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MODULAR FORMS OF WEIGHT $\frac{1}{2}$ DEFINED ON PRODUCTS OF p-ADIC UPPER HALF-PLANES

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ABSTRACT. We continue Stark's study of modular forms defined on products of p-adic upper half-planes. Specifically, we restrict to the case of the number field $\mathbb{Q}$ and one finite prime. In this setting we develop a multiplier system for modular forms of weight $\frac{1}{2}$, and provide an example of such a form.

1. INTRODUCTION

Historically, the relationship between modular forms and Dirichlet series defined over the rational numbers has played a crucial role in number theory. For example, the functional equation of the Riemann zeta function can be derived using a modular form of weight one-half, a classical theta function. A more systematic study of the correspondence between modular forms and Dirichlet series was begun by Hecke in the 1930s. More recently (1967), Weil introduced an extension of this correspondence to Hilbert modular forms and Dirichlet series defined over number fields. The Taniyama-Weil conjecture, which relates elliptic curves and modular forms, extends this correspondence even further (see [Te] for more information). These advancements of the Hecke theory have also occupied a critical position in the study of number theory.

Unfortunately, the relationship between modular forms and Dirichlet series is difficult to extend to number fields, particularly if the class number of the number field is not equal to one. One response to this difficulty involves the use of adeles (see [We] for example). Another approach to this problem is to introduce a ring larger than the ring of integers of the number field. Given a finite set of odd primes, S, including the infinite prime, we take this larger ring to be the ring of S-integers. By defining S appropriately, one can force the ring of S-integers to have class number one. Stark introduced this method of attack in [St1], where he used it to define a new type of modular form of integral weight. Rhodes continued the study of this approach as it pertains to integral weight modular forms in his thesis [Rh]. In this paper, we begin to consider modular forms of half-integral weight for the number field $K = \mathbb{Q}$ and $S = \{\infty, p\}$ by finding the appropriate theta multiplier, and generating an example of such a form.

2. **Background.** We start by introducing the terminology that will be used throughout the paper. In particular, an upper half-plane must be constructed,

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as well as an appropriate discrete group, both of which must depend on our set of primes. We begin by assuming that $\mathbb{Q}$ is the number field and that S is a finite set of primes of $\mathbb{Z}$, including the infinite prime and excluding the prime $p = 2$, say $S = \{\infty, p_1, p_2, \ldots, p_n\}$.

Given the set S, we will need to define two types of numbers that depend on the primes in S. First, we can define an S-integer to be a rational number m/n such that the only primes dividing n are the finite primes in S. In particular, if $S = \{\infty, p\}$, then an S-integer is a rational number of the form mp^j, where m and j are in $\mathbb{Z}$. For a general S, an S-integer is a number of the form $m \cdot \prod_{i=1}^{n} p_i^{j_i}$, where the m and j_i are in $\mathbb{Z}$. We denote the ring of S-integers by $\mathfrak{O}(S)$, and the units of $\mathfrak{O}(S)$, the S-units, by $\mathfrak{O}(S)^{\times} = \{\pm \prod_{p \in S} p^{j_p} | j_p \in \mathbb{Z}\}$.

In addition to the S-integers, we will need to make use of the p-adic numbers, denoted by $\mathbb{Q}_p$, and the p-adic integers, denoted by $\mathbb{Z}_p$. We will take $|\cdot|_p$ to be the valuation on $\mathbb{Q}_p$ normalized by $|p|_p = \frac{1}{p}$.

In order to get a better feel for these different types of numbers, we need to have a better way of describing the numbers and their properties. In particular, we need to define some sort of norm and trace. We will use the definitions given in [St1].

We begin by giving Stark's definition of an absolute norm or S-norm that can be used for S-integers [St1]. Given a rational number α, we define

$$\mathbb{N}(\alpha) = \mathbb{N}_S(\alpha) = \alpha \cdot \prod_{\substack{p \in S \\ p \neq \infty}} |\alpha|_p.$$

This norm does everything a norm should do; $\mathbb{N}(\alpha)$ is always a rational number, $\mathbb{N}(\alpha) = 0$ if and only if $\alpha = 0$, and α is an S-unit if and only if $\mathbb{N}(\alpha) = \pm 1$. It is also multiplicative. We can extend this norm to a direct product $\prod_{p \in S} \mathbb{Q}_p$, where $\mathbb{Q}_\infty = \mathbb{R}$. In particular, if $z = (z_p)_{p \in S}$ is in $\prod_{p \in S} \mathbb{Q}_p$, then we define

$$\mathbb{N}(z) = z_\infty \cdot \prod_{\substack{p \in S \\ p \neq \infty}} |z_p|_p.$$

Next, we will define an absolute trace, or S-trace. Let $x = (x_p)_{p \in S}$ be a vector in $\prod_{p \in S} \mathbb{Q}_p$. Since all the x_p's are in different p-adic fields, it is not possible to define $\operatorname{tr}(x)$ by simply adding the components of x. Instead we will combine them $(\operatorname{mod} 1)$. In order to accomplish this for a finite prime, p, we denote by $\operatorname{tr}_p(x_p)$ the fractional part of x_p. Therefore $x_p - \operatorname{tr}_p(x_p)$ is in $\mathbb{Z}_p$. For $p = \infty$, we let $\operatorname{tr}_\infty(x_\infty) = x_\infty$, a real number. Now we can define our S-trace by

$$\operatorname{tr}(x) = \operatorname{tr}_S(x) = \operatorname{tr}_\infty(x_\infty) - \sum_{\substack{p \in S \\ p \neq \infty}} \operatorname{tr}_p(x_p).$$

This S-trace has several interesting properties. In particular, we will need that it is additive, and that a "trace formula" exists to partially describe its behavior.

Lemma 2.1. *If a and b are in $\prod_{p \in S} \mathbb{Q}_p$, then*

$$\operatorname{tr}(a + b) \equiv \operatorname{tr}(a) + \operatorname{tr}(b) \pmod 1.$$

The "trace formula", which appears in [St1], addresses the question of whether or not the trace of an S-integer is actually a rational integer, as it should be.

Lemma 2.2 (the trace formula, preliminary version). *Let ν be an S-integer. Then* $\operatorname{tr}(\nu) \equiv 0 \pmod 1$.

3. The upper half-plane and the discrete group

In this section we want to generalize the Poincaré upper half-plane, $\mathfrak{h}_{\mathbb{R}} = \mathfrak{h}_\infty = \{x_\infty + iy_\infty | x_\infty, y_\infty \in \mathbb{R}, \text{ and } y_\infty > 0\}$. We have altered the standard notation slightly here, to distinguish it from the other upper half-planes with which we shall deal. There are two generalizations that we will consider. In each we will replace the real numbers, considered as the completion of $\mathbb{Q}$ with respect to the valuation $|\cdot|_\infty$, with $\mathbb{Q}_p$, the completion of $\mathbb{Q}$ with respect to the valuation $|\cdot|_p$.

First, given a nonsquare unit Δ_p in $\mathbb{Q}_p$, we define the upper half-plane

$$\mathfrak{h}_p = \{x_p + y_p\sqrt{\Delta_p} | x_p, y_p \in \mathbb{Q}_p, \text{ and } y_p \neq 0\}.$$

Unfortunately, this is not a true "upper" half-plane, but rather a union of upper and lower half-planes.

