $\mathbf{F}_{p^s}$. For each orbit of $(\mathbf{F}_{p^s})^*$ under this action, choose a representative ω^i, and let I be the set of exponents occurring in these representatives. Let z_i be the number of elements in the orbit of ω^i:

$$z_i = \min\{z > 0 \mid \omega^{ip^z} = \omega^i\}.$$

For $i \in I$, let $\overline{V}_i = \bigoplus_{k=0}^{z_i - 1} \overline{W}_{ip^k}$. Then the character of $\overline{V}_i$ lies in $\mathbf{F}_p$, so there is an $\mathbf{F}_p$-representation V_i with $V_i \otimes \mathbf{F}_{p^s} \cong \overline{V}_i$ [HB, 1.17].

V_i is an irreducible representation of $\mathbf{F}_p[C]$. In fact, V_i is a projective indecomposable isomorphic to $\mathbf{F}_p[C]f_i$ for a unique primitive idempotent f_i in $\mathbf{F}_p[C] : \mathbf{F}_p[C] \cong \bigoplus_{i \in I} \mathbf{F}_p[C]f_i$. (This is just the familiar decomposition of the commutative semisimple ring $\mathbf{F}_p[C]$ as a direct sum of fields.) Note that $z_i = \dim_{\mathbf{F}_p} \hom_{\mathbf{F}_p[C]}(V_i, V_i)$.

To fix an inclusion $\rho\colon \mathbf{F}_p[C] \to \mathbf{F}_p[\mathrm{M}_{s,s}(\mathbf{Z}/p)]$, let c act as left multiplication by ω on $\mathbf{F}_{p^s}$ with basis $\{1, \omega, \dots, \omega^{s-1}\}$. For f_i, the idempotent associated to V_i we have by Proposition 1.8,

$$\mathbf{F}_p[\mathrm{M}_{s,s}(\mathbf{Z}/P)]\rho(f_i) \cong \bigoplus_{\mu \in \Lambda} z_i b_{i\mu} P_\mu,$$

where $b_{i\mu}$ is the multiplicity of V_i in $\rho_*(V_\mu)$, the restriction of V_μ to $\mathbf{F}_p[C]$. Since $V_i \otimes \mathbf{F}_{p^s} \cong \overline{V}_i \cong \bigoplus_{k=0}^{z_i-1} \overline{W}_{ip^k}$, we have that $b_{i\mu}$ also equals the multiplicity of $\overline{W}_i$ in $(\overline{\rho})_*(V_\mu \otimes \mathbf{F}_{p^s})$, where $\overline{\rho}\colon \mathbf{F}_{p^s}[C] \to \mathbf{F}_{p^s}[\mathrm{M}_{s,s}(\mathbf{Z}/p)]$ is induced by ρ.

Definition 5.1. For each $i \in I$, let $\widehat{M}_s(i) = \bigoplus_\mu z_i b_{i\mu} H_\mu^{\otimes s}$, and let $M_s(j) = \bigoplus_\mu b_{i\mu} H_\mu^{\otimes s}$ for $j \equiv ip^k \pmod{p^s - 1}$.

Theorem 5.2 [S2]. $T(M_s(j)) \cong M_s(j) \oplus H^{\otimes s}$.

Proof. Consider the map $\tau \circ \rho\colon \mathbf{F}_p[C] \to M_{p^s, p^s}(\mathbf{F}_p) \otimes \mathbf{F}_p[\mathrm{M}_{s,s}(\mathbf{Z}/p)]$ (where τ was defined in §3). The image of the idempotent f_i under this map gives

$$T(\widehat{M}_s(i)) = \bigoplus_\mu z_i c_{i\mu} H_\mu^{\otimes s},$$

where $c_{i\mu}$ is the multiplicity of V_i in $(\tau \circ \rho)_*(V^{p^s} \otimes V_\mu)$. We need the following lemma, which is proved below.

Lemma 5.3. $c_{i\mu} = \sum_{k=0}^{p^s-1} b_{i-k,\mu}$.

We now have the following isomorphisms.

$$\begin{aligned}
T(\widehat{M}_s(i)) &\cong \bigoplus_{\mu} z_i \left(\sum_{k=0}^{p^s-1} b_{i-k,\mu} \right) H_\mu^{\otimes s} \\
&\cong \bigoplus_{\mu} z_i b_{i\mu} H_\mu^{\otimes s} \oplus \bigoplus_{\mu} z_i \left(\sum_{k=1}^{p^s-1} b_{k\mu} \right) H_\mu^{\otimes s} \\
&\cong \widehat{M}_s(i) \oplus z_i \bigoplus_{k=1}^{p^s-1} \bigoplus_{\mu} b_{k\mu} H_\mu^{\otimes s} \\
&\cong \widehat{M}_s(i) \oplus z_i \bigoplus_{k=1}^{p^s-1} M_s(k) \cong \widehat{M}_s(i) \oplus z_i H^{\otimes s}.
\end{aligned}$$

Since $\widehat{M}_s(i) \cong z_i M_s(j)$ for $j = ip^k$, the theorem follows.

