TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 334, Number 2, December 1992

COHOMOLOGY OF THE SYMPLECTIC GROUP $Sp_4(\mathbb{Z})$ PART I: THE ODD TORSION CASE

ALAN BROWNSTEIN AND RONNIE LEE

ABSTRACT. Let h_2 be the degree two Siegel space and $Sp(4, \mathbb{Z})$ the symplectic group. The quotient $Sp(4, \mathbb{Z})\backslash h_2$ can be interpreted as the moduli space of stable Riemann surfaces of genus 2. This moduli space can be decomposed into two pieces corresponding to the moduli of degenerate and nondegenerate surfaces of genus 2. The decomposition leads to a Mayer-Vietoris sequence in cohomology relating the cohomology of $Sp(4, \mathbb{Z})$ to the cohomology of the genus two mapping class group Γ_2^0. Using this tool, the 3- and 5-primary pieces of the integral cohomology of $Sp(4, \mathbb{Z})$ are computed.

INTRODUCTION

Let $Sp_4(\mathbb{Z})$ denote the group of 4-by-4 integral matrices preserving the skew-symmetric pairing $\left[\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right]$. The purpose of this paper is to investigate the cohomology of this group.

In 1978, C. Soulé gave a complete calculation of the cohomology of $SL_3(\mathbb{Z})$. At about the same time some partial calculations of the cohomology of $SL_4(\mathbb{Z})$ and $SL_5(\mathbb{Z})$ were made by Lee and Szczarba (cf. [S, L-S]). Subsequently there were computations on related problems by various authors, notably Ash, Brownstein, Mendoza, Schwermer, and Vogtmann (cf. [A, Br, M, S-V, V]). The method of these authors was based on studying a cellular decomposition of the arithmetic quotient space. In the present paper our approach will rely on the geometry of the moduli space. The basic idea is that the arithmetic quotient space admits an interpretation as the moduli space of stable Riemann surfaces of genus 2. Prior to this, Mumford, Harer, and Lee-Weintraub investigated some of the structures of these moduli spaces and their cohomology (cf. [Mu1, Mu2, H, L-W]). Thus, in some sense, our research can be regarded as a continuation of their work.

Let h_2 denote the Siegel upper half space of genus 2, i.e.,

$$h_2 = \{Z | Z \in \mathcal{M}_2(\mathbb{C}),\ Z = Z^T,\ \operatorname{Im} Z > 0\}.$$

The symplectic group $Sp_4(\mathbb{Z})$ operates totally discontinuously on h_2 via the formula

$$M \cdot Z = (AZ + B) \cdot (CZ + D)^{-1}$$

Received by the editors April 4, 1990.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 11F75; Secondary 11F46, 32G15.

Both authors were partially supported by NSF grants during the period of research.

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. With respect to this action, the Siegel space h_2 admits an equivariant decomposition into two geometric pieces corresponding to the moduli spaces of nondegenerate and degenerate surfaces of genus 2. The orbifold fundamental groups of these spaces give rise respectively to the mapping class group Γ_2^0 and the wreath product $(SL_2(\mathbb{Z}) \times SL_2(\mathbb{Z})) \rtimes \mathbb{Z}/2$ of two copies of $SL_2(\mathbb{Z})$. One of our main results is to express the cohomology of $Sp_4(\mathbb{Z})$ in terms of the cohomology of these groups.

Theorem (5.13). *There exists a Mayer-Vietoris sequence of cohomology groups with coefficients in a ring* R,

$$\cdots \to H^d(Sp_4(\mathbb{Z}); R) \to H^d(SL_2(\mathbb{Z}) \wr \mathbb{Z}/2; R) \oplus H^d(\Gamma_2^0; R) \\ \to H^d(B \rtimes \mathbb{Z}/2; R) \to H^{d+1}(Sp_4(\mathbb{Z}); R) \to \cdots,$$

where the group $B \rtimes \mathbb{Z}/2$ *is defined in* (5.12).

There is a corresponding Mayer-Vietoris sequence for the projective symplectic group $PSp_4(\mathbb{Z})$.

From a different perspective, Benson, Bödigheimer, Cohen, and Peim have made extensive computations of the cohomology of Γ_2^0 (cf. [C, B-C, B-C-P]). The authors are particularly grateful to Fred Cohen for many useful conversations and ideas.

Using the Mayer-Vietoris sequence and knowledge of the cohomology of other groups, we perform explicit computations of the mod 3 and mod 5 cohomology. As the results are too technical to be stated here we refer the reader to (6.3) and (6.10). In a sequel the authors plan to complete the study by calculating the mod 2 and integral cohomology. The organization of the paper is as follows:

Section 1 describes the relationship between the symplectic groups $Sp_4(\mathbb{Z})$, $PSp_4(\mathbb{Z})$ and the mapping class groups Γ_2^0, Γ_0^6. Sections 2 and 3 give a geometric decomposition of the moduli space of principally polarized abelian varieties, $\mathscr{A}_2$, in terms of degenerations of Riemann surfaces. The two pieces in the decomposition correspond to the moduli spaces of degenerate and nondegenerate genus 2 Riemann surfaces. Section 4 contains a calculation of the orbifold fundamental group of the boundary of the moduli space of degenerate surfaces, $SP(\mathscr{M}_1^1)$, in $\mathscr{A}_2$. Section 5 establishes the existence of the Mayer-Vietoris sequence using the Borel construction for classifying spaces. The paper concludes with the computation of the 3- and 5-primary pieces of the integral cohomology of $Sp_4(\mathbb{Z})$.

1

A basic tool in our computation of the cohomology $H^*(Sp_4(\mathbb{Z}))$ of the symplectic group is reducing the problem to calculating the cohomology of the mapping class group Γ_0^6 of 6 points on a 2-sphere. The object of this section is to describe the relationship between the mapping class groups Γ_2^0, Γ_0^6 and the symplectic groups $Sp_4(\mathbb{Z})$, $PSp_4(\mathbb{Z})$.

Let $z_1, \dots, z_6$ denote 6 points on the 2-sphere S^2. The group of orientation-preserving diffeomorphisms of $S^2 - \{z_1, \dots, z_6\}$, $\mathrm{Diff}^+(S^2 - \{z_1, \dots, z_0\})$, has the structure of a topological group with respect to the compact open topology for function spaces. The set $\pi_0(\mathrm{Diff}^+(S^2 - \{z_1, \dots, z_6\}))$ of pathwise-connected components of $\mathrm{Diff}^+(S^2 = \{z_1, \dots, z_6\})$ has an induced group structure and is known as the mapping class group Γ_0^6 of the 2-sphere with 6 punctures. An element of Γ_0^6 can be thought of as an isotopy class of diffeomorphisms preserving the set of six points $\{z_1, \dots, z_6\}$.

Throughout this paper we will denote by S a Riemann surface of genus 2 without punctures. Analogous to the definition of Γ_0^6 above, there is the mapping class group Γ_2^0 of a Riemann surface of genus 2.

By a result of Bergau and Mennicke, and Birman (cf. [BM] and [Bi, p. 184]) Γ_2^0 has the following presentation:

$$
\begin{aligned}
\text{generators: } & \zeta_1, \dots, \zeta_5, \\
\text{relations: } & [\zeta_i, \zeta_j] = 1, \qquad |i-j| > 1, \\
& \zeta_i\zeta_{i+1}\zeta_i = \zeta_{i+1}\zeta_i\zeta_{i+1}, \\
& (\zeta_1\zeta_2\zeta_2\zeta_4\zeta_5)^6 = 1, \\
& \zeta^2 = 1, \\
& [\zeta, \zeta_i] = 1,
\end{aligned}
\tag{1.1}
$$

where $\zeta = \zeta_1\zeta_2\zeta_3\zeta_4\zeta_5^2\zeta_4\zeta_3\zeta_2\zeta_1$. The element ζ is central of order 2. If we form the quotient group $\Gamma_2^0/\langle\zeta\rangle$ then the resulting group is isomorphic to Γ_0^6. Hence there is a central extension

$$1 \to \mathbb{Z}/2\langle\zeta\rangle \to \Gamma_2^0 \to \Gamma_0^6 \to 1. \tag{1.2}$$

A detailed proof of (1.2) can be found in [Bi]. Underlying the proof is the geometric fact that a genus 2 Riemann surface S is a 2-fold branched cover over a 2-sphere with 6 branch points. In other words there exists an involution $i: S \to S$ with 6 fixed points such that the orbit space S/i is a 2-sphere. It follows from Teichmüller theory that given any two genus 2 surfaces with such an involution (S, i), (S', i') they can be deformed from one to the other by a 1-parameter family of genus 2 surfaces with involutions. Thus a diffeomorphism $\varphi: S \to S$ can be deformed to one which preserves the involution i, $i \circ \varphi = \varphi \circ i$. Passing to the orbit space S/i, φ gives rise to a diffeomorphism φ/i preserving the 6 branch points. By mapping φ to φ/i we obtain a homomorphism $\Gamma_2^0 \to \Gamma_0^6$. Clearly this homomorphism is surjective; its kernel is the cyclic group of order 2 generated by the involution i. Thus we have the central extension of (1.2).

The integral homology group $H_1(S, \mathbb{Z})$ of a genus 2 Riemann surface is a free abelian group of rank 4. The intersection number of two cycles provides us with a nonsingular skew-symmetric pairing $H_1(S, \mathbb{Z}) \times H_1(S, \mathbb{Z}) \to \mathbb{Z}$. A diffeomorphism $\varphi: S \to S$ induces an automorphism $\varphi_*: H_1(S, \mathbb{Z}) \to H_1(S, \mathbb{Z})$ on homology preserving the intersection pairing. The automorphism φ_* lies in the integral symplectic group

$$Sp_4(\mathbb{Z}) \cong \mathrm{Aut}(H_1(S, \mathbb{Z});\ \text{intersection pairing}).$$

There is a natural homomorphism $p: \Gamma_2^0 \to Sp_4(\mathbb{Z})$ defined by mapping the isotopy class of φ, $[\varphi]$, to φ^*. Since $p(i) = -I$ we obtain an induced homomorphism $\overline{p}: \Gamma_0^6 \to PSp_4(\mathbb{Z})$.