To get around this inconsistency we will take advantage of the fact that saying y_∞ is greater than zero is equivalent to saying that y_∞ is in $(\mathbb{R}^\times)^2 = \{r^2 | r \in \mathbb{R}^\times\}$. The analogy that we will use in the p-adic case is to require that $\operatorname{ord}_p(y_p) \equiv 0 \pmod 2$. While this is not an exact paralleling of the infinite case, it does provide us with a nice space to use. In particular, we define

$$\mathfrak{h}_p^0 = \{x_p + y_p\sqrt{\Delta_p} | x_p, y_p \in \mathbb{Q}_p, y_p \neq 0 \text{ and } \operatorname{ord}_p(y_p) \equiv 0 \pmod 2\}.$$

Associated to each $x_p + y_p\sqrt{\Delta_p} \in \mathfrak{h}_p^0$ we assign the notation $\operatorname{ord}_p x_p = -l$ and $\operatorname{ord}_p y_p = -2j$. This notation will be used extensively in §4.

Now we return to our set S. We define two direct products of upper half-planes depending on the primes in S:

$$\mathfrak{h}_S = \prod_{p \in S} \mathfrak{h}_p \quad \text{and} \quad \mathfrak{h}_S^0 = \mathfrak{h}_{\mathbb{R}} \times \prod_{\substack{p \in S \\ p \neq \infty}} \mathfrak{h}_p^0.$$

The next issue to consider is what matrix groups will act on $\mathfrak{h}_S$ and $\mathfrak{h}_S^0$. We bring the ring of S-integers back into the picture at this time by introducing two subgroups of the group $G = GL_2(\mathfrak{O}(S))^+$, the group of two-by-two matrices with entries in $\mathfrak{O}(S)$ and positive determinant. We define

$$\Gamma_S = SL_2(\mathfrak{O}(S)) = \{A \in G | \det(A) = 1\}$$

and

$$\Gamma_S^0 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_S | c \equiv 0 \pmod 4 \right\},$$

where $c \equiv 0 \pmod 4$ means that 4 divides c in $\mathfrak{O}(S)$. Note that any matrix in Γ_S^0 can be written as a product of the following matrices:

$$\begin{pmatrix} 0 & -1/2 \\ 2 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1/p \\ 0 & 1 \end{pmatrix}^{\pm 1} \quad \text{and} \quad \begin{pmatrix} p & 0 \\ 0 & p^{-1} \end{pmatrix}^{\pm 1}.$$

The group Γ acts on $\mathfrak{h}_S$, and the group Γ_S^0 acts on $\mathfrak{h}_S^0$, by linear fractional transformation. If $z = (z_\infty, z_{p_1}, \ldots, z_{p_n})$ is in $\mathfrak{h}_S$ (or $\mathfrak{h}_S^0$), and

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is in Γ (or Γ_S^0), then we define

$$A \circ z = \left(\frac{az_\infty + b}{cz_\infty + d}, \frac{az_{p_1} + b}{cz_{p_1} + d}, \ldots, \frac{az_{p_n} + b}{cz_{p_n} + d} \right).$$

We will be especially interested in the action of Γ_S^0 on $\mathfrak{h}_S^0$, and from this point on we will concern ourselves primarily with this action.

4. Modular forms

Now that the scene has been set, we must specify exactly what is meant by a modular form in this setting. For the remainder of this paper, unless otherwise specified, we will assume that we only have one p-adic upper half-plane; that is, $S = \{\infty, p\}$.

We begin by considering modular forms of integral weight on $\mathfrak{h}_S$. In [St1] Stark defined modular forms of integral weight k on the upper half-plane $\mathfrak{h}_S$, as follows.

Definition 4.1. A function $f: \mathfrak{h}_S \to \mathbb{C}$ is called a modular form of integer weight k for the group G on the product upper half-space $\mathfrak{h}_S$ if for z in $\mathfrak{h}_S$

$$f(A \circ z) = \mathbb{N}(cz + d)^k f(z) \quad \text{for all } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ in } G.$$

Stark generates an example of such a form in the same paper; if $f(z_\infty)$ is a classical modular form of integral weight k, then the function $F(z) = f((z_\infty - x_p)/p^j)$, where $z \in \mathfrak{h}_S$, $|y_p|_p = p^{-j}$, and x_p is taken to be in $\mathbb{Z}[p^{-1}] \pmod{p^j}$, is a modular form of weight k for G acting on $\mathfrak{h}_S$.

To define modular forms of weight $1/2$ we will use instead the upper half-plane $\mathfrak{h}_S^0$. Then, in trying to generalize the classical definition of modular forms of weight $\frac{1}{2}$, we expect that the transformation rule will involve some sort of theta multiplier. Recall that the theta multiplier (see [Sh]) for z in $\mathfrak{h}_\mathbb{R}$ and for $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $\Gamma_0(4)$ is defined as

$$j(A, z) = \begin{cases} \varepsilon_d^{-1} \left(\frac{c}{d}\right)(cz + d)^{1/2} & \text{if } c \neq 0, \\ 1 & \text{if } c = 0, \end{cases}$$

where

$$\varepsilon_d = \begin{cases} 1 & \text{if } d \equiv 1 \pmod 4 \\ i & \text{if } d \equiv 3 \pmod 4, \end{cases}$$

$\left(\frac{c}{d}\right)$ is the Kronecker symbol, and we are taking the principal value of the square root. This multiplier system will not work in the new setting because the entries of our matrices are not necessarily integers, but S-integers.

In the p-adic setting we seek a transformation rule of the form

$$f(A \circ z) = \chi_S(A, z_p)\mathbb{N}(cz + d)^{1/2} f(z),$$

where $\chi_S(A, z_p)$ is an eighth root of unity. With this end in mind, we first define a theta function on $\mathfrak{h}_S^0$. This function will ultimately be used to give us an example of a weight $\frac{1}{2}$ form on $\mathfrak{h}_S^0$, as well as the exact multiplier system. To make this definition, we will need to have a way of combining p-adic numbers and real numbers in an exponent. We therefore define for $z = (z_\infty, z_p) \in \mathfrak{h}_S^0$:

$$W_\infty(y_\infty) = \exp(-\pi y_\infty), \qquad W_p(y_p) = \begin{cases} 1 & \text{if } y_p \in \mathbb{Z}_p, \\ 0 & \text{otherwise}, \end{cases}$$

$$e_\infty(x_\infty) = \exp(\pi i x_\infty), \quad \text{and} \quad e_p(x_p) = \exp(-\pi i x_p),$$

where, by an abuse of notation, x_p is taken to be equal to the fractional part of x_p. We also take the fractional part of x_p so that the numerator is even.

We then combine these expressions so that the theta function will have a term dependent on the "real" part of the variable z, and a term dependent on the imaginary part of z. Define

$$W(y) = W_p(y_p) \times W_\infty(y_\infty) \quad \text{and} \quad e(x) = e_p(x_p) \times e_\infty(x_\infty).$$

Now, we use the functions defined above to define a theta function. For $z \in \mathfrak{h}_S^0$, and $u = (u_\infty, u_p)$, $v = (v_\infty, v_p) \in \mathbb{C} \times \mathbb{Q}_p$, let

$$\Theta\left(z, \begin{pmatrix} u \\ v \end{pmatrix}\right) = \sum_{n \in \mathbb{Z}[\frac{1}{p}]} W((n+v)^2 y) e((n+v)^2 x - 2nu - uv). \tag{4.1}$$

This function will also be denoted by $\Theta(z, u, v)$. There are several things to notice about this definition. First, it is reasonable to call it a theta function, as it closely resembles a symplectic theta function. The extra term involving x_p can be combined with the term involving x_∞ to give a single term with a trace in the exponent, just as one would expect of a theta function defined over a number field. In fact, $e(n^2 x)$ can be written as $\exp(\pi i \operatorname{tr}(n^2 x))$.

