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FUNDAMENTAL SOLUTIONS FOR HYPOELLIPTIC DIFFERENTIAL OPERATORS DEPENDING ANALYTICALLY ON A PARAMETER

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ABSTRACT. Let $P(\lambda, D) = \sum_{|\alpha|\leq m} a_\alpha(\lambda)D^\alpha$ be a differential operator with constant coefficients a_α depending analytically on a parameter λ. Assume that each $P(\lambda, D)$ is hypoelliptic and that the strength of $P(\lambda, D)$ is independent of λ. Under this condition we show that there exists a regular fundamental solution of $P(\lambda, D)$ which also depends analytically on λ.

0. INTRODUCTION

We consider a differential polynomial

$$(0.1)\qquad P(\lambda, D) = \sum_{|\alpha|\leq m} a_\alpha(\lambda)D^\alpha, \qquad m \in \mathbb{N},\ D = -i\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right),$$

where $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ denotes a multiindex and $|\alpha| := \sum_{\nu=1}^n |\alpha_\nu|$ the length of α. The coefficients $a_\alpha(\lambda)$ are constant with respect to the variable x but may depend on a parameter $\lambda \in \Lambda$. Assuming that Λ is a complex manifold and the functions a_α are analytic we treat the following problem which has been posed by L. Hörmander [4, II, p. 59]:

(*) Does there exist an analytic function $\mathfrak{f}\colon \Lambda \to \mathscr{D}'$ (the space of all distributions) such that $\mathfrak{f}(\lambda)$ is a regular (cf. [4, 10.2.2]) fundamental solution of the differential operator $P(\lambda, D)$ for each $\lambda \in \Lambda$?

F. Trèves [10] showed that if (*) holds and Λ is connected then necessarily $P(\lambda, D)$ is equally strong for each $\lambda \in \Lambda$. Conversely, if this condition is satisfied then for each $\lambda_0 \in \Lambda$ there exists a neighborhood Λ' of λ_0 and an analytic function $\mathfrak{f}\colon \Lambda' \to \mathscr{D}'$ such that $P(\lambda, D)\mathfrak{f}(\lambda) = \delta$, $\lambda \in \Lambda'$ (δ the Dirac distribution) [11]. The question remained open however whether such a function $\mathfrak{f}$ could be defined globally in Λ.

In the present article we give a positive answer to problem (*) in the case when the $P(\lambda, D)$ are hypoelliptic (see §4 below). The idea is to use a result of J. Leiterer [6] on "BCAF-sheaves", which is a strong tool to attack "local-global"-problems for analytic operator functions. For the applicability of this

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machinery it is necessary to construct suitable Banach spaces of distributions such that the mapping $\lambda \mapsto P(\lambda, D)$ can be considered as an analytic function with values in the bounded, surjective linear operators between these spaces.

The organization of this paper is as follows: In §1 we introduce some distribution spaces $\mathbf{B}_{q,k}^{-\infty}$ and $\mathbf{B}_{q,k}^{-\rho,\eta}$ which are embedded in the well-known spaces $\mathbf{B}_{q,k}^{\mathrm{loc}}$ of Hörmander [4, §10.1]. Our definition of $\mathbf{B}_{q,k}^{-\rho,\eta}$ is motivated by an explicit integral formula for fundamental solutions of $P(\lambda, D)$. In the hypoelliptic case such a representation can be used simultaneously for all λ in a compact subset Λ' of Λ. In order to obtain a global solution of $(*)$ we shall perform a Mittag-Leffler procedure in the Fréchet space $\mathbf{B}_{q,k}^{-\infty}$ which contains $\mathbf{B}_{q,k}^{-\rho,\eta}$ for all values of the parameters ρ, η. The behaviour of a differential operator on $\mathbf{B}_{q,k}^{-\rho,\eta}$ will be clarified in §2. In §3 the definition of parameter-depending differential operators is formalized in preparation for the statements and proofs of our main results. §4 contains our solution of problem $(*)$ which will be a corollary of a more general theorem.

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1. Some distribution spaces

In the sequel, $n \in \mathbb{N}$ denotes a fixed positive integer. We adopt the standard notations for spaces of test functions and distributions (cf. [4]):

$\mathscr{D} = \mathscr{C}_c^\infty(\mathbb{R}^n)$, $\mathscr{C}^\infty$-functions with compact support;
$\mathscr{D}' = \mathscr{D}'(\mathbb{R}^n)$, space of all distributions;
$\mathscr{S} = \mathscr{S}(\mathbb{R}^n)$, space of rapidly decreasing $\mathscr{C}^\infty$-functions;
$\mathscr{S}' = \mathscr{S}'(\mathbb{R}^n)$, space of tempered distributions.

Recall that each of these spaces carries a natural locally convex vector space topology. The scalar product of two vectors $\zeta, \eta \in \mathbb{C}^n$ will be denoted by $[\zeta, \eta] := \sum_{\nu=1}^n \zeta_\nu \overline{\eta}_\nu$. If $\varphi \in \mathscr{S}$ then the Fourier transform $\hat{\varphi}$ of φ is the function

$$\hat{\varphi}(\zeta) := \int_{\mathbb{R}^n} \exp(-i[\zeta, x])\varphi(x)\,dx, \qquad \zeta \in \mathbb{R}^n.$$

The Fourier transform $\hat{u}$ of $u \in \mathscr{S}'$ is defined by the formula

$$\langle \hat{u}, \varphi \rangle := \langle u, \hat{\varphi} \rangle, \qquad \varphi \in \mathscr{S},$$

where $\langle \cdot, \cdot \rangle$ denotes the distribution pairing. The following definitions are taken from Hörmander [4, §10.1].

1.1. **Definition.** (a) A function $k\colon \mathbb{R}^n \to (0, \infty)$ will be called a *temperate weight function* if there exist constants $C, D > 0$ such that

$$k(\xi + \zeta) \le (1 + C|\xi|)^D k(\zeta), \qquad \xi, \zeta \in \mathbb{R}^n.$$

The set of all such functions will be denoted by $\mathscr{K}$.

(b) If $k \in \mathscr{K}$ and $1 \le p \le \infty$ we denote by $\mathbf{B}_{p,k}$ the set of all distributions $u \in \mathscr{S}'$ such that $\hat{u}$ is a function and

$$\|u\|_{p,k} := \left((2\pi)^{-n} \int_{\mathbb{R}^n} |k(\xi)\hat{u}(\xi)|^p\,d\xi\right)^{1/p} < \infty.$$

In the case $p = \infty$ this expression has to be interpreted as $\operatorname{ess\,sup}_{\xi \in \mathbb{R}^n} |k(\xi)\hat{u}(\xi)|$.

By [4, Theorem 10.1.7] we have the following embeddings:

$$\mathscr{S} \hookrightarrow \mathbf{B}_{p,k} \hookrightarrow \mathscr{S}',$$

where $\mathfrak{F} \hookrightarrow \mathfrak{G}$ means that the space $\mathfrak{F}$ is a subspace of $\mathfrak{G}$ carrying a stronger topology than that induced by $\mathfrak{G}$. The spaces $\mathbf{B}_{p,k}$ are Banach spaces which, for $1 \le p < \infty$, contain $\mathscr{D}$ as a dense subset. In this case the dual $(\mathbf{B}_{p,k})'$ of $\mathbf{B}_{p,k}$ is (isometrically) isomorphic to $\mathbf{B}_{p',k'}$, where

$$1/p + 1/p' = 1, \qquad k'(\xi) := 1/k(-\xi).$$

In fact, any continuous linear form on $\mathbf{B}_{p,k}$ is given by continuous extension of a form $\mathscr{D} \ni \varphi \mapsto \langle v, \varphi \rangle$ with $v \in \mathbf{B}_{p',k'}$ and the norm of this functional equals $\|v\|_{p',k'}$. Let

$$\mathbf{B}^{\mathrm{loc}}_{p,k} := \{u \in \mathscr{D}' \colon \psi u \in \mathbf{B}_{p,k}, \ \psi \in \mathscr{D}\}$$

denote the local space associated with $\mathbf{B}_{p,k}$. This is a Fréchet space with the system of seminorms $u \mapsto \|\psi u\|_{p,k}$, $\psi \in \mathscr{D}$. In what follows we shall also require some modifications of the spaces $\mathbf{B}_{p,k}$.

