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NUMBER OF ORBITS OF BRANCH POINTS OF R-TREES

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ABSTRACT. An R-tree is a metric space in which any two points are joined by a unique arc. Every arc is isometric to a closed interval of R. When a group G acts on a tree (Z-tree) X without inversion, then X/G is a graph. One gets a presentation of G from the quotient graph X/G with vertex and edge stabilizers attached. For a general R-tree X, there is no such combinatorial structure on X/G. But we can still ask what the maximum number of orbits of branch points of free actions on R-trees is. We prove the finiteness of the maximum number for a family of groups, which contains free products of free abelian groups and surface groups, and this family is closed under taking free products with amalgamation.

INTRODUCTION

Various geometric constructions have been introduced to build a more comprehensive theory of the structure of infinite groups. Among them, Bass-Serre theory [15] has become a standard tool in the study of infinite groups. It says that each nontrivial action of a group on a tree "is equal to" a combinatorial presentation of the group. There are two natural sources of group actions on trees, one is algebraic, the other is geometric. From the algebraic source [15], given a discrete valuation on a field k, one gets a tree and a $\mathrm{GL}_2(k)$-action on it. From the geometric source [11], given a codimension-1 submanifold of an n-manifold M, one gets a tree and a $\pi_1(M)$-action on it. In the algebraic case (Morgan and Shalen [12]), replacing "discrete valuation" by "nondiscrete valuation" produces a "Λ-tree" rather than an ordinary tree, where Λ is an ordered abelian group. If the valuation takes values in R, the result is an R-tree. In the geometric case (Morgan and Shalen [13]), replacing codimension-1 submanifold by "codimension-1 measured lamination" results in a Λ-tree. If the measure takes values in R, one gets an R-tree. An R-tree is a metric space in which any two points are joined by a unique arc. Every arc is isometric to a closed interval of R.

Alperin and Bass [1] asked the following questions. What group-theoretic information about G can be drawn from its action on a Λ-tree? In particular, how much of Bass-Serre theory goes over for R-tree actions? The first natural case to consider is that of free actions. Morgan [11] and Shalen [6] conjectured

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that the only finitely presented groups which admit free actions on R-trees are those free products of surface groups and free abelian groups. The conjecture is confirmed in several special cases by Morgan and Shalen [13], Morgan [10], Gillet and Shalen [6], Morgan and Skora [14], and Jiang [8]. The conjecture is proved completely by E. Rips.

Following Bass [3], a graph Γ consists of a vertex set $V(\Gamma)$, an edge set $E(\Gamma)$, endpoint maps $\partial_0, \partial_1 : E(\Gamma) \to V(\Gamma)$, and a map of reversal of orientation $e \mapsto \bar{e}$ of $E(\Gamma)$ such that $\partial_i \bar{e} = \partial_{1-i} e$. An edge path of length $n > 0$ is a sequence of edges $e_1, e_2, \dots, e_n$, with $\partial_1 e_i = \partial_0 e_{i+1}$, $i = 1, \dots, n-1$. The path is called closed if $\partial_0 e_1 = \partial_1 e_n$ and reduced if $e_{i+1} \neq \overline{e_i}$ $(1 \leq i < n)$. A forest is a graph without closed reduced path. A tree (Z-tree) is a connected forest.

When a group G acts on a Z-tree X without inversion, then X/G is a graph. One gets a presentation of G from the quotient graph X/G with vertex and edge stabilizers attached. For a general R-tree X, there is no such combinatorial structure on X/G. But there are nevertheless some useful finiteness properties for the orbit structure. Let $\operatorname{ind}_X(x) = \operatorname{Card}(\pi_0(X - \{x\}))$ for all $x \in X$. Given a point x of X, the set $\pi_0(X - \{x\})$ is called the set of the directions of x in X. A point x is called a branch point of X if $\operatorname{ind}_X(x) > 2$. Let $BP(X) = \{x \in X | \operatorname{ind}_X(x) > 2\}$. We are interested in the number of G-orbits of branch points, and also in the maximum number of directions of a branch point. A G-action on X is called minimal if there is no proper G-invariant subtree of X. Since one can attach arbitrarily complicated R-trees to X, it is reasonable to restrict our attention to minimal actions. We make the following conjecture.

Conjecture 1. *Let G be a group with n generators. Suppose that G acts freely and minimally on an R-tree X. Then*

$$\sum_{x \in X/G} (\operatorname{ind}_X(x) - 2) \leq 2n - 2.$$

Conjecture 1 implies that the number of G-orbits of branch points is less than or equal to $2n-2$, and that the maximum number of directions of a branch point is less than or equal to $2n$.

In support of Conjecture 1 we prove

Main Theorem. *Let $G = G_1 *_A G_2$ be a free product with amalgamation. Suppose that G acts freely and minimally on an R-tree X. Let X_i be the minimal G_i-invariant subtree of X, and suppose that*

$$a_i = \sum_{x \in X_i/G_i} (\operatorname{ind}_{X_i}(x) - 2) < \infty, \qquad i = 1, 2.$$

Then

$$\sum_{x \in X/G} (\operatorname{ind}_X(x) - 2) \leq a_1 + a_2 + 2.$$

The following is an obvious consequence of the Main Theorem and Rips' Theorem.

Corollary 1. *Let G be a finitely presented group. Let G act freely and minimally on an R-tree X. Then*

$$\sum_{x\in X/G} (\mathrm{ind}_X(x) - 2) \le 2n - 2,$$

where n is the number of generators of G.

The Main Theorem says that if two groups G_1 and G_2 act "nicely" (namely, $\sum_{x\in X_i/G_i}(\mathrm{ind}_{X_i}(x) - 2) < \infty$ for $i = 1, 2$), then their free product with amalgamation G acts "nicely" also (namely, $\sum_{x\in X/G}(\mathrm{ind}_X(x) - 2) < \infty$).

A group action on an R-tree is called small if the stabilizer of each arc does not contain a free group of rank two. Let S be a compact 2-manifold without boundary and with Euler characteristic $\chi(S) < 0$. Skora [17] proves that if the $\pi_1(S)$-action on an R-tree X is small, then $\sum_{x\in X/\pi_1(S)}(\mathrm{ind}_X(x) - 2) \le 2n - 4$, where n is the number of generators of $\pi_1(S)$.

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1. Definitions

Let $G = G_1 *_A G_2$. Since G acts on an R-tree X, so do G_1 and G_2. Let X_1 and X_2 be G_1-invariant and G_2-invariant subtrees of X respectively. Then

$$X = \bigcup_{g\in G} (\mathrm{span}\{X_1, X_2\})g = \bigcup_{m=1}^{\infty} (\mathrm{span}\{X_1, X_2\}) \underbrace{G_1 G_2 G_1 \cdots G_i}_{m},$$

where $i = 1$ if m is odd, and $i = 2$ otherwise. (See Figures 0, 1, and 2.) This gives us a way to prove the Main Theorem by induction on m. Let $G_i = G_j$ if $i \equiv j \mod 2$ and $G_1 G_2 \cdots G_m = \{\alpha_1 \alpha_2 \cdots \alpha_m | \alpha_i \in G_i, i = 1, 2, \ldots, m\}$. Recall that X_i is the minimal G_i-invariant subtree of X, $i = 1, 2$. Let $X(0) = \mathrm{span}\{X_1, X_2\}$, and let

$$X(m) = X(0) G_1 G_2 \cdots G_m = \{x\alpha_1 \cdots \alpha_m | x \in X(0), \alpha_i \in G_i\}.$$

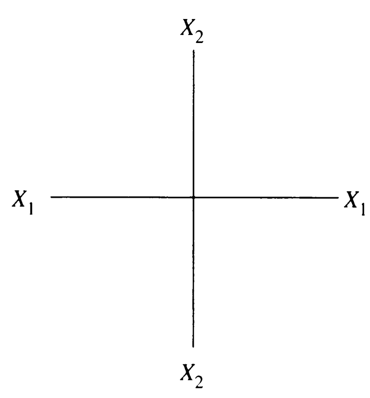

Figure 0

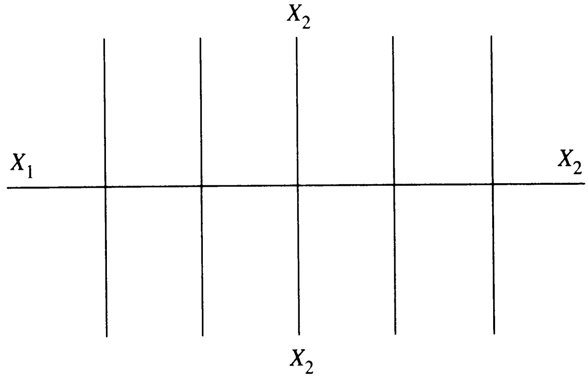

FIGURE 1

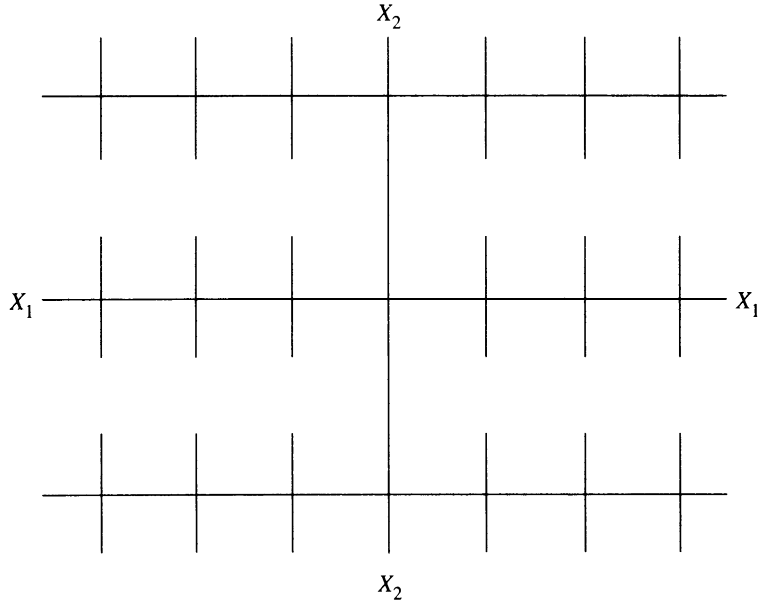

FIGURE 2

It is easy to see that $X(m)$ is a G_m-invariant subtree of X, and $X(m-1) \subset X(m)$ for all m. Note that $\bigcup_{m=1}^{\infty} X(m) = X$. Thus we find a natural decomposition

$$X(m-1) \subset X(m) \subset \bigcup_{m=1}^{\infty} X(m) = X.$$

We want to show the Main Theorem for each $X(m)$. The first problem is to find an analogous statement of the Main Theorem for each $X(m)$. To do this we need to define analogues of a G-orbit and the index of a G-orbit in each $X(m)$, which we denote by $[x]_m$ and $\operatorname{ind}[x]_m$ respectively. Then let $f(m) = \sum(\operatorname{ind}[x]_m - 2)$. We prove that $f(m)$ is bounded uniformly by $a_1 + a_2 + 2$. (Recall that $a_i = \sum_{x \in X_i/G_i}(\operatorname{ind}_{X_i}(x) - 2)$, $i = 1, 2$.) Finally we prove that the uniform bound is an upper bound of $f(\infty) = \sum_{x \in X/G}(\operatorname{ind}_X(x) - 2)$. The Main Theorem will follow immediately.

To get an intuitive picture of $X(m)$, let us consider a simple example. Let

G be a rank 2 free group generated by a and b. Let X_1 and X_2 be two lines intersecting at a point. So $X(0) = X_1 \cup X_2$, $X(1) = X_1 \cup (\bigcup_{n=-\infty}^{+\infty} X_2 a^n)$, and $X(2) = (\bigcup_{m=-\infty}^{+\infty} X_1 b^m) \cup (\bigcup_{p,q=-\infty}^{+\infty} X_2 a^p b^q)$. They are presented in Figures 0, 1 and 2 respectively.

What is a proper analogue of a G-orbit in $X(m)$? Recall that $X(m) = X(0)G_1G_2\cdots G_m$. Since $G_1G_2\cdots G_m$ is just a subset rather than a subgroup of G, thus we cannot discuss "$(G_1G_2\cdots G_m)$-orbits" in $X(m)$. Although G_m acts on $X(m)$, if we let the G_m-orbits be the analogues of the G-orbits in $X(m)$, when m goes to infinity, the G_m-orbits will never get close to the G-orbits. Then there is no hope to get a bound for $f(\infty)$ from the uniform bound (if we can find one) of $f(m)$. We define an "m-equivalence relation" among the points of $X(m)$, and let $[x]_m$ be the m-equivalence class containing x. For a G-orbit in X, its analogue in $X(m)$ is $[x]_m$. The ideal "m-equivalence relation" should satisfy the following conditions:

(1) For $u, v \in X_i$ $(1 \le i \le 2)$, if u and v are in the same G_i-orbit, then u and v are 0-equivalent.

(2) For $u, v \in X(m)$ $(m > 0)$, if u and v are in the same G_m-orbit, then they are m-equivalent.

(3) For $u, v \in X(m-1) \subset X(m)$, if u and v are $(m-1)$-equivalent, then they are m-equivalent.

We call the smallest equivalence relation generated by the above three conditions the m-equivalence relation.

Definition 1.1. For $u, v \in X_i$ $(1 \le i \le 2)$, if there exists an $\alpha \in G_i$ such that $u\alpha = v$, then u and v are called 0-related.

For $u, v \in X(m)$ $(m > 0)$, if there exists an $\alpha \in G_m$ such that $u\alpha = v$, then u and v are called m-related.

For $u, v \in X(m)$ $(m \ge 0)$, if there exist $u_i \in X(m)$, $i = 0, 1, \dots, k$, such that $u_0 = u$, $u_k = v$, and u_{i-1} and u_i are j_i-related for some $j_i \le m$, $i = 1, \dots, k$, then u and v are said to be m-equivalent.