We are now ready to determine the transformation properties of this theta function. In particular, we want to know what the relationship is between $\Theta(A \circ z, A\binom{u}{v})$ and $\Theta(z, \binom{u}{v})$ for A in the subgroup Γ_S^θ of $SL(2, \mathbb{Z}[\frac{1}{p}])$ consisting of all matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ such that $b \equiv c \equiv 0 \pmod 2$. In analogy with the classical theta function, we hope that there are nice transformation formulas for the theta function under translation and inversion of the variable z. These properties are developed in the following theorem of Stark [St2], which gives a transformation rule similar to that for the classical theta function.

Theorem 4.2. *For*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_S^\theta, \qquad z \in \mathfrak{h}_S^0, \text{ and } u, v \in \mathbb{C} \times \mathbb{Q}_p,$$

$\Theta(z, u, v)$ *satisfies the following*:

$$\Theta\left(A \circ z, A\begin{pmatrix} u \\ v \end{pmatrix}\right) = \chi \cdot [\mathbb{N}(cz + d)]^{1/2} \Theta\left(z, \begin{pmatrix} u \\ v \end{pmatrix}\right), \tag{4.2}$$

where χ *is an eighth root of unity depending on* A *and* z.

Proof. Every matrix in Γ_S^θ can be written as a product of the following matrices:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} p & 0 \\ 0 & \frac{1}{p} \end{pmatrix}^{\pm 1} \quad \text{and} \quad \begin{pmatrix} 1 & 2t \\ 0 & 1 \end{pmatrix}$$

where $t \in \mathbb{Z}[\frac{1}{p}]$. Note that while $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ is not in the group Γ_S^θ, it is a generator of a group containing Γ_S^θ. Therefore it is enough to check that (4.2) holds for these matrices.

We begin this process by considering $A = \left(\begin{smallmatrix} 1 & 2t \\ 0 & 1 \end{smallmatrix}\right)$. The action of A only affects x and u. Therefore, $W((n+v)^2 y)$ is unchanged. The part of the summand involving x, $e((n+v)^2 x - 2nu - uv)$, is replaced by

$$e((n+v)^2(x+2t) - 2n(u+2tv) - (u+2tv)v).$$

But, with a little algebra and Lemma 2.2, this expression simplifies to $e((n+v)^2 x - 2nu - uv)$. Thus, $\Theta(A \circ z, A\binom{u}{v}) = \Theta(z, \binom{u}{v})$. Also, notice that for this matrix, $\mathbb{N}(cz+d) = 1$, so with $\chi = 1$, (4.2) holds.

Next, we let

$$A = \begin{pmatrix} p & 0 \\ 0 & \frac{1}{p} \end{pmatrix}^{\pm 1}.$$

The effect of these actions is that z is replaced by $p^{\pm 2} z$, and $\binom{u}{v}$ by $\binom{pu}{v/p}$ or $\binom{u/p}{vp}$. In either case, it is easy to see that $\Theta(A \circ z, A\binom{u}{v}) = \Theta(z, \binom{u}{v})$. Thus, since $\mathbb{N}(cz+d) = 1$, (4.2) holds with $\chi = 1$.

Finally, we consider inversion (which is significantly more complicated). We begin by rewriting $\Theta(z, u, v)$ using Poisson summation:

$$\Theta\left(z, \binom{u}{v}\right) = \sum_{n \in \mathbb{Z}[\frac{1}{p}]} \int_{\mathbb{R} \times \mathbb{Q}_p} W((t+v)^2 y) e((t+v)^2 x - 2tu - uv - 2tn)\, dt,$$

where $dt = dt_\infty dt_p$, and the measure is normalized so that $\int_{\mathbb{Z}_p} dt_p = 1$. With the change of variable $t \to (t - v)$, we can rewrite the integral as

$$\begin{aligned} \Theta\left(z, \binom{u}{v}\right) &= \sum_{n \in \mathbb{Z}[\frac{1}{p}]} \int_{\mathbb{R} \times \mathbb{Q}_p} W(t^2 y) e(t^2 x - 2(t-v)u - uv - 2(t-v)n)\, dt \\ &= \sum_{n \in \mathbb{Z}[\frac{1}{p}]} \left[e(uv + 2vn) \int_{\mathbb{R} \times \mathbb{Q}_p} W(t^2 y) e(t^2 x - 2t(u+n))\, dt \right]. \end{aligned}$$

Now we want to consider the real piece of this integral:

$$\int_{\mathbb{R}} W_\infty(t_\infty^2 y_\infty) e^{\pi i (t_\infty^2 x_\infty - 2t_\infty(u_\infty + n))}\, dt_\infty.$$

One can complete the square in the exponent of the integrand, and also use the fact that $W_\infty(t_\infty^2 y_\infty)$ is its own Fourier transform to simplify this integral to

$$\int_{\mathbb{R}} W_\infty(t_\infty^2 y_\infty) e^{\pi i (t_\infty^2 x_\infty - 2t_\infty(u_\infty + n))}\, dt_\infty = \frac{1}{\sqrt{-i z_\infty}} e^{\pi i (n + u_\infty)^2 (-1/z_\infty)}.$$

Next, we must consider the p-adic piece of the integral,

$$\int_{\mathbb{Q}_p} W_p(t_p^2 y_p) e_p(t_p^2 x_p - 2t_p(u_p + n))\, dt_p.$$

Before we begin, recall the notation that we have been using: $z_p = x_p + y_p \Delta_p$, and $|y_p|_p = p^{2j}$. Also, since we will be working with this integral for quite a while before it is simplified, we drop most of the p-adic subscripts.

Then the change of variable $t \to p^j t$ enables us to simplify the integral:

$$\int_{\mathbb{Q}_p} W_p(t^2 y) e_p(t^2 x - 2t(u+n))\, dt$$
$$= \frac{1}{p^j} \int_{\mathbb{Z}_p} e_p(t^2 p^{2j} x - 2tp^j(u+n))\, dt.$$

At this point, we must break into two cases; we will find that the integral simplifies differently depending on whether $|x_p|_p \le |y_p|_p$, or $|x_p|_p > |y_p|_p$.

In the first case, because $t^2 p^{2j} x \in \mathbb{Z}_p$, the integral simplifies to

$$\frac{1}{p^j} \int_{\mathbb{Z}_p} e_p(-2tp^j(u+n))\, dt = \frac{1}{p^j} \begin{cases} 1 & \text{if } p^j(u+n) \in \mathbb{Z}_p, \\ 0 & \text{otherwise.} \end{cases}$$

The next step in this case is to take this evaluation of the p-adic integral, and combine it with the real integral inside a sum over $\mathbb{Z}[\frac{1}{p}]$. Before we are able to do this successfully, we must consider $-1/z_p$ a little more carefully. We can write

$$-\frac{1}{z_p} = \frac{-x_p}{x_p^2 - y_p^2 \Delta_p} + \frac{y_p}{x_p^2 - y_p^2 \Delta_p} \sqrt{\Delta_p} = r + s\sqrt{\Delta_p},$$

where $|r|_p \le |s|_p$ and $|s|_p = p^{-2j}$. Therefore we can replace the integral $(1/p^j) \int_{\mathbb{Z}_p} e_p(-2tp^j(u+n))\, dt$ by $(1/p^j) W_p((u+n)^2 s)$. Also, notice that $e_p((u+n)^2 r) = 1$ whenever $w_p((u+n)^2 s) \ne 0$.