It is well known that the map p is surjective. In the literature the kernel, $\ker(p)$, is called the Torelli group T_2^0. An obvious element in T_2^0 is the Dehn twist $D(\gamma)$ along the null-homologous curve γ as in Figure (1.3). It is known

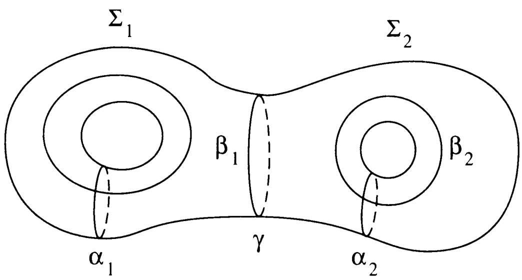

FIGURE (1.3)

that in the genus 2 case the Torelli group is the normal closure in Γ_2^0 of $[D(\gamma)]$. Summarizing the above discussion, we have a commutative diagram of groups:

$$\begin{array}{ccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathbb{Z}/2\langle\zeta\rangle & \xrightarrow{\cong} & \mathbb{Z}/2\langle -I\rangle & & \\
 & & & & \downarrow & & \downarrow & & \\
1 & \to & T_2^0 & \to & \Gamma_2^0 & \xrightarrow{p} & Sp_4(\mathbb{Z}) & \to & 1 \\
 & & \downarrow\cong & & \downarrow & & \downarrow & & \\
1 & \to & T_2^0 & \to & \Gamma_0^6 & \xrightarrow{\bar{p}} & PSp_4(\mathbb{Z}) & \to & 1 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array}$$

2

The various groups Γ_2^0, Γ_0^6, $Sp_4(\mathbb{Z})$, and $PSp_4(\mathbb{Z})$ discussed in §1 all have natural interpretations as fundamental groups of moduli spaces in algebraic geometry. Our object in this section is to explain this correspondence.

Let $\mathscr{M}_2^0$ denote the coarse moduli space of genus 2 Riemann surfaces without punctures, i.e., points in $\mathscr{M}_2^0$ are in one-to-one correspondence with complex structures on a surface S of genus 2. If we have an analytical fibration of such complex surfaces over a base space B, then there exists a unique complex analytic map from B onto $\mathscr{M}_2^0$ (see [Mu1, p. 26]). This moduli space $\mathscr{M}_2^0$ can also be constructed as the quotient of the Teichmüller space T_2^0 under the action of the mapping class group Γ_2^0. The Teichmüller space T_2^0 is contractible, but Γ_2^0 does not act freely. Hence the moduli space has the structure of an orbifold and its orbifold fundamental group $\pi_1^{\mathrm{orb}}(\mathscr{M}_2^0)$ is isomorphic to Γ_2^0.

The Torelli Theorem states that a Riemann surface of genus greater than 0 is determined by its Jacobian variety. In the present genus 2 situation, the Jacobian of S, $\mathrm{Jac}(S)$, is a complex torus of complex dimension 2, i.e., $\mathbb{C}^2/L$, for some lattice L. The lattice L associated to S is obtained very concretely; it is the lattice of all period matrices for S. If $\{dz_1, dz_2\}$ is a basis of the space $H^{1,0}(S)$ of holomorphic 1-forms on S then

$$L = \left\{ \left(\int_\alpha dz_1, \int_\alpha dz_2 \right) \middle| \alpha \text{ is a closed curve on } S \right\}.$$

The above objects form a moduli space known as the (coarse) moduli space $\mathscr{A}_2$ of principally polarized abelian varieties of complex dimension 2 (cf. [Mu1,

p. 70]). An explicit construction of $\mathscr{A}_2$ follows. Let h_2 denote the Siegel upper half space of degree 2,

$$h_2 = \{Z \in \mathscr{M}_4(\mathbb{C}) | Z = Z^T, \ \operatorname{Im} Z > 0\}.$$

The integral symplectic group $Sp_4(\mathbb{Z})$ operates discontinuously on h_2 via the formula $g \cdot Z = (AZ + B)(CZ + D)^{-1}$, where $g \in Sp_4(\mathbb{Z})$ and $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ in block form. The quotient $Sp_4(\mathbb{Z})\backslash h_2$ gives us a model for the moduli space $\mathscr{A}_2$.

To see that $\mathscr{A}_2$ is isomorphic to the orbit space $Sp_4(\mathbb{Z})\backslash h_2$ notice that a point $\Omega \in h_2$ determines a lattice $L = \mathbb{Z}^2 \oplus \mathbb{Z}^2(\Omega)$ in $\mathbb{C}^2$ and hence an abelian variety $\mathbb{C}^2/L$. This gives rise to a map $h_2 \to \mathscr{A}_2$ which is surjective, because every lattice in $\mathbb{C}^2$ is equivalent to one of the form $\mathbb{Z}^2 \oplus \mathbb{Z}^2(\Omega)$. This map fails to be injective because a choice of basis for $\mathbb{C}^2$ is implicitly involved. To be free of these choices we consider automorphisms of L which preserve a skew-symmetric pairing on L, i.e. polarizations. Thus $\Omega \in h_2$ is identified with all its translates under the action of $Sp_4(\mathbb{Z})$.

In terms of $\mathscr{A}_2$, the Torelli Theorem states that the period map $\mathscr{T}: \mathscr{M}_2^0 \to \mathscr{A}_2$, which assigns to a Riemann surface S its Jacobian $\operatorname{Jac}(S)$, is an injection. The complex dimension of the moduli spaces $\mathscr{M}_2^0$ and $\mathscr{A}_2$ are both 3. Hence, by the principle of invariance of domain, the image $\mathscr{T}(\mathscr{M}_2^0)$ is an open subspace of $\mathscr{A}_2$. Note that this situation is atypical since for genus $g > 3$ the dimension of the moduli space $\mathscr{A}_g$ is larger than that of the moduli space $\mathscr{M}_g^0$.

We now give an explicit description of the open subspace $\mathscr{T}(\mathscr{M}_2^0) \in \mathscr{A}_2$. The space h_2 contains the subspace of diagonal matrices

$$h_1 \times h_1 = \left\{ \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix} \middle| \operatorname{Im} z_i > 0, \ i = 1, 2 \right\},$$

which is clearly isomorphic to the product of two Siegel spaces of degree 1. Let $g(h_1 \times h_1)$ denote the translate of $h_1 \times h_1$ by an element $g \in Sp_4(\mathbb{Z})$. We denote by U the union of all these translates and denote by V the complement of U in h_2,

$$U = \bigcup_{g \in Sp_4(\mathbb{Z})} g(h_1 \times h_1), \qquad V = h_2 - \bigcup_{g \in Sp_4(\mathbb{Z})} g(h_1 \times h_1). \tag{2.1}$$

By construction U and V are disjoint $Sp_4(\mathbb{Z})$-invariant subspaces whose union is h_2. Passing to the quotient we obtain two complementary subspaces of $\mathscr{A}_2$,

$$\begin{aligned} Sp_4(\mathbb{Z})\backslash U &= Sp_4(\mathbb{Z})\backslash \bigcup_{g \in Sp_4(\mathbb{Z})} g(h_1 \times h_1), \\ Sp_4(\mathbb{Z})\backslash V &= Sp_4(\mathbb{Z})\backslash h_2 - \bigcup_{g \in Sp_4(\mathbb{Z})} g(h_1 \times h_1), \end{aligned} \tag{2.2}$$

where the former is closed and the latter is open. The following theorem, which is a consequence of a general result of W. Hoyt [Ho], shows that the open subspace $Sp_4(\mathbb{Z})\backslash V$ can be identified with the moduli space $\mathscr{M}_2^0$.

Theorem (2.3). *The open subspace $Sp_4(\mathbb{Z})\backslash V$ in $\mathscr{A}_2$ coincides with the image of the period map $\mathscr{T}$.*

In [Mu1], Mumford indicated a proof of Hoyt's Theorem, using the idea of compactification. Since this is particularly relevant to our discussion, we review the argument here.

There is a natural compactification $(\mathscr{M}_2^0)^*$ of $\mathscr{M}_2^0$ obtained by enlarging $\mathscr{M}_2^0$ to include all stable genus 2 Riemann surfaces with nodes. By a Riemann surface with nodes, we mean a singular surface S_0 such that every point in S_0 has a local neighborhood isomorphic either to the unit disk $\{|z| < 1\}$ in $\mathbb{C}$, or the cone $\{(z_1, z_2) \in \mathbb{C}^2 | z_1 z_2 = 0, \ |z_1| < 1, \ |z_2| < 1\}$. The first case represents a generic point in S_0 which is locally Euclidean, and the second an isolated singular point called a node, which can be obtained by collapsing a meridian circle in a cylinder to a point. The complement of all the nodes, $S_0 - \{\text{nodes}\}$, is a nonsingular surface, whose connected components are called the parts $\{\Sigma\}$. A surface S_0 with nodes is said to be stable if it does not contain any part which is a sphere with 0, 1, or 2 punctures.

There is a related compactification $\mathscr{A}_2^*$ of the moduli space $\mathscr{A}_2$ known as the Satake compactification. This is obtained by adding to $\mathscr{A}_2$ the moduli space of abelian varieties of rank one, and a base point $\{\mathscr{A}_0\}$, i.e.,

$$\mathscr{A}_2^* = \mathscr{A}_2 \amalg \mathscr{A}_1 \amalg \mathscr{A}_0 = Sp_4(\mathbb{Z}) \backslash h_2 \amalg Sp_2(\mathbb{Z}) \backslash h_1 \amalg \{\text{point}\}.$$

According to Namikawa in [N], the period map $\mathscr{T}$ can be extended to an analytical map $\mathscr{T} : (\mathscr{M}_2^0)^* \to \mathscr{A}^*$ on the compactifications. We now describe $\mathscr{A}^*$ explicitly. Given a Riemann surface S_0 with nodes, we first desingularize the surface by replacing each node by two nonsingular points and replacing a neighborhood of the node $z_1 z_2 = 0$ by two disjoint affine neighborhoods. The result is a nonsingular surface $N(S_0)$ called the normalization of S_0. The period mapping is extended to $(\mathscr{M}_2^0)^*$ by assigning to S_0 the period matrix $\text{Jac}(N(S_0))$.

In the genus 2 situation there are "essentially" two different ways to form a stable Riemann surface S_0 as the limit of a family of nonsingular surfaces S_t. The first is to shrink a nonseparating curve α_1 so that its arclength approaches zero, and the second is to perform the same operation to a separating curve γ. (See Figure (2.4).)

We now describe the normalizations in each of the two cases above. The normalization $N(S/\alpha_1)$ is a complex torus of complex dimension 1. Under the extended period map $\mathscr{T}^*$ it is sent to the boundary component $\mathscr{A}_1$ of the Satake compactification $\mathscr{A}_2^*$. In the separating curve case the normalization $N(S/\gamma)$ consists of two complex tori $\overline{\Sigma}_1 \amalg \overline{\Sigma}_2$. Its image under $\mathscr{T}^*$ is the product of two Jacobians $\text{Jac}(\overline{\Sigma}_1) \times \text{Jac}(\overline{\Sigma}_2)$. The period matrix for a single complex torus $\overline{\Sigma}_i$ lies in the upper half plane h_1. Thus considering the product as a complex torus of dimension 2, the period matrix lies in $h_1 \times h_1$. In other words, $\text{Jac}(N(S/\gamma))$ is parametrized by the subspace $Sp_4(\mathbb{Z}) \backslash U$ in (2.2).