Proof of Lemma 5.3. By tensoring with $\mathbf{F}_{p^s}$, we have that $c_{i\mu}$ is the multiplicity of $\overline{W}_i$ in $(\overline{\tau \circ \rho})_*(V^{p^s} \otimes V_\mu \otimes \mathbf{F}_{p^s})$. Since $V^{p^s} \otimes V_\mu \otimes \mathbf{F}_{p^s} \cong (V^{p^s} \otimes \mathbf{F}_{p^s}) \otimes_{\mathbf{F}_{p^s}} (V_\mu \otimes \mathbf{F}_{p^s})$, it follows that $c_{i\mu}$ is the multiplicity of $\overline{W}_i$ in $(\overline{\beta \circ \rho})_*(V^{p^s}) \otimes_{\mathbf{F}_{p^s}} \overline{\rho}_*(V_\mu \otimes \mathbf{F}_{p^s})$. It is easy to see that $(\overline{\beta \circ \rho})_*(V^{p^s}) \cong \bigoplus_{k=0}^{p^s-1} \overline{W}_k$. Also, for any $\mathbf{F}_{p^s}[C]$-representation $\overline{U}$, the multiplicity of $\overline{W}_i$ in $\overline{W}_k \otimes_{\mathbf{F}_{p^s}} \overline{U}$ equals the multiplicity of $\overline{W}_{i-k}$ in $\overline{U}$ (this uses the fact that the $\overline{W}_i$ are one-dimensional group representations). Putting these facts together, we get

$$c_{i\mu} = \sum_{k=0}^{p^s-1} \text{multiplicity of } \overline{W}_{i-k} \text{ in } (\overline{\rho})_*(V_\mu \otimes \mathbf{F}_{p^s}),$$

which equals $\sum_{k=0}^{p^s-1} b_{i-k,\mu}$. □

6. Topological applications

Let V denote the elementary abelian p-group $(\mathbf{Z}/p)^s$. As shown in [HK] (and reviewed below), the $\mathscr{U}$-decompositions of $H^*(BV)$ can be topologically realized by wedge decompositions of the space $\Sigma(B\dot{V}_+)$. In this section, we use the T functor to determine the homotopy type of map $(B(\mathbf{Z}/p), X)$, where X is a wedge summand of ΣBV.

For technical reasons we work in the category of simplicial sets, and the mapping space will be the simplicial function space. Our principle tool is a theorem due to Lannes which allows us to determine the mapping space if a suitable candidate can be found. This theorem involves the $\mathbf{F}_p$-completion, $(\mathbf{F}_p)_\infty$, defined by Bousfield and Kan [BK].

Theorem 6.1 [L, 7.2.1]. *Suppose X and Y are spaces with H^*X and H^*Y of finite type. Further suppose that there is map $B(\mathbf{Z}/p) \times Y \xrightarrow{\omega} X$ such that $T(H^*X) \xrightarrow{\mathrm{ad}(\omega^*)} H^*Y$ is an isomorphism. Then the map*

$$(\mathbf{F}_p)_\infty Y \to \mathrm{map}(B(\mathbf{Z}/p), (\mathbf{F}_p)_\infty X)$$

induced by ω is a homotopy equivalence.

We now describe the space $\mathrm{map}(B(\mathbf{Z}/p), BV)$.

Proposition 6.2. (i) $\operatorname{map}(B(\mathbf{Z}/p), BV)$ *is homotopy equivalent to a disjoint union of* p^s *copies of* BV.

(ii) $\Sigma \operatorname{map}(B(\mathbf{Z}/p), BV_+) \simeq p^s\Sigma(BV_+)$.

Proof. (ii) follows directly from (i).

The components of $\operatorname{map}(B(\mathbf{Z}/p), BV)$ correspond to homotopy classes of maps from $B(\mathbf{Z}/p)$ to BV. These homotopy classes in turn correspond to group homomorphisms from $\mathbf{Z}/p$ to V. Since there are p^s elements of V, there are p^s components in $\operatorname{map}(B(\mathbf{Z}/p), BV)$.

Now fix a map, $B(\mathbf{Z}/p) \xrightarrow{\phi} BV$, and consider the component of

$$\operatorname{map}(B(\mathbf{Z}/p), BV),$$

containing ϕ. Denote this space by $\operatorname{map}(B(\mathbf{Z}/p), BV)_\phi$ and take ϕ to be its base point. There is a map, $\operatorname{map}(B(\mathbf{Z}/p), BV)_\phi \xrightarrow{\tilde{e}} BV$, which takes a function and evaluates it at the base point, giving an element of BV. We will show that $\tilde{e}$ induces an isomorphism on homotopy groups and is therefore a homotopy equivalence.

A map $S^m \xrightarrow{\alpha} \operatorname{map}(B(\mathbf{Z}/p), BV)_\phi$, representing a class in

$$\pi_m \operatorname{map}(B(\mathbf{Z}/p), BV)_\phi,$$

gives a map $S^m \times B(\mathbf{Z}/p) \to BV$, whose restriction to $S^m \vee B(\mathbf{Z}/p)$ is $(\tilde{e}\circ\alpha)\vee\phi$. The cofibration

$$S^m \vee B(\mathbf{Z}/p) \to S^m \times B(\mathbf{Z}/p) \to S^m \wedge B(\mathbf{Z}/p)$$

gives a split short exact sequence in cohomology with coefficients in V. As a result

$$[S^m, BV] \times [B(\mathbf{Z}/p), BV] \cong [S^m \times B(\mathbf{Z}/p), BV].$$

Therefore $\pi_m \operatorname{map}(B(\mathbf{Z}/p), BV)_\phi \cong \pi_m BV$ with the isomorphism induced by $\tilde{e}$. □

For any space Y, there is an evaluation map from $B(\mathbf{Z}/p) \times \operatorname{map}(B\mathbf{Z}/p), Y)$ to Y. If we apply mod-p cohomology and then take the adjoint we get a natural transformation, ψ, with

$$T(H^*(Y)) \xrightarrow{\psi_Y} H^*(\operatorname{map}(B(\mathbf{Z}/p), Y)).$$

For sufficiently nice Y, ψ_Y is an isomorphism.