Regrouping, we now have

$$\Theta\left(z, \begin{pmatrix} u \\ v \end{pmatrix}\right) = \sum_{n \in \mathbb{Z}[\frac{1}{p}]} \left[e(uv + 2vn) \frac{1}{\sqrt{-iz_\infty}} e^{\pi i (n+u_\infty)^2 (-1/z_\infty)} \right.$$
$$\left. \times \frac{1}{p^j} W_p((u_p+n)^2 s) e_p((u_p+n)^2 r) \right]$$
$$= \frac{1}{\sqrt{-iz_\infty}} \frac{1}{p^j} \Theta\left(\frac{-1}{z}, \begin{pmatrix} -v \\ u \end{pmatrix}\right).$$

Finally, since $(1/\sqrt{-iz_\infty})1/p^j$ is equal to an eighth root of unity times $[\mathbb{N}(z)]^{-1/2}$, we have shown that (4.2) holds when $|x_p|_p \le |y_p|_p$.

Now, we must consider the second case, when $|x_p|_p > |y_p|_p$. We recall the convention that $|x_p|_p = p^l$. Again, we look at the p-adic integral,

$$\frac{1}{p^j} \int_{\mathbb{Z}_p} e_p(t^2 p^{2j} x - 2tp^j(n+u))\, dt.$$

Expanding $(t + p^{l-2j})^2$, we see that $t^2 p^{2j} x \equiv (t + p^{l-2j})^2 p^{2j} x \pmod 1$. It is also true that $2(t + p^{l-2j}) p^j (n+u) = 2tp^j(n+u) + 2p^{l-j}(n+u)$. Applying these relations, as well as the change of variable $t \to (t + p^{l-2j})$, we rewrite the integral as follows:

$$\frac{1}{p^j} \int_{\mathbb{Z}_p} e_p(t^2 p^{2j} x - 2tp^j(n+u))\, dt$$
$$= e_p(-2p^{l-j}(n+u)) \frac{1}{p^j} \int_{\mathbb{Z}_p} e_p(t^2 p^{2j} x - 2tp^j(n+u))\, dt.$$

The only way that this equality can hold is if $e_p(-2p^{l-j}(n+u))$ is equal to one, or the integral is equal to zero. However, this integral is not identically zero. Therefore we need only consider the first possibility. Using the fact that if $-1/z_p = r + s\sqrt{\Delta_p}$, then $|s|_p = 2j - 2l$, we see that $e_p(-2p^{l-j}(n+u)) = 1$ if and only if $W_p(s(n+u)^2) = 1$. Returning to the integral, we now have

$$\frac{1}{p^j}\int_{\mathbb{Z}_p} e_p(t^2p^{2j}x - 2tp^j(n+u))\,dt$$
$$= e_p\left(-\frac{1}{x}(n+u)^2\right)\frac{1}{p^j}\int_{\mathbb{Z}_p} e_p\left(\frac{1}{x}[tp^jx - (n+u)]^2\right)dt.$$

Next we would like to bring r back into the picture. With this in mind, we notice that

$$r = -\frac{1}{x}\left(\frac{x^2}{x^2 - y^2\Delta_p}\right) = -\frac{1}{x}\left(\frac{1}{1 - (\frac{y}{x})^2\Delta_p}\right)$$
$$= -\frac{1}{x}\left[1 + \left(\frac{y}{x}\right)^2\Delta_p + \cdots\right] = -\frac{1}{x} - \frac{1}{x}(I),$$

where I is a p-adic integer of order $2l - 4j$. Therefore we can replace $e_p(-\frac{1}{x}(n+u)^2)$ by $e_p(r(n+u)^2)$, as $-\frac{1}{x}I(n+u)^2 \in \mathbb{Z}_p$. Thus, the p-adic integral has been reduced to

$$e_p(r(n+u)^2)\frac{1}{p^j}\int_{\mathbb{Z}_p} e_p\left(\frac{1}{x}[tp^jx - (n+u)]^2\right)dt.$$

We continue by isolating the integral; we rearrange the powers of p, and use the fact that $p^lx \in \mathbb{Z}_p^\times$, and $p^{l-j}(n+u) \in \mathbb{Z}_p$ to get

$$\frac{1}{p^j}\int_{\mathbb{Z}_p} e_p\left(\frac{1}{x}[tp^jx - (n+u)]^2\right)dt = \frac{1}{p^j}\int_{\mathbb{Z}_p} e_p\left(\frac{1/x}{p^{2l-2j}}t^2\right)dt$$
$$= \frac{1}{p^j}\int_{\mathbb{Z}_p} e_p\left(\frac{1/p^lx}{p^{l-2j}}t^2\right)dt.$$

Notice that we have eliminated the dependence on n.

To simplify the integral even further, we replace the integral by a sum. To do this, we notice that $e_p((1/p^lx)t^2/p^{l-2j})$ depends only on t modulo p^{l-2j}. Therefore we can replace the integral by a sum over congruences classes, taking into account that the measure of each congruence class is $1/p^{l-2j}$. This gives us

$$\frac{1}{p^j}\int_{\mathbb{Z}_p} e_p\left(\frac{1/p^lx}{p^{l-2j}}t^2\right)dt = \frac{1}{p^j}\sum_{a(\mathrm{mod}\,p^{l-2j})} e_p\left(\frac{1/p^lx}{p^{l-2j}}a^2\right)\frac{1}{p^{l-2j}}.$$

We would also like to eliminate the p-adic presence so that we will be dealing solely with rational numbers in the exponent. Therefore we approximate the p-adic element in the exponent; assume that $1/p^lx \equiv -2b \pmod{p^{l-2j}}$, where $b \in \mathbb{Z}$. Then $e_p((1/p^lx)a^2/p^{l-2j}) = e^{(2\pi iba^2/p^{l-2j})}$. Now we have simplified the integral to the simple sum

$$\frac{1}{p^{l-j}}\sum_{a(\mathrm{mod}\,p^{l-2j})} e^{(2\pi iba^2/p^{l-2j})}.$$

This sum can be reduced using standard Gaussian sum reductions:

$$\begin{aligned}
&\frac{1}{p^{l-j}} \sum_{a(\bmod p^{l-2j})} e^{(2\pi i b a^2/p^{l-2j})} \\
&\quad = \frac{1}{p^{l-j}} \begin{cases} p^{(l-2j)/2} & \text{if } l \text{ is even} \\ p^{(l-2j-1)/2} \displaystyle\sum_{a(\bmod p)} e^{2\pi i b a^2/p} & \text{if } l \text{ is odd} \end{cases} \\
&\quad = \frac{1}{p^{l/2}} \begin{cases} 1 & \text{if } l \text{ is even} \\ (\frac{b}{p})\varepsilon_p & \text{if } l \text{ is odd} \end{cases}
\end{aligned}$$

where

$$\varepsilon_p = \begin{cases} 1 & \text{if } p \equiv 1 \ (\bmod 4) \\ i & \text{if } p \equiv 3 \ (\bmod 4), \end{cases}$$

and $(\frac{b}{p})$ is the Kronecker symbol.