1.2. **Definition.** Let $\mu \in \mathbb{N}$ and

$$\gamma_\mu(x) := \exp(\gamma'_\mu(x) \cdot \sqrt{1 + [x, x]}), \qquad x \in \mathbb{R}^n, \tag{1.1}$$

where $\gamma'_\mu \in \mathscr{E}^\infty(\mathbb{R}^n)$ is defined recursively by

$$\gamma'_0(x) := [x, x], \qquad \gamma'_\mu(x) := \log(1 + \gamma'_{\mu-1}(x)).$$

Further, let $1 \le p \le \infty$ and $k \in \mathscr{K}$. We define the distribution spaces

$$\mathbf{B}^{\mu}_{p,k} := \{u = 1/\gamma_\mu \cdot \tilde{u} \colon \tilde{u} \in \mathbf{B}_{p,k}\}, \quad \mathbf{B}^{-\mu}_{p,k} := \{v = \gamma_\mu \cdot \tilde{v} \colon \tilde{v} \in \mathbf{B}_{p,k}\}.$$

Obviously these are Banach spaces with the norms $\|u\|^{\mu}_{p,k} := \|\gamma_\mu \cdot u\|_{p,k}$ resp. $\|v\|^{-\mu}_{p,k} := \|1/\gamma_\mu \cdot v\|_{p,k}$.

Remark. Since $\gamma_\mu, 1/\gamma_\mu \in \mathscr{E}^\infty(\mathbb{R}^n)$ we have $\mathbf{B}^{\pm\mu}_{p,k} \subseteq \mathbf{B}^{\mathrm{loc}}_{p,k}$ by [4, Theorem 10.1.23]. If $p < \infty$ then $\mathscr{D}$ is dense in $\mathbf{B}^{\mu}_{p,k}$. In this case there corresponds to any continuous linear form l on $\mathbf{B}^{\mu}_{p,k}$ a unique distribution $\tilde{v} \in \mathbf{B}_{p',k'}$, which satisfies $l(\varphi) = \langle \tilde{v}, \gamma_\mu \varphi \rangle$, $\varphi \in \mathscr{D}$. Clearly, the mapping $l \mapsto v := \gamma_\mu \cdot \tilde{v}$ defines an (isometric) isomorphism of $(\mathbf{B}^{\mu}_{p,k})'$ onto $\mathbf{B}^{-\mu}_{p',k'}$.

1.3. **Lemma.** *Let* $1 < q \le \infty$, $k \in \mathscr{K}$ *and* $0 < \mu < \nu$ *(integers). Then we have the embeddings*

$$\mathbf{B}^{-\nu}_{q,k} \hookrightarrow \mathbf{B}^{-\mu}_{q,k} \hookrightarrow \mathbf{B}^{\mathrm{loc}}_{q,k}. \tag{1.2}$$

Proof. Let $v = \gamma_\nu \cdot \tilde{v}_1 \in \mathbf{B}^{-\nu}_{q,k}$ with $\tilde{v}_1 \in \mathbf{B}_{q,k}$. Then we have

$$v = \gamma_\mu \cdot (\gamma_\nu/\gamma_\mu) \cdot \tilde{v}_1 =: \gamma_\mu \cdot \tilde{v}_2.$$

Since $\gamma_\nu/\gamma_\mu \in \mathscr{S}$ it follows from [4, Theorem 10.1.15] that $\tilde{v}_2 \in \mathbf{B}_{q,k}$ and $\|\tilde{v}_2\|_{q,k} \le c\|\tilde{v}_1\|_{q,k}$ where $c < \infty$ only depends on μ, ν and k. Hence $v \in \mathbf{B}^{-\mu}_{q,k}$ and

$$\|v\|^{-\mu}_{q,k} = \|\tilde{v}_2\|_{q,k} \le c\|\tilde{v}_1\|_{q,k} = c\|v\|^{-\nu}_{q,k},$$

which proves the first embedding in (1.2). The second one can be proved similarly: Fix $\psi \in \mathscr{D}$ and let $v = \gamma_\mu \cdot \tilde{v} \in \mathbf{B}_{q,k}^{-\mu}$, $\tilde{v} \in \mathbf{B}_{q,k}$, be arbitrary. Noting that $\psi \cdot \gamma_\mu \in \mathscr{D} \subseteq \mathscr{S}$ we obtain as above

$$\|\psi v\|_{q,k} = \|\psi\gamma_\mu\tilde{v}\|_{q,k} \le c'\|\tilde{v}\|_{q,k} = c'\|v\|_{q,k}^{-\mu},$$

with $c' < \infty$ depending only on μ, k, and ψ. Since the topology of $\mathbf{B}_{q,k}^{\mathrm{loc}}$ is given by the seminorms $v \mapsto \|\psi v\|_{q,k}$ we are done. □

1.4. **Definition.** For any $1 < q \le \infty$ and $k \in \mathscr{K}$ let

$$\mathbf{B}_{q,k}^{-\infty} := \bigcap_{\mu\in\mathbb{N}} \mathbf{B}_{q,k}^{-\mu}$$

be endowed with the topology which is given by the system of seminorms $\{\|\cdot\|_{q,k}^{-\mu} : \mu \in \mathbb{N}\}$.

Remark. It is not hard to see that $\mathbf{B}_{q,k}^{-\infty}$ is a Fréchet space, and from (1.2) we obtain the embedding

$$\mathbf{B}_{q,k}^{-\infty} \hookrightarrow \mathbf{B}_{q,k}^{\mathrm{loc}}. \tag{1.3}$$

Note that for any $\mu \in \mathbb{N}$ the mapping $v \mapsto 1/\gamma_\mu \cdot v$ embeds $\mathbf{B}_{q,k}^{-\infty}$ continuously into $\mathbf{B}_{q,k}$. The reason why we have introduced the spaces $\mathbf{B}_{q,k}^{-\infty}$ is that they give quite precise information on the growth at infinity of solutions of the equation $P(\lambda, D)\mathfrak{f}(\lambda) = \delta$ when $P(\lambda, D)$ depends analytically on λ (cf. the remark at the end of §4). In order to solve this equation we have to consider certain subspaces of $\mathbf{B}_{q,k}^{-\infty}$:

1.5. **Definition.** Let $1 \le p < \infty$, $k \in \mathscr{K}$, $\rho \ge 0$, and $\eta \in \mathbb{R}^n\setminus\{0\}$. For any $\varphi \in \mathscr{D}$ we put

$$\|\varphi\|_{p,k}^{\rho,\eta} := \left((2\pi)^{-n}\int_{|\xi|\le\rho}\int_{|z|=1}|k(\xi)\hat{\varphi}(\xi+z\eta)|^p\frac{|dz|}{2\pi}\,d\xi + (2\pi)^{-n}\int_{|\xi|\ge\rho}|k(\xi)\hat{\varphi}(\xi)|^p\,d\xi\right)^{1/p}, \tag{1.4}$$

where $\xi \in \mathbb{R}^n$, $z \in \mathbb{C}$.