Proposition 6.3. ψ_{BV} *and* ψ_{BV_+} *are isomorphisms.*

Proof. For ψ_{BV}, we make use of the description of $\operatorname{map}(B(\mathbf{Z}/p), BV)$ in Proposition 6.2 and the description of $T(H^*(BV))$ in Proposition 2.2. Note that both $T(H^*(BV))$ and $H^* \operatorname{map}(B(\mathbf{Z}/p), BV)$ are isomorphic to $p^s H^*(BV)$ as objects in $\mathscr{U}$.

We can consider ψ_{BV} as an algebra endomorphism of $B(s) \otimes H^*(BV)$ (see Proposition 2.2). Using the splitting of the T functor which comes from splitting H as $\widetilde{H} \oplus \mathbf{F}_p$ it is not hard to show that ψ_{BV} takes $1 \otimes H^*(BV)$ to $1 \otimes H^*(BV)$. Thus it is sufficient to consider the restriction of ψ_{BV} to $B(s)$. However ψ_{BV} induces a bijection on the set of algebra maps from $B(s)$ to $\mathbf{F}_p$. This is just the correspondence between augmentations of $T(H^*(BV))$ and the components of $\operatorname{map}(B(\mathbf{Z}/p), BV)$. Furthermore the algebra maps from $B(s)$

to $\mathbf{F}_p$ form a basis for the dual vector space $B(s)^*$. Thus ψ_{BV} is an isomorphism.

The result for ψ_{BV_+} follows directly from the result for ψ_{BV}. □

We now study the space $\mathrm{map}(B(\mathbf{Z}/p), \Sigma(BV_+))$. Consider the map

$$\Sigma\,\mathrm{map}(B(\mathbf{Z}/p), X) \xrightarrow{\phi_X} \mathrm{map}(B(\mathbf{Z}/p), \Sigma X)$$

defined by $\phi_X(t \wedge f)(x) = t \wedge f(x)$.

Lemma 6.4. *If ψ_X is an isomorphism, then $\phi_X^* \circ \psi_{\Sigma X}$ is an isomorphism.*

Proof. Define ω to be the composition of $1 \times \phi_X$ with evaluation. Then $\mathrm{ad}(\omega^*) = \phi_X^* \circ \psi_{\Sigma X}$. By direct calculation the following diagram commutes.

$$\begin{array}{ccc}
(S^1 \vee \mathrm{map}(B(\mathbf{Z}/p), X)) \times B(\mathbf{Z}/p) & \longrightarrow & S^1 \vee X \\
\downarrow & & \downarrow \\
(S^1 \times \mathrm{map}(B(\mathbf{Z}/p), X)) \times B(\mathbf{Z}/p) & \xrightarrow{1\times e} & S^1 \times X \\
\downarrow & & \downarrow \\
(S^1 \wedge \mathrm{map}(B(\mathbf{Z}/p), X)) \times B(\mathbf{Z}/p) & \xrightarrow{\omega} & S^1 \wedge X
\end{array}$$

Now we apply reduced cohomology to the above diagram and take the appropriate adjoints.

$$\begin{array}{ccc}
T(\widetilde{H}^*(\Sigma X)) & \xrightarrow{\mathrm{ad}(\omega^*)} & \widetilde{H}^*(\Sigma\,\mathrm{map}(B(\mathbf{Z}/p), X)) \\
\downarrow & & \downarrow \\
T(\widetilde{H}^*(S^1 \times X)) & \longrightarrow & \widetilde{H}^*(S^1 \times \mathrm{map}(B(\mathbf{Z}/p), X)) \\
\downarrow & & \downarrow \\
T(\widetilde{H}^*(S^1 \vee X)) & \longrightarrow & \widetilde{H}^*(S^1 \vee \mathrm{map}(B(\mathbf{Z}/p), X))
\end{array}$$

Note that both columns are split short exact sequences. Also the T functor commutes with tensor products and direct sums. Furthermore, since $H^*(S^1)$ is finite, $T(H^*(S^1)) = H^*(S^1)$. Using these facts it is not hard to show that the middle horizontal map in the above diagram is $1 \otimes \psi_X$ while the lower map is $1 + \psi_X$. By hypothesis, ψ_X is an isomorphism, thus $\mathrm{ad}(\omega^*)$ is an isomorphism. □

Corollary 6.5. *$\phi_{BV}^* \circ \psi_{\Sigma BV}$ and $\phi_{BV_+}^* \circ \psi_{\Sigma(BV_+)}$ are isomorphisms.*

Proof. Propositions 6.3 and 6.4. □

Theorem 6.6. $\mathrm{map}(B(\mathbf{Z}/p), (\mathbf{F}_p)_\infty \Sigma(BV_+)) \simeq (\mathbf{F}_p)_\infty(p^s \Sigma(BV_+))$.

Proof. By Proposition 6.2, $p^s\Sigma(BV_+) \simeq \Sigma\,\mathrm{map}(B(\mathbf{Z}/p), BV_+)$. The result follows by applying Theorem 6.1 with ω defined as in the proof of Lemma 6.4. □

Remark 6.7. (a) Note that $\Sigma(BV_+) \simeq S^1 \vee \Sigma BV$. Since S^1 is not $\mathbf{F}_p$-complete, the completions in the above theorem are necessary. However, ΣBV is $\mathbf{F}_p$-complete (its homotopy groups are finite p-torsion groups). By the same arguments as above, we can show that $\Sigma\,\mathrm{map}(B(\mathbf{Z}/p), BV)$ is homotopy equivalent to $(p^s - 1)S^1 \vee p^s\Sigma BV$ and that

$$\mathrm{map}(B(\mathbf{Z}/p), \Sigma BV) \simeq (\mathbf{F}_p)_\infty((p^s - 1)S^1 \vee p^s\Sigma BV),$$

where we do not need a completion on the left.