Lemma 1.1. *Suppose that u_{i-1} and u_i are j_i-related, $j_i \le m$. Then there exists a unique $\alpha_i \in G_1 \cup G_2$, such that $u_{i-1}\alpha_i = u_i$.*

If $j_i > 0$, then $\alpha_i \in G_{j_i}$ and $u_{i-1}, u_i \in X(j_i)$.

If $j_i = 0$, then $\alpha_i \in G_t$ and $u_{i-1}, u_i \in X_t$, $t = 1$ or 2.

If $m > 1$, and if $\alpha_i \notin G_m$, then $j_i \le m-1$ and $u_{i-1}, u_i \in X(j_i) \subset X(m-1)$.

If $m = 1$, and if $\alpha_i \notin G_1$, then $j_i = 0$ and $\alpha_i \in G_2$ and $u_{i-1}, u_i \in X_2$.

If $m = 0$, then $j_i = 0$. If $\alpha_i \in G_t - A$, then $u_{i-1}, u_i \in X_t$.

Proof. The proof is left to the reader.

If u and v are m-equivalent, the relation between u and v is denoted by

$$u = u_0 \xrightarrow{\alpha_1} u_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_k} u_k = v,$$

where u_{i-1} and u_i are j_i-related for some $j_i \le m$. When m goes to infinity, $[x]_m$ will "approach" the G-orbit xG in the following sense: If u and v are in the same m-equivalence class then they are in the same G-orbit; conversely if they are in the same G-orbit then there exists $m \ge 0$ such that u and v are m-equivalent.

The next step is to define "index" for $[x]_m$. For the G-orbit, the problem is simple, because $\mathrm{ind}_X(u) = \mathrm{ind}_X(v)$ for all $u, v \in xG$. We let the index of xG be $\operatorname{ind}_X(u)$ for any u in xG. But the story is different for $X(m)$, that $u, v \in [x]_m$ does not necessarily imply that $\operatorname{ind}_{X(m)}(u) = \operatorname{ind}_{X(m)}(v)$. We define an " m-equivalence relation" among the directions of the points of $[x]_m$, and let $\operatorname{ind}[x]_m$ be the cardinality of the m-equivalence class of the directions, namely

$$\operatorname{ind}[x]_m = \operatorname{Card}(\{C \in \pi_0(X(m) - \{u\}) | u \in [x]_m\}/m\text{-relation}).$$

The "m-equivalence relation" among directions should satisfy the following conditions:

(1) For $u, v \in X_i$, $C \in \pi_0(X_i - \{u\})$, and $C' \in \pi_0(X_i - \{v\})$ $(1 \le i \le 2)$, if there exists an $\alpha \in G_i$, so that $u\alpha = v$ and $C\alpha = C'$, then C and C' are 0-equivalent.

(2) For $u, v \in X(m)$, $C \in \pi_0(X(m) - \{u\})$, and $C' \in \pi_0(X(m) - \{v\})$ $(m > 0)$, if there exists an $\alpha \in G_m$, such that $u\alpha = v$ and $C\alpha = C'$, then C and C' are m-equivalent.

(3) If $C \cap X(m-1)$ and $C' \cap X(m-1)$ are $(m-1)$-equivalent, then C and C' are m-equivalent.

Let the m-equivalence relation among directions be the smallest equivalence relation generated by the three conditions above.

Definition 1.2. For $u, v \in X_i$ $(1 \le i \le 2)$, $C \in \pi_0(X_i - \{u\})$, and $C' \in \pi_0(X_i - \{v\})$, if there exists an $\alpha \in G_i$, so that $u\alpha = v$ and $C\alpha = C'$, then C and C' are called 0-related.

For $u, v \in X(m)$, $(m > 0)$, $C \in \pi_0(X(m) - \{u\})$, and $C' \in \pi_0(X(m) - \{v\})$, if there exists an $\alpha \in G_m$, such that $u\alpha = v$ and $C\alpha = C'$, then C and C' are called m-related.

Let $u, v \in X(m)$ $C \in \pi_0(X(m) - \{u\})$, and $C' \in \pi_0(X(m) - \{v\})$ $(m \ge 0)$. We call C and C' m-equivalent if there exist $u_i \in X(m)$ and $C_i \in \pi_0(X(m) - \{u_i\})$, $i = 0, 1, \ldots, n$, such that $u_0 = u$, $u_k = v$, $C_0 = C$, $C_k = C'$, u_{i-1} and u_i are j_i-related, $C_{i-1} \cap X(j_i) \ne \varnothing$, $C_i \cap X(j_i) \ne \varnothing$, and $C_{i-1} \cap X(j_i)$ and $C_i \cap X(j_i)$ are j_i-related, for some $j_i \le m$, $i = 1, \ldots, k$.

Lemma 1.2. *Suppose that u_{i-1} and u_i are j_i-related, and suppose that $C_{i-1} \cap X(j_i)$ and $C_i \cap X(j_i)$ are j_i-related for some $j_i \le m$. Then there exists a unique $\alpha_i \in G_1 \cup G_2$, so that $u_{i-1}\alpha_i = u_i$ and $(C_{i-1} \cap X(j_i))\alpha_i = C_i \cap X(j_i)$. In particular, $C_{i-1}\alpha_i \cap C_i \ne \varnothing$.*

If $m > 1$ and $\alpha_i \notin G_m$, then $j_i \le m-1$, $C_{i-1} \cap X(m-1) \ne \varnothing$, and $C_i \cap X(m-1) \ne \varnothing$.

If $m = 1$ and $\alpha_i \notin G_1$, then $j_i = 0$, $C_{i-1} \cap X_2 \ne \varnothing$, and $C_i \cap X_2 \ne \varnothing$.

If $m = 0$ and $\alpha_i \in G_t - A$, then $j_i = 0$, $C_{i-1} \cap X_t \ne \varnothing$, and $C_i \cap X_t \ne \varnothing$.

Proof. The proof is left to the reader.

We denote the m-equivalence relation between C and C' by

$$C = C_0 \xrightarrow{\alpha_1} C_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_k} C_k = C'.$$

Given $x, y \in X$, let $[x, y]$ be the closed segment in X between x and y and let $(x, y] = [x, y] - \{x\}$. Given a point x in X and a closed subtree X' of X, let $[x, X']$ be the closed segment in X between X' and x. Following Alperin and Bass [1], we write $w = Y(x, y, z)$ if, for $x, y, z \in X$, $[x, y] \cap [x, z] = [x, w]$ for a unique w. For any $x \in X$, if $\operatorname{ind}_X(x) = 1$, then

x is called a closed end point of X. For any $x \in X(m)$, let $[x]_m$ be the m-equivalence class containing x. Let

$$\text{ind}[x]_m = \text{Card}(\{C \in \pi_0(X(m) - \{y\}) | y \in [x]_m\}/m\text{-relation}),$$

and let

$$f(m) = \sum_{[x]_m} (\text{ind}[x]_m - 2),$$

where $[x]_m \in X(m)/m$-relation. The theorem says that

$$f(\infty) = \sum_{x \in X/G} (\text{ind}_X(x) - 2) \le a_1 + a_2 + 2.$$

In the next section, we prove that $f(m) \le k$ for all k implies that $f(\infty) \le k$. In §§3 and 4, we shall prove that $f(m) \le a_1 + a_2 + 2$ for all m.

2. Reducing the proof of the theorem to proving $f(m) \le a_1 + a_2 + 2$

To show that a uniform bound for $f(m)$ is a bound for $f(\infty)$, we need to prove $\text{ind}[x]_m \ge \text{ind}_{X(m)}(x)$. We say that a product $\alpha = \alpha_1 \alpha_2 \cdots \alpha_n \in G_1 *_A G_2$ is a reduced word if either $n = 1$ and $\alpha \in A$ or else $n \ge 1$, each $\alpha_i \in G_{t_i} - A$ for some t_i $(1 \le t_i \le 2)$, and no $t_i = t_{i+1}$.

Lemma 2.1. (1) *If u and v are m-equivalent, and k is the minimal number such that*

$$u = u_0 \xrightarrow{\alpha_1} u_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_k} u_k = v,$$

then $\alpha_1 \cdots \alpha_k$ is a reduced word of $G = G_1 *_A G_2$.

(2) *Let $C \in \pi_0(X(m) - \{x\})$ and $C' \in \pi_0(X(m) - \{y\})$. If C and C' are m-equivalent, and k is the minimal number such that $C = C_0 \xrightarrow{\alpha_1} C_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_k} C_k = C'$, then $\alpha_1 \cdots \alpha_k$ is a reduced word of $G_1 *_A G_2$.*

Proof. (1) Suppose that we have $u = u_0 \xrightarrow{\alpha_1} u_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_k} u_k = v$. If α_i and α_{i+1} are both in G_1 or both in G_2, then we can replace $u_{i-1} \xrightarrow{\alpha_i} u_i \xrightarrow{\alpha_{i+1}} u_{i+1}$ by $u_{i-1} \xrightarrow{\alpha_i\alpha_{i+1}} u_{i+1}$, and we get a shorter chain. We need to check that u_{i-1} and u_{i+1} are j-related for some $j \le m$. This is easy, and it is left to the reader.

(2) If (2) is not the case, without loss of generality, we can assume that $\alpha_i\alpha_{i+1} \in G_1 - A$ and m is an even number (thus $G_{m-1} = G_1$). By (1) we can replace $u_{i-1} \xrightarrow{\alpha_i} u_i \xrightarrow{\alpha_{i+1}} u_{i+1}$ by $u_{i-1} \xrightarrow{\alpha_i\alpha_{i+1}} u_{i+1}$, and assume that $u_{i-1}, u_{i+1} \in X(m-1)$. We will prove that

$$C = C_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_{i-1}} C_{i-1} \xrightarrow{\alpha_i\alpha_{i+1}} C_{i+1} \xrightarrow{\alpha_{i+2}} \cdots \xrightarrow{\alpha_k} C_k = C'$$

satisfies the conditions of Definition 1.2. Since this is a shorter chain, we get a contradiction. To check Definition 1.2, we need to show that $C_{i-1}\alpha_i\alpha_{i+1} \cap C_{i+1} \ne \varnothing$, $C_{i-1} \cap X(m-1) \ne \varnothing$, and $C_{i+1} \cap X(m-1) \ne \varnothing$.

Claim 1. If $C_i \in \pi_0(X(m_i) - \{u\})$, $i = 1, 2, 3$, $C_1 \cap C_2 \ne \varnothing$, and $C_2 \cap C_3 \ne \varnothing$, then $C_1 \cap C_3 \ne \varnothing$.

First suppose $m_1 \le m_2 \ge m_3$. $m_1 \le m_2$ implies that $X(m_1) \subset X(m_2)$, which together with $C_1 \cap C_2 \ne \varnothing$ implies $C_1 \subset C_2$. Thus $C_1 \subset C_2 \supset C_3$. If $C_1 \cap C_3 = \varnothing$, let $x \in C_1$ and $y \in C_3$; then there is a path $[x, y]$ in X joining

x and y. Since C_2 is connected and $x, y \in (C_1 \cup C_3) \subset C_2$, then this implies that $[x, y] \subset C_2 \in \pi_0(X(m_2) - \{u\})$, thus $u \notin [x, y]$. On the other hand, combining $C_1 \in \pi_0(X(m_1) - \{u\})$, $C_3 \in \pi_0(X(m_3) - \{u\})$, and $C_1 \cap C_3 = \varnothing$ yields that $u \in [x, y]$, a contradiction. The proofs for the other cases are the same. This proves Claim 1.

By Claim 1, we have that $C_{i-1}\alpha_i\alpha_{i+1} \cap C_{i+1} \neq \varnothing$. Next we prove that $C_{i-1} \cap X(m-1) \neq \varnothing$ and $C_{i+1} \cap X(m-1) \neq \varnothing$. If $\alpha_i, \alpha_{i+1} \in G_1 - A$, then this follows from Lemma 1.2. If $\alpha_i \in A$ and $\alpha_{i+1} \in G_1 - A$, then $C_i\alpha_i^{-1} \cap X(m-1) = (C_i \cap X(m-1))\alpha_i^{-1} \neq \varnothing$ and $C_{i-1} \cap C_i\alpha_i^{-1} = (C_{i-1}\alpha_i \cap C_i)\alpha_i^{-1} \neq \varnothing$. Thus $C_{i-1} \cap X(m-1) \neq \varnothing$ follows from an argument similar to the one given in Claim 1. The case of $\alpha_i \in G_1 - A$ and $\alpha_{i+1} \in A$ can be treated in the same way. This completes the proof of Lemma 2.1. □

Lemma 2.2. *If $C, C' \in \pi_0(X(m) - \{x\})$ and C and C' are m-equivalent, then $C = C'$.*

Proof. By Definition 1.2

$$C = C_0 \xrightarrow{\alpha_1} C_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_k} C_k = C'.$$

Since

$$x\alpha_1 \cdots \alpha_k = u_0\alpha_1 \cdots \alpha_k = u_k = x$$

and the action is free, thus $\alpha_1 \cdots \alpha_k = 1$. So $k = 0$ and $C = C'$. □

Corollary 2.3. $\operatorname{ind}[x]_m \geq \operatorname{ind}_{X(m)}(x)$. □

If $f(m) \leq k$ for all $m \geq 1$, and if $f(\infty) > k$, then let $x_1, \ldots, x_p \in BP(X)$ such that $\sum_{i=1}^{p}(\operatorname{ind}_X(x_i) - 2) > k$, and $x_iG \neq x_jG$ for all $i \neq j$. Since $\bigcup_{m=1}^{\infty} X(m) = X$, then there exists m such that $x_i \in X(m)$ and $\operatorname{ind}_{X(m)}(x_i) = \operatorname{ind}_X(x_i)$ for all i. Since $x_iG \neq x_jG$, x_i and x_j are not m-equivalent for $i \neq j$. Therefore,

$$f(m) \geq \sum_{i=1}^{p}(\operatorname{ind}[x_i]_m - 2) \geq \sum_{i=1}^{p}(\operatorname{ind}_{X(m)}(x_i) - 2) > k,$$

which is a contradiction. Thus $f(m) \leq m$ for all $m \geq 1$ implies that $f(\infty) \leq k$.