The theorem of Paley-Wiener-Schwartz [4, §7.3] ensures that $\|\varphi\|_{p,k}^{\rho,\eta}$ is finite for each $\varphi \in \mathscr{D}$. Obviously, $\|\cdot\|_{p,k}^{\rho,\eta}$ is a norm on $\mathscr{D}$. We consider the space

$$\mathbf{B}_{p',k'}^{-\rho,\eta} := \{v \in \mathbf{B}_{p',k'}^{\mathrm{loc}} : \|v\|_{p',k'}^{-\rho,\eta} := \sup\{|\langle v, \varphi\rangle| : \varphi \in \mathscr{D},\ \|\varphi\|_{p,k}^{\rho,\eta} \le 1\} < \infty\}$$

endowed with the norm $\|\cdot\|_{p',k'}^{-\rho,\eta}$. Note that $\|\cdot\|_{p,k}^{0,\eta} = \|\cdot\|_{p,k}$ and thus $\mathbf{B}_{p',k'}^{-0,\eta} = \mathbf{B}_{p',k'}$ (isometrically).

Definition (1.4) is motivated by the fact that for any hypoelliptic polynomial P and with suitable $\rho > 0$, $\eta \in \mathbb{R}^n$, the formula

$$\langle \mathfrak{f}_P, \varphi\rangle := (2\pi)^{-n}\left(\int_{|\xi|\le\rho}\int_{|z|=1}\frac{\hat{\varphi}(-\xi-z\eta)}{P(\xi+z\eta)}\frac{dz}{2\pi i z}\,d\xi + \int_{|\xi|\ge\rho}\frac{\hat{\varphi}(-\xi)}{P(\xi)}\,d\xi\right)$$

defines a fundamental solution $\mathfrak{f}_P$ of $P(D)$. The idea of using such a representation goes back to L. Hörmander. It can be shown that $\mathfrak{f}_P \in \mathbf{B}_{\infty,\widetilde{P}}^{-\rho,\eta}$, where $\widetilde{P} \in \mathscr{K}$ is defined as in §2 below.

1.6. **Lemma.** *Let* $1 \le p < \infty$, $k \in \mathscr{K}$, $0 \le \rho_1 \le \rho_2$, *and* $\eta_1, \eta_2 \in \mathbb{R}^n \backslash \{0\}$ *such that* $\eta_2 = t\eta_1$ *with some* $t \ge 1$. *Then the estimate*

$$\|\varphi\|_{p,k}^{\rho_1,\eta_1} \le \|\varphi\|_{p,k}^{\rho_2,\eta_2}, \qquad \varphi \in \mathscr{D}, \tag{1.5}$$

holds, and

$$\mathbf{B}_{p',k'}^{-\rho_1,\eta_1} \hookrightarrow \mathbf{B}_{p',k'}^{-\rho_2,\eta_2}. \tag{1.6}$$

Proof. With the notation

$$w_\tau(\xi) := \int_{|z|=1} |\hat{\varphi}(\xi + z\tau\eta_1)|^p \frac{|dz|}{2\pi}, \qquad \varphi \in \mathscr{D},$$

we have $|\hat{\varphi}(\xi)|^p = w_0(\xi) \le w_1(\xi) \le w_t(\xi)$, $\xi \in \mathbb{R}^n$, (cf. [8, E.II, §3.3]). Using this, the verification of (1.5) is immediate. The embedding (1.6) follows directly from (1.5) and the definition of $\mathbf{B}_{p',k'}^{-\rho,\eta}$. □

1.7. **Lemma.** *Let* $1 \le p < \infty$, $k \in \mathscr{K}$, $\mu \in \mathbb{N}$, $\rho \ge 0$, *and* $\eta \in \mathbb{R}^n \backslash \{0\}$. *Then there exists a constant* $c < \infty$ *such that*

$$\|\varphi\|_{p,k}^{\rho,\eta} \le c\|\varphi\|_{p,k}^{\mu}, \qquad \varphi \in \mathscr{D}. \tag{1.7}$$

Proof. First we note that for any $\varphi \in \mathscr{D}$

$$\begin{aligned}(\|\varphi\|_{p,k}^{\rho,\eta})^p &\le (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{|z|=1} |k(\xi)\hat{\varphi}(\xi + z\eta)|^p \frac{|dz|}{2\pi}\, d\xi \\ &= (2\pi)^{-n} \int_{|z|=1} \int_{\mathbb{R}^n} |k(\xi)\mathscr{F}(\exp(-i[z\eta, \cdot])\varphi)(\xi)|^p\, d\xi \frac{|dz|}{2\pi} \\ &= \int_{|z|=1} (\|\exp(-i[z\eta, \cdot])\varphi\|_{p,k})^p \frac{|dz|}{2\pi}\end{aligned} \tag{1.8}$$

(cf. the proof of Lemma 1.6). Here we have used the notation $\mathscr{F}(v) := \hat{v}$. Now consider the functions $\chi_z(x) := \exp(-i[z\eta, x])/\gamma_\mu(x)$, $|z| = 1$. It is an easy exercise to check that $\{\chi_z : |z| = 1\}$ is a bounded subset of $\mathscr{S}$. With the weight function $M_k \in \mathscr{K}$,

$$M_k(\xi) := \sup_{\zeta \in \mathbb{R}^n} k(\xi + \zeta)/k(\zeta), \qquad \xi \in \mathbb{R}^n,$$

(cf. [4, §10.1]) we have $\mathscr{S} \hookrightarrow \mathbf{B}_{1,M_k}$ [4, Theorem 10.1.7], hence $\sup_{|z|=1} \|\chi_z\|_{1,M_k} =: c < \infty$. It follows from [4, Theorem 10.1.15] that

$$\sup_{|z|=1} \|\chi_z \psi\|_{p,k} \le \sup_{|z|=1} \|\chi_z\|_{1,M_k} \|\psi\|_{p,k} = c\|\psi\|_{p,k}, \qquad \psi \in \mathscr{D}.$$

From (1.8) we thus obtain

$$\begin{aligned}\|\varphi\|_{p,k}^{\rho,\eta} &\le \left(\int_{|z|=1} (\|\chi_z \gamma_\mu \varphi\|_{p,k})^p \frac{|dz|}{2\pi} \right)^{1/p} \\ &\le c \cdot \|\gamma_\mu \varphi\|_{p,k} = c \cdot \|\varphi\|_{p,k}^{\mu}. \quad \square\end{aligned}$$

1.8. **Corollary.** *Under the assumptions of Lemma* 1.7 *the mapping* $v \mapsto \langle v, \cdot \rangle$ *identifies* $\mathbf{B}_{p',k'}^{-\rho,\eta}$ *isometrically with the dual of the normed space* $(\mathscr{D}, \|\cdot\|_{p,k}^{\rho,\eta})$. *In particular* $\mathbf{B}_{p',k'}^{-\rho,\eta}$ *is complete. Furthermore we have*

$$\mathbf{B}_{p',k'}^{-\rho,\eta} \hookrightarrow \mathbf{B}_{p',k'}^{-\mu}. \tag{1.9}$$

Proof. Clearly, $v \mapsto \langle v, \cdot\rangle$ defines an isometric embedding of $\mathbf{B}_{p',k'}^{-\rho,\eta}$ into $(\mathscr{D}, \|\cdot\|_{p,k}^{\rho,\eta})'$. We have to show that it is onto. So let l be a continuous linear form on $(\mathscr{D}, \|\cdot\|_{p,k}^{\rho,\eta})$. By Lemma 1.7 we have for any $\varphi \in \mathscr{D}$

$$(1.10)\qquad |l(\varphi)| \leq \|l\| \cdot \|\varphi\|_{p,k}^{\rho,\eta} \leq c\|l\| \cdot \|\varphi\|_{p,k}^{\mu}.$$