Finally, we combine all the pieces to obtain

$$\begin{aligned}
\Theta\left(z, \begin{pmatrix} u \\ v \end{pmatrix}\right) &= \sum_{n \in \mathbb{Z}[\frac{1}{p}]} \frac{1}{\sqrt{-iz_\infty}} e^{\pi i (n+u_\infty)^2(-1/z_\infty)} e(uv + 2vn) \\
&\qquad \times e_p(r(n+u_p)^2) W_p(s(n+u_p)^2) \frac{1}{p^{l/2}} \begin{cases} 1 & \text{if } l \text{ is even} \\ (\frac{b}{p})\varepsilon_p & \text{if } l \text{ is odd} \end{cases} \\
&= \frac{1}{\sqrt{-iz_\infty}} \frac{1}{p^{l/2}} \left[\begin{cases} 1 & \text{if } l \text{ is even} \\ (\frac{b}{p})\varepsilon_p & \text{if } l \text{ is odd} \end{cases} \right] \sum_{n \in \mathbb{Z}[\frac{1}{p}]} e^{\pi i (n+u_\infty)^2(-1/z_\infty)} \\
&\qquad \times e_p(r(n+u_p)^2) W_p(s(n+u_p)^2) e(uv + 2vn) \\
&= \chi \frac{1}{[\mathbb{N}(z)]^{1/2}} \Theta\left(\frac{-1}{z}, \begin{pmatrix} -v \\ u \end{pmatrix}\right).
\end{aligned}$$

This completes the proof. □

Now that we have established the existence of some sort of "theta multiplier", we would like to be able to write it down explicitly. Once this has been accomplished, we will be able to define modular forms in this p-adic setting. We will also have an example of such a form, so it will not be a totally vacuous definition.

As a preliminary step, we must define a slightly different theta function, a "$2\pi i$" version. This is necessary so that we can have the group Γ_S^0, rather than Γ_S^θ, acting on $\mathfrak{h}_S^0$. The transition is quite straightforward; we will consider, for $z \in \mathfrak{h}_S^0$,

$$\vartheta(z) = \Theta\left(2z, \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right).$$

Then the action of $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_S^0$ on z is equivalent to the action of $\left(\begin{smallmatrix} a & 2b \\ c/2 & d \end{smallmatrix}\right) \in \Gamma_S^\theta$ on $2z$.

It is at this point that the impact of the p-adic variable becomes most apparent, and the theory begins to diverge from the classical theory. As we noticed in the proof of the previous theorem, the relative 'sizes' of x_p and y_p are very important. This will be quite noticable in the development of the multiplier system, as presented in the proof of the theorem below. First, though, we require a technical lemma concerning quadratic Gauss sums.

Lemma 4.3. *For* $c, d \in \mathbb{Z}$, $c \neq 0$, *and* $(c, d) = 1$, *let*

$$G_c(d) = \sum_{l(\mathrm{mod}\, c)} e^{2\pi i l^2 d/c}.$$

Then

$$G_c(-d) = \sqrt{2}(ic)^{-1/2}|c| \left(\frac{c}{d}\right) \varepsilon_d, \tag{4.3}$$

where we take the principal value of the square root.

Proof. First, we notice that $G_c(d)$ is periodic in d with period c. Therefore, without loss of generality, we can assume that d is an odd prime.

Now we apply the reciprocity formula for quadratic Gauss sums [He, p. 209, Theorem 161] to get

$$G_c(-d) = \sqrt{2}(ic^{-1}d)^{-1/2} G_d(c/4). \tag{4.4}$$

$G_d(c/4)$ can be evaluated since we are assuming d is an odd prime. In particular, from Lang [La], we obtain

$$G_d(c/4) = \varepsilon_d \left(\frac{c/4}{d}\right) \sqrt{d}. \tag{4.5}$$

Combining (4.4) and (4.5), and simplifying completes the proof. □

Theorem 4.4. *For* $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_S^0$ *and* $z \in \mathfrak{h}_S^0$, $\vartheta(z)$ *satisfies*

$$\vartheta(A \circ z) = \chi_S(A, z_p)[\mathbb{N}(cz + d)]^{1/2} \vartheta(z). \tag{4.6}$$

The multiplier $\chi_S(A, z_p)$ *is defined as follows*: *If* $c = 0$, *then* $\chi_S(A, z_p) = 1$. *If* $c \neq 0$, *let*

$$c' = c \cdot p^{-\min\{\mathrm{ord}_p(c), \mathrm{ord}_p(dp^{2j})\}},$$

and

$$d' = d \cdot p^{-\min\{\mathrm{ord}_p(c), \mathrm{ord}_p(dp^{2j})\}},$$

and let $w \in \mathbb{Z}$ *be such that* $p^{2j-l} w \equiv p^{2j} x_p \pmod 1$. *Then*

(1) *If* $2j \geq l$, *then*

$$\chi_S(A, z_p) = \varepsilon_{d'}^{-1} \left(\frac{c'}{d' p^{2j}}\right).$$

(2) *If* $l > 2j$, *let* $c' = c_0 p^t$, *where* $c_0 \in \mathbb{Z}$, *and* $(c_0, p) = 1$. *Then there are three possibilities*:

(i) *If* $l - 2j > t \geq 0$, *so* $t = 0$, *or* $t = \mathrm{ord}_p(c) - \mathrm{ord}_p(d) - 2j$, *then*

$$\chi_S(A, z_p) = \varepsilon_{p^{l-t}d'}^{-1} \left(\frac{c_0 p^{l-2j}}{p^{l-t} d' + w c_0}\right).$$

(ii) *If* $l - 2j < t$, *so* $t = \mathrm{ord}_p(c) - \mathrm{ord}_p(d) - 2j$, *then*

$$\chi_S(A, z_p) = \varepsilon_{p^{2j}d'}^{-1} \left(\frac{c'}{p^{2j} d' + p^{2j-l} w c'}\right).$$

(iii) *If* $l - 2j = t$, *let* $r = \mathrm{ord}_p(p^{2j} d' + w c_0)$. *There are two possibilities*: *if* $r \leq l - 2j$, *then*

$$\chi_S(A, z_p) = \varepsilon_{p^r d'}^{-1} \left(\frac{c_0 p^{l-2j-r}}{p^{-r}(p^{2j} d' + w c_0)}\right),$$

and if $r > l - 2j$, *then*

$$\chi_S(A, z_p) = \varepsilon_{p^l d'}^{-1} \left(\frac{c_0}{p^{2j-l}(p^{2j}d' + wc_0)} \right).$$

Remark. In the case that $x_p = 0$, $\operatorname{ord}_p(x_p)$ is undefined. Therefore we take l to be $-\infty$, so that $2j > l$ always. We are then in the first case of the theorem, which does not depend on l.

Proof. If $c = 0$, then $A \circ z$ is a translation, so $\vartheta(z)$ is unchanged.