Now we have reduced the proof of the theorem to proving that $f(m) \leq a_1 + a_2 + 2$ for all m.

3. A BOUND FOR $f(0)$

We prove that $f(m) \leq a_1 + a_2 + 2$ by induction on m. This section is devoted to the initial case $m = 0$. First we rule out two trivial cases.

Case 1. $X_1 \cap X_2 = \varnothing$. Let $[X_1, X_2] = [x_1, x_2]$ be the bridge between X_1 and X_2. Then

$$\operatorname{ind}[x]_0 = \begin{cases} \operatorname{ind}_{X_1}(x) & \text{if } x \in X_1 - \{x_1G_1\}, \\ \operatorname{ind}_{X_2}(x) & \text{if } x \in X_2 - \{x_2G_2\}, \\ \operatorname{ind}_{X_1}(x) + 1 & \text{if } x \in x_1G_1, \\ \operatorname{ind}_{X_2}(x) + 1 & \text{if } x \in x_2G_2, \\ 2 & \text{if } x \in (x_1, x_2). \end{cases}$$

Therefore, $f(0) = \sum_{[x]_0 \in X(0)/0\text{-relation}} (\text{ind}[x]_0 - 2) = a_1 + a_2 + 2$.

Case 2. $X_1 \cap X_2 = \{z\}$. Then

$$\text{ind}[x]_0 = \begin{cases} \text{ind}_{X_1}(x) & \text{if } x \in X_1 - zG_1, \\ \text{ind}_{X_2}(x) & \text{if } x \in X_2 - zG_2, \\ \text{ind}_{X_1}(x) + \text{ind}_{X_2}(x) & \text{if } x \in zG_1 \cup zG_2, \end{cases}$$

and $f(0) = \sum_{[x]_0 \in X(0)/0\text{-relation}} (\text{ind}[x]_0 - 2) = a_1 + a_2 + 2$.

Remark. Cases 1 and 2 do not use the freeness hypothesis.

From now on we can assume that $X_1 \cap X_2$ contains more than one point. We sketch the basic idea of the proof first. Recall that $a_i = \sum_{x \in X_i/G_i} (\text{ind}_{X_i}(x) - 2)$, $i = 1, 2$, and $f(0) = \sum_{[x]_0 \in X(0)/0\text{-relation}} (\text{ind}[x]_0 - 2)$. To show $f(0) \le a_1 + a_2 + 2$, let us consider $[x]_0$ first. We want to know the difference between $\text{ind}_{X_i}(x)$ and $\text{ind}[x]_0$. Let $u \in X_1$ and $v \in X_2$, and suppose u and v are 0-equivalent. Then by Lemma 1.1,

$$u = u_0 \xrightarrow{\alpha_1} u_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_k} u_k = v,$$

where $\alpha_i \in G_i - A$ and $u_{i-1}, u_i \in X_i$, $i = 1, \dots, k$. Since $u_i \alpha_i^{-1} = u_{i-1}$, the point $u_{i-1} \in u_{i-1}G_{i-1} \cap u_i G_i \neq \varnothing$ for all i. The inverse is also true. If there are $u_i \in X_i$, such that $u_{i-1}G_{i-1} \cap u_i G_i \neq \varnothing$ for all i and $u \in u_1 G_1$, $v \in u_k G_k$, then u and v are 0-equivalent. This associates a graph Γ with $X(0)$. The vertices of Γ are G_1-orbits and G_2-orbits. If $xG_1 \cap yG_2 \neq \varnothing$, then xG_1 and yG_2 are adjacent vertices, and they correspond to an edge of Γ. Note that $[x]_0$ is a connected component of Γ. For a graph Γ, let $V(\Gamma)$ be the set of vertices of Γ.

Note that

$$\begin{aligned} a_1 + a_2 &= \sum_{x \in X_1/G_1} (\text{ind}_{X_1}(x) - 2) + \sum_{x \in X_2/G_2} (\text{ind}_{X_2}(x) - 2) \\ &= \sum_{[x]_0} \left(\sum_{yG_i \in V[x]_0)} (\text{ind}_{X_i}(y) - 2) \right); \end{aligned}$$

the last summation is over all connected components $[x]_0$ of Γ. To show

$$f(0) = \sum_{[x]_0 \in X(0)/0\text{-relation}} (\text{ind}[x]_0 - 2) \le a_1 + a_2 + 2,$$

it is equivalent to showing that

$$\sum_{[x]_0} \left((\text{ind}[x]_0 - 2) - \sum_{xG_i \in V[x]_0)} (\text{ind}_{X_i}(x) - 2) \right) \le 2.$$

Lemma 3.1 shows that Γ is a forest. So each connected component of Γ is a tree. Suppose $x \in X_1$, then xG_1 is a vertex of Γ, and $[x]_0$ is a connected component containing xG_1. Let $T_0, T_1, \dots$ be an ascending sequence of subtrees of Γ. Let T_0 be the single vertex xG_1. Suppose $T_{i-1} \subset T_i$ and $T_i - T_{i-1}$ is a single vertex for all i. Recall that

$$\text{ind}[x]_0 = \text{Card}(\{C \in \pi_0(X(0) - \{y\}) | y \in [x]_0\}/0\text{-relation}).$$

Here we denote $y \in \bigcup_{zG_i \in V[x]_0)} zG_i$ by $y \in [x]_0$. We think of $[x]_0$ as a connected component of Γ. We can define "index" for subtrees of $[x]_0$ in a similar way. Let $\mathrm{ind}(T_n) = \mathrm{Card}(\{C \in \pi_0(X(0) - \{y\}) | y \in T_n\}/0\text{-relation})$. Lemma 3.2 shows that

$$(\mathrm{ind}(T_n) - 2) - \sum_{yG_i \in V(T_n)} (\mathrm{ind}_{X_i}(y) - 2) \leq \sum_{\substack{z \in y_1G_1 \cap y_2G_2 \\ (y_1G_1, y_2G_2) \in E(T_n)}} (2 - \mathrm{ind}_{X_1 \cap X_2}(z))$$

for all n. Lemma 3.5 shows that

$$f(0) - (a_1 + a_2) \leq \sum_{[x]_0} \left(\sum_{\substack{z \in y_1G_1 \cap y_2G_2 \\ (y_1G_1, y_2G_2) \in E[x]_0)}} (2 - \mathrm{ind}_{X_1 \cap X_2}(z)) \right).$$

Lemmas 3.3 and 3.4 show that

$$\sum_{\substack{z \in y_1G_1 \cap y_2G_2 \\ (y_1G_1, y_2G_2) \in E(T_n)}} (2 - \mathrm{ind}_{X_1 \cap X_2}(z)) = \sum_{\substack{z \in y_1G_1 \cap y_2G_2 \\ (y_1G_1, y_2G_2) \in E[x]_0)}} (2 - \mathrm{ind}_{X_1 \cap X_2}(z)).$$

Since $a_i = \sum_{y \in X_i/G_i} (\mathrm{ind}_{X_i}(y) - 2)$, $i = 1, 2$, then $\sum_{yG_i \in V[x]_0)} (\mathrm{ind}_{X_i}(y) - 2) < \infty$. Thus

$$\sum_{yG_i \in V[x]_0)} (\mathrm{ind}_{X_i}(y) - 2) = \sum_{yG_i \in V(T_n)} (\mathrm{ind}_{X_i}(y) - 2)$$

for large n. Therefore

$$(\mathrm{ind}[x]_0) - 2) - \sum_{yG_i \in V[x]_0)} (\mathrm{ind}_{X_i}(y) - 2) \leq \sum_{\substack{z \in y_1G_1 \cap y_2G_2 \\ (y_1G_1, y_2G_2) \in E[x]_0)}} (2 - \mathrm{ind}_{X_1 \cap X_2}(z)).$$

Summing over both sides of the above inequality, we get

$$\begin{aligned} f(0) - (a_1 + a_2) &= \sum_{[x]_0} (\mathrm{ind}[x]_0 - 2) - \sum_{[x]_0} \left(\sum_{yG_i \in V[x]_0)} (\mathrm{ind}_{X_i}(y) - 2) \right) \\ &= \sum_{[x]_0} \left((\mathrm{ind}[x]_0 - 2) - \sum_{yG_i \in V[x]_0)} (\mathrm{ind}_{X_i}(y) - 2) \right) \\ &\leq \sum_{[x]_0} \left(\sum_{\substack{z \in y_1G_1 \cap y_2G_2 \\ (y_1G_1, y_2G_2) \in E[x]_0)}} (2 - \mathrm{ind}_{X_1 \cap X_2}(z)) \right). \end{aligned}$$

Lemma 3.6 shows that

$$\sum_{[x]_0} \left(\sum_{\substack{z \in y_1G_1 \cap y_2G_2 \\ (y_1G_1, y_2G_2) \in E[x]_0)}} (2 - \mathrm{ind}_{X_1 \cap X_2}(z)) \right) = \sum_{z \in (X_1 \cap X_2)/A} (2 - \mathrm{ind}_{X_1 \cap X_2}(z)).$$

Lemma 3.7 shows that $\sum_{z \in (X_1 \cap X_2)/A} (2 - \mathrm{ind}_{X_1 \cap X_2}(z)) \leq 2$. This proves that $f(0) \leq a_1 + a_2 + 2$.

Definition 3.1. Let Γ be a graph. It is defined by the following:

$$V(\Gamma) = (X_1/G_1) \cup (X_2/G_2),$$
$$E(\Gamma) = \{(xG_1, yG_2) | xG_1 \cap yG_2 \neq \varnothing\}.$$

A connected component of Γ is exactly a 0-class of $X(0)$.

Lemma 3.1. *Γ is a forest.*

Proof. If this is not the case, let $u_1G_1, u_2G_2, \dots, u_{2k}G_{2k}$ be vertices of a minimal circle in Γ. Let $u_{2k+1}G_{2k+1} = u_1G_1$ and suppose $u_{i-1}G_{i-1} \cap u_iG_i \neq \varnothing$ for all i. Let $x_i \in u_{i-1}G_{i-1} \cap u_iG_i$ for all i. Since $x_i, x_{i+1} \in u_iG_i$, there exists an $\alpha_i \in G_i$ such that $x_i\alpha_i = x_{i+1}$. Since the circle is minimal, we can assume that $u_{i-1}G_{i-1} \cap u_{i+1}G_{i+1} = \varnothing$. Note that $G_{i-1} = G_{i+1}$ and $x_i \in u_{i-1}G_{i-1}$. If $\alpha_i \in A$, it follows that $x_i\alpha_i = x_{i+1} \in u_{i-1}G_{i_1}\alpha_i \cap u_{i+1}G_{i+1} \neq \varnothing$, a contradiction. So $\alpha_i \notin A$ for all i. Since $x_1\alpha_1 \cdots \alpha_k = x_1$, it follows that $\alpha_1 \cdots \alpha_k = 1$. But $\alpha_i \in G_i - A$ for all i, which contradicts $G = G_1 *_A G_2$. Therefore, Γ is a forest. □

Lemma 3.2. *For any finite subtree T_n of Γ,*

$$(\mathrm{ind}(T_n) - 2) - \sum_{yG_i \in V(T_n)} (\mathrm{ind}_{X_i}(y) - 2) \leq \sum_{\substack{z \in y_1G_1 \cap y_2G_2 \\ (y_1G_1, y_2G_2) \in E(T_n)}} (2 - \mathrm{ind}_{X_1 \cap X_2}(z)).$$

Proof. We prove this lemma by induction on n. The case $n = 1$ is trivial. Suppose the lemma holds for a finite tree T_n. We consider a finite tree $T_{n+1} = T_n \cup \{v\}$, where v is a vertex adjacent to T_n. There exists a unique edge e joining v and T_n. Suppose $e = (y_1G_1, y_2G_2)$ and $z \in y_1G_1 \cap y_2G_2$. Note that $v = zG_i$ has $\mathrm{ind}_{X_i}(z)$ directions, but at least $\mathrm{ind}_{X_1 \cap X_2}(z)$ of them are equivalent to the directions in $\{C \in \pi_0(X(0) - \{y\}) | y \in T_n\}$. Therefore, v brings in at most $\mathrm{ind}_{X_i}(z) - \mathrm{ind}_{X_1 \cap X_2}(z)$ many new directions. Then

$$\mathrm{ind}(T_n) - 2 + \mathrm{ind}_{X_i}(z) - \mathrm{ind}_{X_1 \cap X_2}(z) \geq \mathrm{ind}(T_{n+1}) - 2.$$

Note that

$$\sum_{yG_i \in V(T_n)} (\mathrm{ind}_{X_i}(y) - 2) + (\mathrm{ind}_{X_i}(z) - 2) = \sum_{yG_i \in V(T_{n+1})} (\mathrm{ind}_{X_i}(y) - 2),$$

thus

$$\begin{aligned}
(\mathrm{ind}(T_{n+1}) - 2) &- \sum_{yG_i \in V(T_{n+1})} (\mathrm{ind}_{X_i}(y) - 2) \\
&\leq (\mathrm{ind}(T_n) - 2 + \mathrm{ind}_{X_i}(z) - 2 + 2 - \mathrm{ind}_{X_1 \cap X_2}(z)) \\
&\quad - \sum_{yG_i \in V(T_n)} (\mathrm{ind}_{X_i}(y) - 2) + (\mathrm{ind}_{X_i}(z) - 2) \\
&\leq \sum_{\substack{z \in y_1G_1 \cap y_2G_2 \\ (y_1G_1, y_2G_2) \in E(T_{n+1})}} (2 - \mathrm{ind}_{X_1 \cap X_2}(z)).
\end{aligned}$$