Since $\mathbf{B}_{p',k'}^{-\mu}$ is the dual space of $\mathbf{B}_{p,k}^{\mu}$ there exists a distribution $v \in \mathbf{B}_{p',k'}^{-\mu} \subseteq \mathbf{B}_{p',k'}^{\mathrm{loc}}$ such that $l(\varphi) = \langle v, \varphi\rangle$, $\varphi \in \mathscr{D}$. By (1.10) we have $v \in \mathbf{B}_{p',k'}^{-\rho,\eta}$ which proves the first assertion. Next we show the embedding (1.9): If $v \in \mathbf{B}_{p',k'}^{-\rho,\eta}$ it follows from Lemma 1.7 that

$$|\langle v/\gamma_\mu, \varphi\rangle| \leq \|v\|_{p',k'}^{-\rho,\eta}\|\varphi/\gamma_\mu\|_{p,k}^{\rho,\eta} \leq c\|v\|_{p',k'}^{-\rho,\eta}\|\varphi\|_{p,k}, \qquad \varphi \in \mathscr{D}.$$

Since $\mathscr{D}$ is dense in $\mathbf{B}_{p,k}$ this implies that $v/\gamma_\mu \in \mathbf{B}_{p',k'}$ and $\|v\|_{p',k'}^{-\mu} = \|v/\gamma_\mu\|_{p',k'} \leq c\|v\|_{p',k'}^{-\rho,\eta}$. □

We conclude this section by observing that (1.9) yields

$$(1.11)\qquad \mathbf{B}_{q,k}^{-\rho,\eta} \hookrightarrow \mathbf{B}_{q,k}^{-\infty}$$

for any $1 < q \leq \infty$, $k \in \mathscr{K}$, $\rho \geq 0$, and $\eta \in \mathbb{R}^n \backslash \{0\}$.

2. Differential operators in $\mathbf{B}_{q,k}^{-\rho,\eta}$

In this section we investigate how a differential operator with constant coefficients acts in the spaces $\mathbf{B}_{q,k}^{-\rho,\eta}$. If $P(x) = \sum_{|\alpha|\leq m} a_\alpha x^\alpha$ is a polynomial in $x \in \mathbb{R}^n$ we consider the differential expression $P(D) := \sum_{|\alpha|\leq m} a_\alpha D^\alpha$ where $D := -i\partial$, $\partial := (\partial/\partial x_1, \dots, \partial/\partial x_n)$. With a polynomial $P(x)$ we also associate the function

$$\widetilde{P}(\xi) := \left(\sum_{\beta} |P^{(\beta)}(\xi)|^2\right)^{1/2}, \qquad \xi \in \mathbb{R}^n,$$

where $P^{(\beta)} := \partial^\beta P$ and the sum extends over all multi-indices β. Note that $\widetilde{P} \in \mathscr{K}$ for any polynomial P (cf. [4, §10.1]). Now let $P, Q \in \mathbb{C}[x_1, \dots, x_n]$ be any two polynomials. We say that Q is *weaker* than P if $\sup_{\xi\in\mathbb{R}^n} \widetilde{Q}(\xi)/\widetilde{P}(\xi) < \infty$. In this case we write $Q < P$. P and Q are said to be *equally strong* (notation: $P \approx Q$) if $Q < P$ and $P < Q$. Obviously, $\approx$ is an equivalence relation on $\mathbb{C}[x_1, \dots, x_n]$. For the properties of $<$ and $\approx$ see [4, §10.1].

2.1. **Proposition.** *Let $P, P_0 \in \mathbb{C}[x_1, \dots, x_n]\backslash\{0\}$ with $P < P_0$. Let $1 < q \leq \infty$, $k \in \mathscr{K}$, $\rho > 0$, and $\eta \in \mathbb{R}^n\backslash\{0\}$. Then the operator $P(D)$ maps $\mathbf{B}_{q,k\widetilde{P}_0}^{-\rho,\eta}$ continuously into $\mathbf{B}_{q,k}^{-\rho,\eta}$.*

Proof. With $p := q'$ and $k_0 := k\widetilde{P}_0$ we have for any $\varphi \in \mathscr{D}$:

$$(2.1)\quad \begin{aligned} &(2\pi)^n(\|P(-D)\varphi\|^{\rho,\eta}_{p,k_0'})^p \\ &= \int_{|\xi|\le\rho}\int_{|z|=1} |k_0'(\xi)P(-\xi - z\eta)\hat{\varphi}(\xi+z\eta)|^p \frac{|dz|}{2\pi}\,d\xi \\ &\quad + \int_{|\xi|\ge\rho} |k_0'(\xi)P(-\xi)\hat{\varphi}(\xi)|^p\,d\xi \\ &\le c\cdot\int_{|\xi|\le\rho}\int_{|z|=1} |k_0'(\xi)\widetilde{P}(-\xi)\hat{\varphi}(\xi+z\eta)|^p\frac{|dz|}{2\pi}\,d\xi \\ &\quad + \int_{|\xi|\ge\rho} |k_0'(\xi)\widetilde{P}(-\xi)\hat{\varphi}(\xi)|^p\,d\xi \\ &\le (2\pi)^n(c'\|\varphi\|^{\rho,\eta}_{p,k'})^p . \end{aligned}$$

Here we have used the fact that $k_0'(\xi)\widetilde{P}(-\xi) = k'(\xi)\widetilde{P}(-\xi)/\widetilde{P}_0(-\xi) \le c_0k'(\xi)$ and

$$\begin{aligned} |P(-\xi-\zeta)| &= \left|\sum_\beta \frac{1}{\beta!}P^{(\beta)}(-\xi)(-\zeta)^\beta\right| \\ &\le c_1(1+|\zeta|)^m\widetilde{P}(-\xi) \le c_2(1+|\zeta|)^m\widetilde{P}_0(-\xi), \end{aligned}$$

if m is the degree of P. Hence

$$(2.2)\qquad \|P(-D)\varphi\|^{\rho,\eta}_{q',(k\widetilde{P}_0)'} \le c'\|\varphi\|^{\rho,\eta}_{q',k'}, \qquad \varphi\in\mathscr{D}.$$

Now, if $v \in \mathbf{B}^{-\rho,\eta}_{q,k\widetilde{P}_0} \subseteq \mathbf{B}^{\mathrm{loc}}_{q,k\widetilde{P}_0}$ it follows from [4, Theorem 10.1.22] that $P(D)v \in \mathbf{B}^{\mathrm{loc}}_{q,k}$. Furthermore, (2.2) implies that

$$\begin{aligned} |\langle P(D)v, \varphi\rangle| = |\langle v, P(-D)\varphi\rangle| &\le \|v\|^{-\rho,\eta}_{q,k\widetilde{P}_0}\|P(-D)\varphi\|^{\rho,\eta}_{q',(k\widetilde{P}_0)'} \\ &\le c'\|v\|^{-\rho,\eta}_{q,k\widetilde{P}_0}\|\varphi\|^{\rho,\eta}_{q',k'} \end{aligned}$$

for any $\varphi\in\mathscr{D}$. In particular this means that $P(D)v \in \mathbf{B}^{-\rho,\eta}_{q,k}$ and $\|P(D)v\|^{-\rho,\eta}_{q,k} \le c'\|v\|^{-\rho,\eta}_{q,k\widetilde{P}_0}$. □