(b) Bousfield [B, §11] has shown that $S^1 \vee S^1$ is $\mathbf{F}_p$-bad (i.e., the completion map $S^1 \vee S^1 \longrightarrow (\mathbf{F}_p)_\infty(S^1 \vee S^1)$ does not induce an isomorphism on mod-p

homology). It follows that $p^s\Sigma(BV_+)$, which has p^sS^1 as a retract, if $\mathbf{F}_p$-bad. It also follows that $\psi_{\Sigma(BV_+)}$ is not an isomorphism.

We now wish to find the analog of Theorem 6.6 when $\Sigma(BV_+)$ is replaced by one of its wedge summands. To review the wedge decomposition of $\Sigma(BV_+)$ we first recall the following general method for determining wedge decompositions of ΣX for a connected p-local space X (see [Co]). Given a self-map f of ΣX, define the telescope $\mathrm{tel}_f(\Sigma X)$ as $(\coprod_{n\in N}(\Sigma X \times [2n, 2n+1]))/\sim$, where $(x, 2n+1) \sim (f(x), 2n+2)$ for $x \in \Sigma X$ and $n \in \mathbf{N}$. Define $\Sigma X \xrightarrow{\theta_f} \mathrm{tel}_f(\Sigma X)$ by $x \mapsto (x, 0)$. Given self-maps $f_1, \ldots, f_N$ of ΣX, we get a map

$$\theta\colon \Sigma X \xrightarrow{\text{pinch}} \Sigma X \vee \cdots \vee \Sigma X \xrightarrow{\theta_{f_1}\vee\cdots\vee\theta_{f_N}} \bigvee_{i=1}^{N} \mathrm{tel}_{f_i}(\Sigma X).$$

If the f_i induce an orthogonal idempotent decomposition of the identity $1 = f_1^* + \cdots + f_N^*$ in $\mathrm{end}_{\mathscr{U}}(\widetilde{H}^*(\Sigma X))$, then θ is a homology equivalence. If X is p-local then so is ΣX, thus θ is a homotopy equivalence and gives a wedge decomposition of ΣX.

Since BV is connected and p-local we can apply this procedure to ΣBV. The coproduct on ΣBV induces a group structure on the set of homotopy classes of base point preserving self-maps, $[\Sigma BV, \Sigma BV]$. While this group is in general nonabelian, the map from $[\Sigma BV, \Sigma BV]$ to $\mathrm{end}_{\mathscr{U}}(\widetilde{H}^*\Sigma BV)$ is a group homomorphism. It is a consequence of the work of Adams, Gunawardena, and Miller [AGM] that this homomorphism is surjective. As a result any decomposition of $\widetilde{H}^*\Sigma BV$ into $\mathscr{U}$-summands can be realized as a decomposition of ΣBV as a bouquet simply by choosing elements in $[\Sigma BV, \Sigma BV]$ which induce the appropriate idempotents in $\mathrm{end}_{\mathscr{U}}(\widetilde{H}^*\Sigma BV)$.

The indecomposable decomposition of $\widetilde{H}^*BV_+$ in Proposition 2.4 gives an indecomposible decomposition of $\widetilde{H}^*\Sigma(BV_+)$. Since ΣBV is simply connected and $\widetilde{H}^*S^1$ is concentrated in degree one, $\mathrm{end}_{\mathscr{U}}(\widetilde{H}^*\Sigma(BV_+))$ is isomorphic to $\mathrm{end}_{\mathscr{U}}(\widetilde{H}^*S^1) \oplus \mathrm{end}_{\mathscr{U}}(\widetilde{H}^*\Sigma BV)$ as $\mathrm{end}_{\mathscr{U}}(\widetilde{H}^*\Sigma(BV_+))$-modules. Furthermore any primitive idempotent in $\mathrm{end}_{\mathscr{U}}(\widetilde{H}^*\Sigma(BV_+))$ must lie in either $\mathrm{end}_{\mathscr{U}}(\widetilde{H}^*S^1)$ or $\mathrm{end}_{\mathscr{U}}(\widetilde{H}^*\Sigma BV)$. In fact, $\widetilde{H}^*S^1$ is the indecomposable summand $\Sigma H^{\otimes s}_{(0,\ldots,0)}$. Let $X_{(0,\ldots,0)} = S^1$. For $\lambda \neq (0, \ldots, 0)$, the primitive idempotents e_λ lie in $\mathrm{end}_{\mathscr{U}}(\widetilde{H}^*\Sigma BV)$. For these λ, let $f_\lambda \in [\Sigma BV, \Sigma BV]$ be a map which induces $e_\lambda \in \mathrm{end}(\widetilde{H}^*\Sigma BV)$, and let $X_\lambda = \mathrm{tel}_{f_\lambda}\Sigma BV$. Thus $\Sigma(BV_+) \simeq \bigvee_{\lambda\in\Lambda} \dim_{\mathbf{F}_p}(V_\lambda)X_\lambda$. It follows that $p^s\Sigma(BV_+) \simeq \bigvee_{\lambda\in\Lambda} p^s \dim_{\mathbf{F}_p}(V_\lambda)X_\lambda$; we fix such a decomposition. For the space $\bigvee_\lambda c_\lambda X_\lambda$ with $c_\lambda \le p^s \dim_{\mathbf{F}_p}(V_\lambda)$, we denote the obvious inclusion (mapping $c_\lambda X_\lambda$ onto the first c_λ X_λ's in $p^s\Sigma(BV_+)$) by i and the projection by π.

