

THE MARTIN KERNEL AND INFIMA OF POSITIVE HARMONIC FUNCTIONS

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ABSTRACT. Let D be a bounded Lipschitz domain in $\mathbf{R}^n$ and let $K(x, z)$, $x \in D$, $z \in \partial D$, be the Martin kernel based at $x_0 \in D$. For $x, y \in D$, let $k(x, y) = \inf\{h(x) : h \text{ positive harmonic in } D, h(y) = 1\}$. We show that the function k completely determines the family of positive harmonic functions on D . Precisely, for every $z \in \partial D$, $\lim_{y \rightarrow z} k(x, y)/k(x_0, y) = K(x, z)$. The same result is true for second-order uniformly elliptic operators and Schrödinger operators.

1. INTRODUCTION

Positive harmonic functions on a bounded domain D in $\mathbf{R}^n$ enjoy two important and well-known properties: they satisfy the Harnack inequality and allow an integral representation via a kernel function. The Harnack inequality is usually expressed in two forms. The first one is more local and states that if a ball $B(x_0, r)$ is contained in D , then for any positive harmonic function h on D , and any $x \in B(x_0, r)$,

$$r^{n-2} \frac{r^2 - |x - x_0|^2}{(r + |x - x_0|)^n} h(x_0) \leq h(x) \leq r^{n-2} \frac{r^2 - |x - x_0|^2}{(r - |x - x_0|)^n} h(x_0).$$

The second form follows from the first by the usual chain argument: For any compact subset K of D there is a constant c such that for any positive harmonic function h on D , $h(x) \leq ch(y)$ for all $x, y \in K$. An immediate consequence of the inequality is that an arbitrary infimum of positive harmonic functions in D is a continuous function satisfying the same inequality. Therefore, one can form the family of continuous functions

$$\mathcal{H}^{\text{inf}} = \left\{ u : u = \inf_{\alpha} h_{\alpha}, \ h_{\alpha} \text{ positive and harmonic in } D \right\}.$$

Every function in $\mathcal{H}^{\text{inf}}$ is superaveraging, so $\mathcal{H}^{\text{inf}}$ consists of positive superharmonic function which satisfy the Harnack inequality. In §2 we show that $\mathcal{H}^{\text{inf}}$ is a convex cone stable for arbitrary infima and closed for the pointwise convergence.

The second essential property of positive harmonic functions is the integral representation. Following [Doo], let G denote the Green function on D (for

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the Laplacian) and let ν be a measure with compact support $\text{Supp } \nu \subset D$. The function $K_\nu(x, y) = G(x, y)/G\nu(y)$ is called the Martin function (or the Martin kernel) based on ν . There is a unique metrizable compactification D_M of D such that each Martin function K_ν has a continuous extension (denoted also by K_ν) to $D \times (D_M \setminus \text{Supp } \nu)$, and $K_\nu(\cdot, y_1) = K_\nu(\cdot, y_2)$ if and only if $y_1 = y_2$. The boundary $\partial_M D = D_M \setminus D$ is called the Martin boundary. Let $\partial_M^\circ D$ denote the set of minimal points $z \in \partial_M D$. Then for each positive harmonic function h in D there is a unique measure μ on $\partial_M^\circ D$ such that $h = \int_{\partial_M^\circ D} K_\nu(\cdot, z) \mu(dz)$. If the measure ν is the point mass at $x_0 \in D$, the Martin kernel is said to be based at x_0 . In this case the continuous extension of the Martin kernel to $\partial_M D$ means that as $y \rightarrow z$ in the Martin topology, where $y \in D$, $z \in \partial_M D$, we have

$$(1.1) \quad \lim_{y \rightarrow z} G(x, y)/G(x_0, y) = K_{x_0}(x, z).$$

The close relation between the Harnack inequality and the Martin kernel is revealed in a simple case of the unit ball $B = B(0, 1)$ centered at the origin. The Martin boundary of B is its Euclidean boundary and the Martin kernel is the Poisson kernel $P(x, z) = (1 - |x|^2)/(|z - x|^n)$, $x \in B$, $z \in \partial B$. The Harnack inequality for B is derived from the representation formula and reads

$$(1.2) \quad \frac{1 - |x|^2}{(1 + |x|)^n} h(0) \leq h(x) \leq \frac{1 - |x|^2}{(1 - |x|)^n} h(0), \quad x \in B.$$

This inequality is sharp and the bounds are attained for $h = P(\cdot, z)$ where $z = -x/|x|$ for the left inequality, and $z = x/|x|$ for the right inequality. In this sense, the Harnack inequality is tailor-made for balls and, moreover, it distinguishes the center of the ball. It is not optimal for other domains. To obtain optimal lower bounds for an arbitrary domain D , it is natural to introduce the function

$$(1.3) \quad k(x, y) = \inf\{h(x) : h \text{ positive harmonic in } D, h(y) = 1\},$$

where $x, y \in D$. We will show that k is continuous function on $D \times D$. Obviously, $k(x, y)$ is the greatest lower bound for a positive harmonic function in D which is 1 at y . As in the case of the unit ball, this lower bound is attained by the Martin kernel: there exists $z = z(x, y)$ in the minimal Martin boundary $\partial_M^\circ D$ such that $k(x, y) = K(x, z)/K(y, z)$ where K is the Martin kernel based at some point x_0 in D . Therefore, the Martin kernel K completely determines the function k .