Since the Satake compactification $\mathscr{A}_2^*$ is irreducible, $\mathscr{T}^*$ must be onto and so a point in A_2^* can either be the Jacobian of a nonsingular Riemann surface or the limit of the Jacobians of a family of nonsingular Riemann surfaces. From the discussion in the previous paragraph it follows that a point in $\mathscr{A}_2$, away from the boundary component $\overline{\mathscr{A}}_1$, can lie either in $\mathscr{T}(\mathscr{M}_2^0)$ or $Sp_4(\mathbb{Z}) \backslash U$. Hence we have $\mathscr{A}_2 = \mathscr{T}(\mathscr{M}_2^0) \amalg Sp_4(\mathbb{Z}) \backslash U$, i.e., $\mathscr{T}(\mathscr{M}_2^0) = \mathscr{A}_2 - Sp_4(\mathbb{Z}) \backslash U = Sp_4(\mathbb{Z}) \backslash V$. This proves Theorem 2.3.

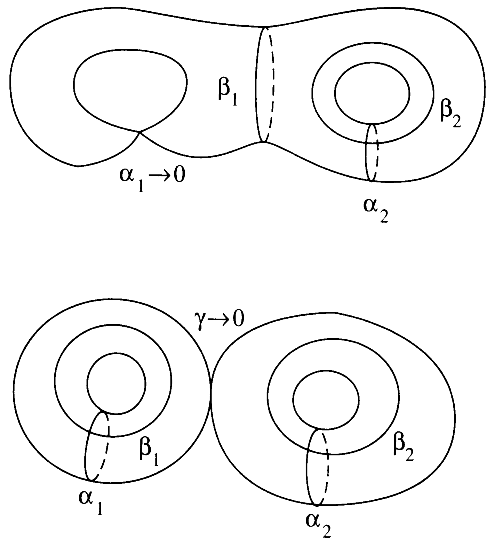

FIGURE (2.4)

3

Before proceeding further we need a better description of U and its quotient $Sp_4(\mathbb{Z})\backslash U$.

The subspace $h_1 \times h_1 = \{(\begin{smallmatrix} z_1 & 0 \\ 0 & z_2 \end{smallmatrix}) | \operatorname{Im} z_i > 0\}$ in h_2 is the fixed point set of the involution $\tau = \operatorname{diag}(1, -1, 1, -1)$ in $Sp_4(\mathbb{Z})$. Let $N(\tau)$ denote the normalizer of $\pm\tau$ in $Sp_4(\mathbb{Z})$. Then $N(\tau)$ consists of the following matrices:

$$(3.1) \qquad \begin{pmatrix} a_1 & 0 & b_1 & 0 \\ 0 & a_2 & 0 & b_2 \\ c_1 & 0 & d_1 & 0 \\ 0 & c_2 & 0 & d_2 \end{pmatrix}, \quad \begin{pmatrix} a_1 & 0 & b_1 & 0 \\ 0 & a_2 & 0 & b_2 \\ c_1 & 0 & d_1 & 0 \\ 0 & c_2 & 0 & d_2 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \in SL_2(\mathbb{Z}).$$

In other words $N(\tau) \cong (SL_2(\mathbb{Z}) \times SL_2(\mathbb{Z})) \rtimes \mathbb{Z}/2$, where the $\mathbb{Z}/2$ action switches the two copies of $SL_2(\mathbb{Z})$.

Clearly the action $(N(\tau), h_2)$ of $N(\tau)$ on the Siegel space h_2 keeps the subspace $h_1 \times h_1$ invariant. In fact $N(\tau)$ is the stabilizer of $h_1 \times h_1$ in $Sp_4(\mathbb{Z})$ for the following reason. An element $g \in Sp_4(\mathbb{Z})$ sends the fixed point subvariety $h_1 \times h_1$ of τ to $g(h_1 \times h_1)$ which is the fixed point subvariety of $\tau^g = g\tau g^{-1}$. If $\tau^g \neq \pm\tau$ then the defining equations for the fixed point subvarieties show that $g(h_1 \times h_1) \neq h_1 \times h_1$. Hence the two subvarieties coincide precisely when $g \in N(\tau)$.

Since $N(\tau)$ is the stabilizer of $h_1 \times h_1$, it follows that there exists a one-to-one correspondence between the fixed point subvarieties $(h_2)^{\tau^g} = g(h_1 \times h_1)$ and the right coset space $Sp_4(\mathbb{Z})/N(\tau)$. This allows us to rewrite U as a union of $g(h_1 \times h_1)$ indexed over $Sp_4(\mathbb{Z})/N(\tau)$,

$$U = \bigcup_{g \in Sp_4(\mathbb{Z})} g(h_1 \times h_1) = \bigcup_{g \in Sp_4(\mathbb{Z})/N(\tau)} g(h_1 \times h_1).$$

In fact this is a disjoint union.

Lemma (3.2). *If the cosets $gN(\tau) \neq hN(\tau)$ then $g(h_1 \times h_1)$ and $h(h_1 \times h_1)$ are disjoint.*

Proof. Suppose we have a point Z in the intersection $h_1 \times h_1$ and $g(h_1 \times h_1)$ with $g \notin N(\tau)$. Then τ and τ^g are distinct elements of the isotropy subgroup of Z, $I(Z)$, in $Sp_4(\mathbb{Z})$. In [G1, G2], Gottschling classified all the fixed point subvarieties X in h_2, and their isotropy subgroups $I(X)$. Table (3.3) contains the 8 cases when X is a subvariety of $h_1 \times h_1$, together with the isotropy subgroup $I(X)/\{\pm I\}$ and the center of $I(X)/\{\pm I\}$.

From Table (3.3) we see that τ lies in the center of $I(X)/\{\pm I\}$ for each of these subvarieties. Since conjugation by g induces an automorphism of $I(X)$, τ^g must also lie in the center. In the cases 3.3(b)–(e), (h), the center has a unique element of order 2, so $\tau = \pm\tau^g$. In the remaining two cases we note that the other matrices of order 2 are not conjguate to $\pm\tau$. This is easily seen by reducing the matrices mod 2. Thus in all cases $\tau = \pm\tau^g$, i.e., g lies in the normalizer of τ.

Definition (3.4). Let X be a space and $i: X \times X \to X \times X$ be the involution $i(x_1, x_2) = (x_2, x_1)$. Then the symmetric product $SP(X)$ is the orbit space $\langle i \rangle \backslash X \times X$.

Corollary (3.5). *The subvariety of $Sp_4(\mathbb{Z})\backslash U$ is isomorphic to the symmetric product $SP(\mathscr{A}_1)$ of $\mathscr{A}_1 = SL_2(\mathbb{Z})/h_1$.*

Proof. From the preceding discussion we have

$$\begin{aligned} Sp_4(\mathbb{Z})\backslash U &= Sp_4(\mathbb{Z})\backslash \coprod_{g \in Sp_4(\mathbb{Z})/N(\tau)} g(h_1 \times h_1) \\ &= N(\tau)\backslash h_1 \times h_1 \\ &= (SL_2(\mathbb{Z}) \times SL_2(\mathbb{Z})) \rtimes \mathbb{Z}/2 \backslash h_1 \times h_1 \\ &= \mathbb{Z}/2\backslash(SL_2(\mathbb{Z})/h_1 \times SL_2(\mathbb{Z})/h_1) \\ &= SP(\mathscr{A}_1). \end{aligned}$$

As shown in (3.2) the subspaces in the collection $\{g(h_1 \times h_1) | g \in Sp_4(\mathbb{Z})/N(\tau)\}$ are pairwise disjoint. As a result, we can pass to the orbit space $N(\tau)\backslash h_1 \times h_1$ in $\mathscr{A}_2$, form its tubular neighborhood, and lift this neighborhood back to h_2. This gives a $N(\tau)$-equivariant tubular neighborhood $\eta(h_1 \times h_1)$ which is disjoint from any translate $g\eta(h_1 \times h_1)$, $g \notin N(\tau)$.

Let W denote the union of these tubular neighborhoods, i.e.,

$$W = \coprod_{g \in Sp_4(\mathbb{Z})/N(\tau)} g\eta(h_1 \times h_1).$$

Then h_2 can be written as the union of two $Sp_4(\mathbb{Z})$-invariant subspaces W and V. The intersection $W \cap V$ has the equivariant homotopy type of the union

TABLE (3.3)

a	$X = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & \sqrt{-1} \end{pmatrix}$	$I(X)/\{\pm I\}$ is generated by $\alpha = \begin{pmatrix} & & -1 & \\ & 1 & & \\ 1 & & & \\ & & & 1 \end{pmatrix}, \quad B = \begin{pmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{pmatrix}$ with relations $\alpha^4 = \beta^2 = 1$, $[\alpha^2, \beta] = 1$.	Center of $I(X)/\pm I$ consists of $I, \tau, \begin{pmatrix} & & -1 & \\ & & & -1 \\ 1 & & & \\ & 1 & & \end{pmatrix}, \begin{pmatrix} & & 1 & \\ & & & -1 \\ -1 & & & \\ & 1 & & \end{pmatrix}$ and is isomorphic to $\mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$.
b	$X = \begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix}$ $\omega^3 = 1$, $\operatorname{Im}\omega > 0$	$I(X)/\{\pm I\}$ is generated by α, β, γ with the relation $\alpha^2 = \beta^2 = \gamma^3 = (\alpha\beta)^6 = 1$, $[\alpha, \gamma] = [\beta, \gamma] = 1$.	Center of $I(X)/\{\pm I\}$ is cyclic of order 6. The element $\tau = (\alpha, \beta)^2$ is the unique element of order 2.
c	$X = \begin{pmatrix} \omega & 0 \\ 0 & \sqrt{-1} \end{pmatrix}$ $\omega^3 = 1$, $\operatorname{Im}\omega > 0$	$I(X)/\{\pm I\}$ is cyclic of order 12.	$I(X)/\{\pm I\}$ is abelian, and contains τ as the unique element of order 2.
d	$X = \begin{pmatrix} \omega & 0 \\ 0 & z \end{pmatrix}$ $\omega^3 = 1$, $\operatorname{Im}\omega > 0$ $\operatorname{Im} z > 0$	$I(X)/\{\pm I\}$ is cyclic of order 6.	$I(X)/\{\pm I\}$ is abelian, and contains τ as the unique element of order 2.
e	$X = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & z \end{pmatrix}$	$I(X)/\{\pm I\}$ is cyclic of order 4.	$I(X)/\{\pm I\}$ is abelian, and contains τ as the unique element of order 2.
f	$X = \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}$ $\operatorname{Im} z > 0$	$I(X)/\{\pm I\} \cong \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$ and is given by $I, \tau, \begin{pmatrix} & & & -1 \\ & & -1 & \\ & 1 & & \\ 1 & & & \end{pmatrix}, \begin{pmatrix} & & & -1 \\ & & 1 & \\ & 1 & & \\ -1 & & & \end{pmatrix}.$	$I(X)/\{\pm I\}$ is abelian.
g	$X = \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}$ $\operatorname{Im} z > 0$	$I(X)/\{\pm I\} \cong \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$ and is given by $I, \tau, \begin{pmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{pmatrix}, \begin{pmatrix} & -1 & & \\ 1 & & & \\ & & & -1 \\ & & 1 & \end{pmatrix}.$	$I(X)/\{\pm I\}$ is abelian.
h	$X = \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix}$	$I(X)/\{\pm I\} \cong \mathbf{Z}/2\mathbf{Z}$ and is given by I, τ.	$I(X)/\{\pm I\}$ is abelian.

of the boundary of each tubular neighborhood $g\eta(h_1 \times h_1)$,

$$W \cap V \sim \partial V \sim \coprod_{g \in Sp_4(\mathbb{Z})/N(\tau)} g \cdot \partial\eta(h_1 \times h_1).$$

In order to understand the structure of ∂V, we return to the moduli space $(\mathscr{M}_2^0)^*$ of stable genus 2 Riemann surfaces with nodes. Inside $(\mathscr{M}_2^0)^*$ there is a subspace $SP(\mathscr{M}_1^1)$ consisting of stable Riemann surfaces $\Sigma_1 \cup \Sigma_2$, where $\Sigma_1 \cap \Sigma_2 = \{x\}$, a single point, and Σ_1, Σ_2 are smooth Riemann surfaces of genus 1. As the notation suggests, the subspace can be constructed as follows (cf. [Mu2, p. 319]). Form the symmetric product $SP(\mathscr{M}_1^1)$ of the moduli space of genus 1 Riemann surfaces with one puncture. Each of the factors in the product parameterizes one part of the stable Riemann surface. We must mod out by the action which switches the factors, as this does not affect the complex structure.