Proposition 6.8. *Any direct summand of $\widetilde{H}^*(p^s\Sigma(BV_+))$ can be realized by a wedge summand of $p^s\Sigma(BV_+)$. More precisely, let $B \xrightarrow{j} \widetilde{H}^*(p^s\Sigma(BV_+))$ be the inclusion of a summand. So $B \cong \bigoplus_\lambda c_\lambda \Sigma H_\lambda^{\otimes s}$ with $c_\lambda \le p^s \dim_{\mathbf{F}_p}(V_\lambda)$. Let $Y = \bigvee_\lambda c_\lambda X_\lambda$ and let $Y \xrightarrow{i} p^s\Sigma(BV_+)$ be the inclusion as above. Then there is a homotopy equivalence h of $p^s\Sigma(BV_+)$ such that $B \xrightarrow{i^*\circ(h^{-1})^*\circ j} \widetilde{H}^*(Y)$ is an isomorphism.*

Proof. Let $\widetilde{H}^*(p^s\Sigma(BV_+)) \xrightarrow{k} B$ be the projection. The endomorphisms $e = j \circ k$ and $d = (i \circ \pi)^*$ are the idempotents in $\mathrm{end}_{\mathscr{U}}(\widetilde{H}^* p^s\Sigma(BV_+))$ determining B and $\widetilde{H}^*(Y)$. Since this endomorphism ring is a finite-dimensional algebra, we can refine the idempotent decompositions $e+(1-e)$ and $d+(1-d)$ to primitive orthogonal decompositions and any two primitive orthogonal decompositions are conjugate. It follows that there exists a unit, u, such that $u^{-1}eu = d$.

We wish to choose u so that $u = h^*$ for some homotopy equivalence h. Since $p^s\Sigma BV$ is simply connected and $\widetilde{H}^* p^s S^1$ is concentrated in degree one, we have

$$\mathrm{end}_{\mathscr{U}}(\widetilde{H}^* p^s\Sigma(BV_+)) \cong \mathrm{end}_{\mathscr{U}}(\widetilde{H}^* p^s S^1) \oplus \mathrm{end}_{\mathscr{U}}(\widetilde{H}^* p^s \Sigma BV).$$

Thus we can choose u_1, the restriction of u to $\widetilde{H}^* p^s S^1$, independently of u_2, the restriction of u to $\widetilde{H}^*\Sigma BV$. Since the map from $[p^s\Sigma BV, p^s\Sigma BV]$ to $\mathrm{end}_{\mathscr{U}}(\widetilde{H}^* p^s\Sigma BV)$ is surjective and $p^s\Sigma BV$ is simply connected and p-local, any unit in $\mathrm{end}_{\mathscr{U}}(\widetilde{H}^* p^s\Sigma BV)$ is induced by some homotopy equivalence of $p^s\Sigma BV$. Thus for any choice of u_2, there is a homotopy equivalence h_2 with $h_2^* = u_2$.

There are, however, units in $\mathrm{end}_{\mathscr{U}}(\widetilde{H}^* p^s S^1)$ which are not induced by homotopy equivalences of $p^s S^1$. As a result we must choose u_1 with care.

Let e_1 and d_1 be the restrictions of e and d to $\widetilde{H}^* p^s S^1$. The image of d_1 is the subspace spanned by the first $c_{(0,\dots,0)}$ elements of the standard basis. We can chooses a basis for the image of e_1 and extend to a basis for $\widetilde{H}^* p^s S^1$ so that the change of basis matrix u_1 (having $u_1^{-1} e_1 u_1 = d_1$) is, up to a permutation of the columns, lower triangular with ones on the diagonal. By Lemma 6.9 below, there is a homotopy equivalence h_1 in $[p^s S^1, p^s S^1]$ with $(h_1)^* = u_1$.

Finally, let $h = h_1 \vee h_2$ in $[p^s\Sigma(BV_+), p^s\Sigma(BV_+)]$ By the choice of h we have $(h^{-1})^* jkh^* = \pi^* i^*$. Multiplying this formula on the right by $(h^{-1})^* j$ yields $(h^{-1})^* j = \pi^* i^* (h^{-1})^* j$. It follows that $i^*(h^{-1})^* j : B \to \widetilde{H}^*(Y)$ is a monomorphism and hence an isomorphism since B and $\widetilde{H}^*(Y)$ have the same dimension in each degree. □

Lemma 6.9. *Suppose $u \in \mathrm{end}_{\mathscr{U}}(\widetilde{H}^* mS^1) \cong M_{m,m}(\mathbf{F}_p)$ is a lower triangular matrix whose diagonal entries are* 1. *Then there is a homotopy equivalence of mS^1, say h, with $h^* = u$.*

Proof. An element in $[mS^1, mS^1]$ is determined by the map induced on $\pi_1(mS^1)$, the free group on m generators. Take $\{b_r\}$ to be the standard generating set and let u be the matrix (u_{st}). Define h by $h(b_r) \equiv b_1^{u_{r1}} \cdots b_{r-1}^{u_{r(r-1)}} b_r$. Clearly $h^* = u$. Define g recursively starting with $g(b_1) \equiv b_1$ and setting $g(b_r) \equiv (g(b_1^{u_{r1}} \cdots b_{r-1}^{u_{r(r-1)}}))^{-1} b_r$. Direct calculation shows that $h \circ g$ and $g \circ h$ are both the identity. Thus h is a homotopy equivalence. □

Remark 6.10. Note that the map $\mathrm{M}_{m,m}(\mathbf{Z}) \to [mS^1, mS^1]$ does not necessarily send invertible matrices to homotopy equivalences.