For the unit ball B it is possible to explicitly compute the function k . First note that from (1.2) it follows that $k(x, 0) = (1 - |x|^2)/(1 + |x|)^n$. In order to compute $k(x, y)$ for any $y \in D$, some results on conformal maps are needed. These can be found in [Ahl]. Let $J : x \mapsto x^*$, $x^* = x/|x|^2$ be the reflection through the unit sphere. The full Möbius group $\widehat{M}(\mathbb{R}^n)$ is the group generated by all similarities on $\mathbb{R}^n$ together with J . Following [Ahl], for any $\gamma \in \widehat{M}(\mathbb{R}^n)$, let $|\gamma'|$ be the positive number such that $\gamma'/|\gamma'| \in O(n)$. Here $O(n)$ denotes the orthogonal group and γ' the Jacobian of γ at the point x . For a positive harmonic function h on B and $\gamma \in \widehat{M}(\mathbb{R}^n)$, let $(T_\gamma h)(x) = |\gamma'|^{(n-2)/2} h(\gamma x)$. It is proved in Lemma 2.1 of [Leu] that T_γ preserves positivity and harmonicity.

For $y \in B$, $y \neq 0$, let

$$\gamma_y(x) = \frac{(1 - |x|^2)(x - y) - |x - y|^2 y}{1 - 2xy + |x|^2 |y|^2}$$

where xy denotes the inner product of x and y . Then γ_y maps B onto itself and $\gamma_y(y) = 0$. Now it easily follows that

$$k(x, y) = \frac{|y - y^*|^{n-2}}{|x - y^*|^{n-2}} k(\gamma_y(x), 0).$$

Such computations are essentially done in [Leu] (see Theorem 3.2 there). After simplifications, k can be written in the following form that we find convenient:

$$(1.4) \quad k(x, y) = (1 - |y|^2)^{n-2} \frac{(1 - |y|^2)(1 - |x|^2)}{(|y||x - y^*| + |x - y|)^n}, \quad y \neq 0.$$

Regarded as a function of x , $k(x, y)$ is a potential for every $y \in D$. The normalized functions $k(x, y)/k(0, y)$ are equal to 1 at the origin. Direct computation shows that as y approaches the point z on the boundary of B , these normalized functions converge to the Poisson kernel: $\lim_{y \rightarrow z} k(x, y)/k(0, y) = P(x, z)$. Hence, the function k carries all informations about the cone of positive harmonic functions on B .

We show that the same is true for a bounded Lipschitz domain: knowledge of the function k is sufficient to recover the Martin kernel. More precisely, let D be a bounded Lipschitz domain. Then the Martin boundary $\partial_M D$ of D is its Euclidean boundary and all boundary points are minimal (see [Hun]). Let K be the Martin kernel based at $x_0 \in D$. We prove that if $x \in D$ and $z \in \partial D$, then

$$(1.5) \quad \lim_{y \rightarrow z} \frac{k(x, y)}{k(x_0, y)} = K(x, z)$$

and the limit is uniform on compact subsets of D . This can be regarded as an analogue of (1.1), the difference being that functions $x \mapsto k(x, y)/k(x_0, y)$, $y \in D$, satisfy the Harnack inequality and are not harmonic near the boundary ∂D .

A similar result can be proved for more general second-order elliptic differential operators. To unify exposition, we work in an abstract setting which is described in the next section. Formula (1.5) for this setting is proved in §3. In §4 we recall some of the known results which provide examples for the situation studied in §§2 and 3. In §5 we show that the function k (as in most potential-theoretical results) reflects a dichotomy between the two-dimensional case and higher-dimensional cases. In dimension two, k is symmetric while the symmetry is lost in higher dimensions. This symmetry provides a result similar to (1.5) in case infimum in relation (1.3) is replaced by supremum. In §6 we give an interesting example of a function which is an infimum of positive harmonic functions.

2. CONVEX CONES OF POSITIVE CONTINUOUS FUNCTIONS

Let D be a locally compact topological space with countable basis. Then D is metrizable and let d denote a metric compatible with the topology on D . By $\mathcal{C}(D)$ we denote the space of continuous functions on D with the topology of

uniform convergence on compact subsets. Let $\mathcal{H}$ be a closed convex cone of strictly positive continuous functions on D containing the function identically equal to zero. We assume that $\mathcal{H}$ has a compact basis, i.e., there exists a hyperplane $\mathcal{L}$ in $\mathcal{C}(D)$ such that the family $\mathcal{G} = \mathcal{L} \cap \mathcal{H}$ is compact in $\mathcal{C}(D)$ and generates $\mathcal{H} : \mathcal{H} = \{\lambda u : \lambda \geq 0, u \in \mathcal{G}\}$.