2.2. **Proposition.** *Let* $P, P_0 \in \mathbb{C}[x_1,\dots,x_n]\backslash\{0\}$ *with* $P \approx P_0$. *Let* $1 < q \le \infty$ *and* $k \in \mathscr{K}$. *Assume that* $\rho > 0$, $\eta \in \mathbb{R}^n\backslash\{0\}$ *are chosen such that with some* $c > 0$

$$\begin{aligned} &\widetilde{P}_0(-\xi) \le c\cdot|P(-\xi)| \quad \text{if } \xi\in\mathbb{R}^n,\ |\xi|\ge\rho; \\ &P(-\xi - z\eta) \ne 0 \quad \text{if } \xi\in\mathbb{R}^n,\ |\xi|\le\rho,\ z\in\mathbb{C},\ |z| = 1. \end{aligned}$$

Then the operator $P(D)\colon \mathbf{B}^{-\rho,\eta}_{q,k\widetilde{P}_0} \to \mathbf{B}^{-\rho,\eta}_{q,k}$ *is surjective.*

Proof. Since $\inf\{|P(-\xi - z\eta)|/\widetilde{P}_0(-\xi)\colon |\xi|\le\rho,\ |z| = 1\} =: c_1 > 0$ a similar calculation as in (2.1) yields

$$(2.3)\qquad \|P(-D)\varphi\|^{\rho,\eta}_{q',(k\widetilde{P}_0)'} \ge c_2\|\varphi\|^{\rho,\eta}_{q',k'}, \qquad \varphi\in\mathscr{D},$$

with $c_2 := \min\{c_1, c^{-1}\}$. Now let $w \in \mathbf{B}^{-\rho,\eta}_{q,k}$ be given. Then by (2.3) the mapping $P(-D)\varphi \mapsto \langle w, \varphi\rangle$ is a well-defined continuous linear form on the subspace $P(-D)\mathscr{D}$ of $(\mathscr{D}, \|\cdot\|^{\rho,\eta}_{q',(k\widetilde{P}_0)'})$. By Corollary 1.8 and the Hahn-Banach theorem there exists a distribution $v \in \mathbf{B}^{-\rho,\eta}_{q,k\widetilde{P}_0}$ such that

$$\langle P(D)v, \varphi\rangle = \langle v, P(-D)\varphi\rangle = \langle w, \varphi\rangle, \qquad \varphi \in \mathscr{D},$$

i.e., $P(D)v = w$. □

We shall see in §4 that the assumptions of Proposition 2.2 can be satisfied if P is hypoelliptic.

3. Hypoelliptic families of differential operators

For any fixed polynomial $P_0 \in \mathbb{C}[x_1, \dots, x_n]$ we consider the sets

$$\mathbb{W}(P_0) := \{P \in \mathbb{C}[x_1, \dots, x_n] : P < P_0\},$$
$$\mathbb{E}(P_0) := \{P \in \mathbb{C}[x_1, \dots, x_n] : P \approx P_0\}.$$

It is clear that $\mathbb{W}(P_0)$ only contains polynomials of degree less than or equal to the degree of P_0. Hence $\mathbb{W}(P_0)$ is a finite-dimensional complex vector space and $\mathbb{E}(P_0) \subseteq \mathbb{W}(P_0)$. The following result is due to Hörmander.

3.1. **Lemma.** *The set* $\mathbb{E}(P_0)$ *is a domain of holomorphy in* $\mathbb{W}(P_0)$.

Proof. According to Hörmander [4, Theorem 10.4.7] there exists a compact subset $\mathscr{L}_0$ of $\mathbb{W}(P_0)'\backslash\{0\}$ such that

$$\mathbb{E}(P_0) = \{P \in \mathbb{W}(P_0) : l(P) \neq 0 \text{ for each } l \in \mathscr{L}_0\}.$$

The compactness of $\mathscr{L}_0$ at once implies that $\mathbb{E}(P_0)$ is open. Of course, $\mathbb{E}(P_0) \neq \varnothing$ since $P_0 \in \mathbb{E}(P_0)$. For any $l \in \mathscr{L}_0$ set

$$\mathbb{E}_l := \{P \in \mathbb{W}(P_0) : l(P) \neq 0\}.$$

This is a domain of holomorphy in $\mathbb{W}(P_0)$ since the function $P \mapsto 1/l(P)$ is analytic in $\mathbb{E}_l$ and singular everywhere on the boundary of $\mathbb{E}_l$. Because of $\mathbb{E}(P_0) = \bigcap_{l\in\mathscr{L}_0} \mathbb{E}_l = \text{interior}(\mathbb{E}(P_0))$ we conclude from [3, 2.5.7] that $\mathbb{E}(P_0)$ is also a domain of holomorphy. □

Remark. In general, the set $\mathbb{E}(P_0)$ is not connected as can be seen from the following example: Consider the polynomials $P_0(x) := x + iy$, $P_1(x) := x - iy$ in two variables. We have $P_0 \approx P_1$. However it is not possible to find a path from P_0 to P_1 in $\mathbb{W}(P_0)$ without leaving the set $\mathbb{E}(P_0)$. Hence P_0 and P_1 belong to different components of $\mathbb{E}(P_0)$.

Now consider a polynomial $P(\lambda, x) = \sum_{|\alpha|\leq m} a_\alpha(\lambda)x^\alpha$ where the coefficients a_α (constant with respect to x) are functions of a parameter $\lambda \in \Lambda$.

3.2. **Definition.** We say that the family $\{P(\lambda, \cdot) : \lambda \in \Lambda\}$ is an $\mathbb{E}(P_0)$-*family*, P_0 a fixed polynomial, if $P(\lambda, \cdot) \approx P_0$ for each $\lambda \in \Lambda$. It is called a *continuous* resp. *analytic* (etc.) $\mathbb{E}(P_0)$-*family* if the functions a_α are continuous resp. analytic in Λ. Note that by Lemma 3.1 the family $\{P : P \in \mathbb{E}(P_0)\}$ itself is an analytic $\mathbb{E}(P_0)$-family with parameter manifold $\Lambda = \mathbb{E}(P_0)$ which is a domain of holomorphy in $\mathbb{W}(P_0)$.

3.3. *Remark.* Let $\{P(\lambda, \cdot) : \lambda \in \Lambda\}$ be an $\mathbb{E}(P_0)$-family. With any fixed basis $\{P_0, P_1, \dots, P_{r-1}\}$ of $\mathbb{W}(P_0)$ we can write

$$P(\lambda, x) = \sum_{s=0}^{r-1} b_s(\lambda)P_s(x) \tag{3.1}$$

where the b_s are complex-valued functions of λ. Since the family $\{P_0, P_1, \dots, P_{r-1}\}$ is linearly independent the function b_s are continuous, analytic (etc.) iff each a_α has this property.

3.4. **Definition.** A polynomial $P \in \mathbb{C}[x_1, \dots, x_n]$ is called *hypoelliptic* if one of the following equivalent conditions [4, Theorem 11.1.3] is satisfied.

(i) If $d_P(\xi)$ is the distance of $\xi \in \mathbb{R}^n$ to the set $\{\zeta \in \mathbb{C}^n : P(\zeta) = 0\}$ then $d_P(\xi) \to \infty$ when $\xi \to \infty$ in $\mathbb{R}^n$.

(ii) $P^{(\alpha)}(\xi)/P(\xi) \to 0$ when $\xi \to \infty$ in $\mathbb{R}^n$ and $\alpha \neq 0$.

(iii) There exist constants $c, C > 0$ such that

$$|P^{(\alpha)}(\xi)|/|P(\xi)| \leq C|\xi|^{-c|\alpha|}$$

if $\xi \in \mathbb{R}^n$ and $|\xi|$ is sufficiently large.

An $\mathbb{E}(P_0)$-family will be called hypoelliptic if P_0 is hypoelliptic. In this case each polynomial $P \in \mathbb{E}(P_0)$ is hypoelliptic, too [4, Theorem 11.1.9]. On the other hand, if P_0 and P are both elliptic and have the same degree then $P \in \mathbb{E}(P_0)$ [4, Theorem 10.4.9].