If $c \neq 0$, we begin by assuming that $2j \geq l$. Then we can simplify the theta function

$$\begin{aligned}\vartheta(z) &= \sum_{n \in \mathbb{Z}[\frac{1}{p}]} e^{-2\pi n^2 y_\infty} W_p(n^2 y_p) e^{2\pi i n^2 x_\infty} e^{-2\pi i n^2 x_p} \\ &= \sum_{n \in \mathbb{Z}} e^{-2\pi n^2 p^{2j} y_\infty} e^{2\pi i n^2 p^{2j} x_\infty}.\end{aligned}$$

Now fix z_p, and let $x_\infty = -d'/c'$. Replacing x_∞ in the summation, we see that

$$\begin{aligned}\vartheta(z) &= \sum_{n \in \mathbb{Z}} e^{-2\pi n^2 p^{2j} y_\infty} e^{-2\pi i n^2 p^{2j} d'/c'} \\ &= \sum_{k \in \mathbb{Z}} \sum_{h (\operatorname{mod} c')} e^{-2\pi (h+kc')^2 p^{2j} y_\infty} e^{-2\pi i (h+kc')^2 p^{2j} d'/c'} \\ &= \sum_{k \in \mathbb{Z}} \sum_{h (\operatorname{mod} c')} e^{-2\pi (h+kc')^2 p^{2j} y_\infty} e^{-2\pi i h^2 p^{2j} d'/c'} \\ &= \sum_{h (\operatorname{mod} c')} e^{-2\pi i h^2 p^{2j} d'/c'} \sum_{k \in \mathbb{Z}} e^{-2\pi (h/c' + k)^2 (c')^2 p^{2j} y_\infty} \\ &= \sum_{h (\operatorname{mod} c')} e^{-2\pi i h^2 p^{2j} d'/c'} \theta\left(2(c')^2 p^{2j} y_\infty i, 0, \frac{h}{c'}\right),\end{aligned}$$

where $\theta(z, u, v)$ is the symplectic theta function. Then we apply the inversion formula for the symplectic theta function (see [St1], for example). This gives

$$(4.7) \quad \vartheta(z) = (2(c')^2 p^{2j} y_\infty)^{-1/2} \sum_{h (\operatorname{mod} c')} e^{-2\pi i h^2 p^{2j} d'/c'} \theta\left(\frac{i}{2(c')^2 p^{2j} y_\infty}, \frac{-h}{c'}, 0\right).$$

But we are not interested in $\vartheta(z)$ by itself, but rather how it relates to $\vartheta(A \circ z)$. We showed in the previous theorem that the relationship is of the form $\vartheta(A \circ z) = \chi_S(A, z_p)[\mathbb{N}(cz+d)]^{1/2} \vartheta(z)$, where $\chi_S(A, z_p)$ is an eighth root of unity. Let us now use this last relationship, for the x_∞ that we have chosen, and the fixed z_p, as well as equation (4.7), to determine $\chi_S(A, z_p)$. Also, note that $A \circ z_\infty = i/c^2 y_\infty + a/c$ (using the fact that $d'/c' = d/c$). Substituting into

the transformation, we see that

$$\begin{aligned}
&\vartheta\left(\left(\frac{i}{c^2 y_\infty}+\frac{a}{c}, A\circ z_p\right)\right)\\
&\quad=\chi_S(A, z_p)[\mathbb{N}(cz+d)]^{1/2}(2(c')^2p^{2j}y_\infty)^{-1/2}\\
&\qquad\times\sum_{h(\mathrm{mod}\,c')} e^{-2\pi i h^2 p^{2j}\frac{d'}{c'}}\theta\left(\frac{i}{2(c')^2p^{2j}y_\infty}, \frac{-h}{c'}, 0\right)\\
&\quad=\chi_S(A, z_p)\left(c\left(\frac{-d'}{c'}+iy_\infty\right)+d\right)^{1/2}|cz_p+d|_p^{1/2}(2(c')^2p^{2j}y_\infty)^{-1/2}\\
&\qquad\times\sum_{h(\mathrm{mod}\,c')} e^{-2\pi i h^2 p^{2j}\frac{d'}{c'}}\theta\left(\frac{i}{2(c')^2p^{2j}y_\infty}, \frac{-h}{c'}, 0\right).
\end{aligned}$$

Also, in the spirit of simplification, notice that $(c(-d'/c'+iy_\infty)+d)=ciy_\infty$.

We would now like to eliminate the dependence on y_∞. In order to do this we will take the limit as $y_\infty\to 0^+$ of both sides of the last equality. Carrying this step out, we see that both of the theta functions approach one, thus reducing our equality to

$$1=\chi_S(A, z_p)|cz_p+d|_p^{1/2}(ic)^{1/2}(2(c')^2p^{2j})^{-1/2}\sum_{h(\mathrm{mod}\,c')} e^{-2\pi i h^2 p^{2j}d'/c'}. \tag{4.8}$$

The next order of business is to evaluate the summation. This is quite simple in this case because we are dealing with a quadratic Gauss sum. Applying Lemma 4.3 to our sum, we get

$$\sum_{h(\mathrm{mod}\,c')} e^{-2\pi i h^2 p^{2j}\frac{d'}{c'}}=\sqrt{2}(ic')^{-1/2}|c'|\left(\frac{c'}{p^{2j}d'}\right)\varepsilon_{p^{2j}d'}.$$

It is also necessary to evaluate the p-adic valuation $|cz_p+d|_p$. Because $2j\geq l$, $\mathrm{ord}_p(cz_p+d)=\min\{\mathrm{ord}_p(d), \mathrm{ord}_p(c)-2j\}$. Therefore $|cz_p+d|_p=p^{-\min\{\mathrm{ord}_p(d), \mathrm{ord}_p(c)-2j\}}$.

As a final step we make use of these evaluations to condense (4.8). Simplifying as we go along, we see that

$$1=\chi_S(A, z_p)\left(\frac{c'}{p^{2j}d'}\right)\varepsilon_{d'},$$

or, solving for $\chi_S(A, z_p)$,

$$\chi_S(A, z_p)=\varepsilon_{d'}^{-1}\left(\frac{c'}{p^{2j}d'}\right).$$

This proves the first part of the theorem. It also helps to establish the credibility of $\vartheta(z)$ as a potential modular form of weight $\frac{1}{2}$ because it so closely resembles the classical theta multiplier.

Now the real work begins. We assume that $l>2j$ and see what happens. As in the previous case, we will begin by rewriting $\vartheta(z)$. We will see immediately that this case will be more complicated due to the presence of the x_p. However, the path we take will follow that of the previous case; we start by obtaining a sum over the rational integers, and then try to extract a Gauss sum.

Using the definition of $\vartheta(z)$ and $W_p(y_p)$ we can easily accomplish the first task:

$$\vartheta(z) = \sum_{n\in\mathbb{Z}} e^{-2\pi n^2 p^{2j} y_\infty} e^{2\pi i n^2 p^{2j} x_\infty} e^{-2\pi i n^2 p^{2j} x_p}.$$