Let $k : D \times D \rightarrow \mathbf{R}$ be defined by

$$(2.1) \quad k(x, y) = \inf_{u \in \mathcal{H}} \frac{h(x)}{h(y)}.$$

The infimum over $\mathcal{H}$ in the definition above can be replaced by the infimum over the basis $\mathcal{G}$. For x, y, z in D ,

$$(2.2) \quad k(x, z)k(z, y) \leq k(x, y).$$

An easy argument using compactness of the basis shows that $k(x, y) > 0$ for all $x, y \in D$. We also need the following simple lemma.

Lemma 2.1. *The function k is continuous on $D \times D$.*

Proof. It is easy to see that k is continuous in x and y separately. Let $\{(x_n, y_n)\}$ be a sequence in $D \times D$ converging to (x, y) . From (2.2),

$$k(y, y_n) \leq \frac{k(x_n, y_n)}{k(x_n, y)} \leq \frac{1}{k(y_n, y)}.$$

Therefore,

$$k(y, y_n) \frac{k(x_n, y)}{k(x, y)} \leq \frac{k(x_n, y_n)}{k(x, y)} \leq \frac{1}{k(y_n, y)} \frac{k(x_n, y)}{k(x, y)}$$

Let $n \rightarrow \infty$ in inequalities above. Separate continuity gives

$$1 \leq \liminf \frac{k(x_n, y_n)}{k(x, y)} \leq \limsup \frac{k(x_n, y_n)}{k(x, y)} \leq 1$$

which proves the lemma. $\square$

Let $\mathcal{S}$ be a family of all functions u on D that satisfy

$$(2.3) \quad u(x) \geq k(x, y)u(y)$$

for all x and y in D . By using continuity of k and the fact that $k(x, x) = 1$ for all $x \in D$, it follows that $\mathcal{S}$ is a convex cone of strictly positive continuous functions (unless identically zero) closed for arbitrary infima and suprema. Furthermore, it is closed in the topology of pointwise convergence. If $\mathcal{T}$ is a subset of $\mathcal{S}$ that is bounded at a point $x \in D$, then it is bounded in $\mathcal{C}(D)$ and locally uniformly equicontinuous. In particular, if $\mathcal{T}$ is a closed cone and $\mathcal{T}_x = \{u \in \mathcal{T} : u(x) = 1\}$, then $\mathcal{T}_x$ is a compact basis for $\mathcal{T}$. Besides being closed in $\mathcal{C}(D)$, $\mathcal{T}$ is also closed in the topology of pointwise convergence. Indeed, if $u_n(x) \rightarrow u(x)$ for every $x \in D$, then boundedness of $\{u_n\}$ at every point implies that $\{u_n\}$ is relatively compact. A convergent subsequence converges to a function in $\mathcal{T}$ which is evidently equal to u .

Let $\mathcal{F}$ denote the closure in $\mathcal{C}(D)$ of the family $\{u_1 \wedge u_2 \wedge \cdots \wedge u_n : u_j \in \mathcal{H}, j = 1, 2, \dots, n, n \in \mathbf{N}\}$. Then $\mathcal{F}$ is the smallest closed convex cone stable under finite minima containing $\mathcal{H}$. It is also closed for countable infima. Let

$$(2.4) \quad \mathcal{H}^{\text{inf}} = \{u : u = \inf u_\alpha, u_\alpha \in \mathcal{H}\}.$$

Since each function in $\mathcal{H}$ satisfies (2.3) by definition of k , this is also true of $\mathcal{H}^{\text{inf}}$. Hence $\mathcal{H}^{\text{inf}} \subset \mathcal{S}$ and, in particular, each function in $\mathcal{H}^{\text{inf}}$ is continuous. By Choquet's lemma, every function in $\mathcal{H}^{\text{inf}}$ is an infimum of a countable family of functions. Since $\mathcal{S}$ is closed under countable infima, $\mathcal{H}^{\text{inf}} \subset \mathcal{S}$. To show the converse inclusion, first note that every function in $\mathcal{S}$ is an increasing limit of functions in $\mathcal{H}^{\text{inf}}$. Indeed, if $u \in \mathcal{S}$, then $u = \lim u_n$ where $u_n \in \mathcal{H}^{\text{inf}}$. If $v_k = \inf_{n \geq k} u_n$, then $\{v_k\}$ is an increasing sequence in $\mathcal{H}^{\text{inf}}$ converging to u . Next we need the following lemma.

Lemma 2.2. *Let $u \in \mathcal{H}^{\text{inf}}$ and $x \in D$. Then there exists $v \in \mathcal{H}$ such that $v \geq u$ in D and $v(x) = u(x)$.*

Proof. Let $u = \inf u_\alpha$. There is a sequence $\{u_n\} \subset \{u_\alpha\}$ such that $u_n(x) \downarrow u(x)$. Being bounded at x , u_n is relatively compact. Therefore, a subsequence $\{u_{n_i}\}$ converges to a function v in $\mathcal{E}(D)$. In particular, $u_{n_i}(x) \rightarrow v(x)$, so $u(x) = v(x)$. Since $u_n \geq u$ in D for all $n \in \mathbb{N}$, it follows that $v \geq u$ in D . $\square$