4. The main results

For any complex manifold Λ and a locally convex vector space $\mathscr{E}$ we denote by $\mathscr{H}(\Lambda, \mathscr{E})$ the set of all analytic functions on Λ with values in $\mathscr{E}$ (cf. [5, §16.7]). This space will be endowed with the topology of uniform convergence on compact subsets of Λ.

Let $P_0 \neq 0$ be a fixed hypoelliptic polynomial, Λ a complex manifold and $\{P(\lambda, \cdot) : \lambda \in \Lambda\}$ an analytic hypoelliptic $\mathbb{E}(P_0)$-family.

4.1. **Theorem.** *Let* $1 < q \leq \infty$ *and* $k \in \mathscr{K}$. *Assume that* Λ *is a Stein manifold. Then for any* $\mathfrak{g} \in \mathscr{H}(\Lambda, \mathbf{B}_{q,k})$ *there exists* $\mathfrak{f} \in \mathscr{H}(\Lambda, \mathbf{B}^{\mathrm{loc}}_{q,k\widetilde{P}_0})$ *such that*

(1) $P(\lambda, D)\mathfrak{f}(\lambda) = \mathfrak{g}(\lambda)$, $\lambda \in \Lambda$;

(2) *for each* $\mu \in \mathbb{N}$ *the function* $\lambda \mapsto f(\lambda)/\gamma_\mu$ *is analytic with values in* $\mathbf{B}_{q,k\widetilde{P}_0}$ *where* γ_μ *is given by* (1.1).

Condition 4.1(2) means precisely that $\mathfrak{f} \in \mathscr{H}(\Lambda, \mathbf{B}^{-\infty}_{q,k\widetilde{P}_0})$. If $\mathfrak{g}$ is a constant then Λ may be an arbitrary complex manifold:

4.2. **Corollary.** *Let* $1 < q \leq \infty$ *and* $k \in \mathscr{K}$. *Let* Λ *be an arbitrary complex manifold. Then for any* $\mathfrak{g}_0 \in \mathbf{B}_{q,k}$ *there exists* $\mathfrak{f} \in \mathscr{H}(\Lambda, \mathbf{B}^{\mathrm{loc}}_{q,k\widetilde{P}_0})$ *such that* $P(\lambda, D)\mathfrak{f}(\lambda) \equiv \mathfrak{g}_0$. *Additionally,* 4.1(2) *is satisfied.*

Proof. By Lemma 3.1 we may consider the family $\{P : P \in \mathbb{E}(P_0)\}$ as an analytic $\mathbb{E}(P_0)$-family over itself and $\Lambda = \mathbb{E}(P_0)$ is a domain of holomorphy in $\mathbb{W}(P_0)$, hence Stein. From Theorem 4.1 we obtain a function $\tilde{\mathfrak{f}} \in \mathscr{H}(\mathbb{E}(P_0), \mathbf{B}^{-\infty}_{q,k\widetilde{P}_0})$ such that $P(D)\tilde{\mathfrak{f}}(P) = \mathfrak{g}_0$, $P \in \mathbb{E}(P_0)$. Since the mapping $\lambda \mapsto \mathfrak{p}(\lambda) := P(\lambda, \cdot)$ is analytic with values in $\mathbb{E}(P_0)$ we have $\mathfrak{f} := \tilde{\mathfrak{f}} \circ \mathfrak{p} \in \mathscr{H}(\Lambda, \mathbf{B}^{-\infty}_{q,k\widetilde{P}_0})$ and $P(\lambda, D)\mathfrak{f}(\lambda) \equiv \mathfrak{g}_0$. □

Since the Dirac distribution δ belongs to $\mathbf{B}_{\infty,1}$ (1 denotes the weight function which is identically 1) we immediately obtain a solution of problem $(*)$ posed in the introduction:

4.3. **Corollary.** *Let* $q = \infty$, $k \equiv 1$ *and* $\mathfrak{g}(\lambda) \equiv \delta$. *Let* Λ *be an arbitrary complex manifold. Then there exists* $\mathfrak{f} \in \mathscr{H}(\Lambda, \mathbf{B}^{\mathrm{loc}}_{\infty, \widetilde{P}_0})$ *such that* $P(\lambda, D)\mathfrak{f}(\lambda) \equiv \delta$. *Additionally,* 4.1(2) *is satisfied.*

4.4. *Remark.* If Λ is an open subset of $\mathbb{R}^d$ then the analogues of Theorem 4.1 and its corollaries hold with "analytic" replaced by "real analytic."

Proof. Let $\Lambda_1 \subseteq \mathbb{C}^d$ be a complex neighborhood of Λ such that the coefficients b_s in the representation (3.1) of $P(\lambda, x)$ and the function $\mathfrak{g}$ are analytic on Λ_1. Since $\mathbb{E}(P_0)$ is an open subset of $\mathbb{W}(P_0)$ we can choose Λ_1 in such a way that $\{P(\lambda, \cdot): \lambda \in \Lambda_1\}$ is still an $\mathbb{E}(P_0)$-family. By a result of Grauert [1, §3] there exists a complex Stein neighborhood Λ_2 of Λ such that $\Lambda \subseteq \Lambda_2 \subseteq \Lambda_1$. Now it suffices to apply Theorem 4.1 for the parameter manifold Λ_2 and take the restriction of the solution $\mathfrak{f}$ to Λ. □

If $\mathfrak{F}$, $\mathfrak{G}$ are Banach spaces we denote by $\mathscr{L}(\mathfrak{F}, \mathfrak{G})$ the space of all bounded linear operators from $\mathfrak{F}$ to $\mathfrak{G}$ equipped with the operator norm topology. The following result is due to J. Leiterer [6, Theorems 2.3(iv) and 5.1, and Corollary 5.4]. It will play an essential role in the proof of Theorem 4.1.

4.5. **Theorem.** *Let* $\mathfrak{F}$, $\mathfrak{G}$ *be Banach spaces and* Λ *a complex Stein manifold. Let* $\mathfrak{T} \in \mathscr{H}(\Lambda, \mathscr{L}(\mathfrak{F}, \mathfrak{G}))$ *such that* $\mathfrak{T}(\lambda)\mathfrak{F} = \mathfrak{G}$ *for each* $\lambda \in \Lambda$. *Then*

(a) *There exists for each function* $\mathfrak{g} \in \mathscr{H}(\lambda, \mathfrak{G})$ *a function* $\mathfrak{f} \in \mathscr{H}(\Lambda, \mathfrak{F})$ *such that* $\mathfrak{T}(\lambda)\mathfrak{f}(\lambda) = \mathfrak{g}(\lambda)$, $\lambda \in \Lambda$.

(b) *For any open subset* Λ' *of* Λ *let* $\mathscr{N}(\Lambda') := \{\mathfrak{f} \in \mathscr{H}(\Lambda', \mathfrak{F}): \mathfrak{T}(\lambda)\mathfrak{f}(\lambda) \equiv 0\}$. *If* Λ' *is holomorphically convex then the set* $\mathscr{N}(\Lambda)_{|\Lambda'}$ *of restrictions to* Λ' *of functions in* $\mathscr{N}(\Lambda)$ *is dense in* $\mathscr{N}(\Lambda')$.

It is our aim to apply this theorem in the case where the $\mathfrak{T}(\lambda)$ are differential operators generated by the formal expressions $P(\lambda, D)$. The following lemma is quoted from F. Trèves [9, p. 23].