In the proof of (2.3) we have shown that the period map $\mathscr{T}: \mathscr{M}_2^0 \to \mathscr{A}_2$ extends to a surjection $\mathscr{T}^*: \mathscr{M}_2^0 \amalg SP(\mathscr{M}_1^1) \to \mathscr{A}_2$, making the following diagram commute:

$$\begin{array}{ccc} \mathscr{M}_2^0 & \xrightarrow{\mathscr{T}} & \mathscr{T}(\mathscr{M}_2^0) \\ \cap & & \cap \\ \mathscr{M}_2^0 \amalg SP(\mathscr{M}_1^1) & \xrightarrow{\mathscr{T}^*} & \mathscr{A}_2 \\ \cup & & \cup \\ SP(\mathscr{M}_1^1) & \xrightarrow{\mathscr{T}^*} & N(\tau)\backslash h_1 \times h_1 \end{array}.$$

Lemma (3.6). *The above map, $\mathscr{T}^*: \mathscr{M}_2^0 \amalg SP(\mathscr{M}_1^1) \to \mathscr{A}_2$, is an isomorphism of orbifolds.*

Proof. By Corollary (3.4) the quotient $N(\tau)\backslash h_1 \times h_1$ is isomorphic to the symmetric product $SP(\mathscr{A}_1)$ of $\mathscr{A}_1 = SL_2(\mathbb{Z})/h_1$. On the other hand the moduli space $\mathscr{M}_1^1$ is isomorphic to $SL_2(\mathbb{Z})/h_1$ under the period mapping (cf. [Mu2, p. 34]). Hence taking symmetric products we have an induced isomorphism $SP(\mathscr{M}_1^1) \xrightarrow{\simeq} SP(\mathscr{A}_1)$ on the bottom line of the preceding diagram. As the top line also gives an isomorphism, the result follows.

4

In the preceding section we examined the correspondence between the decomposition of the Siegel space $h_2 = V \cup W$ and certain moduli spaces via the extended period map $\mathscr{T}^*$. We now determine the structure and orbifold fundamental group of the intersection $V \cap W$ by looking at its pre-image under $\mathscr{T}^*$.

By the result of Lemma (3.6) we can pull back, via $\mathscr{T}^*$, the tubular neighborhood $N(\tau)\backslash\eta(h_1 \times h)$ in $Sp_4(\mathbb{Z})/h_2$ to a corresponding tubular neighborhood $\eta(SP(\mathscr{M}_1^1))$ of $SP(\mathscr{M}_1^1)$. Let $\partial\eta(SP(\mathscr{M}_1^1))$ denote the boundary of this tubular neighborhood. Then $\mathscr{T}^*$ induces isomorphisms between the orbifold fundamental groups $\pi_1^{\text{orb}}(N(\tau)\backslash\partial\eta(h_1 \times h_1))$ and $\pi_1^{\text{orb}}(\partial\eta(SP(\mathscr{M}_1^1)))$.

The boundary $\partial\eta(h_1 \times h_1)$ is a circle fibration over a contractible base space $h_1 \times h_1$, and therefore has the homotopy type of a circle. Taking the quotient under $N(\tau)$ we get a corresponding fibration for the orbit space $N(\tau)\backslash\partial\eta(h_1 \times h_1)$. This leads to a group extension of orbifold fundamental groups:

$$1 \to \mathbb{Z} \to \pi_1^{\text{orb}}(N(\tau)\backslash\partial\eta(h_1 \times h_1)) \to \pi_1^{\text{orb}}(N(\tau)\backslash h_1 \times h_1) \to 1 . \tag{4.1}$$

The infinite cyclic generator of $\pi_1(\partial\eta(h_1 \times h_1))$ is determined by the orientation of a fiber circle, which in turn is determined by the complex orientation of a complex line bundle; hence it remains unchanged as we vary over the base space. Thus (4.1) is a central extension, which we will determine in this section.

Let $St_2(\mathbb{Z})$ denote the Steinberg group associated to $SL_2(\mathbb{Z})$ (cf. [Mi, p. 82]). The group $St_2(\mathbb{Z})$ has presentation

$$St_2(\mathbb{Z}) = \langle \sigma_1 , \sigma_2 | \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle .$$

There is a homomorphism $\psi\colon St_2(\mathbb{Z}) \to SL_2(\mathbb{Z})$ given by $\psi(\sigma_1) = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, $\psi(\sigma_2) = \left(\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right)$. The kernel of ψ is an infinite cyclic subgroup generated by $\sigma = (\sigma_1\sigma_2\sigma_1)^4$. Since the center of $St_2(\mathbb{Z})$ is generated by $(\sigma_1\sigma_2\sigma_1)^2$, $\ker(\psi)$ is central, i.e. there is a central extension

$$1 \to \mathbb{Z}[\sigma] \to St_2(\mathbb{Z}) \xrightarrow{\psi} SL_2(\mathbb{Z}) \to 1 . \tag{4.2}$$

Lemma (4.3). *The orbifold fundamental group of $\partial\eta(SP(\mathscr{M}_1^1))$,*

$$\pi_1^{\text{orb}}(\partial\eta(SP(\mathscr{M}_1^1)))$$

is isomorphic to

$$\frac{St_2(\mathbb{Z}) \times St_2(\mathbb{Z})}{\mathbb{Z}\langle \sigma \times \sigma^{-1} \rangle} \rtimes \mathbb{Z}/2 ,$$

where the $\mathbb{Z}/2$ acts by switching the two $St_2(\mathbb{Z})$ factors.

A point μ in $SP(\mathscr{M}_1^1)$ is represented by a stable Riemann surface $S_0 = \Sigma_1 \cup \Sigma_2$, $\Sigma_1 \cap \Sigma_2 = \{x\}$. Let ν_μ denote a normal slice of the neighborhood $\eta(SP(\mathscr{M}_1^1))$ at the point μ. Then ν_μ represents a one-parameter family of genus 2 Riemann surfaces S_t, $t \in \nu_\mu$, degenerating to S_0. Near the singular point x in S_0, the degeneration can be described by the equation $z_1 z_2 = t$. For every $t \neq 0$, the surface S_t contains a hyperboloid, which degenerates to the cone $z_1 z_2 = 0$ as $t \to 0$ (cf. [Mu2, p. 305] and Diagram (4.4) below).

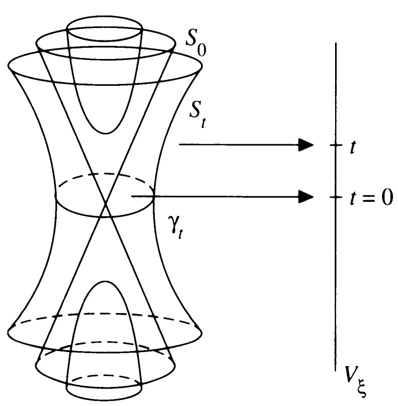

DIAGRAM (4.4)

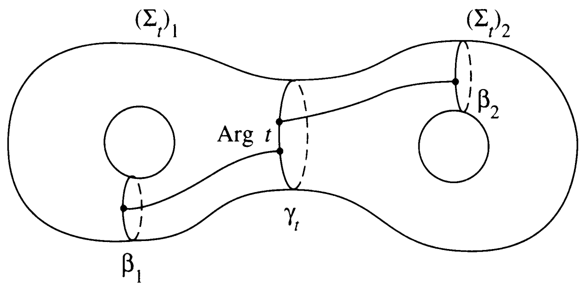

DIAGRAM (4.5)

For each t, the hyperboloid contains a geodesic curve γ_t which separates S_t into two pieces $(\Sigma_t)_1$ and $(\Sigma_t)_2$. In fact t can be chosen so that it becomes part of the Fenchel-Nielsen coordinates for Teichmüller space (see [A, p. 84]). The absolute value $|t|$ of t measures the arc-length of this separating geodesic curve γ_t, and the angle $\operatorname{Arg}(t)$ appears as the angle parameter in the Fenchel-Nielsen coordinates. If we fix $|t|$ and vary $\operatorname{Arg}(t)$ this has the effect of rotating the parts $(\Sigma_t)_1$ and $(\Sigma_t)_2$ so that on γ_t they move the angle $\operatorname{Arg}(t)$ with respect to the previous position. See Diagram (4.5).

As we vary over points μ in the base space $SP(\mathcal{M}_1^1)$ and points t in the fiber $\partial\eta_\mu$ we have the same picture: a Riemann surface $S_{(\mu,t)}$ with a separating geodesic curve γ_t. Hence we have a differential fibration over $\partial\eta(SP(\mathcal{M}_1^1))$ whose fiber consists of a pair (S, γ), where S is a genus 2 Riemann surface and γ is a separating curve. The structure group of this fibration is $\operatorname{Diff}^+(S, \gamma)$, i.e. orientation-preserving diffeomorphisms which leave the separating curve γ invariant. The holonomy map of this fibration gives us a map

$$\mathrm{hol}\colon \pi_1^{\mathrm{orb}}(SP(\mathcal{M}_1^1)) \to \Gamma(S, \gamma)$$

where $\Gamma(S, \gamma) = \pi_0(\operatorname{Diff}^+(S, \gamma))$.