Let u be a function in $\mathcal{S}$. Then $u = \uparrow \lim u_n$ where u_n are in $\mathcal{H}^{\text{inf}}$. Fix $x \in D$. By the lemma above, for each $n \in \mathbb{N}$, there is $v_n \in \mathcal{H}$ such that $v_n \geq u_n$ in D and $v_n(x) = u_n(x)$. A subsequence $\{v_{n_i}\}$ converges to a function v^x in $\mathcal{E}(D)$. Moreover, $v^x(x) = u(x)$ and, since $v_{n_i} \geq u_{n_i}$, it follows that $v^x \geq u$ on D . Let $R_u = \inf\{v \in \mathcal{H} : v \geq u\}$. Then $R_u \in \mathcal{H}^{\text{inf}}$ and $R_u \geq u$. But, for $x \in D$, $v^x \geq u$ and $v^x(x) = u(x)$. Therefore, $R_u = u$, so $u \in \mathcal{H}^{\text{inf}}$. By putting the preceding together, we obtain

Proposition 2.3. *Let $\mathcal{H}$ be a closed convex cone of strictly positive (unless zero) continuous functions. Assume that $\mathcal{H}$ has a compact basis. Then the family $\mathcal{H}^{\text{inf}} = \{u : u = \inf u_\alpha, u_\alpha \in \mathcal{H}\}$ is a closed convex cone, stable for arbitrary infima and closed in the topology of pointwise convergence.*

For $y \in D$, let $\mathcal{H}_y = \{u \in \mathcal{H} : u(y) = 1\}$. Then $\mathcal{H}_y$ is a compact basis for $\mathcal{H}$; hence $k(x, y) = \inf\{u(x) : u \in \mathcal{H}, u(y) = 1\}$. Let us record two corollaries.

Corollary 2.4. *Assume that $1 \in \mathcal{H}^{\text{inf}}$. If $\phi : (0, \infty) \rightarrow \mathbf{R}$ is a positive, increasing, concave function, then $\phi \circ u \in \mathcal{H}^{\text{inf}}$ for each $u \in \mathcal{H}^{\text{inf}}$.*

Corollary 2.5. *Let Σ be a locally compact topological space. Assume that $u : D \times \Sigma \rightarrow \mathbf{R}_+$ has the following two properties: (i) $x \mapsto u(x, \sigma) \in \mathcal{H}^{\text{inf}}$ for each $\sigma \in \Sigma$, (ii) $\sigma \mapsto u(x, \sigma)$ is continuous for each $x \in D$. If μ is a positive Radon measure on the Borel σ -algebra of Σ , then the function $\int_\Sigma u(\cdot, \sigma) \mu(d\sigma)$ belongs to $\mathcal{H}^{\text{inf}}$.*

Proof. It is enough to notice that the measure μ can be approximated by positive linear combinations of point-mass measures and use the fact that $\mathcal{H}^{\text{inf}}$ is closed for pointwise topology. $\square$

3. KERNEL FUNCTION

Let D and $\mathcal{H}$ be as in the previous section. In addition, we assume that D is contained in a compact metrizable space denoted by $\bar{D}$, such that D is the interior of $\bar{D}$ and the metric of $\bar{D}$ restricted to D is d . Let ∂D denote $\bar{D} \setminus D$; we call ∂D the boundary of D . Recall that $\mathcal{H}_x = \{h \in \mathcal{H} : h(x) = 1\}$.

We assume that there is $u_0 \in \mathcal{H}$ satisfying $m \leq u_0 \leq M$ for some positive constants m and M . Let x_0 be an arbitrary, but fixed point in D . In this section we assume the existence of a function on $D \times \partial D$, which we call the kernel function. The basic hypothesis is:

(H₁) There exists a function $K: D \times \partial D \rightarrow \mathbf{R}$ such that

(i) for each $z \in \partial D$, $x \mapsto K(x, z)$ belongs to $\mathcal{H}_{x_0}$,

(ii) for each $x \in D$, $z \mapsto K(x, z)$ is continuous on ∂D .

Note that if μ is a finite measure on Borel sets of ∂D , then the function u defined by

$$(3.1) \quad u(x) = \int_{\partial D} K(x, z) \mu(dz)$$

belongs to $\mathcal{H}$ (see Corollary 2.5). We shall assume that all functions in $\mathcal{H}$ arise in this way.

(H₂) For each $u \in \mathcal{H}$, there exists a unique Borel measure μ on ∂D such that (3.1) holds.

For $x, y \in D$ let $\mathcal{H}_{x,y} = \{u \in \mathcal{H} : u(y) = 1, u(x) = k(x, y)\}$. By Lemma 2.2, $\mathcal{H}_{x,y}$ is nonempty. Furthermore, it is convex, compact and closed for pointwise convergence. We show that it contains functions of the form $x \mapsto K(x, z)$, $z \in \partial D$ (properly normalized).