This proves Lemma 3.2. □

Lemma 3.3. *Suppose that a group H acts freely on an R-tree X. Let H_i be subgroups of H, let Y_i be H_i-invariant subtrees of Y, and put*

$$a_i = \sum_{x\in Y_i} (\mathrm{ind}_{Y_i}(x) - 2) < \infty, \qquad i = 1, 2.$$

Then

$$\mathrm{Card}(BP(Y_1 \cap Y_2)/H_1 \cap H_2) < \infty.$$

Proof. If this is not the case, suppose that there exist $x_i \in BP(Y_1 \cap Y_2)$, $i = 1, 2, \dots$, such that $x_i(H_1 \cap H_2) \cap x_j(H_1 \cap H_2) = \varnothing$ for all $i \neq j$. Since $\sum_{x\in Y_1/H_1}(\mathrm{ind}_{Y_1}(x) - 2) = a_1$, there are only finitely many H_1-orbit of branch points. We can assume that all points x_i are in the same H_1-orbit. By the same token, all points x_i are in the same H_2-orbit. Then for any $i \neq j$ there exist an $\alpha_{i,j} \in H_1$ and $\beta_{i,j} \in H_2$ such that $x_i\alpha_{i,j} = x_j$ and $x_i\beta_{i,j} = x_j$. Hence $\alpha_{i,j} = \beta_{i,j} \in H_1 \cap H_2$, and $x_i(H_1 \cap H_2) = x_j(H_1 \cap H_2)$, which contradicts $x_i(H_1 \cap H_2) \cap x_j(H_1 \cap H_2) = \varnothing$. □

Lemma 3.4. *Let a group H act freely on an R-tree Y. Suppose* $\mathrm{Card}(BP(Y)/H) < \infty$ *and* $\mathrm{ind}_Y(x) < \infty$ *for all* $x \in Y$. *Then*

$$\mathrm{Card}(\{x \in Y | \mathrm{ind}_Y(x) = 1\}/H) < \infty.$$

Proof. Lemma 3.4 says that there are only finitely many H-orbits of closed end points. If $H = \{1\}$, it follows that $\mathrm{Card}(BP(Y)) = \mathrm{Card}(BP(Y)/H) < \infty$. For each $x \in \{y \in Y | \mathrm{ind}_Y(y) = 1\}$ there exists a unique branch point $s(x) \in BP(Y)$ such that $d(x, s(x)) = \min\{d(x, y) | y \in BP(Y)\}$. There are only finitely many different $s(x)$. If $x \neq y \in \{z \in Y | \mathrm{ind}_Y(z) = 1\}$ and $s(x) = s(y)$, then x and y are in the different components of $Y - \{s(x)\}$. Since $\mathrm{ind}_Y(s(x)) < \infty$, then $\mathrm{Card}\{x \in Y | \mathrm{ind}_Y(x) = 1\} < \infty$.

Suppose $H \neq \{1\}$ and let Y_H be the minimal H-invariant subtree of Y. First we prove that $\mathrm{Card}(\pi_0(Y - Y_H)/H) < \infty$. For each $C \in \pi_0(Y - Y_H)$ there exists a unique base point $x_c \in Y_H$ such that $C \in \pi_0(Y - \{x_c\})$. In particular, $x_c \in BP(Y)$. Since $\mathrm{Card}(BP(Y)/H) < \infty$, it follows that

$$\mathrm{Card}(\{x_c | C \in \pi_0(Y - Y_H)\}/H) < \infty.$$

Since $\mathrm{ind}_Y(x_c) < \infty$ for all x_c, it follows that $\mathrm{Card}(\pi_0(Y - Y_H)/H) < \infty$. Since Y_H is a minimal H-invariant subtree, it follows that $\mathrm{ind}_{Y_H}(x) \geq 2$ for all $x \in Y_H$. Thus $\{x \in Y | \mathrm{ind}_Y(x) = 1\} \subset Y - Y_H$. For any $C \in \pi_0(Y - Y_H)$, since $\mathrm{Card}(BP(Y)/H) < \infty$ and $C \cap C(H - \{1\}) = \varnothing$, we have $\mathrm{Card}(BP(C)) < \infty$. We have already shown that the lemma is valid when $H = \{1\}$. Applying this result to C shows that

$$\mathrm{Card}\{x \in C | \mathrm{ind}_Y(x) = 1\} = \mathrm{Card}\{x \in C | \mathrm{ind}_C(x) = 1\} < \infty.$$

Therefore, $\mathrm{Card}(\{x \in CH | \mathrm{ind}_Y(x) = 1\}/H) < \infty$. Since $\mathrm{Card}(\pi_0(Y - Y_H)/H) < \infty$, it follows that

$$\mathrm{Card}(\{x \in Y | \mathrm{ind}_Y(x) = 1\}/H) < \infty. \quad \square$$

Lemma 3.5.

$$f(0) - (a_1 + a_2) \leq \sum_{[x]_0} \left(\sum_{\substack{z\in y_1G_1\cap y_2G_2 \\ (y_1G_1, y_2G_2)\in E[x]_0}} (2 - \mathrm{ind}_{X_1\cap X_2}(z)) \right).$$

Proof. Applying Lemma 3.4 to X_i and G_i for $i = 1, 2$, we get that $\mathrm{Card}(BP(X_1 \cap X_2)/A) < \infty$. Applying Lemma 3.5 to A and $X_1 \cap X_2$, we get that $\mathrm{Card}(\{x \in X_1 \cap X_2 | \mathrm{ind}_{X_1 \cap X_2}(x) = 1\}/A) < \infty$. Therefore

$$\sum_{\substack{z \in y_1 G_1 \cap y_2 G_2 \\ (y_1 G_1, y_2 G_2) \in E(T_n)}} (2 - \mathrm{ind}_{X_1 \cap X_2}(z)) = \sum_{\substack{z \in y_1 G_1 \cap y_2 G_2 \\ (y_1 G_1, y_2 G_2) \in E[x]_0)}} (2 - \mathrm{ind}_{X_1 \cap X_2}(z))$$

for large n. Since $a_i = \sum_{y \in X_i/G_i} (\mathrm{ind}_{X_i}(y) - 2)$, $i = 1, 2$, it follows that $\sum_{yG_i \in V[x]_0)} (\mathrm{ind}_{X_i}(y) - 2) < \infty$. Thus

$$\sum_{yG_i \in V[x]_0)} (\mathrm{ind}_{X_i}(y) - 2) = \sum_{yG_i \in V(T_n)} (\mathrm{ind}_{X_i}(y) - 2)$$

for large n. Therefore

$$\begin{aligned}(\mathrm{ind}[x]_0) - 2) &- \sum_{yG_i \in V[x]_0)} (\mathrm{ind}_{X_i}(y) - 2) \\ &\leq \sum_{\substack{z \in y_1 G_1 \cap y_2 G_2 \\ (y_1 G_1, y_2 G_2) \in E[x]_0)}} (2 - \mathrm{ind}_{X_1 \cap X_2}(z)).\end{aligned}$$

Summing over both sides of the above inequality, we get

$$\begin{aligned} f(0) - (a_1 + a_2) &= \sum_{[x]_0} (\mathrm{ind}[x]_0 - 2) - \sum_{[x]_0} \left(\sum_{yG_i \in V[x]_0)} (\mathrm{ind}_{X_i}(y) - 2) \right) \\ &= \sum_{[x]_0} \left((\mathrm{ind}[x]_0 - 2) - \sum_{yG_i \in V[x]_0)} (\mathrm{ind}_{X_i}(y) - 2) \right) \\ &\leq \sum_{[x]_0} \left(\sum_{\substack{z \in y_1 G_1 \cap y_2 G_2 \\ (y_1 G_1, y_2 G_2) \in E[x]_0)}} (2 - \mathrm{ind}_{X_1 \cap X_2}(z)) \right). \quad \square \end{aligned}$$

Lemma 3.6.

$$\sum_{[x]_0} \left(\sum_{\substack{z \in y_1 G_1 \cap y_2 G_2 \\ (y_1 G_1, y_2 G_2) \in E[x]_0)}} (2 - \mathrm{ind}_{X_1 \cap X_2}(z)) \right) = \sum_{z \in (X_1 \cap X_2)/A} (2 - \mathrm{ind}_{X_1 \cap X_2}(z)).$$

Proof. Let $\psi : E(\Gamma) \to (X_1 \cap X_2)/A$ be defined as follows: for each $e = (uG_1, vG_2) \in E(\Gamma)$, if $x \in uG_1 \cap vG_2$, then let $\psi(e) = xA$. First we prove that ψ is well defined. Suppose $x, y \in uG_1 \cap vG_2$. Then there exist an $\alpha \in G_1$ and a $\beta \in G_2$ such that $x\alpha = y$ and $x\beta = y$. Therefore, $\alpha = \beta \in G_1 \cap G_2 = A$ and $xA = yA$.

Next we prove that ψ is injective. Suppose $xA = \psi(u_1 G_1, v_1 G_2) = \psi(u_2 G_1, v_2 G_2) = yA$. Then $x \in u_1 G_1 \cap v_1 G_2$, $y \in u_2 G_1 \cap v_2 G_2$, and $xA = yA$. Thus there exists an $\alpha \in A$ such that $x\alpha = y$. This implies that

$$x\alpha = y \in u_1 G_1 \alpha \cap u_2 G_1 = u_1 G_1 \cap u_2 G_2 \neq \varnothing$$

and $u_1 G_1 = u_2 G_1$. Similarly $v_1 G_2 = v_2 G_2$. Therefore, ψ is injective.

Finally we prove that ψ is surjective. For all $x \in X_1 \cap X_2$, let $e = (xG_1, xG_2)$. Then $e \in E(\Gamma)$ and $\psi(e) = xA$. So ψ is bijective. Therefore

$$\sum_{[x]_0} \left(\sum_{\substack{z \in y_1 G_1 \cap y_2 G_2 \\ (y_1 G_1, y_2 G_2) \in E[x]_0)}} (2 - \mathrm{ind}_{X_1 \cap X_2}(z)) \right) = \sum_{z \in (X_1 \cap X_2)/A} (2 - \mathrm{ind}_{X_1 \cap X_2}(z)).$$

This proves Lemma 3.6. □

Lemma 3.7. *Let a group H act freely on an R-tree Y. Suppose* $\mathrm{Card}(BP(Y)/H) < \infty$ *and* $\mathrm{ind}_Y(x) < \infty$ *for all* $x \in Y$. *Then* $\sum_{x \in Y/H}(2 - \mathrm{ind}_Y(x)) \le 2$.

Proof. Note that $\mathrm{ind}_Y(x) \ge 1$ for all $x \in Y$. By Lemma 3.4 we can assume that

$$\{x \in Y | \mathrm{ind}_Y(x) = 1\}/H = \{y_1 H, \dots, y_k H\}.$$

If $k \le 2$, then we are done. Suppose $k > 2$ and let

$$Y(p) = \mathrm{span}\{y_i H | i = 1, \dots, p\} \quad \text{for all } p.$$

Note that $Y(p)$ is a H-invariant subtree and

$$\mathrm{Card}(BP(Y(p))/H) < \infty \quad \text{for all } p.$$

We first prove that $\sum_{x \in Y(p)/H}(2 - \mathrm{ind}_{Y(p)}(x)) \le 2$ by induction on p. Since $\{y_i H | i = 1, \dots, p\}$ are the only closed end points of $Y(p)$, it follows that $y_{p+1} \notin Y(p)$. Let $[y_{p+1}, Y(p)] = [y_{p+1}, y']$. Then

$$2 - \mathrm{ind}_{Y(p+1)}(x) = \begin{cases} 1 & \text{if } x \in y_{p+1}H, \\ 0 & \text{if } x \in (y_{p+1}, y')H, \\ 2 - \mathrm{ind}_{Y(p)}(x) - 1 & \text{if } x \in y'H, \\ 2 - \mathrm{ind}_{Y(p)}(x) & \text{if } x \in Y(p) - y'H. \end{cases}$$

Thus

$$\begin{aligned} &\sum_{x \in Y(p+1)/H} (2 - \mathrm{ind}_{Y(p+1)}(x)) \\ &\quad = \sum_{x \in (Y(p)/H) - \{y'H\}} (2 - \mathrm{ind}_{Y(p)}(x)) + (2 - \mathrm{ind}_{Y(p)}(y') - 1) + 1 \\ &\quad = \sum_{x \in Y(p)/H} (2 - \mathrm{ind}_{Y(p)}(x)) \le 2. \end{aligned}$$

The inequality follows from induction. Thus $\sum_{x \in Y(k)/H}(2 - \mathrm{ind}_{Y(k)}(x)) \le 2$. Since $Y(k)$ contains all closed end points of Y,

$$\sum_{x \in Y/H} (2 - \mathrm{ind}_Y(x)) \le \sum_{x \in Y(k)/H} (2 - \mathrm{ind}_{Y(k)}(x)) \le 2.$$

This proves Lemma 3.7. □

Lemmas 3.5, 3.6, and 3.7 yield that $f(0) - a_1 + a_2 \le 2$.