The mapping class group $\Gamma(S, \gamma)$ has been studied by Charney and Lee in the following manner [C-L]. Associated to the stable Riemann surface $S/\gamma = \Sigma_1 \cup \Sigma_2$ there is a related mapping class group $\Gamma(S/\gamma) = \pi_0(\operatorname{Diff}^+(S/\gamma))$. Given a diffeomorphism $\varphi\colon (S, \gamma) \to (S, \gamma)$ there is an induced diffeomorphism $\varphi/\gamma\colon S/\gamma \to S/\gamma$ which fixes the point $[\gamma]$ in the quotient. This induces a surjective group homomorphism $\Gamma(S, \gamma) \to \Gamma(S/\gamma)$ whose kernel is the infinite cyclic group $\mathbb{Z}D(\gamma)$ given by a Dehn twist about γ, i.e.

$$1 \to \mathbb{Z}D(\gamma) \to \Gamma(S, \gamma) \to \Gamma(S/\gamma) \to 1. \tag{4.6}$$

If we let t move around the fiber circle of $\partial\eta_\mu$ once, then the effect on the complex structure of S_t amounts to rotating the part $(\Sigma_1)_t$ by an angle of 2π. In terms of the holonomy, this is, up to sign, the same as performing a Dehn twist along the curve γ. Thus the group extension (4.6) is compatible with the group extension (4.1),

$$\begin{array}{ccccccccc} 1 & \to & \mathbb{Z} & \to & \pi_1^{\mathrm{orb}}(\partial\eta(SP(\mathcal{M}_1^1))) & \to & \pi_1^{\mathrm{orb}}(SP(\mathcal{M}_1^1)) & \to & 1 \\ & & \downarrow \mathrm{hol} & & \downarrow \mathrm{hol} & & \downarrow \mathrm{hol} & & \\ 1 & \to & \mathbb{Z}D(\gamma) & \to & \Gamma(S, \gamma) & \to & \Gamma(S/\gamma) & \to & 1 \end{array} \tag{4.7}$$

Claim. The map $\mathrm{hol}\colon \Gamma(S/\gamma) \to \pi_1^{\mathrm{orb}}(SP(\mathcal{M}_1^1))$ in (4.7) is an isomorphism.

A homeomorphism $\varphi \in \Gamma(S/\gamma)$ fixes the point $\{x\} = \Sigma_1 \cap \Sigma_2$, thus it must either map each part to itself or switch the two parts. Furthermore, φ can be decomposed into its restriction to each part $\varphi|\Sigma_i$ and each restriction represents an element of the mapping class group $\Gamma(\Sigma_i, x) \cong \Gamma_1^1$. Hence we see that

$$\Gamma(S/\gamma) \cong (\Gamma(\Sigma_1, x) \times \Gamma(\Sigma_2, x)) \rtimes \mathbb{Z}/2.$$

Any diffeomorphism of a torus is isotopic to one which fixes a point, thus $\Gamma_1^1 \cong \Gamma_1^0 \cong SL(2, \mathbb{Z})$. As $\pi_1^{\text{orb}}(SP(\mathscr{M}_1^1)) \cong SL(2, \mathbb{Z}) \wr \mathbb{Z}/2$ and the holonomy map gives an isomorphism on each factor, the claim follows.

The above claim implies that the holonomy map gives an isomorphism

$$\pi_1^{\text{orb}}(SP(\mathscr{M}_1^1)) \xrightarrow{\cong} \Gamma(S, \gamma).$$

It remains to determine the structure of the latter group.

Let $S_{1,1}$ be a genus one surface with one boundary component. Then the mapping class group $\Gamma_{1,1}$ is defined to be the group of isotopy classes of orientation-preserving diffeomorphisms of $S_{1,1}$ which fix the boundary pointwise. It is well known that $\Gamma_{1,1}$ is isomorphic to the Steinberg group $St_2(\mathbb{Z})$. The generators σ_1, σ_2 correspond to Dehn twists about the meridian and longitude, respectively, and the central element $\sigma = (\sigma_1\sigma_2\sigma_1)^4$ corresponds to a Dehn twist parallel to the boundary.

The genus 2 surface S is the union, along γ, of two genus 1 surfaces with boundary component γ. Call these two pieces $\overline{\Sigma}_1$ and $\overline{\Sigma}_2$. A diffeomorphism of S preserving γ must either preserve or switch $\overline{\Sigma}_1$ and $\overline{\Sigma}_2$. Let $\Gamma(S, \gamma)'$ denote the index 2 subgroup of mapping classes which map the two halves to themselves. If $\psi: S \to S$ represents an element of $\Gamma(S, \gamma)'$ we can assume ψ fixes γ pointwise, since a Dehn twist about γ is isotopic to a Dehn twist about another simple closed curve parallel to γ. Thus ψ can be decomposed into its restrictions $\psi_i: \overline{\Sigma}_i \to \overline{\Sigma}_i$ and there is an epimorphism $\pi: St_2(\mathbb{Z}) \times St_2(\mathbb{Z}) \to \Gamma(S, \gamma)'$ given by pasting together two mapping classes in $\Gamma_{1,1} \cong St_2(\mathbb{Z})$. The map π fails to be injective since a Dehn twist parallel to γ in $\overline{\Sigma}_1$ is isotopic to a Dehn twist parallel to γ in $\overline{\Sigma}_2$. Identifying these gives the kernel of π, which is an infinite cyclic group generated by $\sigma \times \sigma^{-1}$. Therefore we have an isomorphism

$$\Gamma(S, \gamma)' \cong \frac{St_2(\mathbb{Z}) \times St_2(\mathbb{Z})}{\mathbb{Z}\langle \sigma \times \sigma^{-1} \rangle}.$$

In general, however, we can exchange the $\overline{\Sigma}_i$'s, so we must form the wreath product, giving

$$\Gamma(S, \gamma) \cong \frac{St_2(\mathbb{Z}) \times St_2(\mathbb{Z})}{\mathbb{Z}\langle \sigma \times \sigma^{-1} \rangle} \rtimes \mathbb{Z}/2.$$

This concludes the proof of (4.3).

5

From the results of §2 it is easily seen that the moduli space $\mathscr{M}_2^0$ and $\mathscr{A}_2$ have orbifold fundamental groups Γ_2^0 and $Sp_4(\mathbb{Z})$, respectively. The period map $\mathscr{T}$ is a map of orbifolds which induces the homomorphism $p: \Gamma_2^0 \to Sp_4(\mathbb{Z})$ on the orbifold fundamental groups of these spaces. For our application it is convenient to express these orbifold fundamental groups as fundamental groups of an appropriate Borel construction.

Definition (5.1). Let the pair (Γ, X) consist of a group Γ and a space X on which Γ acts. Let $E\Gamma$ denote a universal Γ-space, i.e. a contractible space with a free Γ-action. Then Γ acts via the diagonal action on the product. The quotient space $E\Gamma \times_\Gamma X$ is called the Borel construction $B[\Gamma, X]$ associated to (Γ, X).

In the above situation, if X is contractible then $B[\Gamma, X]$ is a $K(\Gamma, 1)$, i.e. the Borel construction can be regarded as the classifying space $B\Gamma$ of Γ. In particular this holds if we take $X = h_2$ and Γ to be $Sp_4(\mathbb{Z})$ or $PSp_4(\mathbb{Z})$. If the space X is not contractible, some additional care is necessary to compute the fundamental group.

Proposition (5.2). *The Borel construction* $B[Sp_4(\mathbb{Z}), V]$ *is a* $K(\pi, 1)$*-space with fundamental group* $\pi = \Gamma_2^0$.

Proof. Recall that $V = h_2 - \bigcup_g g(h_1 \times h_1)$ and $\mathscr{M}_2^0$ is the quotient of the Teichmüller space under the action of Γ_2^0. By Theorem 2.3, $\mathscr{M}_2^0$ can be identified with the quotient $Sp_4(\mathbb{Z})\backslash V$, so V can be thought of as the branched cover of $\mathscr{M}_2^0$ associated to the homomorphism $p\colon \Gamma_2^0 \to Sp_4(\mathbb{Z})$. By covering space theory, V is the quotient of Teichmüller space modulo the action of the Torelli group $T_2^0 = \ker p$. Since the Torelli group has no elements of finite order, it operates freely on the Teichmüller space and so V is a $K(\pi, 1)$ with fundamental group $\pi = T_2^0$.

Forming the Borel construction $B[Sp_4(\mathbb{Z}), V]$, we have a fibration

$$V \to B[Sp_4(\mathbb{Z}), V] \to B[Sp_4(\mathbb{Z})] \tag{5.3}$$

with fiber V and base $B[Sp_4(\mathbb{Z})]$. Since both fiber and base are $K(\pi, 1)$'s it follows that the total space is a $K(\pi, 1)$ and there is a group extension

$$1 \to \pi_1(V) \to \pi_1(B[Sp_4(\mathbb{Z}), V]) \to \pi_1(B[Sp_4(\mathbb{Z})]) \to 1. \tag{5.4}$$

There is a projection

$$\mathrm{pr}\colon B[Sp_4(\mathbb{Z}), V] \to Sp_4(\mathbb{Z})\backslash V = \mathscr{M}_2^0$$

which induces a natural homomorphism on orbifold fundamental groups

$$\mathrm{pr}_*\colon \pi_1(B[Sp_4(\mathbb{Z}), V]) \to \pi_1^{\mathrm{orb}}(\mathscr{M}_2^0).$$

Clearly the inclusion maps $\pi_1(V) \hookrightarrow \pi_1(B[Sp_4(\mathbb{Z}), V])$ and $\pi_1(V) \hookrightarrow \pi_1^{\mathrm{orb}}(\mathscr{M}_2^0)$ are compatible with pr_* hence there is an isomorphism of exact sequences

$$\begin{array}{ccccccccc}
1 & \to & \pi_1(V) & \to & \pi_1(B[Sp_4(\mathbb{Z}), V]) & \to & \pi_1(B[Sp_4(\mathbb{Z})]) & \to & 1 \\
 & & \downarrow\cong & & \downarrow & & \downarrow\cong & & \\
1 & \to & \pi_1(V) & \to & \pi_1^{\mathrm{orb}}(\mathscr{M}_2^0) & \to & Sp_4(\mathbb{Z}) & \to & 1 \\
 & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \\
1 & \to & T_2^0 & \to & \Gamma_2^0 & \to & Sp_4(\mathbb{Z}) & \to & 1
\end{array} \tag{5.5}$$

We have $\pi_1(B[Sp_4(\mathbb{Z}), V]) \cong \Gamma_2^0$, proving (5.2).

Applying an identical argument to $(PSp_4(\mathbb{Z}), V)$ we have

Proposition (5.6). *The Borel construction* $B[PSp_4(\mathbb{Z}), V]$ *is a* $K(\pi, 1)$ *space with fundamental group* Γ_0^6.