Let $u \in \mathcal{H}_{x,y}$ and μ the measure representing u . Then

$$k(x, y) = u(x) = \int_{\partial D} K(x, z) \mu(dz).$$

Assume that Δ' and Δ'' are disjoint Borel subsets of ∂D , such that $\Delta' \cup \Delta'' = \partial D$ and $\mu(\Delta') > 0$, $\mu(\Delta'') > 0$. Let $\alpha' = \int_{\Delta'} K(y, z) \mu(dz)$ and $\alpha'' = \int_{\Delta''} K(y, z) \mu(dz)$. Then both α' and α'' are positive, so one can define measures ν' and ν'' on ∂D by

$$\nu' = \frac{1}{\alpha'} \mu|_{\Delta'} \quad \text{and} \quad \nu'' = \frac{1}{\alpha''} \mu|_{\Delta''}.$$

Let $u' = \int_{\partial D} K(\cdot, z) \nu'(dz)$ and $u'' = \int_{\partial D} K(\cdot, z) \nu''(dz)$. Then u' and u'' are in $\mathcal{H}$ and a simple computation shows that $u'(y) = u''(y) = 1$. Hence, $u' \geq k(x, y)$ and $u'' \geq k(x, y)$. Therefore, since $\alpha' + \alpha'' = 1$,

$$\begin{aligned} u(x) &= \alpha' u'(x) + \alpha'' u''(x) \\ &\geq \alpha' k(x, y) + \alpha'' k(x, y) \\ &= k(x, y) = u(x). \end{aligned}$$

This implies that $u'(x) = u''(x) = k(x, y)$. Hence, u' and u'' are in $\mathcal{H}_{x,y}$ and representing measures have smaller support.

Proposition 3.1. *For y and x in D , there exists $z = z(x, y) \in \partial D$ such that $K(\cdot, z)/K(y, z) \in \mathcal{H}_{x,y}$.*

Proof. Let $u \in \mathcal{H}_{x,y}$ and $u = \int_{\partial D} K(\cdot, \zeta) \mu(d\zeta)$. If μ is a multiple of a point mass at z , there is nothing to prove. If μ charges some point $z \in \partial D$, take $\Delta' = \{z\}$ in the construction preceding the statement. Then the function u' from above is precisely $K(\cdot, z)/K(y, z)$. So assume that μ does not charge points. Let $z \in \text{Supp } \mu$. Then μ charges every neighborhood of z in ∂D . Let $\{\Delta_n\}$ be a decreasing sequence of neighborhoods of z in ∂D such that

$\bigcap_{n=1}^{\infty} \Delta_n = \{z\}$. We assume, without loss of generality, that $\mu(\partial D \setminus \Delta_n) > 0$ for every $n \in \mathbb{N}$. Let

$$\alpha_n = \int_{\Delta_n} K(y, \zeta) \mu(d\zeta), \quad \nu_n = \frac{1}{\alpha_n} \mu|_{\Delta_n}, \quad u_n = \int_{\partial D} K(\cdot, \zeta) \nu_n(d\zeta).$$

Then $u_n \in \mathcal{H}_{x,y}$ for each n . By using continuity of $K(y, \cdot)$ it is easy to see that for all but finitely many $n \in \mathbb{N}$, $1/(2K(y, z)) \leq \nu_n(\partial D) \leq 2/K(y, z)$. Hence, the sequence $\{\nu_n\}$ is bounded. Without loss of generality, assume that $\{\nu_n\}$ weakly converges to a positive Borel measure ν on ∂D . Let $v = \int_{\partial D} K(\cdot, \zeta) \nu(d\zeta) = \lim_n \int_{\partial D} K(\cdot, \zeta) \nu_n(d\zeta) = \lim_n u_n$. Since $\mathcal{H}_{x,y}$ is closed for pointwise convergence, $v \in \mathcal{H}_{x,y}$. It is easy to see that ν is concentrated on $\{z\}$. Therefore, $\nu = c\delta_z$ for some positive constant c . Further, $1 = u_n(y) = \int_{\partial D} K(y, \zeta) \nu_n(d\zeta) \rightarrow \int_{\partial D} K(y, \zeta) \nu(d\zeta) = cK(y, z)$. Hence, $v = K(\cdot, z)/K(y, z) \in \mathcal{H}_{x,y}$. $\square$

The proposition above shows that the kernel function K completely determines k . To show the converse, i.e., that K is determined by k , an additional hypothesis is needed. In view of the examples discussed in §4, this hypothesis is more restrictive than the first two.

(H₃) For all $z_1, z_2 \in \partial D$ such that $z_1 \neq z_2$,

$$\lim_{(x,z) \rightarrow (z_1, z_2)} K(x, z) = 0,$$

where $(x, z) \in D \times \partial D$ and $(x, z) \rightarrow (z_1, z_2)$ un $\overline{D} \times \overline{D}$.

Let us fix $x \in D$ (x_0 is still fixed). Recall that we have assumed the existence of the function $u_0 \in \mathcal{H}$ satisfying $m \leq u_0 \leq M$. Hence, for any $y \in D$, $u_0(x)/u_0(y) \leq M/m$. Therefore, $k(x, y) \leq M/m$. For $y \in D$, let $z(x, y)$ denote a point on ∂D such that $K(\cdot, z(x, y))/K(y, z(x, y)) \in \mathcal{H}_{x,y}$.