Next we begin the process of finding a Gauss sum and a symplectic theta function in the above sum. As before, this will be accomplished by replacing z_∞ with a specific value, $-d'/c' + iy_\infty$, and breaking our sum up into a sum over congruence classes:

$$\begin{aligned}
\vartheta(z) &= \sum_{\substack{k\in\mathbb{Z} \\ h(\mathrm{mod}\, c'p^{l-2j})}} e^{-2\pi(h+kc'p^{l-2j})^2 p^{2j} y_\infty} \\
&\quad \times e^{-2\pi i(h+kc'p^{l-2j})^2 p^{2j} d'/c'} e^{-2\pi i(h+kc'p^{l-2j})^2 p^{2j} x_p} \\
&= \sum_{h(\mathrm{mod}\, c'p^{l-2j})} e^{-2\pi i h^2 p^{2j} d'/c'} e^{-2\pi i h^2 p^{2j} x_p} \sum_{k\in\mathbb{Z}} e^{-2\pi(h+kc'p^{l-2j})^2 p^{2j} y_\infty} \\
&= \sum_{h(\mathrm{mod}\, c'p^{l-2j})} e^{-2\pi i h^2 p^{2j}(d'/c'+x_p)} \sum_{k\in\mathbb{Z}} e^{-2\pi((h/c'p^{l-2j})+k)^2 (c'p^{l-2j})^2 p^{2j} y_\infty} \\
&= \sum_{h(\mathrm{mod}\, c'p^{l-2j})} e^{-2\pi i h^2 p^{2j}(d'/c'+x_p)} \theta\left(2(c'p^{l-2j})^2 p^{2j} y_\infty i,\, 0,\, \frac{h}{c'p^{l-2j}}\right).
\end{aligned}$$

At this point we invert the symplectic theta function. This yields

$$\begin{aligned}
\vartheta(z) = &\sum_{h(\mathrm{mod}\, c'p^{l-2j})} e^{-2\pi i h^2 p^{2j}(d'/c'+x_p)} (2(c'p^{l-2j})^2 p^{2j} y_\infty)^{-1/2} \\
&\times \theta\left(\frac{i}{2(c'p^{l-2j})^2 p^{2j} y_\infty},\, -\frac{h}{c'p^{l-2j}},\, 0\right).
\end{aligned} \tag{4.9}$$

Now we need to return to the big picture; we must fit this last relationship into the transformation formula. To do this, we combine (4.2) with (4.9), with x_∞ specified as always. We then consider what happens in the limit as $y_\infty \to 0^+$. The net result is that

$$\begin{aligned}
1 = &\chi_S(A,\, z_p)(ic)^{1/2} |cz_p + d|_p^{1/2} [\sqrt{2}|c'|p^{l-j}]^{-1} \\
&\times \sum_{h(\mathrm{mod}\, p^{l-2j}c')} e^{-2\pi i h^2 p^{2j}(d'/c'+x_p)}.
\end{aligned} \tag{4.10}$$

The nice thing about this last equality is that it contains a quadratic Gauss sum. However, to evaluate the sum, we must worry about the numerator and the denominator in the exponent being relatively prime. Therefore we must make some simplifications in the summation. In particular, we would like to eliminate the p-adic references. We would also like to deal with integers whenever possible, so we want to extract any powers of p. Thus, we set w to be an integer such that $p^{2j-l} w \equiv p^{2j} x_p \pmod 1$. Also, we write $c' = c_0 p^t$, where $c_0 \in \mathbb{Z}$, $t \geq 0$, and $(p, c_0) = 1$. These definitions allow us to rewrite (4.10) as

$$\begin{aligned}
1 = &\chi_S(A,\, z_p)(ic)^{1/2} |cz_p + d|_p^{1/2} [\sqrt{2}|c'|p^{l-j}]^{-1} \\
&\times \sum_{h(\mathrm{mod}\, p^{l-2j}c')} e^{-2\pi i h^2 (p^{l-t} d' + wc_0)/c_0 p^{l-2j}}.
\end{aligned} \tag{4.11}$$

The evaluation of the sum above will vary depending on the relationship between c_0p^{l-2j} and $p^{l-t}d' + wc_0$. This is what gives us the many different flavors of theta multiplier. Considering this relationship as a function of t, we will break our study into three separate cases. Each of these cases will be stated in two different ways. In particular, we have

(i) $l - 2j > t \geq 0 \Leftrightarrow \operatorname{ord}_p(p^{l-t}d') > 0$,

(ii) $l - 2j < t \Leftrightarrow \operatorname{ord}_p(p^{l-t}d') < 0$,

(iii) $l - 2j = t \Leftrightarrow \operatorname{ord}_p(p^{l-t}d') = 0$.

In case (i) $(c_0p^{l-2j}, p^{l-t}d' + c_0w) = 1$ so we can evaluate the summation in (4.11) immediately as the product of a quadratic Gauss sum and p^t. This, combined with Lemma 4.3, yields the following equality:

$$1 = \chi_S(A, z_p)(ic)^{1/2}|cz_p + d|_p^{1/2}[\sqrt{2}|c'|p^{l-j}]^{-1}$$
$$\times p^t\sqrt{2}(ic_0p^{l-2j})^{-1/2}|c_0p^{l-2j}|\varepsilon_{p^{l-t}d'+wc_0}\left(\frac{c_0p^{l-2j}}{p^{l-t}d' + wc_0}\right).$$

This last equation is very easy to simplify. Using the relationships between c, c', and c_0, as well as the actual value of the p-adic valuation that occurs, we can eliminate all of the powers of the prime that occur. In particular, $c' = c \cdot p^{-\min\{\operatorname{ord}_p(c), \operatorname{ord}_p(d)+2j\}} = c_0p^t$. Also, using the conditions that $l - 2j > t \geq 0$, and $l > 2j$, we determine that $|cz_p+d|_p^{1/2} = p^{-(\operatorname{ord}_p(c)-l)/2}$. These relationships, along with a little algebra, give us the desired result, that

$$\chi_S(A, z_p) = \varepsilon_{p^{l-t}d'}^{-1}\left(\frac{c_0p^{l-2j}}{p^{l-t}d' + wc_0}\right).$$

Thus we have established the first part of the second case of the theorem.

Now we pursue the second part; we assume that $l - 2j < t$. In this case we can rewrite

$$p^{l-t}d' + wc_0 = p^{l-t-2j}(p^{2j}d' + p^{t+2j-l}wc_0).$$

The reason for this is that now $p^{2j}d' + p^{t+2j-l}wc_0$ is an integer, and is also relatively prime to p. This enables us to evaluate the Gauss sum. In particular, we are looking at

$$1 = \chi_S(A, z_p)(ic)^{1/2}|cz_p + d|_p^{1/2}[\sqrt{2}|c'|p^{l-j}]^{-1}$$
$$\times \sum_{h(\bmod p^{l-2j+t}c_0)} e^{-2\pi i h^2(p^{2j}d' + p^{t+2j-l}wc_0)/p^tc_0}$$
$$= \chi_S(A, z_p)(ic)^{1/2}|cz_p + d|_p^{1/2}[\sqrt{2}|c'|p^{l-j}]^{-1}p^{l-2j}$$
$$\times \sum_{h(\bmod c')} e^{-2\pi i h^2(p^{2j}d' + p^{2j-l}wc')/c'}.$$

We now have to simplify a Gauss sum for which the numerator and denominator in the exponent are relatively prime. Therefore we apply Lemma 4.3. This step and some algebra leave us with

$$1 = \chi_S(A, z_p)(ic)^{1/2}|cz_p + d|_p^{1/2}\frac{1}{|c'|}p^{-j}[i(c')^{-1}]^{-1/2}\varepsilon_{p^{2j}d'}\left(\frac{c'}{p^{2j}d' + p^{2j-l}wc'}\right).$$

To simplify the right side of this equality, we make use of the conditions that are unique to this case; because $0 < t = \operatorname{ord}_p(c) - \operatorname{ord}_p(d) - 2j$ and $l - 2j < t$,

we see that $|cz_p + d|_p = p^{-\operatorname{ord}_p(d)}$. It is also true that $c' = c_0 p^t = cp^{-\operatorname{ord}_p(d)-2j}$. Thus, we arrive at the desired relation, namely that

$$\chi_S(A, z_p) = \varepsilon_{p^{2j}d'}^{-1}\left(\frac{c'}{p^{2j}d' + p^{2j-l}wc'}\right).$$