We now prove the main result of this section.

Theorem 6.11. *Let X be a wedge summand of $\Sigma(BV_+)$, and let $Y = \bigvee_\lambda c_\lambda X_\lambda$, where $T(\widetilde{H}^* X) \cong \bigoplus_\lambda c_\lambda \Sigma H_\lambda^{\otimes s}$. Then $\mathrm{map}(B(\mathbf{Z}/p), (\mathbf{F}_p)_\infty X) \simeq (F_p)_\infty Y$. If X*

is a wedge summand of ΣBV, *then* X *is* $\mathbf{F}_p$*-complete and* $\operatorname{map}(B(\mathbf{Z}/p), X) \simeq (\mathbf{F}_p)_\infty Y$.

Proof. Let $\rho\colon \Sigma(BV_+) \longrightarrow X$ be the projection. By Corollary 6.5(ii) and Proposition 6.2(ii), the composition

$$T(\widetilde{H}^*X) \xrightarrow{T(\rho^*)} T(\widetilde{H}^*\Sigma(BV_+)) \xrightarrow{\phi^*_{BV_+}\circ\psi_{\Sigma(BV_+)}} \widetilde{H}^*(\Sigma\operatorname{map}(B(\mathbf{Z}/p), BV_+)) \xrightarrow{(\simeq)^*} \widetilde{H}^*(p^n\Sigma(BV_+))$$

is an isomorphism onto a direct summand $B \xrightarrow{j} \widetilde{H}^*(p^s\Sigma(BV_+))$. Take h as in Lemma 6.8 and let $f = h^{-1}\circ i$ from Y to $p^s\Sigma BV$ (so $B \xrightarrow{f^*\circ j} \widetilde{H}^*(Y)$ is an isomorphism).

Consider the map $\omega\colon B\mathbf{Z}/p \times Y \longrightarrow X$ given by the following diagram:

$$\begin{array}{ccc}
B(\mathbf{Z}/p)\times Y & & \\
\downarrow 1\times f & & \\
B(\mathbf{Z}/p)\times p^s\Sigma(BV_+) & & \\
\downarrow \simeq & & \\
B(\mathbf{Z}/p)\times \Sigma\operatorname{map}(B(\mathbf{Z}/p), BV_+) & & \\
\downarrow 1\times\phi_{BV_+} & & \\
B(\mathbf{Z}/p)\times \operatorname{map}(B(\mathbf{Z}/p), \Sigma(BV_+)) & \xrightarrow{e} & \Sigma(BV_+) \\
\downarrow 1\times\operatorname{map}(B(\mathbf{Z}/p),\rho) & & \\
B(\mathbf{Z}/p)\times \operatorname{map}(B(\mathbf{Z}/p), X) & \xrightarrow{e} & X
\end{array}$$

By construction, $\operatorname{ad}(\omega^*)$ is an isomorphism. Hence the result follows from Theorem 6.1. □

The following special case follows from the algebraic results of the previous sections.

Theorem 6.12. $\operatorname{map}(B(\mathbf{Z}/p), (\mathbf{F}_p)_\infty X_\lambda) \simeq (\mathbf{F}_p)_\infty(\bigvee_{\mu\in\Lambda} a_{\lambda\mu}X_\mu)$.

Proof. We apply Theorem 6.11 with $X = X_\lambda$ and $Y = \bigvee_\mu a_{\lambda\mu}X_\mu$. This is the appropriate Y since $T(\widetilde{H}^*(X_\lambda)) \cong \bigoplus_\mu a_{\lambda\mu}\Sigma H_\mu^{\otimes s}$ by Theorem 3.2. □

We conclude with a comment about the $\mathbf{F}_p$-completions in Theorem 6.12. $S^1 = X_{(0,\dots,0)}$ is the only indecomposable wedge summand of $\Sigma(BV_+)$ which fails to be $\mathbf{F}_p$-complete. So unless $\lambda = (0, \dots, 0)$ the first completion is redundant. Since $p^s\Sigma BV$ is $\mathbf{F}_p$-complete, the second completion is also redundant unless $a_{\lambda(0,\dots,0)} \neq 0$. Recall from Example 3.12 that

$$a_{\lambda(0,\dots,0)} = \begin{cases} 1 & \text{if } \lambda = \lambda[d], \\ 0 & \text{otherwise}. \end{cases}$$

Corollary 6.13. *If* $\lambda \neq \lambda[d]$, *then* $\operatorname{map}(B(\mathbf{Z}/p), X_\lambda) \simeq \bigvee_{\mu\in\Lambda} a_{\lambda\mu}X_\mu$.

APPENDIX. $a_{\lambda\mu}$ FOR SMALL p AND s

We tabulate the values for the coefficients $a_{\lambda\mu}$ occurring in Theorem 3.2 for small values of p and s in Tables 1–3. The coefficient $a_{\lambda\mu}$ occurs in the $c(\lambda)$th row and $c(\mu)$th column, where $c(\lambda) = \sum_{i=1}^{s} \lambda_i p^i$, and rows (columns) are numbered from 0 to $p^s - 1$. This ordering is chosen so as to make the consequences of the results of the paper easy to recognize. For example, the upper left $p^{s-1} \times p^{s-1}$ subblock of each table gives the $\{a_{\lambda\mu}\}$ for $H^{\otimes(s-1)}$, and by Theorem 4.11, this subblock added to the lower left $p^{s-1} \times p^{s-1}$ subblock gives the lower right $p^{s-1} \times p^{s-1}$ subblock. Also note that the lower right $(p-1)p^{s-1} \times (p-1)p^{s-1}$ subblock gives the $\{a'_{\lambda\mu}\}$; Example 4.8 shows that the $p^{s-1} \times p^{s-1}$ subblocks of this are the same along diagonals.