4. A uniform upper bound of $f(m)$

We prove that $f(m) \le f(m-1)$ by induction on m. The following example shows when and where the increase of index occurs. Let $G = \langle a \rangle * \langle b \rangle$ be a free group of rank two. Let $X_1 = X_a$ and $X_2 = X_b$. Suppose that $X_1 \cap X_2 = [u, v]$. Let $X(0) = X_1 \cup X_2$ and let $X(0) - X_1 = X_2 - [u, v] = C \sqcup C'$. Suppose that $ua = v$ and $Ca \cap C' = (v, v']$. Note that u and v are 0-related. Since $C \cap X_1 = \varnothing$ and $C' \cap X_1 = \varnothing$, C and C' are not 0-equivalent. Thus $\text{ind}[v]_0 = 4$. Note that $\text{ind}_{X(0)}(v) = 3 = \text{ind}_{X(0)}(u)$. Since $Ca \cap C' \neq \varnothing$, C and C' are 1-equivalent. Thus $\text{ind}[v]_1 = 3$. So the index is decreased at v. On the other hand, $\text{ind}[v']_0 = 2$, and $\text{ind}[v']_1 = 3$. The index is increased here. If we consider v and v' together, then $(\text{ind}[v]_0 - 2) + (\text{ind}[v']_0 - 2) = (4-2) + (2-2) = 2$ and $(\text{ind}[v]_1 - 2) + (\text{ind}[v']_1 - 2) = (3-1) + (3-1) = 2$. Thus the sum of weighted indices stays the same. This is the basic idea for the theorem. (See Figure 3.)

We sketch the idea of the proof first. The basic idea is similar to the one in the proof of $f(0) - (a_1 + a_2)$. Since the situation is more complicated here, several changes must be made. Lemma 4.1 shows that to compare $f(m-1)$ and $f(m)$ it is enough to consider $X(m-1)$. Then we study where the index will increase. Lemma 4.2 shows that the increase of index only occurs in a finite subset of $(\pi_0(X(m-1) - X(m-2))/A)$. Suppose this finite set contains k elements. Next we insert k equivalent relations, called the $(m-1, p)$-relations, $p = 1, 2, \ldots, k$, between the $(m-1)$-equivalence relation and the m-equivalence relation. We define $f(m-1, p)$ to be the analogue of $f(m)$ for $(m-1, p)$-relations. Lemma 4.3 shows that to prove $f(m) \leq f(m-1)$ it is enough to prove $f(m-1, p) \leq f(m-1, p-1)$ for all p. The rest of §4 is quite similar to §3. We define a graph $\Gamma(p)$, which is an analogue of Γ in §3. Lemma 4.5 shows that $\Gamma(p)$ is a forest, so each component of $\Gamma(p)$ is a tree. Each connected component of $\Gamma(p)$ is an $(m-1)$-equivalent class of $X(m-1)$. We define an index for a subtree of $\Gamma(p)$ in the same way we defined $\text{ind}[x]_m$. Lemma 4.7 shows that

$$(\text{ind}(U) - 2) - \sum_{[y]_{(m-1,p-1)} \in V(U)} (\text{ind}[y]_{(m-1,p-1)} - 2) \leq \sum_{e \in E(U)} (2 - \beta\psi(e))$$

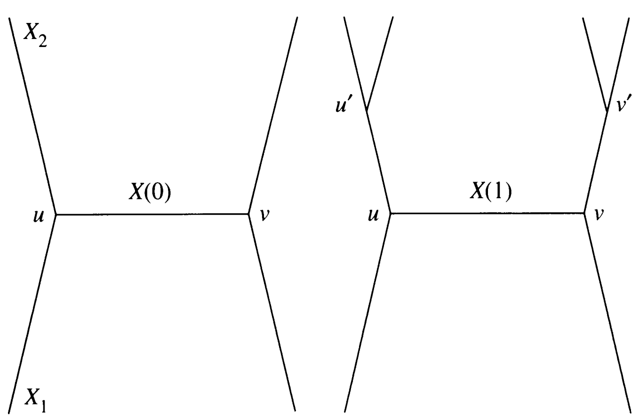

FIGURE 3

for any finite subtree U. Definitions of β and ψ are given in Definitions 4.6 and 4.7. This is an analogue of Lemma 3.2. Lemma 4.8 shows that $f(m-1, p) - f(m-1, p-1) \leq \sum_{e\in E(\Gamma(p))}(2-\beta\psi(e))$. This is an analogue of Lemma 3.5. Lemma 4.4 shows that ψ is bijective, thus

$$\sum_{e\in E(\Gamma(p))}(2-\beta\psi(e)) = \sum_{x\in C_p''\cap(\cup_{j=i}^{p-1}C_j''\alpha_j)}(2-\beta(x)),$$

which is an analogue of Lemma 3.6. Lemma 4.9 shows that

$$\sum_{x\in C_p''\cap(\bigcup_{j=i}^{p-1}C_j''\alpha_j)}(2-\beta(x)) \leq 1,$$

which is an analogue of Lemma 3.7. This will complete the proof of the Main Theorem. Suppose that $m \geq 1$ and $f(m-1) \leq a_1 + a_2 + 2$. Lemma 4.1 shows that we can restrict our attention to $X(m-1)$.

Lemma 4.1.

$$\begin{aligned}\operatorname{Card}(\{C \in \pi_0(X(m) - \{v\}) \mid v \in [u]_m \cap X(m-1),\\ C \cap X(m-1) \neq \varnothing\}/m\text{-relation})\\ = \operatorname{Card}(\{C \in \pi_0(X(m) - \{v\}) \mid v \in [u]_m\}/m\text{-relation})\end{aligned}$$

for any $u \in X(m)$.

Proof. Lemma 4.1 says that for any $u \in X(m)$ and $C \in \pi_0(X(m) - \{u\})$, there exist a $v \in [u]_m \cap X(m-1)$ and a $C' \in \pi_0(X(m) - \{v\})$ such that $C' \cap X(m-1) \neq \varnothing$ and C and C' are m-equivalent. It suffices to show that there is an $h \in G_m$ such that $Ch \cap C' \neq \varnothing$ and $uh = v$.

Case 1. There exists $u_1 \in C$ such that $u \in [u_1, X(m-2)]$.

Let $[u_1, X(m-2)] = [u_1, u_1']$. Since $X(m) = X(m-1)G_m$ and $u_1 \in X(m)$, there exists an $h \in G_m$ such that $u_1 h \in X(m-1)$. Since $u_1' h \in X(m-2)h = X(m-2) \subset X(m-1)$, it follows that

$$uh \in [u_1, u_1']h = [u_1 h, u_1' h] \subset X(m-1).$$

Let $v = uh$ and $C' \in \pi_0(X(m) - \{v\})$ such that $(v, u_1 h] = (u, u_1]h \subset C'$. Then $uh = v$ and $\varnothing \neq (u, u_1]h \subset Ch \cap C'$.

Case 2. There does not exist $u_1 \in C$ such that $u \in [u_1, X(m-2)]$.

Suppose $[u, X(m-2)] = [u, u_1']$. Let $u_1 \in C$ and $u_2 = Y(u_1, u, u_1')$. If $u_2 = u$, then

$$u \in [u_1, u_1'] = [u_1, X(m-2)],$$

which contradicts the above assumption. So $u_2 \neq u$ and $u_2 \in (u, u_1] \subset C$. Since $u \in X(m) = X(m-1)G_m$, there exists an $h \in G_m$ such that $uh \in X(m-1)$. Thus

$$u_2 h \in [u, X(m-2)]h = [uh, X(m-2)] \subset X(m-1).$$

Let $v = uh$ and $C' \in \pi_0(X(m-1) - \{v\})$ such that $(v, u_2 h] \subset C'$. Then $uh = v$ and $\varnothing \neq (u, u_2]h \subset Ch \cap C'$. For the case $m = 1$, we replace $X(m-1)$ by X_1. The rest of the proof remains unchanged. □

From the example given in the beginning of this section we can see that the increase of the index only occurs in $X(m-1) - X(m-2)$, thus we focus on

$X(m-1)-X(m-2)$. Note that $X(m-1)-X(m-2)$ is A-invariant. If $C, C' \in \pi_0(X(m-1)-X(m-2))$, and $Ca \cap C' \neq \varnothing$ for some $a \in A$, then $C = C'$. So the increase of the index only occurs in those $C, C' \in \pi_0(X(m-1)-X(m-2))$ where there is $g \in G_m - A$ such that $Cg \cap C' \neq \varnothing$. Furthermore, we only need to consider $(X(m-1)-X(m-2))/A$. For any $C \in \pi_0(X(m-1) \dot{-} X(m-2))$, let CA denote the A-orbit containing C.

Definition 4.1. For any $C, C' \in \pi_0(X(m-1)-X(m-2))$, CA and $C'A$ are called G_m-equivalent if there exists an $\alpha \in G_m$ such that $CA\alpha \cap C'A \neq \varnothing$.

Note that the index increases only in those G_m-equivalence classes which contain more than one element. Lemma 4.2 shows the finiteness of G_m-equivalence classes.

Lemma 4.2. (a) *There are only finitely many G_m-equivalence classes containing more than one element.*

(b) *Each G_m-class contains only finitely many elements.*

Proof. (a) If this is not the case, let S_i be G_m-classes such that $\mathrm{Card}(S_i) \geq 2$, $i = 1, 2, \ldots$. Let $C_iA \in S_i$ and suppose that $C_i \in \pi_0(X(m-1)-\{v_i\})$ and $v_i \in X(m-2)$ for all i. Since $X(m-2)$ has no closed end point, then $v_i \in BP(X(m-1))$. If

$$\mathrm{Card}(\{C_i | i = 1, 2, \ldots\}/(m-1)\text{-relation}) < \infty,$$

then it contradicts the hypothesis $f(m-1) \leq a_1 + a_2 + 2$. So without loss of generality, we can assume that C_i and C_j are $(m-1)$-equivalent for all i and j. Then choose $C_i'A \in S_i - \{C_iA\}$ for all i. We can also assume that all C_i' and C_j' are $(m-1)$-equivalent. Then we get C_1, C_1', C_2, and C_2'. They are distinct. Note that C_i and C_i' are G_m-equivalent for $i = 1, 2$, C_1 and C_2 are $(m-1)$-equivalent, and C_1' and C_2' are $(m-1)$-equivalent. Thus we have the following diagram:

$$\begin{array}{ccccccc} C_1 & \xrightarrow{\beta_1} & \cdots & \xrightarrow{\beta_p} & C_2 \\ \big\downarrow{\scriptstyle \alpha_1} & & & & \big\downarrow{\scriptstyle \alpha_2} \\ C_1' & \xrightarrow{\gamma_1} & \cdots & \xrightarrow{\gamma_q} & C_2' \end{array}$$

Note that $C_1 \in \pi_0(X(m-1)-X(m-2))$ and $C_1 \cap X(m-2) = \varnothing$. Recall that C_1 and C_2 are $(m-1)$-equivalent. If $\beta \in G_m - A = G_{m-2} - A$, then $C_1 \cap X(m-2) \neq \varnothing$ by Lemma 1.2, a contradiction. So by the same reasoning, $\beta_1, \beta_p, \gamma_1, \gamma_q \notin G_m - A$. On the other hand, $\alpha_1, \alpha_2 \in G_m - A$, and $\beta_1 \cdots \beta_p$ and $\gamma_1 \cdots \gamma_q$ are reduced words of G by Lemma 2.1. Therefore, $\alpha_1\gamma_1 \cdots \gamma_q\alpha_2^{-1}\beta_p^{-1} \cdots \beta_1^{-1} \neq 1$. But $v_1\alpha_1\gamma_1 \cdots \gamma_q\alpha_2^{-1}\beta_p^{-1} \cdots \beta_1^{-1} = v_1$, which contradicts that the G-action is free. This proves (a).

For the proof of (b) we need Claim 1.

Claim 1. Let $C, C' \in \pi_0(X(m-1)-X(m-2))$. If $CA \neq C'A$ and they are G_m-equivalent, then C and C' are not $(m-1)$-equivalent.

Suppose that $C \in \pi_0(X(m-1)-\{v\})$, $C' \in \pi_0(X(m-1)-\{v'\})$, $C\alpha \cap C' \neq \varnothing$, and $v\alpha = v'$ for some $\alpha \in G_m - A$. If C and C' are $(m-1)$-equivalent, then

$$C = C_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_k} C_k = C'.$$

By Lemma 2.1, we can assume that $\alpha_1 \cdots \alpha_k$ is a reduced word. On the other hand, $v\alpha_1 \cdots \alpha_k = v'$ and $v\alpha = v'$, hence $\alpha_1 \cdots \alpha_k = \alpha$. So $k = 1$ and

$\alpha_1 = \alpha \in G_m - A$. By Lemma 1.2, $\alpha \in G_m - A = G_{m-2} - A$ implies that $C \cap X(m-2) \neq \varnothing$ and $C' \cap X(m-2) \neq \varnothing$, which contradicts $C, C' \in \pi_0(X(m-1) - X(m-2))$. This proves Claim 1.

(b) For any $C \in \pi_0(X(m-1) - X(m-2))$ there is a $v \in X(m-2)$ such that $C \in \pi_0(X(m-1) - \{v\})$. Since $X(m-2)$ has no closed end point, thus $v \in BP(X(m-1))$. If there are infinitely many elements in a G_m-class, then they are in distinct $(m-1)$-classes by Claim 1. This contradicts the hypothesis $f(m-1) \leq a_1 + a_2 + 2$. This proves (b) and completes the proof of Lemma 4.2. □

Let $S_1, S_2, \dots, S_t$ all be G_m-classes containing more than one element. Let

$$S_j = \{C''_p A | i_{j-1} < p \leq i_j\}, \qquad j = 1, \dots, t, \; 0 = i_0 < i_1 < \cdots < i_t = k.$$

So $\bigcup_{j=1}^t S_j = \{C_1 A, \dots, C_k A\}$. We divide the proof of $f(m) \leq f(m-1)$ into k steps. At each step we just consider one more $C''_i A$. We insert k equivalent relations, called $(m-1, p)$-relations, $p = 1, 2, \dots, k$, between the $(m-1)$-equivalence relation and the m-equivalence relation. The $(m-1, p)$-relations involve only $C''_1 A, \dots, C''_p A$. The reason for defining $(m-1, p)$-relations is to make $\Gamma(p)$ a forest (see Definition 4.5). We have the following definitions.