4.6. **Lemma.** *Let* Λ *be a compact topological space and* $\{Q(\lambda, \cdot): \lambda \in \Lambda\}$ *a continuous hypoelliptic* $\mathbb{E}(Q_0)$*-family,* $Q_0 \neq 0$. *Then there exists a constant* $c > 0$ *such that*

$$(4.1) \quad c^{-1}(1 + |Q_0(\xi)|) \leq 1 + |Q(\lambda, \xi)| \leq c(1 + |Q_0(\xi)|), \qquad \lambda \in \Lambda, \ \xi \in \mathbb{R}^n.$$

4.7. **Corollary.** *Same assumptions as in Lemma* 4.6. *Then there exist constants* $\rho, c > 0$ *such that*

$$(4.2) \qquad c^{-1}\widetilde{Q}_0(\xi) \leq |Q(\lambda, \xi)| \leq c\widetilde{Q}_0(\xi), \qquad \lambda \in \Lambda, \ \xi \in \mathbb{R}^n, \ |\xi| \geq \rho.$$

Proof. Choose $c_1 > 0$ such that (4.1) holds with $c = c_1$. Since by 3.4(ii) we have $|Q_0(\xi)| \to \infty$ $(\xi \to \infty$ in $\mathbb{R}^n)$ one can find $\rho > 0$ with the property that $|Q_0(\xi)| \geq 2c_1^{-1}$, $|\xi| \geq \rho$; hence

$$1 + |Q_0(\xi)| - c_1 \geq \tfrac{1}{2}(1 + |Q_0(\xi)|), \qquad |\xi| \geq \rho.$$

Now (4.1) yields

$$|Q(\lambda, \xi)| \geq c_1^{-1}(1 + |Q_0(\xi)|) - 1 \geq (2c_1)^{-1}(1 + |Q_0(\xi)|), \qquad \lambda \in \Lambda, \ |\xi| \geq \rho.$$

By 3.4(iii) we have $\widetilde{Q}_0(\xi) \leq c_2(1+|Q_0(\xi)|)$, $\xi \in \mathbb{R}^n$, with some constant $c_2 > 0$. Hence, $\widetilde{Q}_0(\xi) \leq 2c_1c_2|Q(\lambda, \xi)|$, $\lambda \in \Lambda$, $|\xi| \geq \rho$. The second inequality in (4.2) follows from the second inequality in (4.1). □

4.8. **Lemma.** *Same assumptions as in Lemma* 4.6. *Further, let* $\eta \in \mathbb{R}^n\backslash\{0\}$ *and* $\rho > 0$ *be fixed. Then there exists* $t > 0$ *such that*

$$(4.3)\qquad Q(\lambda, \xi + z\eta) \neq 0, \qquad \lambda \in \Lambda,\ \xi \in \mathbb{R}^n,\ |\xi| \le \rho,\ z \in \mathbb{C},\ |z| \ge t.$$

Proof. Consider the polynomial $q_{\lambda,\xi}(z) := Q(\lambda, \xi+z\eta)$, $z \in \mathbb{C}$. Since $|q_{\lambda,\xi}(z)| \to \infty$ $(z \to \infty$ in $\mathbb{R})$ by 3.4(ii), $q_{\lambda,\xi}$ is nonconstant. (4.1) implies that the degree of $q_{\lambda,\xi}$ is independent of $\lambda \in \Lambda$. Since for fixed $\xi \in \mathbb{R}^n$ the polynomial $Q(\lambda, \xi + \cdot)$ is equally strong as $Q(\lambda, \cdot)$, the same argument shows that the degree of $q_{\lambda,\xi}$ is independent of $\xi \in \mathbb{R}^n$ too. Since $\Lambda \times \{\xi \in \mathbb{R}^n : |\xi| \le \rho\}$ is compact there exists $t > 0$ such that the zeros z of $q_{\lambda,\xi}$ satisfy $|z| < t$ for $\lambda \in \Lambda$, $|\xi| \le \rho$. □

Proof of Theorem 4.1. First we choose an exhausting sequence of open submanifolds Λ_ν of Λ such that each Λ_ν is holomorphically convex, $\overline{\Lambda_\nu}$ is compact and $\overline{\Lambda_\nu} \subseteq \Lambda_{\nu+1}$, $\nu \in \mathbb{N}$. Note that $\{Q(\lambda, \cdot) : \lambda \in \Lambda\}$ is an analytic hypoelliptic $\mathbb{E}(Q_0)$-family where we have set $Q(\lambda, x) := P(\lambda, -x)$, $Q_0(x) := P_0(-x)$. Hence by Corollary 4.7 there exists an increasing sequence $0 < \rho_1 \le \rho_2 \le \cdots$ such that with some constants $c_\nu > 0$,

$$(4.4)\qquad c_\nu^{-1}\widetilde{Q}_0(\xi) \le |Q(\lambda, \xi)| \le c_\nu \widetilde{Q}_0(\xi), \qquad \lambda \in \overline{\Lambda_\nu},\ \xi \in \mathbb{R}^n,\ |\xi| \ge \rho_\nu.$$

Fix any vector $\eta \in \mathbb{R}^n$, $|\eta| = 1$. From Lemma 4.8 we get the existence of a sequence $0 < t_1 \le t_2 \le \cdots$ such that

$$(4.5)\quad Q(\lambda, \xi + zt_\nu\eta) \neq 0, \qquad \lambda \in \overline{\Lambda_\nu},\ \xi \in \mathbb{R}^n,\ |\xi| \le \rho_\nu,\ z \in \mathbb{C},\ |z| = 1.$$

We set $\eta_\nu := t_\nu\eta$ and define the spaces

$$\mathfrak{F}_\nu := \mathbf{B}^{-\rho_\nu, \eta_\nu}_{q, k\widetilde{P}_0}, \quad \mathfrak{G}_\nu := \mathbf{B}^{-\rho_\nu, \eta_\nu}_{q, k}, \qquad \nu \in \mathbb{N}.$$

By (1.3), (1.6), and (1.11) we have

$$(4.6)\qquad \mathfrak{F}_\nu \hookrightarrow \mathfrak{F}_{\nu+1} \hookrightarrow \mathfrak{F} := \mathbf{B}^{-\infty}_{q, k\widetilde{P}_0} \hookrightarrow \mathbf{B}^{\text{loc}}_{q, k\widetilde{P}_0},$$

$$(4.7)\qquad \mathbf{B}_{q,k} \hookrightarrow \mathfrak{G}_\nu \hookrightarrow \mathfrak{G}_{\nu+1} \hookrightarrow \mathbf{B}^{\text{loc}}_{q,k}$$

(recall that $\mathbf{B}_{q,k} = \mathbf{B}^{-0,\eta}_{q,k}$). Now let $\{P_0, P_1, \dots, P_{r-1}\}$ be a basis of $\mathbb{W}(P_0)$. According to Remark 3.3 we can write $P(\lambda, x) = \sum_{s=0}^{r-1} b_s(\lambda)P_s(x)$ with $b_s \in \mathscr{H}(\Lambda, \mathbb{C})$, $s = 0, \dots, r-1$. By Proposition 2.1 each $P_s(D)$ induces an operator $\mathfrak{T}_{s,\nu} \in \mathscr{L}(\mathfrak{F}_\nu, \mathfrak{G}_\nu)$ such that $\mathfrak{T}_{s,\nu}u = P_s(D)u$, $u \in \mathfrak{F}_\nu$, in the distribution sense. Hence the mapping $\lambda \mapsto \mathfrak{T}_\nu(\lambda) := \sum_{s=0}^{r-1} b_s(\lambda)\mathfrak{T}_{s,\nu} = P(\lambda, D)_{|\mathfrak{F}_\nu}$ is analytic with values in $\mathscr{L}(\mathfrak{F}_\nu, \mathfrak{G}_\nu)$. It is clear that

$$(4.8)\qquad \mathfrak{T}_{\nu+1}(\lambda)_{|\mathfrak{F}_\nu} = \mathfrak{T}_\nu(\lambda), \qquad \lambda \in \Lambda_\nu.$$