In general, the entries were found using the methods of Carlisle sketched in the second part of §4. The entries following from Corollary 3.10 and Theorem 4.11 are printed in bold face.

TABLE 1. $a_{\lambda\mu}$ for $p = 2$ and $s \leq 4$

$a_{\lambda\mu}$	0 0 0 0	1 0 0 0	0 1 0 0	1 1 0 0	0 0 1 0	1 0 1 0	0 1 1 0	1 1 1 0	0 0 0 1	1 0 0 1	0 1 0 1	1 1 0 1	0 0 1 1	1 0 1 1	0 1 1 1	1 1 1 1
0000	**1**															
1000	**1**	**2**														
0100	**1**	2	**2**	2												
1100	**0**	**1**	**1**	**3**												
0010	**1**	1	1	2	**2**	1	1	2								
1010	**0**	**1**	2	5	**1**	**3**	2	5								
0110	**0**	**0**	**1**	3	**1**	2	**3**	5								
1110	**0**	**0**	**0**	**1**	**0**	**1**	**1**	**4**								
0001	**1**	2	2	6	2	4	6	20	**2**	2	2	6	2	4	6	20
1001	**0**	**1**	1	2	1	1	2	7	**1**	**3**	1	2	1	1	2	7
0101	**0**	**0**	**1**	2	2	2	4	12	**1**	2	**3**	4	2	2	4	12
1101	**0**	**0**	**0**	**1**	**0**	2	3	9	**0**	**1**	**1**	**4**	**0**	2	3	9
0011	**0**	**0**	**0**	0	**1**	0	1	5	**1**	1	1	2	**3**	1	2	7
1011	**0**	**0**	**0**	**0**	**0**	**1**	3	9	**0**	**1**	2	5	**1**	**4**	5	14
0111	**0**	**0**	**0**	**0**	**0**	**0**	**1**	4	**0**	**0**	**1**	3	**1**	2	**4**	9
1111	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**1**	**0**	**0**	**0**	**1**	**0**	**1**	**1**	**5**

TABLE 2. $a_{\lambda\mu}$ for $p = 3$ and $s \leq 3$

$a_{\lambda\mu}$	0 0 0	1 0 0	2 0 0	0 1 0	1 1 0	2 1 0	0 2 0	1 2 0	2 2 0	0 0 1	1 0 1	2 0 1	0 1 1	1 1 1	2 1 1	0 2 1	1 2 1	2 2 1	0 0 2	1 0 2	2 0 2	0 1 2	1 1 2	2 1 2	0 2 2	1 2 2	2 2 2
000	**1**																										
100	**1**	**2**	**1**																								
200	**1**	**1**	**2**																								
010	**0**	1	1	**2**	1	2	**0**	1	1																		
110	**1**	2	3	**1**	**3**	3	**1**	2	3																		
210	**0**	1	1	**1**	**1**	**3**	**0**	1	1																		
020	**1**	1	2	**0**	1	1	**2**	1	2																		
120	**0**	**1**	2	**1**	2	3	**1**	**3**	3																		
220	**0**	**0**	**1**	**0**	1	1	**1**	**1**	**3**																		
001	**0**	1	3	2	4	7	3	6	13	**2**	2	2	1	2	6	2	7	10	**0**	1	3	2	4	7	3	6	13
101	**0**	0	0	1	1	3	1	2	4	**1**	**3**	2	1	1	2	1	3	5	**0**	0	0	1	1	3	1	2	4
201	**0**	0	0	1	0	2	0	1	2	**1**	**1**	**3**	**0**	1	1	2	2	5	**0**	0	0	1	0	2	0	1	2
011	**1**	1	1	0	1	2	2	3	5	**1**	1	1	**3**	1	3	0	2	4	**1**	1	1	0	1	2	2	3	5
111	**0**	1	1	2	2	6	1	5	8	**1**	2	3	**1**	**4**	5	3	6	11	**0**	1	1	2	2	6	1	5	8
211	**0**	**0**	1	**0**	2	3	2	3	6	**0**	1	1	**1**	**1**	**4**	**1**	3	4	**0**	**0**	1	**0**	2	3	2	3	6
021	**0**	**0**	0	1	0	2	0	1	2	**1**	1	2	**1**	1	1	**3**	2	5	**0**	**0**	0	1	0	2	0	1	2
121	**0**	**0**	**0**	**0**	1	3	1	3	4	**0**	**1**	2	**1**	2	3	**1**	**4**	6	**0**	**0**	**0**	**0**	1	3	1	3	4
221	**0**	**0**	**0**	**0**	**0**	1	**0**	1	1	**0**	**0**	**1**	**0**	1	1	**1**	**1**	**4**	**0**	**0**	**0**	**0**	**0**	1	**0**	1	1

TABLE 2 (continued)