Now we have arrived at the last case. This time, we assume that $l - 2j = t > 0$. This allows us to rewrite the equality given in (4.11) as follows:

$$1 = \chi_S(A, z_p)(ic)^{1/2}|cz_p + d|_p^{1/2}[\sqrt{2}|c'|p^{l-j}]^{-1} \times \sum_{h(\operatorname{mod} p^{l-2j}c')} e^{-2\pi i h^2(p^{2j}d' + wc_0)/p^t c_0}.$$

Notice that $p \nmid p^{2j}d'wc_0$. Therefore, it is possible that $p \mid (p^{2j}d' + wc_0)$. In any case, we can write $p^{2j}d' + wc_0 = p^r m$, where r is a nonnegative integer, and m is an integer relatively prime to p. This substitution enables us to consider

$$1 = \chi_S(A, z_p)(ic)^{1/2}|cz_p + d|_p^{1/2}[\sqrt{2}|c'|p^{l-j}]^{-1} \times \sum_{h(\operatorname{mod} p^{l-2j}c')} e^{-2\pi i h^2 p^r m/p^{l-2j}c_0}.$$

Fortunately, $(c_0, m) = 1$, so we will just need to worry about powers of p when we try to simplify the summation. However, these powers of p prove to be quite troublesome; they may be in either the numerator or denominator of the exponent, leaving us with yet another pair of cases to consider. These cases are determined by the relative sizes of r and $l - 2j$.

We begin by assuming that $r \leq l - 2j$. Then we are looking at

$$\begin{aligned} 1 &= \chi_S(A, z_p)(ic)^{1/2}|cz_p + d|_p^{1/2}[\sqrt{2}|c'|p^{l-j}]^{-1} \times \sum_{h(\operatorname{mod} p^{l-2j}c')} e^{-2\pi i h^2 m/p^{l-2j-r}c_0} \\ &= \chi_S(A, z_p)(ic)^{1/2}|cz_p + d|_p^{1/2}[\sqrt{2}|c'|p^{l-j}]^{-1}p^{r+t} \times \sum_{h(\operatorname{mod} p^{l-2j-r}c_0)} e^{-2\pi i h^2 m/p^{l-2j-r}c_0}. \end{aligned}$$

We now have a Gauss sum that we can simplify. We again use Lemma 4.3 to deal with the Gauss sum, transforming this last equality into

$$1 = \chi_S(A, z_p)(ic)^{1/2}|cz_p + d|_p^{1/2}\frac{1}{|c'|}p^{r-j}(ic_0^{-1})^{-1/2}p^{-j-r/2+l/2}\varepsilon_m\left(\frac{p^{l-2j-r}c_0}{m}\right).$$

The next step is to use the conditions that we have been given to relate all of the different c's, and also to come up with a value for the valuation. First, we notice that $c' = c_0 p^t = cp^{-\operatorname{ord}_p(d)-2j}$. Then we find that $|cz_p + d|_p = p^{-\operatorname{ord}_p(d)-r}$.

This leaves us with

$$\begin{aligned} \chi_S(A, z_p) &= \varepsilon_m^{-1}\left(\frac{p^{l-2j-r}c_0}{m}\right) \\ &= \varepsilon_{p^{-r}(p^{2j}d' + wc_0)}^{-1}\left(\frac{p^{l-2j-r}c_0}{p^{-r}(p^{2j}d' + wc_0)}\right) \\ &= \varepsilon_{p^r d'}^{-1}\left(\frac{c_0 p^{l-2j-r}}{p^{-r}(p^{2j}d' + wc_0)}\right). \end{aligned}$$

In the case that $r > l - 2j$ things work out slightly differently because all the powers of p are in the numerator of the exponent. In particular, we must somehow transform

$$\begin{aligned} 1 &= \chi_S(A, z_p)(ic)^{1/2}|cz_p + d|_p^{1/2}[\sqrt{2}|c'|p^{l-j}]^{-1} \\ &\quad \times \sum_{h(\mathrm{mod}\, p^{l-2j}c')} e^{-2\pi i h^2 m p^{r-l+2j}/c_0} \\ &= \chi_S(A, z_p)(ic)^{1/2}|cz_p + d|_p^{1/2}[\sqrt{2}|c'|p^{l-j}]^{-1}p^{l-2j+t} \\ &\quad \times \sum_{h(\mathrm{mod}\, c_0)} e^{-2\pi i h^2 m p^{r-l+2j}/c_0} \end{aligned}$$

into a somewhat normal looking theta multiplier.

First we use our standard tools to evaluate the summation. This yields

$$1 = \chi_S(A, z_p)(ic)^{1/2}|cz_p + d|_p^{1/2}\frac{1}{|c'|}p^{l-3j}(ic_0^{-1})^{-1/2}\varepsilon_{p^{r-l+2j}m}\left(\frac{c_0}{p^{r-l+2j}m}\right).$$

Then we notice that $c' = c_0 p^t = cp^{-\mathrm{ord}_p(d)-2j}$, and $|cz_p + d|_p = p^{-\mathrm{ord}_p(c)+2j}$.

Putting this all together, we see that

$$\begin{aligned} 1 &= \chi_S(A, z_p)\varepsilon_{p^{r-l+2j}m}\left(\frac{c_0}{p^{r-l+2j}m}\right) \\ &= \chi_S(A, z_p)\varepsilon_{p^{r-l+2j}p^{-r}(p^{2j}d'+wc_0)}\left(\frac{c_0}{p^{2j-l}(p^{2j}d'+wc_0)}\right) \\ &= \chi_S(A, z_p)\varepsilon_{p^l d'}\left(\frac{c_0}{p^{2j-l}(p^{2j}d'+wc_0)}\right). \end{aligned}$$

Rearranging this last equality gives us the final statement of the theorem:

$$\chi_S(A, z_p) = \varepsilon_{p^l d'}^{-1}\left(\frac{c_0}{p^{2j-l}(p^{2j}d'+wc_0)}\right). \quad \square$$

We will now take the multiplier constructed in the proof of Theorem 4.4, and use it to define a modular form of weight $\frac{1}{2}$ on $\mathfrak{h}_S^0$ for $S = \{\infty, p\}$.

Definition 4.5. A function $f\colon \mathfrak{h}_S^0 \to \mathbb{C}$ is called a modular form of weight $\frac{1}{2}$ on $\mathfrak{h}_S^0$ if $f(z)$ satisfies

$$f(A \circ z) = \chi_S(A, z_p)\mathbb{N}(cz + d)^{1/2}f(z),$$

for $z \in \mathfrak{h}_S^0$, and $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_S^0$.

In the context of proving Theorem 4.4, we have established that there is a modular form of the type that we are looking for.

Theorem 4.6. *The function $\vartheta(z)$ is a modular form of weight $\frac{1}{2}$ defined on $\mathfrak{h}_S^0$.*

It remains to extend the definition of a modular form used in this paper from $S = \{\infty, p\}$ to an arbitrarily large set of primes. It would also be nice to generate more examples in order to get a better picture of the space of such forms. Finally, an analog of Hecke theory for these modular forms is waiting to be developed.

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