Lemma 3.2. *Let $y \rightarrow z$, $z \in \partial D$. Then $z(x, y) \rightarrow z$.*

Proof. Let $\{y_n\}$ be a sequence in D converging to z and let us denote the corresponding points $z(x, y_n)$ on the boundary by z_n . Since ∂D is compact, we may assume that $\{z_n\}$ converges to some point $z_0 \in \partial D$. If $z_0 \neq z$, then by (H₃), $\limsup K(y_n, z_n) = 0$. By the continuity of K , $\lim K(x, z_n) = K(x, z_0) < \infty$. Therefore, the sequence $\{K(x, z_n)/K(y_n, z_n)\}$ is unbounded. On the other hand, $K(x, z_n)/K(y_n, z_n) = k(x, y_n) \leq M/m$ which yields contradiction. Hence $z_0 = z$. $\square$

Theorem 3.3. *For every $z \in \partial D$,*

$$\lim_{y \rightarrow z} \frac{k(\cdot, y)}{k(x_0, y)} = K(\cdot, z).$$

Proof. Let us fix $x \in D$. Let v_y and z_y be points on ∂D such that

$$K(\cdot, v_y)/K(y, v_y) \in \mathcal{H}_{x_0,y}$$

and

$$K(\cdot, z_y)/K(y, z_y) \in \mathcal{H}_{x,y}, \quad y \in D.$$

Thus,

$$(3.2) \quad \frac{K(x, z_y)}{K(y, z_y)} = k(x, y) \quad \text{and} \quad \frac{K(x_0, v_y)}{K(y, v_y)} = k(x_0, y).$$

By definition of $k(\cdot, y)$,

$$(3.3) \quad \frac{K(x, v_y)}{K(y, v_y)} \geq k(x, y) \quad \text{and} \quad \frac{K(x_0, z_y)}{K(y, z_y)} \geq k(x_0, y).$$

From (3.2) and (3.3) it follows that

$$(3.4) \quad \frac{K(y, z_y)}{K(y, v_y)} \geq \frac{K(x, z_y)}{K(x, v_y)} \quad \text{and} \quad \frac{K(y, z_y)}{K(y, v_y)} \leq 1.$$

As $y \rightarrow z$, Lemma 3.2 gives that $z_y \rightarrow z$ and $v_y \rightarrow z$. The first inequality above and continuity of K give

$$\liminf_{y \rightarrow z} \frac{K(y, z_y)}{K(y, v_y)} \geq \liminf_{y \rightarrow z} \frac{K(x, z_y)}{K(x, v_y)} = \frac{K(x, z)}{K(x, z)} = 1.$$

From the second inequality in (3.4) it follows that

$$\limsup_{y \rightarrow z} \frac{K(y, z_y)}{K(y, v_y)} \leq 1.$$

Hence,

$$(3.5) \quad \lim_{y \rightarrow z} \frac{K(y, z_y)}{K(y, v_y)} = 1.$$

By (3.2),

$$\frac{k(x, y)}{k(x_0, y)} = \left[\frac{K(x, z_y)}{K(y, z_y)} \right] \left[\frac{K(x_0, v_y)}{K(y, v_y)} \right]^{-1} = K(x, z_y) \frac{K(y, v_y)}{K(y, z_y)}.$$

Therefore, by (3.5) and continuity of K ,

$$\lim_{y \rightarrow z} \frac{k(x, y)}{k(x_0, y)} = \lim_{y \rightarrow z} K(x, z_y) \frac{K(y, v_y)}{K(y, z_y)} = K(x, z). \quad \square$$

Remark. Let u be an arbitrary strictly positive continuous function on D , and let ${}_u\mathcal{H} = \{h/u : h \in \mathcal{H}\}$. Then ${}_u\mathcal{H}$ is another closed convex cone with compact basis. Let ${}_uk(x, y) = \inf\{v(x) : v \in {}_u\mathcal{H}, v(y) = 1\}$. Then it easily follows that ${}_uk(x, y) = (u(y)/u(x))k(x, y)$. If K is the kernel function for $\mathcal{H}$, then ${}_uK$ defined by ${}_uK(x, z) = (u(x_0)/u(x))K(x, z)$ is the kernel function for ${}_u\mathcal{H}$. Moreover,

$$\lim_{y \rightarrow z} \frac{{}_uk(x, y)}{{}_uk(x_0, y)} = {}_uK(x, z),$$

so Theorem 3.3 holds for the cone ${}_u\mathcal{H}$. We note that the kernel function ${}_uK$ need not satisfy (H_3) .

4. POSITIVE SOLUTIONS OF THE SCHRÖDINGER EQUATION

The motivating example for the results in the previous section was the cone of positive harmonic functions in a bounded Lipschitz domain. Some recent results from [Chi] and [Cra] show that the cone of positive solutions of the Schrödinger equation also satisfies hypotheses from §3. Here we give a brief review of these results.