$a_{\lambda\mu}$	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2
	0	0	0	1	1	1	2	2	2	0	0	0	1	1	1	2	2	2	0	0	0	1	1	1	2	2	2
	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2	2	2
002	1	2	2	1	2	6	2	7	10	0	1	3	2	4	7	3	6	13	2	2	2	1	2	6	2	7	10
102	0	1	1	1	1	2	1	3	5	0	0	0	1	1	3	1	2	4	1	3	2	1	1	2	1	3	5
202	0	0	1	0	1	1	2	2	5	0	0	0	1	0	2	0	1	2	1	1	3	0	1	1	2	2	5
012	0	0	0	1	0	1	0	1	3	1	1	1	0	1	2	2	3	5	0	1	1	3	1	3	0	2	4
112	0	0	0	0	1	2	2	4	8	0	1	1	2	2	6	1	5	8	1	2	3	1	4	5	3	6	11
212	0	0	0	0	0	1	0	2	3	0	0	1	0	2	3	2	3	6	0	1	1	1	1	4	0	3	4
022	0	0	0	0	0	0	1	1	3	0	0	0	1	0	2	0	1	2	1	1	2	0	1	1	3	2	5
122	0	0	0	0	0	0	0	1	3	0	0	0	0	1	3	1	3	4	0	1	2	1	2	3	1	4	6
222	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	1	0	0	1	0	1	1	1	1	4

TABLE 3. $a_{\lambda\mu}$ for $p = 5$ and $s \leq 2$

	0	1	2	3	4	0	1	2	3	4	0	1	2	3	4	0	1	2	3	4	0	1	2	3	4
$a_{\lambda\mu}$	0	0	0	0	0	1	1	1	1	1	2	2	2	2	2	3	3	3	3	3	4	4	4	4	4
00	**1**																								
10	**1**	**2**	**1**	**1**	**1**																				
20	**1**	**1**	**2**	**1**	**1**																				
30	**1**	**1**	**1**	**2**	**1**																				
40	**1**	**1**	**1**	**1**	**2**																				
01	**0**	1	1	1	1	**2**	1	1	1	2	**0**	0	0	1	1	**0**	0	1	1	1	**0**	1	1	1	1
11	**0**	1	2	2	2	**1**	**3**	2	2	3	**1**	1	1	2	3	**0**	0	1	2	2	**0**	1	2	2	2
21	**0**	1	1	2	2	**1**	**1**	**3**	2	3	**0**	1	1	2	2	**1**	1	2	2	3	**0**	1	1	2	2
31	**1**	2	2	2	3	**1**	**1**	**1**	**3**	3	**0**	**0**	1	2	2	**0**	1	2	2	2	**1**	2	2	2	3
41	**0**	1	1	1	1	**1**	**1**	**1**	**1**	**3**	**0**	**0**	**0**	1	1	**0**	**0**	1	1	1	**0**	1	1	1	1

TABLE 3 (continued)

	0	1	2	3	4	0	1	2	3	4	0	1	2	3	4	0	1	2	3	4	0	1	2	3	4
$a_{\lambda\mu}$	0	0	0	0	0	1	1	1	1	1	2	2	2	2	2	3	3	3	3	3	4	4	4	4	4
02	**0**	0	1	1	1	**0**	1	1	1	1	**2**	1	1	1	2	**0**	0	0	1	1	**0**	0	1	1	1
12	**0**	0	1	2	2	**0**	1	2	2	2	**1**	**3**	2	2	3	**1**	1	1	2	2	**0**	0	1	2	2
22	**1**	1	2	2	3	**0**	1	1	2	2	**1**	**1**	**3**	2	3	**0**	1	1	2	2	**1**	1	2	2	3
32	**0**	1	1	1	1	**1**	2	2	2	3	**1**	**1**	**1**	**3**	3	**0**	**0**	1	2	2	**0**	1	2	2	2
42	**0**	**0**	1	1	1	**0**	1	1	1	1	**1**	**1**	**1**	**1**	**3**	**0**	**0**	**0**	1	1	**0**	**0**	1	1	1
03	**0**	0	0	1	1	**0**	0	1	1	1	**0**	1	1	1	1	**2**	1	1	1	2	**0**	0	0	1	1
13	**1**	1	1	2	3	**0**	0	1	2	2	**0**	1	2	2	2	**1**	**3**	2	2	3	**1**	1	1	2	3
23	**0**	1	1	2	2	**1**	1	2	2	3	**0**	1	1	2	2	**1**	**1**	**3**	2	3	**0**	1	1	2	2
33	**0**	**0**	1	2	2	**0**	1	2	2	2	**1**	2	2	2	2	**1**	**1**	**1**	**3**	3	**0**	**0**	1	2	2
43	**0**	**0**	**0**	1	1	**0**	**0**	1	1	1	**0**	1	1	1	1	**1**	**1**	**1**	**1**	**3**	**0**	**0**	**0**	1	1
04	**1**	1	1	1	2	**0**	0	0	1	1	**0**	0	1	1	1	**0**	1	1	1	1	**2**	1	1	1	2
14	**0**	**1**	1	1	2	**1**	1	1	2	3	**0**	0	1	2	2	**0**	1	2	2	2	**1**	**3**	2	2	3
24	**0**	**0**	**1**	1	2	**0**	1	1	2	2	**1**	1	2	2	3	**0**	1	1	2	2	**1**	**1**	**3**	2	3
34	**0**	**0**	**0**	**1**	2	**0**	**0**	1	2	2	**0**	1	2	2	2	**1**	2	2	2	3	**1**	**1**	**1**	**3**	3
44	**0**	**0**	**0**	**0**	**1**	**0**	**0**	**0**	1	1	**0**	**0**	1	1	1	**0**	1	1	1	1	**1**	**1**	**1**	**1**	**3**

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