Definition 4.2. For any $u, v \in X(m-1)$, u and v are called $(m-1, p)$-equivalent if the following conditions hold.

(1) u and v are m-equivalent.

(2) Suppose $u = u_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_k} u_k = v$. If u_{i-1} and u_i are not $(m-1)$-equivalent, then $u_{i-1}, u_i \in \bigcup_{j=1}^p C''_j A$.

Definition 4.3. Suppose that u and v are $(m-1, p)$-equivalent, $C \in \pi_0(X(m) - \{u\})$, $C' \in \pi_0(X(m) - \{u\})$, $C \cap X(m-1) \neq \varnothing$, and $C' \cap X(m-1) \neq \varnothing$. Then C and C' are called $(m-1, p)$-equivalent if the following conditions hold.

(1) C and C' are m-equivalent.

(2) Suppose $C = C_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_k} C_k = C'$. If $C_{i-1} \cap X(m-1)$ and $C_i \cap X(m-1)$ are not $(m-1)$-equivalent, then $u_{i-1}, u_i \in \bigcup_{j=1}^p \overline{C}''_j A$, where $\overline{C}''_j = C''_j \cup \{v_j\}$ if $C''_j \in \pi_0(X(m-1) - \{v_j\})$.

Lemma 4.3. (1) *The* $(m-1, 1)$*-relation is the same as the* $(m-1)$*-relation.*

(2) *The* $(m-1, k)$*-relation is the same as the* m*-relation.*

Proof. *Claim* 1. If u and v are $(m-1, 1)$-equivalent, then they are $(m-1)$-equivalent.

By Definition 4.2, u and v are m-equivalent. Suppose that

$$u = u_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_k} u_k = v.$$

If Claim 1 is not true, suppose that u_{i-1} and u_i are not $(m-1)$-equivalent for some i. By Lemma 2.1, if $k > 1$, we can assume that $\alpha_j \notin A$ for all j. If $k = 1$ and $\alpha_1 \in A$, then u_0 and u_1 are $(m-1)$-equivalent, which contradicts the assumption that u_{i-1} and u_i are not $(m-1)$-equivalent for some i. So we can assume that $\alpha_j \notin A$ for all j. If $\alpha_i \in G_{m-1} - A$, then by Lemma 1.1, $u_{i-1}, u_i \in X(m-1)$. Thus u_{i-1} and u_i are $(m-1)$-equivalent, which contradicts the assumption. So $\alpha_i \in G_m - A$. Since u and v are

$(m-1, 1)$-equivalent, by part (2) of Definition 4.2, $u_{i-1}, u_i \in C_1''A$. Suppose $u_{i-1} \in C \in C_1''A$; then $u_i = u_{i-1}\alpha_i \in C\alpha_i \notin C_1''A$ since $\alpha_i \in G_m - A$. This contradicts $u_{i-1}, u_i \in C_1''A$, and it completes the proof of Claim 1.

Claim 2. If C and C' are $(m-1, 1)$-equivalent, then $C \cap X(m-1)$ and $C' \cap X(m-1)$ are $(m-1)$-equivalent.

The proof is similar to that of Claim 1 and is left to the reader.

Note that $(m-1)$-equivalence implies $(m-1, 1)$-equivalence, and $(m-1, k)$-equivalence implies m-equivalence.

Claim 3. If $u, v \in X(m-1)$ and u and v are m-equivalent, then they are $(m-1, k)$-equivalent.

Suppose that $u = u_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_k} u_k = v$. If $k > 1$, we can assume that $\alpha_i \notin A$ for all i. If $k = 1$ and $\alpha_1 \in A$, then u and v are $(m-1)$-equivalent. Therefore, u and v are $(m-1, k)$-equivalent. So we can assume that all $\alpha_i \notin A$.

First we prove that $u_i \in X(m-1)$ for all i. For any i, either α_i or α_{i+1} is in $G_{m-1} - A$. Say $\alpha_{i+1} \in G_{m-1} - A$, then by Lemma 1.1, $u_i, u_{i+1} \in X(m-1)$. So $u_i \in X(m-1)$ for all i.

Next we prove that if u_{j-1} and u_j are not $(m-1)$-equivalent, then they are in $X(m-1) - X(m-2)$. Suppose u_{j-1} and u_j are not $(m-1)$-equivalent; then $\alpha_j \notin G_{m-1} - A$. So $\alpha_j \in G_m - A$. If one of u_{j-1} and u_j is in $X(m-2)$, so is the other. Since $u_{j-1}\alpha_j = u_j$ and $\alpha_j \in G_m - A = G_{m-2} - A$, if $u_{j-1}, u_j \in X(m-2)$, then they are $(m-1)$-equivalent. This contradicts the assumption that u_{j-1} and u_j are not $(m-1)$-equivalent. So $u_{j-1}, u_j \in X(m-1) - X(m-2)$.

Finally we prove that if u_{j-1} and u_j are not $(m-1)$-equivalent, then they are in $\bigcup_{j=1}^k C_j''A$. Suppose $u_{j-1} \in C \in \pi_0(X(m-1) - X(m-2))$ and $u_j \in C' \in \pi_0(X(m-1) - X(m-2))$. Since $u_{j-1}\alpha_j = u_j$ and $\alpha_j \in G_m - A$, it follows that CA and $C'A$ are G_m-equivalent. Since $\alpha_j \notin A$, it follows that $CA \neq C'A$. Therefore, $CA, C'A \in \{C_1''A, \dots, C_k''A\}$ and $u_{j-1}, u_j \in \bigcup_{j=1}^k C_j''A$. This proves Claim 3.

Claim 4. If $C \in \pi_0(X(m) - \{u\})$, $C' \in \pi_0(X(m) - \{v\})$, $C \cap X(m-1) \neq \varnothing$, $C' \cap X(m-1) \neq \varnothing$, and C and C' are m-equivalent, then C and C' are $(m-1, k)$-equivalent.

Suppose $C = C_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_k} C_k = C'$. If $k > 1$, then $\alpha_i \notin A$ and $C_i \cap X(m-1) \neq \varnothing$ for all i. If $k = 1$ and $\alpha_1 \in A$, then $C \cap X(m-1)$ and $C' \cap X(m-1)$ are $(m-1)$-equivalent. Therefore, C and C' are $(m-1, k)$-equivalent. So we can assume that $\alpha_i \notin A$ and $C_i \cap X(m-1) \neq \varnothing$ for all i. If $C_{i-1} \cap X(m-1)$ and $C_i \cap X(m-1)$ are not $(m-1)$-equivalent, then $\alpha_i \in G_m - A$. Since $\alpha_i \in G_m - A$ and $u_{i-1}\alpha_i = u_i$, then u_{i-1} and u_i are either both in $X(m-2)$ or both in $X(m-1) - X(m-2)$. We prove that $u_{i-1}, u_i \in \bigcup_{j=1}^k \overline{C}_j''A$.

First suppose $u_{i-1}, u_i \in X(m-2)$. Since $C_{i-1}\alpha_i \cap C_i \neq \varnothing$, it follows that $C_{i-1} \cap X(m-2)$ and $C_i \cap X(m-2)$ are either both empty or both nonempty. Suppose $C_{i-1} \cap X(m-2) \neq \varnothing$ and $C_i \cap X(m-2) \neq \varnothing$. Since $C_{i-1}\alpha_i \cap C_i \neq \varnothing$, it follows $C_{i-1} \cap X(m-2)$ and $C_i \cap X(m-2)$ are $(m-2)$-equivalent, which contradicts that $C_{i-1} \cap X(m-1)$ and $C_i \cap X(m-1)$ are not $(m-1)$-equivalent. If $C_{i-1} \cap X(m-2) = \varnothing$ and $C_i \cap X(m-2) = \varnothing$, then $C_{i-1} \cap X(m-1)$ and

$C_i \cap X(m-1)$ are in $\pi_0(X(m-1) - X(m-2))$. Since $C_{i-1}\alpha_i \cap C_i \neq \varnothing$,

$$C_{i-1} \cap X(m-1),\ C_i \cap X(m-1) \in \bigcup_{j=1}^{k} C_j''A$$

and $u_{i-1}, u_i \in \bigcup_{j=1}^{k} \overline{C}_j''A$.

Now suppose $u_{i-1}, u_i \in X(m-1) - X(m-2)$. Let $u_{i-1} \in C$ and $u_i \in C'$, where $C, C' \in \pi_0(X(m-1) - X(m-2))$. Since $u_{i-1}\alpha_i = u_i$, it follows that $C\alpha \cap C' \neq \varnothing$. Since $\alpha_i \in G_m - A$, it follows that $CA \neq C'A$ and CA and $C'A$ are G_m-equivalent. Thus $CA, C'A \in \{C_1''A, \dots, C_k''A\}$. So $u_{i-1}, u_i \in \bigcup_{j=1}^{k} \overline{C}_j''A$. This proves that u and v are $(m-1, k)$-equivalent. Lemma 4.3 follows. □

Definition 4.4. Let $[u]_{(m-1,p)}$ denote the $(m-1, p)$-equivalence class containing u. Let

$$\operatorname{ind}[u]_{(m-1,p)} = \operatorname{Card}(\{C \in \pi_0(X(m) - \{v\}) | v \in [u]_{(m-1,p)},\ C \cap X(m-1) \neq \varnothing\}/(m-1, p)\text{-relation}).$$

Let

$$f(m-1, p) = \sum_{[u]_{(m-1,p)}} (\operatorname{ind}[u]_{(m-1,p)} - 2),$$

where $[u]_{(m-1,p)} \in X(m-1)/(m-1, p)$-relation.

To prove that $f(m) \leq f(m-1)$, it suffices to prove that $f(m-1, p) \leq f(m-1, p-1)$ for all p. We prove this by induction on p; the hypothesis is that $f(m-1, p-1) \leq a_1 + a_2 + 2$.

Definition 4.5. Let $\Gamma(p)$ be a graph, $p = 1, 2, \dots$. It is defined by the following:

$V(\Gamma(1)) = X(m-1)/(m-1)$-relation,

$E(\Gamma(1)) = \varnothing$,

$V(\Gamma(p)) = \{\text{connected components of } \Gamma(p-1)\}$,

$E(\Gamma(p)) = \{([u], [v]) | [u] \neq [v] \in V(\Gamma(p)), \exists x \in [u], y \in [v] \text{ and } \alpha \in G_m - A$ such that $x\alpha = y$ and $x, y \in \bigcup_{j=1}^{p} C_j''A\}$.

Note that a connected component of $\Gamma(p)$ is an $(m-1, p)$-equivalence class of $X(m-1)$. Suppose $i_t < p \leq i_{t+1}$, we now use the notation set up in the paragraph before Definition 4.2. For $i_t + 1 \leq j \leq p-1$, there exists an $\alpha_j \in G_m - A$ such that $C_j''\alpha_j \cap C_p'' \neq \varnothing$. If $j \leq i_t$, then there is no such α_j such that $C_j''\alpha_j \cap C_p'' \neq \varnothing$.

Suppose $p = i_t + 1$. We prove that the $(m-1, p)$-equivalence relation is the same as the $(m-1, p-1)$-equivalence relation. Therefore, $f(m-1, p-1) = f(m-1, p)$. Recall Definitions 4.2 and 4.3. It suffices to prove that if $u_{i-1}, u_i \in \bigcup_{j=1}^{p} C_j''A$ and $u_{i-1}\alpha_i = u_i$ for some $\alpha_i \in G_m - A$, then $u_{i-1}, u_i \in \bigcup_{j=1}^{p-1} C_j''A$. Since $(C_p''A(G_m - A)) \cap (C_p''A) = \varnothing$, it follows that u_{i-1} and u_i are not both in $C_p''A$. Suppose $u_i \in C_p''A$ and $u_{i-1} \in C_j''A$ for $j < p = i_t + 1$. Then there exists an $\alpha \in G_m - A$ such that $C_j''\alpha \cap C_p'' \neq \varnothing$, which contradicts our assumption. Thus $u_{i-1}, u_i \in \bigcup_{j=1}^{p-1} C_j''A$.

In the rest of the paper we assume that $i_t + 1 < p$ and we denote $i_t + 1$ by i for simplicity. Next we define a bijection ψ between the edges of $\Gamma(p)$ and $C_p'' \cap (\bigcup_{j=i}^{p-1} C_j''\alpha_j)$.

Definition 4.6. Let $\psi : E(\Gamma(p)) \to C_p'' \cap (\bigcup_{j=i}^{p-1} C_j''\alpha_j)$ be defined as follows: for all $e = ([u], [v]) \in E(\Gamma(p))$, let $\psi(e) = \{x, y\} \cap C_p''$, where $x \in [u]$, $y \in [v]$, $\alpha \in G_m$, $x\alpha = y$, and $x, y \in \bigcup_{j=1}^{p} C_j''$. We call $\{x, y\}$ a bridge between $[u]$ and $[v]$.