Let D be a bounded domain in $\mathbf{R}^n$, $n \geq 3$, and let

$$(4.1) \quad A = \sum_{i,j=1}^n D_i(a_{ij}(x)D_j)$$

be an elliptic operator in divergence form with bounded coefficients a_{ij} satisfying $a_{ij} = a_{ji}$. We assume that A is uniformly elliptic on D , i.e., there exist constants λ and Λ , $0 < \lambda < \Lambda$ such that

$$\lambda|\xi|^2 \leq \sum a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2$$

for all $\xi \in \mathbf{R}^n$ and all $x \in D$. Let q be a function on D which belongs to the Kato class $K_n(D)$, i.e.,

$$(4.2) \quad \limsup_{r \rightarrow 0} \sup_{x \in D} \int_{|x-y| \leq r} \frac{|q|(y)}{|x-y|^{n-2}} dy = 0.$$

Let $L = -A + q$ be the Schrödinger operator on D . A weak solution of $Lu = 0$ is a function u in the Sobolev space $H_{\text{loc}}^{1,2}(D)$ satisfying

$$(4.3) \quad - \sum_{i,j=1}^n \int_D a_{ij}(x) D_i u(x) D_j \phi(x) dx = \int_D q(x) u(x) \phi(x) dx$$

for every function $\phi \in \mathcal{C}_c^\infty(D)$. Let $\mathcal{H}^{\text{inf}}$ denote the family of positive solutions of $Lu = 0$. Then $\mathcal{H}^{\text{inf}}$ is a closed convex cone in $\mathcal{E}(D)$ with compact basis $\mathcal{H}_{x_0}$ where x_0 is an arbitrary point in D . This easily follows from the continuity theorem, Harnack's theorem and Lemma 1.1 in [Chi].

To show the existence of the kernel function for $\mathcal{H}^{\text{inf}}$, both the domain D and the function q need to be specialized. We will assume that D is a bounded Lipschitz domain. Let us first take $q = 0$. The existence of the kernel function K on $D \times \partial D$ satisfying hypothesis (H_1) and (H_2) was proved in [Caf, Theorems 3.1 and 4.1]. (See also [Cra].) Here, ∂D is the Euclidean boundary of D . We note that the fact that D is Lipschitz is not crucial for the first two hypotheses. For an arbitrary domain one could take the Martin boundary. The corresponding Martin kernel would satisfy (H_1) and (H_2) . Regularity of D is needed only for (H_3) . That (H_3) holds for a Lipschitz domain follows from Lemma 2.5 in [Caf].

In the general case $q \neq 0$, we must choose q such that $\mathcal{H}^{\text{inf}}$ does not consist of the zero function only. We require that q has a finite gauge. This condition is usually expressed using probabilistic notions. An equivalent analytic condition is that there exists $u \in \mathcal{H}^{\text{inf}}$ with $\inf_D u > 0$ (see [Cra, Theorem 2.23]). With such q there exists the kernel function K_L on $D \times \partial D$ satisfying (H_1) – (H_3) [Cra, Theorem 5.5]. K_L can be expressed in terms of the kernel function K for the operator A and the conditional gauge F (the interested reader is referred to [Cra]).

Let $k(x, y) = \inf\{u(x) : u \in \mathcal{H}^{\text{inf}}, u(y) = 1\}$. From Theorem 3.3 it follows that k suffices to recover all positive solutions of $Lu = 0$.

5. SYMMETRY OF k

In this section we consider positive harmonic functions for the Laplacian. Let us recall the formula for k for the unit ball in $\mathbf{R}^n$ given by (1.4):

$$k(x, y) = (1 - |y|^2)^{n-2} \frac{(1 - |x|^2)(1 - |y|^2)}{(|y||x - y^*| + |x - y|)^n}.$$

Since $|y||x - y^*| = |x||y - x^*|$, it follows that k is symmetric if the dimension is $n = 2$, while for $n \geq 3$ it is not symmetric. For $n = 2$ we identify $\mathbf{R}^2$ with the complex plane $\mathbf{C}$ and let B be the unit disc. If D is a simply-connected region in $\mathbf{C}$ (with at least two boundary points), then there is a conformal mapping $w : D \rightarrow B$. Let $w = u + iv$. For $f : B \rightarrow \mathbf{R}$ of class $\mathcal{C}^2$ we define $g : D \rightarrow \mathbf{R}$ by $g(x) = f(w(x))$, $x \in D$. Then $g \in \mathcal{C}^2(D)$ and $\Delta g(x) = |\nabla u(x)|^2 (\Delta f)(w(x))$ (e.g. [Rao, 6.19]). If f is positive and harmonic in B , i.e., $f \in \mathcal{H}(B)$, then g is positive and harmonic in D , i.e., $g \in \mathcal{H}(D)$. Let $\tilde{k}(x, y) = \inf\{g(x) : g \in \mathcal{H}(D), h(y) = 1\}$, $x, y \in D$. Then $\tilde{k}(x, y) = \inf\{f(w(x)) : f \in \mathcal{H}(B), f(w(y)) = 1\} = k(w(x), w(y))$. This shows that $\tilde{k}$ is symmetric.