Applying the Borel construction simultaneously to h_2, V, W, and $W \cap V$ we obtain a push-out diagram of Borel constructions

$$\begin{array}{ccc} B[Sp_4(\mathbb{Z}), V \cap W] & \to & B[Sp_4(\mathbb{Z}), W] \\ \downarrow & & \downarrow \\ B[Sp_4(\mathbb{Z}), V] & \to & B[Sp_4(\mathbb{Z}), h_2] \end{array}$$

By Van Kampen's Theorem there is a corresponding push-out diagram of fundamental groups and the preceeding results allow us to fill in the bottom line:

$$(5.7)\qquad \begin{array}{ccc} \pi_1(B[Sp_4(\mathbb{Z}), V \cap W]) & \to & \pi_1(B[Sp_4(\mathbb{Z}), W]) \\ \downarrow & & \downarrow \\ \Gamma_2^0 & \to & Sp_4(\mathbb{Z}) \end{array}$$

Proposition (5.8). *The Borel construction $B[Sp_4(\mathbb{Z}), W]$ is a $K(\pi, 1)$-space with fundamental group $\pi = N(\tau)$. The Borel construction $B[PSp_4(\mathbb{Z}), W]$ is a $K(\pi, 1)$-space with fundamental group*

$$\pi = \frac{SL_2(\mathbb{Z}) \times SL_2(\mathbb{Z})}{\mathbb{Z}/2\langle -I, -I\rangle} \rtimes \mathbb{Z}/2.$$

Proof. The space $U = \coprod_{g \in Sp_4(\mathbb{Z})/N(\tau)} g(h_1 \times h_1)$ is an equivariant deformation retract of W. Viewing U as $Sp_4(\mathbb{Z}) \times_{N(\tau)} (h_1 \times h_1)$ we have the following series of homotopy equivalences:

$$\begin{aligned} B[Sp_4(\mathbb{Z}), W] &= E[Sp_4(\mathbb{Z})] \times_{Sp_4(\mathbb{Z})} W \sim E[Sp_4(\mathbb{Z})] \times_{Sp_4(\mathbb{Z})} U \\ &\sim E[Sp_4(\mathbb{Z})] \times_{Sp_4(\mathbb{Z})} (Sp_4(\mathbb{Z}) \times_{N(\tau)} (h_1 \times h_1)) \\ &\sim E[N(\tau)] \times_{N(\tau)} (h_1 \times h_1) = B[N(\tau), h_1 \times h_1]. \end{aligned}$$

Since $h_1 \times h_1$ is contractible, the result follows.

Recall from (3.1) that $N(\tau) \cong (SL_2(\mathbb{Z}) \times SL_2(\mathbb{Z})) \rtimes \mathbb{Z}/2$. Replacing $Sp_4(\mathbb{Z})$ by $PSp_4(\mathbb{Z})$ above, the second claim follows immediately. ▪

It remains only to determine the fundamental group $\pi_1(B[Sp_4(\mathbb{Z}), V \cap W])$.

Proposition (5.9). *The Borel construction $B[Sp_4(\mathbb{Z}), V \cap W]$ is a $K(\pi, 1)$-space with fundamental group π isomorphic to the semidirect product $\frac{St_2(\mathbb{Z}) \times St_2(\mathbb{Z})}{\mathbb{Z}/2\langle\sigma \times \sigma^{-1}\rangle} \rtimes \mathbb{Z}/2$, where the $\mathbb{Z}/2$ acts by switching the two factors in the product.*

Proof. By Lemma (4.3) it suffices to show that

$$\pi_1(B[Sp_4(\mathbb{Z}), V \cap W]) \cong \pi_1^{\text{orb}}(SP(\mathscr{M}_1^1)).$$

As noted in §3, $V \cap W$ has the same homotopy type as

$$\coprod_{g \in Sp_4(\mathbb{Z})/N(\tau)} g\partial\eta(h_1 \times h_1).$$

Hence,

$$\begin{aligned} B[Sp_4(\mathbb{Z}), V \cap W] &\sim E[N(\tau)] \times_{N(\tau)} \partial\eta(h_1 \times h_1) \\ &= B[N(\tau), \partial\eta(h_1 \times h_1)]. \end{aligned}$$

This gives rise to a fibration

$$\partial\eta(h_1 \times h_1) \to B[N(\tau), \partial\eta(h_1 \times h_1)] \to B[N(\tau)]$$

with fiber $\partial\eta(h_1 \times h_1)$ and base $B[N(\tau)]$. The base $B[N(\tau)]$ is a $K(\pi, 1)$, and since the fiber has the homotopy type of a circle it too is a $K(\pi, 1)$. It follows that the total space is a $K(\pi, 1)$ and we have a group extension

$$(5.10)\qquad 1 \to \pi_1(\partial\eta(h_1 \times h_1)) \to \pi_1(B[N(\tau), \partial\eta(h_1 \times h_1)]) \to N(\tau) \to 1.$$

Now there is a projection

$$B[N(\tau), \partial\eta(h_1 \times h_1)] \to N(\tau)\backslash\partial\eta(h_1 \times h_1)$$

which is compatible with the fibration structure of $N(\tau)\backslash\partial\eta(h_1 \times h_1)$ preceding (4.1). It is not hard to see that there is an induced isomorphism of the exact sequences (4.1) and (5.10) giving the desired result.

Proposition (5.11). *The Borel construction* $B[PSp_4(\mathbb{Z}), V \cap W]$ *is a* $K(\pi, 1)$-*space with fundamental group isomorphic to the semidirect product* $\frac{St_2(\mathbb{Z})\times St_2(\mathbb{Z})}{\mathbb{Z}\langle\mu\times\mu^{-1}\rangle} \rtimes \mathbb{Z}/2$, *where* $\mu = (\sigma_1\sigma_2\sigma_1)^2 \in St_2(\mathbb{Z})$.

Note that there is a central extension $1 \to \mathbb{Z}[\mu] \to St_2(\mathbb{Z}) \to PSL_2(\mathbb{Z}) \to 1$ and the image of μ in $SL_2(\mathbb{Z})$ is $-I$. The argument then proceeds as in (4.3) and (5.9), and we omit the details.

To simplify notation it will be convenient to make the following definitions:

$$(5.12)\qquad \begin{aligned} A &:= SL_2(\mathbb{Z}) \times SL_2(\mathbb{Z}), & B &:= \frac{St_2(\mathbb{Z}) \times St_2(\mathbb{Z})}{\mathbb{Z}\langle\sigma \times \sigma^{-1}\rangle}, \\ A' &:= \frac{SL_2(\mathbb{Z}) \times SL_2(\mathbb{Z})}{\mathbb{Z}/2\langle -I, -I\rangle}, & B' &:= \frac{St_2(\mathbb{Z}) \times St_2(\mathbb{Z})}{\mathbb{Z}\langle\mu \times \mu^{-1}\rangle}. \end{aligned}$$

As a direct consequence of the results in this section, we get

Theorem (5.12). *Let* R *be a ring. Then there are Mayer-Vietoris sequences in cohomology*

$$\begin{aligned} \cdots \to H^*(Sp_4(\mathbb{Z}); R) &\to H^*(\Gamma_2^0; R) \oplus H^*(A \rtimes \mathbb{Z}/2; R) \\ &\to H^*(B \rtimes \mathbb{Z}/2; R) \to H^{*+1}(Sp_4(\mathbb{Z}); R) \to \cdots \end{aligned}$$

and

$$\begin{aligned} \cdots H^*(PSp_4(\mathbb{Z}); R) &\to H^*(\Gamma_0^6; R) \oplus H^*(A' \rtimes \mathbb{Z}/2; R) \\ &\to H^*(B' \rtimes \mathbb{Z}/2; R) \to H^{*+1}(PSp_4(\mathbb{Z}); R) \to \cdots. \end{aligned}$$

6

We have shown that there is a Mayer-Vietoris sequence

$$\begin{aligned} \cdots \to H^*(PSp_4(\mathbb{Z}); R) &\to H^*(\Gamma_0^6; R) \oplus H^*(A' \rtimes \mathbb{Z}/2; R) \\ &\to H^*(B' \rtimes \mathbb{Z}/2; R) \to H^{*+1}(PSp_4(\mathbb{Z}); R) \to \cdots. \end{aligned}$$

We now compute the cohomology for the local rings $R = \mathbb{Z}_{(p)}$, p prime, $p > 2$, using this sequence. Since there is a central extension $1 \to \mathbb{Z}/2 \to Sp_4(\mathbb{Z}) \to PSp_4(\mathbb{Z}) \to 1$, we obtain the p-primary part of the integral cohomology of $Sp_4(\mathbb{Z})$ for $p \neq 2$.

Lemma (6.1). *For arbitrary coefficient ring* R *there exist isomorphisms in cohomology*

$$H^*(A' \rtimes \mathbb{Z}/2; R) \overset{\cong}{\leftarrow} H^*\left(\frac{\mathbb{Z}/12 \times \mathbb{Z}/12}{\mathbb{Z}/2\langle 6, -6\rangle} \rtimes \mathbb{Z}/2; R\right),$$

$$H^*(B' \rtimes \mathbb{Z}/2; R) \overset{\cong}{\leftarrow} H^*\left(\frac{\mathbb{Z} \times \mathbb{Z}}{\mathbb{Z}\langle 6, -6\rangle} \rtimes \mathbb{Z}/2; R\right).$$

Furthermore the natural induced map $H^*(A' \rtimes \mathbb{Z}/2) \to H^*(B' \rtimes \mathbb{Z}/2)$ *can be computed via the corresponding map*

$$\rho^*: H^*\left(\frac{\mathbb{Z}/12 \times \mathbb{Z}/12}{\mathbb{Z}/2\langle 6, -6\rangle} \rtimes \mathbb{Z}/2; R\right) \to H^*\left(\frac{\mathbb{Z} \times \mathbb{Z}}{\mathbb{Z}\langle 6, -6\rangle} \rtimes \mathbb{Z}/2; R\right)$$

where ρ *is reduction* mod 12 *in each factor of the semidirect product.*

Proof. Recall that A' is the group $\frac{SL_2(\mathbb{Z})\times SL_2(\mathbb{Z})}{\mathbb{Z}/2\langle -I, -I\rangle}$. It is well known that $SL_2(\mathbb{Z})$ has the structure of an amalgamated free product $\mathbb{Z}/6 *_{\mathbb{Z}/2} \mathbb{Z}/4$ where the cyclic factors are given by $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0\end{smallmatrix}\right)$ respectively. A direct computation shows that $H^*(SL_2(\mathbb{Z})) \cong H^*(\mathbb{Z}/12)$. The natural homomorphism $\phi_1: SL_2(\mathbb{Z}) \to \mathbb{Z}/12$ of $SL_2(\mathbb{Z})$ onto its commutator quotient induces an isomorphism in cohomology. Similarly the map $\phi_2: PSL_2(\mathbb{Z}) \to \mathbb{Z}/6$, given by abelianization, also induces an isomorphism in cohomology.