From (4.4), (4.5), and Proposition 2.2 we conclude that $\mathfrak{T}_\nu(\lambda)\mathfrak{F}_\nu = \mathfrak{G}_\nu$ for each $\lambda \in \Lambda_\nu$. By (4.7) we have $\mathfrak{g} \in \mathscr{H}(\Lambda, \mathfrak{G}_\nu)$. It follows from part (a) of Theorem 4.5 that there exists for each ν a function $\tilde{\mathfrak{f}}_\nu \in \mathscr{H}(\Lambda_\nu, \mathfrak{F}_\nu)$ such that

$$\mathfrak{T}_\nu(\lambda)\tilde{\mathfrak{f}}_\nu(\lambda) = \mathfrak{g}(\lambda), \qquad \lambda \in \Lambda_\nu.$$

Now let $\sigma_1 \le \sigma_2 \le \cdots$ be an increasing fundamental sequence of seminorms of the Fréchet space $\mathfrak{F}$. Set $\mathfrak{f}_1 := \tilde{\mathfrak{f}}_1$ and assume that we have already constructed functions $\mathfrak{f}_\nu \in \mathscr{H}(\Lambda_\nu, \mathfrak{F}_\nu)$, $\nu = 1, \dots, \mu$. Consider then $\mathfrak{d}_{\mu+1}(\lambda) :=$

$\tilde{\mathfrak{f}}_{\mu+1}(\lambda) - \mathfrak{f}_\mu(\lambda)$, $\lambda \in \Lambda_\mu$. Obviously, $\mathfrak{d}_{\mu+1} \in \mathscr{H}(\Lambda_\mu, \mathfrak{F}_{\mu+1})$ and we may assume inductively that $\mathfrak{T}_{\mu+1}(\lambda)\mathfrak{d}_{\mu+1}(\lambda) = 0$, $\lambda \in \Lambda_\mu$. By part (b) of Theorem 4.5 there exists for arbitrary $\varepsilon_{\mu+1} > 0$ a function $\mathfrak{c}_{\mu+1} \in \mathscr{H}(\Lambda_{\mu+1}, \mathfrak{F}_{\mu+1})$ with the properties

$$\mathfrak{T}_{\mu+1}(\lambda)\mathfrak{c}_{\mu+1}(\lambda) = 0, \qquad \lambda \in \Lambda_{\mu+1},$$
$$\sup_{\lambda\in\Lambda_{\mu-1}} \|\mathfrak{d}_{\mu+1}(\lambda) - \mathfrak{c}_{\mu+1}(\lambda)\|_{\mathfrak{F}_{\mu+1}} \le \varepsilon_{\mu+1},$$

where for convenience we put $\Lambda_0 := \varnothing$. Since $\mathfrak{F}_{\mu+1} \hookrightarrow \mathfrak{F}$ we can choose $\varepsilon_{\mu+1}$ so small that $\sup_{\lambda\in\Lambda_{\mu-1}} \sigma_\mu(\mathfrak{d}_{\mu+1}(\lambda) - \mathfrak{c}_{\mu+1}(\lambda)) \le 2^{-\mu}$. With this choice of $\mathfrak{c}_{\mu+1}$ we set $\mathfrak{f}_{\mu+1}(\lambda) := \tilde{\mathfrak{f}}_{\mu+1}(\lambda) - \mathfrak{c}_{\mu+1}(\lambda)$, $\lambda \in \Lambda_{\mu+1}$. In this way one obtains a sequence of functions $\mathfrak{f}_\nu \in \mathscr{H}(\Lambda_\nu, \mathfrak{F}_\nu) \subseteq \mathscr{H}(\Lambda_\nu, \mathfrak{F})$ with the properties

$$\mathfrak{T}_\nu(\lambda)\mathfrak{f}_\nu(\lambda) = P(\lambda, D)\mathfrak{f}_\nu(\lambda) = \mathfrak{g}(\lambda), \qquad \lambda \in \Lambda_\nu, \tag{4.9}$$
$$\sup_{\lambda\in\Lambda_{\nu-1}} \sigma_\nu(\mathfrak{f}_{\nu+1}(\lambda) - \mathfrak{f}_\nu(\lambda)) \le 2^{-\nu}. \tag{4.10}$$

Hence the limit $\mathfrak{f}(\lambda) := \lim_{\nu\to\infty} \mathfrak{f}_\nu(\lambda)$ exists in $\mathfrak{F}$ for each $\lambda \in \Lambda$, and $\mathfrak{f} \in \mathscr{H}(\Lambda, \mathfrak{F}) \subseteq \mathscr{H}(\Lambda, \mathbf{B}^{\mathrm{loc}}_{q,k\tilde{P}_0})$. It follows directly from the definition of $\mathfrak{F}$ that for each $\mu \in \mathbb{N}$ the function $\lambda \mapsto \mathfrak{f}(\lambda)/\gamma_\mu$ is analytic with values in $\mathbf{B}_{q,k\tilde{P}_0}$. Since $P(\lambda, D)\colon \mathbf{B}^{\mathrm{loc}}_{q,k\tilde{P}_0} \to \mathbf{B}^{\mathrm{loc}}_{q,k}$ is continuous [4, Theorem 10.1.22] we conclude from (4.6), (4.7), (4.9), and (4.10) that $P(\lambda, D)\mathfrak{f}(\lambda) \equiv \mathfrak{g}(\lambda)$. The proof is complete. □

Remark. It is known that each differential operator with constant coefficients possesses a tempered fundamental solution [2, 7]. Thus we might have tried to search for a solution $\mathfrak{f}\colon \Lambda \to \mathscr{S}'$ of the equation $P(\lambda, D)\mathfrak{f}(\lambda) \equiv \delta$. However, let us consider the ordinary differential operator $P(\lambda, D) = \frac{d}{dx} - \lambda$ where $\lambda \in \mathbb{C} = \Lambda$ is a parameter. Denoting by $H(x)$ the Heaviside function, each fundamental solution of $P(\lambda, D)$ can be written in the form $\mathfrak{f}_\lambda(x) = (H(x) + c_\lambda)\exp(\lambda x)$, c_λ a constant. If now it is required that $\mathfrak{f}_\lambda \in \mathscr{S}'$ then we have to take $c_\lambda = -1$ for $\mathrm{Re}(\lambda) > 0$, $c_\lambda = 0$ for $\mathrm{Re}(\lambda) < 0$. This example shows that in general it is not even possible to find a continuous function $\mathfrak{f}\colon \Lambda \to \mathscr{S}'$ satisfying $P(\lambda, D)\mathfrak{f}(\lambda) \equiv \delta$. At the same time we see that the solution space $\mathfrak{F} = \mathbf{B}^{-\infty}_{q,k\tilde{P}_0}$ which occurs in the proof of Theorem 4.1 cannot be chosen much smaller, for the space $\mathfrak{F}$ has to contain, for any $a > 0$, distributions in $\mathbf{B}^{\mathrm{loc}}_{q,k\tilde{P}_0}$ which "grow" at infinity like $\exp(a\sqrt{1 + [x, x]})$. However, F. Trèves [11] has shown that in the situation of Corollary 4.3 it is possible locally (in Λ) to choose $\mathfrak{f}(\lambda)$ with arbitrarily small exponential growth.

While this paper was in print the author succeeded in eliminating the assumption of hypoellipticity from Theorem 4.1 and its corollaries. A more general result will appear in Ann. Inst. Fourier (Grenoble).

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