Let D be a Lipschitz, simply-connected domain and denote $\tilde{k}$ simply by k . Let us define the function l on $D \times D$ in the same way as k only replacing infimum by supremum:

$$l(x, y) = \sup_{h \in \mathcal{H}} \frac{h(x)}{h(y)}.$$

It is easy to see that $l(x, y) = 1/k(y, x)$ (note the change of x and y). Since k is symmetric, $l(x, y) = 1/k(x, y)$. Let K be the kernel function for $\mathcal{H}(D)$ based at x_0 . Then for $z \in \partial D$,

$$\lim_{y \rightarrow z} \frac{l(x, y)}{l(x_0, y)} = \lim_{y \rightarrow z} \frac{k(x_0, y)}{k(x, y)} = \frac{1}{K(x, z)}.$$

6. EXAMPLE

We conclude with two simple results for infima of harmonic functions on a bounded domain D in $\mathbf{R}^n$. By $\mathcal{H}(D)$ we denote the cone of positive harmonic functions on D , and let G be the Green function for D .

Proposition 6.1. *Let D be star-shaped and let $u = G\mu$ be a potential from $\mathcal{H}^{\text{inf}}(D)$. Then μ cannot have a compact support.*

Proof. Assume that $\text{Supp } \mu$ is compact. Let U be a relatively compact open set containing $\text{Supp } \mu$ such that $D \setminus U$ is connected. Let x_0 be any point from $D \setminus U$. By Lemma 2.2, there is $h \in \mathcal{H}(D)$ such that $h \geq u$ in D and $h(x_0) = u(x_0)$. Then the function $h - u$ is nonnegative and harmonic in $D \setminus U$ and $(h - u)(x_0) = 0$. Since $D \setminus U$ is connected, $h - u = 0$ in $D \setminus U$. By continuity, $h = u$ on ∂U . Since u is superharmonic, $u \geq h$ in U . Therefore, $u = h$ in D , which contradicts the fact that u is a potential. $\square$

Now we give an interesting example of a function in $\mathcal{H}^{\text{inf}}(D)$. Let τ_D denote the exit time from D of the n -dimensional Brownian motion (X_t, P^x) and let E^x denote the expectation with respect to P^x .

Proposition 6.2. *Let $\phi(x) = E^x(\tau_D)$. Then $\phi \in \mathcal{H}^{\text{inf}}(D)$.*

Proof. First we establish this result for the ball $B = B(0, r)$. Let $\tau_B = \inf\{t > 0 : X_t \notin B\}$. Then $E^x(\tau_B) = (r^2 - |x|^2)/n$ for $x \in B$ (e.g. [Rao, 4.6]). The function $x \mapsto r^2 - |x|^2$ is concave on B , and therefore in $\mathcal{H}^{\text{inf}}(B)$. For the general domain D , let $B = B(0, r)$ be a ball containing D . Let

$\phi(x) = E^x(\tau_D)$, $x \in D$, and $\psi(x) = E^x(\tau_B)$, $x \in B$. It is well known that $\Delta\phi = -1$ in D , and $\Delta\psi = -1$ in B . Hence, $\Delta(\psi - \phi) = 0$ in D , so $\psi - \phi$ is harmonic in D . Obviously, $\phi \leq \psi$ in D . If $h = \psi - \phi$, then h is strictly positive and harmonic. Since $\psi \in \mathcal{H}^{\text{inf}}(B)$, certainly $\psi \in \mathcal{H}^{\text{inf}}(D)$. Hence, $\psi = \inf_{\alpha} h_{\alpha}$, h_{α} harmonic and positive in D . But then $\phi = \psi - h = \inf_{\alpha} (h_{\alpha} - h)$ and each $h_{\alpha} - h$ is positive and harmonic in D . $\square$

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REFERENCES

- [Ahl] L. V. Ahlfors, *Möbius transformation in several dimensions*, Ordway Lectures in Mathematics, University of Minnesota, 1981.
- [Caf] L. Caffarelli, E. Fabes, S. Mortola, and S. Salsa, *Boundary behaviour of nonnegative solutions of elliptic operators in divergence form*, Indiana J. Math. **30** (1981), 621–640.
- [Chi] F. Chiarenza, E. Fabes, and N. Garofalo, *Harnack's inequality for Schrödinger operators and continuity of solutions*, Proc. Amer. Math. Soc. **98** (1986), 415–425.
- [Cra] M. Cranston, E. Fabes, and Z. Zhao, *Conditional gauge and potential theory for Schrödinger operator*, Trans. Amer. Math. Soc. **307** (1988), 171–194.
- [Doo] J. L. Doob, *Classical potential theory and its probabilistic counterpart*, Springer, New York and Heidelberg, 1984.
- [Hun] R. A. Hunt and R. L. Wheeden, *Positive harmonic functions on Lipschitz domains*, Trans. Amer. Math. Soc. **147** (1970), 507–527.
- [Leu] H. Leutwiler, *On a distance invariant under Möbius transformation in $\mathbf{R}^n$* , Ann. Acad. Sci. Fenn. Ser. A I Math. **12** (1987), 3–17.
- [Mar] R. S. Martin, *Minimal positive harmonic functions*, Trans. Amer. Math. Soc. **49** (1941), 137–172.
- [Rao] M. Rao, *Brownian motion and classical potential theory*, Lecture Notes Ser., vol. 667, Aarhus Univ., Aarhus, 1977.

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