Lemma 4.4. *ψ is bijective.*

Proof. Claim 1. *ψ is well defined.* By Definition 4.5, we only know that $x, y \in \bigcup_{j=1}^{p} C_j''A$. Note that $[u]$ and $[v]$ are $(m-1, p-1)$-equivalence classes of $X(m-1)$. If $x \in [u]$ and $\alpha_0 \in A$, then $x\alpha_0$ and x are $(m-1)$-equivalent. Therefore, they are $(m-1, p-1)$-equivalent. Replacing x and y by $x\alpha_0$ and $y\beta_0$, where $\alpha_0, \beta_0 \in A$, we have that $x\alpha_0 \in [u]$ and $y\beta_0 \in [v]$. Since $x\alpha_0\alpha_0^{-1}\alpha\beta_0 = y\beta_0$ and $\alpha_0^{-1}\alpha\beta_0 \in G_m - A$, we can assume, without loss of generality, that $x, y \in \bigcup_{j=1}^{p} C_j''$. Suppose $x_i \in [u]$, $y_i \in [v]$, $\alpha_i \in G_m - A$, $x_i\alpha_i = y_i$, and $x_i, y_i \in \bigcup_{j=1}^{p} C_j''$ for $i = 1, 2$. We have the following diagram:

$$\begin{array}{ccccccc} x_1 & \xrightarrow{\beta_1} & u_1 & \cdots & u_{k-1} & \xrightarrow{\beta_k} & x_2 \\ \big\downarrow{\scriptstyle \alpha_1} & & & & & & \big\downarrow{\scriptstyle \alpha_2} \\ y_1 & \xrightarrow{\gamma_1} & v_1 & \cdots & v_{t-1} & \xrightarrow{\gamma_t} & y_2 \end{array}$$

where $x_1, u_1, \ldots, u_{k-1}$ and x_2 are in the same $(m-1, p-1)$-class $[u]$, and $y_1, v_1, \ldots, v_{t-1}$ and y_2 are in the same $(m-1, p-1)$-class $[v]$. Note that $[u] \neq [v]$. We prove that $\{x_1, y_1\} \cap C_p'' = \{x_2, y_2\} \cap C_p''$. If $\{x_1, y_1\} \cap C_p'' = \varnothing$, then $x_1, y_1 \in \bigcup_{j=1}^{p-1} C_j''$. Since $x_1\alpha_1 = y_1$ and $\alpha_1 \in G_m - A$, it follows that x_1 and y_1 are $(m-1, p-1)$-equivalent, thus $[u] = [v]$, a contradiction. So $\{x_i, y_i\} \cap C_p'' \neq \varnothing$ for $i = 1, 2$.

Case 1. $y_1 \neq y_2$, $y_1, y_2 \in C_p''$. First we prove that $\gamma_1 \in G_{m-1} - A$. If $t = 1$, then $y_1\gamma_1 = y_2$. Since $y_1, y_2 \in C_p''$ and $(C_p''(G_m - \{1\})) \cap C_p'' = \varnothing$, it follows that $\gamma_1 \notin G_m$. So $\gamma_1 \in G_{m-1} - A$. If $t > 1$, we can assume that $\gamma_i \notin A$ for all i. Since y_1 and v_1 are $(m-1, p-1)$-equivalent, either y_1 and v_1 are $(m-1)$-equivalent, or $y_1, v_1 \in \bigcup_{j=1}^{p-1} C_j''A$. Since $y_1 \in C_p''$, it follows that $y_1 \notin \bigcup_{j=1}^{p-1} C_j''A$. So y_1 and v_1 are $(m-1)$-equivalent. But $y_1\gamma_1 = v_1$. If $\gamma_1 \in G_m - A$, then y_1 and v_1 are $(m-2)$-equivalent. In particular, $y_1, v_1 \in X(m-2)$, which contradicts that $y_1 \in C_p'' \in \pi_0(X(m-1) - X(m-2))$. So $\gamma_1 \in G_{m-1} - A$.

Next we prove that $\beta_1^{-1}\alpha_1 \notin A$. If $\beta_1^{-1}\alpha_1 \in A \subset G_{m-1}$, then $u_1\beta_1^{-1}\alpha_1 = y_1$ implies that y_1 and u_1 are $(m-1)$-equivalent. Therefore, $[v] = [u]$, a contradiction.

By the same reasoning, we have that $\gamma_t \notin G_m$ and $\beta_k\alpha_2 \notin A$. We can assume that $\beta_1 \cdots \beta_k$ and $\gamma_1 \cdots \gamma_t$ are reduced words by Lemma 2.1. Since $y_1\gamma_1 \cdots \gamma_t\alpha_2^{-1}\beta_k^{-1} \cdots \beta_1^{-1}\alpha_1 = y_1$, it follows that $\gamma_1 \cdots \gamma_t\alpha_2^{-1}\beta_k^{-1} \cdots \beta_1^{-1}\alpha_1 = 1$. A simple inspection finds that this contradicts $G = G_1 *_A G_2$. So Case 1 cannot happen.

Case 2. $x_2, y_1 \in C_p''$. As we proved in Case 1, $x_2 \in C_p''$ implies that $\beta_k \notin G_m$

and $\alpha_2\gamma_t^{-1} \notin A$, and $y_1 \in C_p''$ implies that $\gamma_1 \notin G_m$ and $\beta_1^{-1}\alpha_1 \notin A$. We still get a contradiction. So Case 2 also cannot happen.

Case 3. $x_1, y_2 \in C_p''$.

Case 4. $x_1 \neq x_2$, $x_1, x_2 \in C_p''$.

Cases 3 and 4 also cannot happen; the proofs are the same.

So the only possibilities are:

(a) $x_1 = x_2 \in C_p''$ and $y_1, y_2 \notin C_p''$.

(b) $y_1 = y_2 \in C_p''$ and $x_1, x_2 \notin C_p''$.

This shows ψ is well defined.

Claim 2. *ψ is injective.* Let $e_i = ([u_i], [v_i]) \in E(\Gamma(p))$ for $i = 1, 2$. Suppose $x_i \in [u_i]$, $y_i \in [v_i]$, and $\alpha_i \in G_m - A$ such that $x_i\alpha_i = y_i$ and $x_i, y_i \in \bigcup_{j=1}^{p} C_j''$ for $i = 1, 2$. Suppose $y_1 = \psi(e_1) = \psi(e_2)$. Since $y_1 \in [v_1] \cap [v_2]$, it follows that $[v_1] = [v_2]$. Since $x_1\alpha_1\alpha_2^{-1} = x_2$ for $\alpha_1\alpha_2 \in G_m$ and $x_1, x_2 \in \bigcup_{j=1}^{p-1} C_j''$, it follows that x_1 and x_2 are $(m-1, p-1)$-equivalent. Therefore, $[u_1] = [u_2]$ and $e_1 = e_2$. So ψ is injective.

Claim 3. *ψ is surjective.* For $y \in C_p'' \cap (\bigcup_{j=i}^{p-1} C_j''\alpha_j)$ there exist an $x \in \bigcup_{j=i}^{p-1} C_j''$ and an $\alpha_j \in G_m - A$ such that $x\alpha_j = y$. Since $\alpha \in G_m - A = G_{m-2} - A$, if x and y are $(m-1)$-equivalent, then by Lemma 1.1, $x, y \in X(m-2)$, which contradicts $y \in C_p'' \in \pi_0(X(m-1) - X(m-2))$. So x and y are not $(m-1)$-equivalent. Combining this fact with that $y \notin \bigcup_{j=1}^{p-1} C_j''A$ yields that x and y are not $(m-1, p-1)$-equivalent. Thus $[x] \neq [y]$ and $([x], [y]) \in E(\Gamma(p))$. Since $\psi([x], [y]) = y$, it follows that ψ is surjective. This completes the proof of Lemma 4.4. □

Lemma 4.5. *$\Gamma(p)$ is a forest.*

Proof. If this is not the case, let $[u_1], [u_2], \dots, [u_k]$ be vertices of a minimal circle of $\Gamma(p)$. Let $[u_{k+1}] = [u_1]$. Suppose $x_{i,j} \in [u_i]$ and $\alpha_i \in G_m - A$ such that $x_{i,2}\alpha_i = x_{i+1,1}$ and $x_{i,2}, x_{i+1,1} \in \bigcup_{j=1}^{p} C_j''$ for $i = 1, \dots, k$ and $j = 1, 2$. Suppose $x_{i,1} \xrightarrow{\beta_{i,1}} \cdots \xrightarrow{\beta_{i,k_i}} x_{i,2}$ for all i. By the argument given in the proof of Claim 1 of Lemma 4.4, we can prove that

$$\beta_{1,1}\cdots\beta_{1,k_1}\alpha_1\beta_{2,1}\cdots\beta_{2,k_2}\cdots\alpha_k \neq 1.$$

On the other hand, since the action is free and

$$x_{1,1}\beta_{1,1}\cdots\beta_{1,k_1}\alpha_1\beta_{2,1}\cdots\beta_{2,k_2}\cdots\alpha_k = x_{1,1},$$

we have

$$\beta_{1,1}\cdots\beta_{1,k_1}\alpha_1\beta_{2,1}\cdots\beta_{2,k_2}\cdots\alpha_k = 1,$$

a contradiction. □

Lemma 4.6. (1) $\mathrm{Card}(BP(C_p'' \cap (\bigcup_{j=i}^{p-1} C_j''\alpha_j))) < \infty$.

(2) $\mathrm{Card}\{x \in C_p'' \cap (\bigcup_{j=i}^{p-1} C_j''\alpha_j) | \beta(x) = 1\} < \infty$,

Proof. (1) If this is not true, then let $x_t \in BP(C_p'' \cap (\bigcup_{j=i}^{p-1} C_j''\alpha_j))$ for $t = 1, 2, \dots$. Since $x_t \in \bigcup_{j=i}^{p-1} C_j''\alpha_j$, there exist $y_t \in \bigcup_{j=i}^{p-1} C_j''$ and $\beta_t \in \{\alpha_i, \dots, \alpha_{p-1}\}$, such that $y_t\beta_t = x_t$ for all t. Since all x_t are branch points and $f(m-1, p-1) \leq a_1 + a_2 + 2$, we can assume that all x_t are in the same

$(m-1, p-1)$-equivalence class. On the other hand, since $\operatorname{ind}_{\bigcup_{j=i}^{p-1} C_j''\alpha_j}(y_t) \geq 3$, it follows that $\operatorname{ind}[y_t]_{(m-1,p-1)} \geq 3$. So we can also assume that all y_t are in the same $(m-1, p-1)$-class. Suppose $y_i\beta_i = x_i$ for $i = 1, 2$, x_1 and x_2 are $(m-1, p-1)$-equivalent, and so are y_1 and y_2. Since $([x_1], [y_1]) = ([x_2], [y_2])$, it follows that $x_1 = \psi[x_1], [y_1]) = \psi([x_2], [y_2]) = x_2$, which contradicts that $x_1 \neq x_2$. This proves (1).

(2) Suppose $\beta(x_t) = 1$ for $t = 1, 2, \ldots$. Since $\beta(x_t) = 1$, it follows that $x_k \notin [x_i, x_j]$ for i, j, and k distinct, thus

$$Y(x_i, x_j, x_k) \in BP\left(C_p'' \cap \left(\bigcup_{j=i}^{p-1} C_j''\alpha_j\right)\right).$$

If there are infinitely many x_t and there are only finitely many distinct $Y(x_i, x_j, x_k)$, then

$$\operatorname{ind}_{C_p''}(Y(x_i, x_j, x_k)) = \infty$$

for some i, j, and k, which contradicts that $f(m-1) \leq a_1 + a_2 + 2$. If there are infinitely many distinct $Y(x_i, x_j, x_k)$, then it contradicts (1). So there are only finitely many x_t, and (2) follows. □

Definition 4.7. Let

$$\beta(x) = \operatorname{Card}\left(\pi_0\left(C_p'' \cap \left(\bigcup_{j=i}^{p-1} C_j''\alpha_j\right) - x\right)\right)$$

for $x \in (C_p'' \cap (\bigcup_{j=i}^{p-1} C_j''\alpha_j))$.

Put $\{x_p\} = \overline{C}_p'' - C_p''$, and let

$$\delta = \begin{cases} 1 & \text{if } x_p \in \bigcup_{[v]\in U}[v], \\ 0 & \text{otherwise.} \end{cases}$$

Let U be a subtree of $\Gamma(p)$, and let

$$\operatorname{ind}(U) = \operatorname{Card}(\{C \in \pi_0(X(m) - \{x\}) | x \in U, \\ C \cap X(m-1) \neq \varnothing\}/(m-1, p)\text{-relation}).$$

This is an analogue of $\operatorname{ind}[x]_m$.

Lemma 4.7. *Let U be a finite subtree of $\Gamma(p)$. Then*

$$(\operatorname{ind}(U) - 2) - \sum_{[y]_{(m-1,p-1)} \in V(U)} (\operatorname{ind}[y]_{(m-1,p-1)} - 2) \\ \leq \sum_{e \in E(U)} (2 - \beta\psi(e)) - \delta.$$

Proof. We prove this lemma by induction on $\operatorname{Card}(V(U)) = k$. If $k = 1$, then U is a single point v. If $\delta = 0$, then

$$\operatorname{Card}(\{C \in \pi_0(X(m) - \{x\}) | x \in [v], \\ C \cap X(m-1) \neq \varnothing\}/(m-1, p-1)\text{-relation}) - 2 \\ \geq \operatorname{Card}(\{C \in \pi_0(X(m) - \{x\}) | x \in [v], \\ C \cap X(m-1) \neq \varnothing\}/(m-1, p)\text{-relation}) - 2$$

and $\operatorname{ind}(U) - \operatorname{ind}[v]_{(m-1,p-1)} \leq 0$.

Suppose $\delta = 1$. Recall that $\overline{C}_p'' = C_p'' \cup \{x_p\}$. Since $\delta = 1$, it follows that $x_p \in [v]$ and $C_p'' \in \pi_0(X(m-1) - \{x_p\})$. Note that C_p'' and C_{p-1}'' are not $(m-1, p-1)$-equivalent, but they are $(m-1, p)$-equivalent. Therefore,

$$\begin{aligned}&\operatorname{Card}(\{C \in \pi_0(X(m) - \{x\}) | x \in [v],\\ &\qquad\qquad C \cap X(m-1) \neq \varnothing\}/(m-1, p-1)\text{-relation})\\ &\quad \geq \operatorname{Card}(\{C \in \pi_0(X(m) - \{x\}) | x \in [v],\\ &\qquad\qquad C \cap X(m-1) \neq \varnothing\}/(m-1, p)\text{-relation}) + 1.\end{aligned}$$

If $k > 1$, then suppose a subtree U satisfies Lemma 4.7 and $\operatorname{Card}(V(U)) = k$. We consider a subtree $U \cup \{[v]\}$. There exists a unique $e \in E(\Gamma(p))$ such that e joining v and U.