Now the above homomorphism ϕ_1 induces a map ϕ',

$$\begin{array}{ccccccccc} 1 & \to & \mathbb{Z}/2\langle -I, -I\rangle & \to & SL_2(\mathbb{Z}) \times SL_2(\mathbb{Z}) & \to & A' & \to & 1 \\ & & \downarrow & & \downarrow \phi_1\times\phi_1 & & \downarrow \phi' & & \\ 1 & \to & \mathbb{Z}/2\langle 6, -6\rangle & \to & \mathbb{Z}/12 \times \mathbb{Z}/12 & \to & \frac{\mathbb{Z}/12\times\mathbb{Z}/12}{\mathbb{Z}\langle 6, -6\rangle} & \to & 1 \end{array}.$$

In addition this is an isomorphism in cohomology because the commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 1 & \to & SL_2(\mathbb{Z}) & \to & A' & \overset{\mathrm{pr}}{\to} & PSL_2(\mathbb{Z}) & \to & 1 \\ & & \downarrow \phi_1 & & \downarrow \phi' & & \downarrow \phi_2 & & \\ 1 & \to & \mathbb{Z}/12 & \to & \frac{\mathbb{Z}/12\times\mathbb{Z}/12}{\mathbb{Z}\langle 6, -6\rangle} & \to & \mathbb{Z}/6 & \to & 1 \end{array}$$

leads to identical LHS spectral sequences. The map ϕ' extends to the semidirect product

$$\begin{array}{ccccccccc} 1 & \to & A' & \to & A' \rtimes \mathbb{Z}/2 & \to & \mathbb{Z}/2 & \to & 1 \\ & & \downarrow \phi' & & \downarrow \phi'' & & \downarrow \cong & & \\ 1 & \to & \frac{\mathbb{Z}/12\times\mathbb{Z}/12}{\mathbb{Z}\langle 6, -6\rangle} & \to & \frac{\mathbb{Z}/12\times\mathbb{Z}/12}{\mathbb{Z}\langle 6, -6\rangle} \rtimes \mathbb{Z}/2 & \to & \mathbb{Z}/2 & \to & 1 \end{array}$$

and we obtain once more identical LHS spectral sequences.

Recall that $B' = \frac{St_2(\mathbb{Z})\times St_2(\mathbb{Z})}{\mathbb{Z}\langle \mu\times\mu^{-1}\rangle}$ where $\mu = (\sigma_1\sigma_2\sigma_1)^2$ is central and the map $St_2(\mathbb{Z}) \to SL_2(\mathbb{Z})$ is given by $\phi_1 \mapsto \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1\end{smallmatrix}\right)$, $\sigma_2 \mapsto \left(\begin{smallmatrix} 1 & 0 \\ -1 & 1\end{smallmatrix}\right)$. The remaining claims in (6.1) follow by similar arguments noting that $St_2(\mathbb{Z}) \cong \pi_1(K)$, where K is the complement of the trefoil knot in S^3. Hence there is an isomorphism $H^*(St_2(\mathbb{Z})) \cong H^*(\mathbb{Z})$.

Theorem (6.2). *Let* $p \geq 5$. *We have*

$$H^*(Sp_4(\mathbb{Z}); \mathbb{Z}_{(p)}) \cong \begin{cases} H^*(\Gamma_0^6; \mathbb{Z}_{(p)}), & * \neq 2, \\ \mathbb{Z}_{(p)} \oplus H^2(\Gamma_0^6; \mathbb{Z}_{(p)}), & * = 2. \end{cases}$$

Proof. The finite group $\frac{\mathbb{Z}/12\times\mathbb{Z}/12}{\mathbb{Z}/2\langle 6, -6\rangle} \rtimes \mathbb{Z}/2$ has only 2 and 3 torsion, thus has trivial cohomology with coefficients in $\mathbb{Z}_{(p)}$. The group $\frac{\mathbb{Z}\times\mathbb{Z}}{\mathbb{Z}\langle 6, -6\rangle} \rtimes \mathbb{Z}/2$ has presentation

$$\langle A, B, C | A^6 = C^2 = 1, \ [A, B] = (CA)^2 = 1, \ [C, B] = A\rangle$$

where $A = (1, -1)$, $B = (0, 1)$, and $C =$ switch. This leads to a central extension

$$1 \to \mathbb{Z}^{AB^2} \to \frac{\mathbb{Z} \times \mathbb{Z}}{\mathbb{Z}\langle 6, -6\rangle} \rtimes \mathbb{Z}/2 \to D_{24} \to 1$$

where D_{24} is the dihedral group of order 24. The associated LHS spectral sequence degenerates, giving $H^*(\frac{\mathbb{Z}\times\mathbb{Z}}{\mathbb{Z}\langle 6,-6\rangle} \rtimes \mathbb{Z}/2; \mathbb{Z}_{(p)}) \cong H^*(\mathbb{Z}; \mathbb{Z}_{(p)})$. Plugging the above information into the Mayer-Vietoris sequence the result follows.

The computation of the cohomology $H^*(\Gamma_0^6; \mathbb{Z}_{(p)})$ can be found in [C]: $H^*(\Gamma_0^6; \mathbb{Z}_{(p)}) \cong H^*(\mathbb{Z}/5; \mathbb{Z}_{(p)})$, $p \neq 2, 3$.

Corollary (6.3). *For* $p \geq 7$ *we have*

$$H^*(Sp_4(\mathbb{Z}); \mathbb{Z}_{(p)}) \cong \begin{cases} \mathbb{Z}_{(p)}, & * = 0, 2, \\ 0, & \textit{otherwise}, \end{cases}$$

and for $p = 5$,

$$H^*(Sp_4(\mathbb{Z}); \mathbb{Z}_{(5)}) \cong \begin{cases} \mathbb{Z}_{(5)}, & * = 0, \\ \mathbb{Z}_{(5)}, & * = 2, \\ \mathbb{Z}/5, & * > 2,\ * \textit{ even}, \\ 0, & \textit{otherwise.} \end{cases}$$

Note that for $p \geq 7$ this agrees with the result in [L-W].

Lemma (6.4). *For* mod 3 *cohomology the map*

$$H^*(A' \rtimes \mathbb{Z}/2; \mathbb{F}_3) \to H^*(B' \rtimes \mathbb{Z}/2; \mathbb{F}_3)$$

is onto.

Proof. As in (6.2) there is a central extension

$$1 \to \mathbb{Z}^{AB^2} \to \frac{\mathbb{Z}\times\mathbb{Z}}{\mathbb{Z}\langle 6,-6\rangle} \rtimes \mathbb{Z}/2 \to D_{24} \to 1.$$

The associated LHS spectral sequence has the following E_2-terms:

$$\begin{aligned} E_2^{**} &\cong H^*(D_{24}; \mathbb{F}_3) \otimes H^*(\mathbb{Z}; \mathbb{F}_3) \\ &\cong H^*(\Sigma_3; \mathbb{F}_3) \otimes H^*(\mathbb{Z}; \mathbb{F}_3) \\ &\cong \mathbb{F}_3[\alpha^2] \otimes \Lambda[\beta] \otimes \Lambda[x], \end{aligned}$$

$|x| = 1$, $|\alpha^2| = 4$, $|\beta| = 3$. The spectral sequence abuts at E_2 and so $H^*(B' \rtimes \mathbb{Z}/2; \mathbb{F}_3) \cong H^2(\Sigma_3; \mathbb{F}_3) \otimes H^*(\mathbb{Z}; \mathbb{F}_3)$. (See Diagram 6.5.)

There is also a central extension

$$1 \to \mathbb{Z}/6 \xrightarrow{\overline{(1,1)}} \frac{\mathbb{Z}/12\times\mathbb{Z}/12}{\mathbb{Z}/2\langle 6,-6\rangle} \rtimes \mathbb{Z}/2 \to D_{24} \to 1.$$

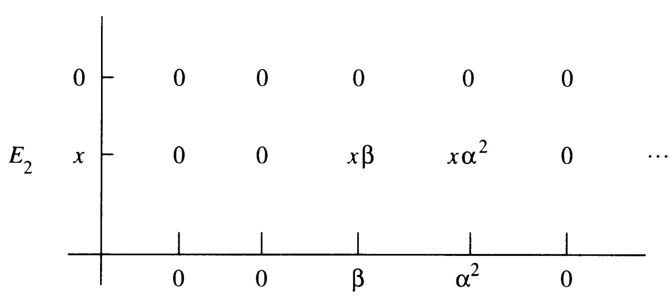

Diagram (6.5)

We get a spectral sequence with E_2-terms:

$$E_2^{*,*} \cong H^*(\Sigma_3\,;\mathbb{F}_3) \otimes H^*(\mathbb{Z}/3\,;\mathbb{F}_3)$$
$$\cong \mathbb{F}_3[\hat{\alpha}^2] \otimes \Lambda[\hat{\beta}] \otimes \Lambda[\hat{x}] \otimes \mathbb{F}_3[\hat{y}]\,,$$

$|\hat{\alpha}|^2 = 4\,,\ |\hat{\beta}| = 3\,,\ |\hat{x}| = 1\,,\ |\hat{y}| = 2$. The d_2-differential is trivial giving the following E_3-terms:

E_3						
$\hat{x}\hat{y}$	0	0	$\hat{x}\hat{y}\hat{\beta}$	$\hat{x}\hat{y}\hat{\alpha}^2$	0	
$\hat{y}$	0	0	$\hat{y}\hat{\beta}$	$\hat{y}\hat{\alpha}^2$	0	$\cdots$
$\hat{x}$	0	0	$\hat{x}\hat{\beta}$	$\hat{x}\hat{\alpha}^2$	0	
	0	0	$\hat{\beta}$	$\hat{\alpha}^2$	0	

DIAGRAM (6.6)

The element $\hat{y}$ must be a permanent cocycle, because abelianization of the group gives a direct summand isomorphic to $H^*(\mathbb{Z}/3\,;\mathbb{F}_3)$ in the cohomology. Thus we have $H^*(A' \rtimes \mathbb{Z}/2\,;\mathbb{F}_3) \cong H^*(\mathbb{Z}/3\,;\mathbb{F}_3) \otimes H^*(\Sigma_3\,;\mathbb{F}_3)$.

Comparing the above calculations, the natural map $H^*(A' \rtimes \mathbb{Z}/2\,;\mathbb{F}_3) \to H^*(B' \rtimes \mathbb{Z}/2\,;\mathbb{F}_3)$ is given by taking $g \mapsto \hat{g}\,,\ g = x\,,\alpha^2\,,\beta$ and $\hat{y} \mapsto 0$. Therefore it is a surjection and (6.4) follows.

By the results of the previous lemmas, the Mayer-Vietoris sequence breaks up into short exact sequences in mod 3 cohomology.

$$0 \to H^*(Sp_4(\mathbb{Z})\,;\mathbb{F}_3) \to H^*(A' \rtimes \mathbb{Z}/2\,;\mathbb{F}_3) \oplus H^*(\Gamma_0^6\,;\mathbb{F}_3)$$
$$\xrightarrow{\rho^*} H^*(B' \rtimes \mathbb{Z}/2\,;\mathbb{F}_3) \to 0$$

i.e., $H^*(Sp_4(\mathbb{Z})\,;\mathbb{F}_3) \cong H^*(\Gamma_0^6\,;\mathbb{F}_3) \oplus \ker \rho^*$.