Let $\{u', v'\}$ be a bridge between $[v]$ and $[u] \in V(U)$ and $u' \in C_p''$. Suppose that $\beta\psi(e) = t$. Then

$$t = \operatorname{Card}\left(\pi_0\left(\left(C_p'' \cap \left(\bigcup_{j=i}^{p-1} C_j''\alpha_j\right)\right) - \{u'\}\right)\right).$$

Suppose that

$$C_1, \dots, C_t \in \pi_0\left(\left(C_p'' \cap \left(\bigcup_{j=i}^{p-1} C_j''\alpha_j\right)\right) - \{u'\}\right).$$

Then there exist $\beta_j \in G_m - A$ such that $u'\beta_j \in \bigcup_{s=i}^{p-1} C_s''$ for $j = 1, \dots, t$. Furthermore, there exist $C_j' \in \pi_0(X(m-1) - \{u'\beta_j\})$ such that $C_j\beta_j \cap C_j' \neq \varnothing$ for $j = 1, \dots, t$.

Case 1. $\delta = 0$ By Lemma 2.2, C_i and C_j are not m-equivalent for $i \neq j$. Since $C_j\beta_j \cap C_j' \neq \varnothing$ and $u' \in C_p''$, we have that C_j and C_j' are $(m-1, p)$-equivalent but not $(m-1, p-1)$-equivalent. Therefore, C_i' and C_j' are not m-equivalent for $i \neq j$, and neither are C_i and C_j'. So

$$\operatorname{Card}(\{C_1, \dots, C_t, C_1', \dots, C_t'\}/(m-1, p-1)\text{-relation}) = 2t,$$

$$\operatorname{Card}(\{C_1, \dots, C_t, C_1', \dots, C_t'\}/(m-1, p)\text{-relation}) = t.$$

Without loss of generality, we may assume that $u' \in [u]$ and $v' \in [v]$. Since at least t representatives (for example $C_1', \dots, C_t'$) of

$$\{C \in \pi_0(X(m) - \{x\}) | x \in [v], C \cap X(m-1) \neq \varnothing\}/(m-1, p-1)\text{-relation}$$

are $(m-1, p)$-equivalent to t representatives (for example $C_1, \dots, C_t$) of

$$\begin{aligned}&\{C \in \pi_0(X(m) - \{x\}) | x \in [w]_{(m-1,p)} \cap U,\\ &\qquad\qquad C \cap X(m-1) \neq \varnothing\}/(m-1, p)\text{-relation},\end{aligned}$$

then $[v]_{(m-1,p-1)}$ brings in at most $\operatorname{ind}[v]_{(m-1,p-1)} - \beta\psi(e)$ distinct $(m-1, p)$-equivalence classes of directions. Therefore

$$\operatorname{ind}(U \cup \{[v]\}) \leq \operatorname{ind}(U) + (\operatorname{ind}[v]_{(m-1,p-1)} - \beta\psi(e)).$$

By induction,

$$(\operatorname{ind}(U)-2)-\left(\sum_{[u]\in V(U)}(\operatorname{ind}[u]_{(m-1,p-1)}-2)\right)\leq\sum_{e'\in E(U)}(2-\beta\psi(e')).$$

Thus

$$\begin{aligned}
&\operatorname{ind}(U\cup\{[v]\})-2)-\left(\sum_{[u]\in V(U\cup\{[v]\})}(\operatorname{ind}[u]_{(m-1,p-1)}-2)\right)\\
&\quad=(\operatorname{ind}(U\cup\{[v]\})-2)-\left(\sum_{[u]\in V(U)}(\operatorname{ind}[u]_{(m-1,p-1)}-2)\right)\\
&\qquad-(\operatorname{ind}[v]_{(m-1,p-1)}-2)\\
&\quad\leq(\operatorname{ind}(U)+\operatorname{ind}[v]_{(m-1,p-1)}-\beta\psi(e)-2)\\
&\qquad-\left(\sum_{[u]\in V(U)}(\operatorname{ind}[u]_{(m-1,p-1)}-2)\right)-(\operatorname{ind}[v]_{(m-1,p-1)}-2)\\
&\quad\leq(\operatorname{ind}(U)-2+\operatorname{ind}[v]_{(m-1,p-1)}-2+2-\beta\psi(e))\\
&\qquad-\left(\sum_{[u]\in V(U)}(\operatorname{ind}[u]_{(m-1,p-1)}-2)\right)-(\operatorname{ind}[v]_{(m-1,p-1)}-2)\\
&\quad\leq\sum_{e'\in E(U)}(2-\beta\psi(e'))+(2-\beta\psi(e))\leq\sum_{e'\in E(U\cup\{[v]\})}(2-\beta\psi(e')).
\end{aligned}$$

Case 2. $\delta=1$. If C''_p and C''_{p-1} are not $(m-1,p-1)$-equivalent to any C_i and C'_i for $i=1,\dots,t$, then

$$\operatorname{Card}(\{C_1,\dots,C_t,C'_1,\dots,C'_t,C''_p,C''_{p-1}\}/(m-1,p-1)\text{-relation})=2t+2,$$

$$\operatorname{Card}(\{C_1,\dots,C_t,C'_1,\dots,C'_t,C''_p,C''_{p-1}\}/(m-1,p)\text{-relation})\leq t+1.$$

Therefore, the contribution of $[v]$ to $w(U\cup\{[v]\})$ is less than or equal to

$$\operatorname{ind}[v]-((2t+2)-(t+1))=\operatorname{ind}[v]-(t+1).$$

If C''_p is $(m-1,p-1)$-equivalent to some C_i, then C''_{p-1} cannot be $(m-1,p-1)$-equivalent to any C_j or C'_j, $i\neq j$. Otherwise C_i and C_j will be $(m-1,p)$-equivalent, which contradicts Lemma 2.2. If C''_{p-1} is $(m-1,p-1)$-equivalent to C'_i, then C_i and C'_i are $(m-1,p-1)$-equivalent, which also gives a contradiction. Thus

$$\operatorname{Card}(\{C_1,\dots,C_t,C'_1,\dots,C'_t,C''_p,C''_{p-1}\}/(m-1,p-1)\text{-relation})=2t+1.$$

Since $C_1,\dots,C_t$ represent t distinct $(m-1,p)$-equivalence classes,

$$\operatorname{Card}(\{C_1,\dots,C_t,C'_1,\dots,C'_t,C''_p,C''_{p-1}\}/(m-1,p)\text{-relation})=t.$$

Then $[v]$ brings in at most $\operatorname{ind}[v]-((2t+1)-t)=\operatorname{ind}[v]-t-1$ distinct $(m-1,p)$-equivalence classes of directions. Therefore

$$\operatorname{ind}(U\cup\{[v]\})\leq\operatorname{ind}(U)+(\operatorname{ind}[v]_{(m-1,p-1)}-\beta\psi(e)-1).$$

Thus

$$\begin{aligned}
&(\operatorname{ind}(U\cup\{[v]\})-2)-\left(\sum_{[u]\in V(U\cup\{[v]\})}(\operatorname{ind}[u]_{(m-1,p-1)}-2)\right)\\
&\quad=(\operatorname{ind}(U)-2)-\left(\sum_{[u]\in V(U)}(\operatorname{ind}[u]_{(m-1,p-1)}-2)\right)-(\operatorname{ind}[v]_{(m-1,p-1)}-2)\\
&\quad\le(\operatorname{ind}(U)+\operatorname{ind}[v]_{(m-1,p-1)}-\beta\psi(e)-1-2)\\
&\qquad-\left(\sum_{[u]\in V(U)}(\operatorname{ind}[u]_{(m-1,p-1)}-2)\right)-(\operatorname{ind}[v]_{(m-1,p-1)}-2)\\
&\quad\le(\operatorname{ind}(U)-2+\operatorname{ind}[v]_{(m-1,p-1)}-2+2-\beta\psi(e))\\
&\qquad-\left(\sum_{[u]\in V(U)}(\operatorname{ind}[u]_{(m-1,p-1)}-2)\right)-(\operatorname{ind}[v]_{(m-1,p-1)}-2)-1\\
&\quad\le\sum_{e'\in E(U)}(2-\beta\psi(e'))+(2-\beta\psi(e))-1\le\sum_{e'\in E(U)}(2-\beta\psi(e'))-1.
\end{aligned}$$

This proves Lemma 4.7. □

Lemma 4.8. $f(m-1,p)-f(m-1,p-1)\le\sum_{e\in E(\Gamma(p))}(2-\beta\psi(e))-1$.

Proof. Recall that $\psi(e)=\{x,y\}\cap(C_p''\cap(\bigcup_{j=i}^{p-1}C_j''\alpha_j))$. Let U be a finite subtree of $\Gamma(p)$, and let $[x]_{(m-1,p)}$ be the connected component of $\Gamma(p)$ containing U. Since by Lemma 4.6,

$$\operatorname{Card}\{e\in E(\Gamma(p))|2-\beta\psi(e)\neq 0\}<\infty,$$

then for U big enough, $2-\beta\psi(e)=0$ for all $e\in E[x]_{(m-1,p)})-E(U)$. Thus $\operatorname{ind}(U)=\operatorname{ind}[x]_{(m-1,p)}$. We think of $[x]_{(m-1,p)}$ as a connected component of $\Gamma(p)$, not a vertex of $\Gamma(p+1)$. Note that $\operatorname{ind}[x]_{(m-1,p)}<\infty$. If

$$\begin{aligned}
&(\operatorname{ind}[x]_{(m-1,p)}-2)-\left(\sum_{[v]\in V[x]_{(m-1,p-1)})}(\operatorname{ind}[v]_{(m-1,p-1)}-2)\right)\\
&\quad>\sum_{e\in E[x]_{(m-1,p-1)})}(2-\beta\psi(e))-\delta,
\end{aligned}$$

then there exists a big finite subtree U of $[x]_{(m-1,p)}$, such that

$$(\operatorname{ind}(U)-2)-\left(\sum_{[v]\in V(U)}(\operatorname{ind}[v]_{(m-1,p-1)}-2)\right)>\sum_{e\in E(U)}(2-\beta\psi(e))-\delta,$$

which contradicts Lemma 4.7. So

$$\begin{aligned}
&(\operatorname{ind}[x]_{(m-1,p)}-2)-\left(\sum_{[v]\in V[x]_{(m-1,p-1)})}(\operatorname{ind}[v]_{(m-1,p-1)}-2)\right)\\
&\quad\le\sum_{e\in E[x]_{(m-1,p-1)})}(2-\beta\psi(e))-\delta.
\end{aligned}$$

Taking the summation over all $[x]_{(m-1,p)} \in X(m-1)/(m-1,p)$-relation, we get

$$\sum_{[x]_{(m-1,p)}} (\mathrm{ind}[x]_{(m-1,p)} - 2) - \sum_{[x]_{(m-1,p)}} \left(\sum_{[v] \in V[x]_{(m-1,p-1)}} (\mathrm{ind}[v]_{(m-1,,p-1)} - 2) \right)$$

$$\leq \sum_{[x]_{(m-1,p)}} \left(\sum_{e \in E[x]_{(m-1,p-1)}} (2 - \beta\psi(e)) \right) - \delta = \sum_{e \in E(\Gamma(p))} (2 - \beta\psi(e)) - 1,$$

where $[x]_{(m-1,p)} \in X(m-1)/(m-1,p)$-relation. This proves Lemma 4.8. □

Since ψ is a bijection, then

$$\sum_{e \in E(\Gamma(p))} (2 - \beta\psi(e)) = \sum_{x \in C_p'' \cap (\cup_{j=i}^{p-1} C_j'' \alpha_j)} (2 - \beta(x)).$$

To show $f(m-1, p) \leq f(m-1, p-1)$ and to complete the proof of the theorem, we only need the following lemma.

Lemma 4.9. $\sum_{x \in C_p'' \cap (\bigcup_{j=i}^{p-1} C_j'' \alpha_j)} (2 - \beta(x)) \leq 1$.

Proof. We prove this lemma by induction on

$$\mathrm{Card} \left\{ x \in C_p'' \cap \left(\bigcup_{j=i}^{p-1} C_j'' \alpha_j \right) \middle| \beta(x) = 1 \right\} = k.$$

If $k \leq 1$, we are done. Suppose $k > 1$. Let $H = \{1\}$ and $Y = \overline{C}_p'' \cap (\bigcup_{j=i}^{p-1} \overline{C}_j'' \alpha_j)$. Then Lemma 4.6 shows that $\mathrm{Card}(BP(Y)/H) < \infty$. Since $f(m-1) \leq a_1 + a_2 + 2$, it follows that $\mathrm{ind}_Y(x) < \infty$. Applying Lemma 3.7 to H and Y yields that $\sum_{x \in Y} (2 - \mathrm{ind}_Y(x)) \leq 2$. Since

$$Y = \overline{C}_p'' \cap \left(\bigcup_{j=i}^{p-1} \overline{C}_j'' \alpha_j \right) = \left(C_p'' \cap \left(\bigcup_{j=i}^{p-1} C_j'' \alpha_j \right) \right) \cup \{x_p\},$$

we have

$$\sum_{x \in C_p'' \cap (\bigcup_{j=i}^{p-1} C_j'' \alpha_j)} (2 - \beta(x)) + (2 - \mathrm{ind}_Y(x_p)) = \sum_{x \in Y} (2 - \mathrm{ind}_Y(x)) \leq 2.$$

Since $\mathrm{ind}_Y(x_p) = 1$, it follows that $\sum_{x \in C_p'' \cap (\bigcup_{j=i}^{p-1} C_j'' \alpha_j)} (2 - \beta(x)) \leq 1$. This proves Lemma 4.9, and completes the proof of the theorem. □

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