As a consequence, the Poincaré series of $Sp_4(\mathbb{Z})$,

$$P(Sp_4(\mathbb{Z})) = \sum_{k=0}^{\infty} \dim(H^k(Sp_4(\mathbb{Z})\,;\mathbb{Z}/3))t^k\,,$$

can be computed as follows:

(6.7) $$P(Sp_4(\mathbb{Z})) = P(\Gamma_0^6) + P(A' \rtimes \mathbb{Z}/2) - P(B' \rtimes \mathbb{Z}/2)\,.$$

By computations of Benson [BC], $P(\Gamma_0^6) = (1 + t^3 + t^4 + t^5)/(1 - t^4)$ and the other Poincaré series are easily derived from the cohomology calculations in (6.4),

$$P(A' \rtimes \mathbb{Z}/2) = \frac{(1+t)}{(1-t^2)} \cdot \frac{(1+t^3)}{(1-t^4)}\,,$$
$$P(B' \rtimes \mathbb{Z}/2) = (1+t) \cdot \frac{(1+t^3)}{1-t^4}\,.$$

Adding these together, as in (6.6), we have

Theorem (6.8). *The dimension of $H^*(Sp_4(\mathbb{Z});\mathbb{Z}/3)$ in each degree is given by the Poincaré series*

$$\frac{1-2t+3t^2-2t^3+2t^4-t^5}{(1-t)^2(1+t^2)}.$$

To recover the $\mathbb{Z}_{(3)}$-cohomology from the $\mathbb{Z}/3$-cohomology, we utilize the Bockstein operation. First we observe that in the short exact sequence (6.6) the homomorphisms are induced by continuous maps between classifying spaces. As a result, there is a long exact sequence in Bockstein cohomology,

$$\begin{aligned} &\cdots \to H^*(\beta; H^*(Sp_4(\mathbb{Z});\mathbb{Z}/3)) \\ (6.9)\qquad &\to H^*(\beta; H(\Gamma_0^6;\mathbb{Z}/3)) \oplus H^*(\beta; H(A'\rtimes\mathbb{Z}/2;\mathbb{Z}/3)) \\ &\to H^*(\beta; H(B'\rtimes\mathbb{Z}/2;\mathbb{Z}/3)) \to H^{*+1}(\beta; H(Sp_4(\mathbb{Z});\mathbb{Z}/3)) \to \cdots. \end{aligned}$$

From work of F. Cohen, the $\mathbb{Z}_{(3)}$-cohomology of Γ_0^6 is an elementary abelian 3-group in each degree (cf. [C, Proposition 6.1]), and so has Bockstein cohomology $H^*(\beta; H(\Gamma_0^6;\mathbb{Z}/3)) = 0$.

To compute the Bockstein cohomology of the groups $A'\rtimes\mathbb{Z}/2$ and $B'\rtimes\mathbb{Z}/2$ we use the central extensions of Lemma (6.4). Once again, the associated LHS spectral sequences degenerate, giving:

$$\begin{aligned} H^*(B'\rtimes\mathbb{Z}/2;\mathbb{Z}_{(3)}) &\cong \mathbb{Z}_{(3)}[x]/\langle x^2\rangle \otimes H^*(\Sigma_3;\mathbb{Z}_{(3)}), \\ H^*(A'\rtimes\mathbb{Z}/2;\mathbb{Z}_{(3)}) &\cong H^*(\mathbb{Z}/3;\mathbb{Z}_{(3)}) \otimes H^*(\Sigma_3;\mathbb{Z}_{(3)}), \end{aligned}$$

where $|x| = 1$. For degree $* > 0$ the cohomology of both these groups is an elementary abelian 3-group, except for $H^1(B'\rtimes\mathbb{Z}/2;\mathbb{Z}_{(3)}) \cong \mathbb{Z}_{(3)}$. Consequently

$$H^*(\beta; H(B'\rtimes\mathbb{Z}/2);\mathbb{Z}/3) = \begin{cases} \mathbb{Z}/3, & * = 1, \\ 0, & * > 1, \end{cases}$$

$$H^*(\beta; H(A'\rtimes\mathbb{Z}/2;\mathbb{Z}/3)) = 0, \qquad * > 0.$$

Plugging this information into (6.9) we obtain

$$H^*(\beta; H(Sp_4(\mathbb{Z});\mathbb{Z}/3)) = \begin{cases} 0, & * \neq 2, \\ \mathbb{Z}/3, & * = 2. \end{cases}$$

A lemma of Serre applies to this situation [S, Lemma 2]; we conclude that the cohomology $H^*(Sp_4(\mathbb{Z});\mathbb{Z}_{(3)})$ has exponent 3, except at degree $* = 2$. Furthermore, the rank can be derived from the Poincaré series in (6.8).

Finally, the low-dimensional cohomology groups $H^1(Sp_4(\mathbb{Z});\mathbb{Z}_{(3)})$ and $H^2(Sp_4(\mathbb{Z});\mathbb{Z}_{(3)})$ can be computed directly from the Mayer-Vietoris sequence (5.12), using the abelianizations of the relevant groups. Summarizing this information we have

Theorem (6.10). *The 3-primary part of $H^*(Sp_4(\mathbb{Z});\mathbb{Z})$ is given by*

$$H^*(Sp_4(\mathbb{Z});\mathbb{Z}_{(3)}) = \begin{cases} \mathbb{Z}_{(3)}, & k = 0, \\ 0, & k = 1, \\ \mathbb{Z}_{(3)}, & k = 2, \\ \bigoplus_{\lambda_k} \mathbb{Z}/3, & k > 2, \end{cases}$$

where λ_k is given by the formula

$$\left(1+\frac{1}{t}\right)\sum \lambda_k t^k = \frac{(1-2t+3t^2+2t^3+2t^4-t^5)}{(1-t)^2(1+t^2)}.$$

We note that if $k \equiv 0, 3 \bmod 4$, a closed form for k can be derived rather easily using the Mayer-Vietoris sequence. For $k \equiv 0 \bmod 4$ the map $\rho^*: H^*(A' \rtimes \mathbb{Z}/2; \mathbb{Z}_{(3)}) \to H^*(B' \rtimes \mathbb{Z}/2; \mathbb{Z}_{(3)})$ is onto, hence there is a short exact sequence

$$0 \to H^k(Sp_4(\mathbb{Z}); \mathbb{Z}_{(3)}) \to H^k(\Gamma_0^6; \mathbb{Z}_{(3)}) \oplus H^k(A' \rtimes \mathbb{Z}/2; \mathbb{Z}_{(3)}) \\ \to H^k(B' \rtimes \mathbb{Z}/2; \mathbb{Z}_{(3)}) \to 0.$$

Counting the number of copies of $\mathbb{Z}/3$ from the cohomology calculations above, $\lambda_k = \frac{k}{4} + 1$, $k \equiv 0 \bmod 4$, $k > 0$. Trivially, $\lambda_k = 0$ for $k \equiv 3 \bmod 4$.

References

[A] W. Abikoff, *The real analytic theory of Teichmüller space*, Lecture Notes in Math., vol. 820, Springer, Berlin, 1980.

[As] A. Ash, *Cohomology of congruence subgroups of $SL(n, \mathbb{Z})$*, Math. Ann. **249** (1980), 55–73.

[BC] D. Benson and F. Cohen, *Mapping class groups of low genus and their cohomology*, Mem. Amer. Math. Soc. No. 443, 1991.

[BM] P. Bergau and J. Mennicke, *Über topologische Abbildungen de Brezelfläche vom Geschlecht 2*, Math. Z. **74** (1960), 414–435.

[Bi] J. Birman, *Braids, links, and mapping class groups*, Ann. of Math. Stud., Vol. 66, Princeton Univ. Press, Princeton, N.J., 1971.

[BCP] C.-F. Bödigheimer, F. R. Cohen, and M. Peim, *Mapping class groups and function spaces* (to appear).

[Br] A. B. Brownstein, *Homology of Hilbert modular groups*, Thesis, University of Michigan, 1987.

[CL] R. Charney and R. Lee, *Moduli space of stable curves from a homotopy viewpoint*, J. Differential Geom. **20** (1984), 185–235.

[C] F. R. Cohen, *On the mapping class groups for a punctured sphere and a surface of genus 2*, Preprint, University of Rochester, 1989.

[Ha] J. Harer, *Stability of the homology of the mapping class groups of orientable surfaces*, Ann. of Math. (2) **121** (1985), 215–249.

[Ho] W. Hoyt, *On the products and algebraic families of Jacobian varieties*, Ann. of Math. (2) **77** (1963), 415–423.

[G1] E. Gottschling, *Über die Fixpunkte der Siegelschen Modulgruppe*, Math. Ann. **143** (1961), 111–149.

[G2] ——, *Über die Fixpunkte untergruppren der Siegelschen Modulgruppe*, Math. Ann. **143** (1961), 399–430.

[L-S] R. Lee and R. H. Szczarba, *On the torsion in $K_4(\mathbb{Z})$ and $K_5(\mathbb{Z})$*, Duke Math. J. **45** (1978), 101–129.

[L-W] R. Lee and S. Weintraub, *Cohomology of $Sp_4(\mathbb{Z})$ and related groups and spaces*, Topology **24** (1985), 391–410.

[M] E. Mendoza, *Cohomology of PGL_2 over imaginary quadratic number fields*, Bonner Math. Schriften, No. 128, Bonn, 1980.

[Mi] J. Milnor, *Introduction to algebraic K-theory*, Ann. of Math. Stud., No. 72, Princeton Univ. Press, Princeton, N.J., 1971.

[Mu1] D. Mumford, *Curves and their Jacobians*, University of Michigan Press, Ann Arbor, 1975.

[Mu2] ——, *Towards an enumerative geometry of the moduli space of curves*, Arithmetic and Geometry, Vol. II, Birkhäuser, Boston, Mass., 1983, pp. 271–328.

[N] Y. Namikawa, *On the canonical holomorphic map from the moduli space of stable curves to the Igusa monoidal transformation*, Nagoya Math. J. **52** (1973), 197–259.

[S-V] J. Schwermer and K. Vogtmann, *The integral homology of SL_2 and PSL_2 of euclidean imaginary quadratic integers*, Comment. Math. Helv. **58** (1983), 573–598.

[V] K. Vogtmann, *Rational homology of Bianchi groups*, Math. Ann. **272** (1985), 399–419.

Department of Mathematics, The University of Texas at Austin, Austin, Texas 78712
Current address: Department of Mathematics, Rutgers University, Newark, New Jersey 07102
E-mail address: albrowns@gandalf.rutgers.edu

Department of Mathematics, Yale University, New Haven, Connecticut 06520