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REYE CONSTRUCTIONS FOR NODAL ENRIQUES SURFACES

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ABSTRACT. A classical Reye congruence X is an Enriques surface of rational equivalence class $(3, 7)$ in the grassmannian $G(1, 3)$ of lines of $\mathbf{P}^3$. X is the locus of lines of $\mathbf{P}^3$ which are included in two quadrics of $W =$ web of quadrics. A generalization to $G(1, t)$ is given (1) for each $t > 2$ there exist Enriques surfaces X of class $(t, 3t-2)$ in $G(1, t)$, (2) the determinant of the dual of the universal bundle on X is $\mathscr{O}_X(2E+R+K_X)$, with $E =$ isolated elliptic curve, $R^2 = -2$, $E \cdot R = t$, (3) X parameterizes lines of $\mathbf{P}^t$ which are included in a codimension 2 subsystem of W, $W =$ linear system of quadrics of dimension $\binom{t}{2}$. The paper includes a description of the variety of trisecant lines to a smooth Enriques surface of degree 10 in $\mathbf{P}^5$.

1. INTRODUCTION AND PRELIMINARIES

Let X be an Enriques surface over $\mathbf{C}$,

$$L = \mathrm{Num}(X)$$

the group of its numerical equivalence classes. As it is well known

$$L = \mathrm{Pic}(X)/\mathrm{torsion} \cong \mathbf{Z}^{10}$$

and, as a lattice, it is isomorphic to the orthogonal direct sum $E_8 \oplus H$ (where H is a hyperbolic plane and E_8 is defined as usual, cf. [10, p. 105]). In [10] F. Cossec and I. Dolgachev have studied L in all details with the purpose of describing projective models of X; among them the so-called *Fano models* are of particular interest. Let us give first their construction; one has [10]:

(1.1) L contains finitely many (modulo isometries) sets of isotropic vectors $\{e_1, \dots, e_{10}\}$ such that

$$e_i \cdot e_j = 1 - \delta_{ij}, \qquad \frac{1}{3}\sum e_i \in L;$$

these sets always satisfy the following properties: let $|C|$, $|C'|$ be the two linear systems of numerical class $\frac{1}{3}\sum e_i$, $C - C' = K_X =$ canonical divisor of X, then

(1.2) $$C^2 = 10, \qquad p_a(C) = 6, \quad \dim |C| = 5.$$

Assume $|C|$ is irreducible, then

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(1.3) ϕ_C is a morphism of degree 1 and $X_C = \phi_C(X)$ is normal with at most rational double points;

(1.4) each divisor C satisfying (1.2), (1.3) is obtained as in (1.1).

For an isotropic vector $e \in L$ we will say that $2e$ *corresponds to an elliptic pencil* if there exist (exactly) two curves E, E' of class e and such that $|2E| = |2E'|$ is an elliptic pencil. Note that $E - E' \sim K_X$ and $h^0(\mathcal{O}_X(E)) = h^0(\mathcal{O}_X(E')) = 1$; curves E, E' as above are said to be *isolated elliptic curves* on X. Let C be as in (1.2), assume C is very ample then $2e_i$ corresponds to an elliptic pencil $|2E_i| = |E_i'|$. Moreover:

(1.5) E_i, E_i' are contained in X_C as plane cubics; there is no other such curve in X_C.

The same holds for C'. By definition X_C $(X_{C'})$ is a *Fano model*; C (C') a *Fano polarization.* We recall that, on a general X, each Fano polarization is very ample and that C very ample $\Leftrightarrow C'$ very ample. To simply notations we will write X instead of X_C when no confusion arises.

Now let us describe the contents of this paper: our first purpose was to describe the variety $\operatorname{Tris}(X)$ of trisecant lines to a smooth Fano model X. This is done in §§2 and 3: we show that, excluding one exceptional case, a line L is trisecant to X if and only if L is trisecant to one of the twenty plane cubics in X. The exception is the following: $\operatorname{Tris}(X) = \mathbf{P}^3$ blown up in twenty points; $X = X_{C'}$, where $C' = C + K_X$ is a Reye polarization.

We recall that a *Reye polarization* C' is a special Fano polarization, the special condition on it being

$$X_{C'} \subset G = \text{ smooth quadric of } \mathbf{P}^5.$$

If C' is a Reye polarization we will call $X_{C'}$ a *Reye congruence*; the reason is that G is the grassmannian of lines of $\mathbf{P}^3$ and that, traditionally, surfaces in G (i.e. two dimensional families of lines of $\mathbf{P}^3$) were called congruence of lines.

The main modern result on Reye congruences is the following: every Reye congruence X is a nodal Enriques surface and, conversely, on every nodal X admitting a very ample Fano polarization there exists a Reye polarization too [8], [10, vol. II]. We recall that a *nodal Enriques surface* X contains by definition a curve R of arithmetic genus 0 and such that $h^0(\mathcal{O}_R) = 1$. Following [10] we will say that R is an *indecomposable nodal cycle.* By [10, p. 25] a curve R in X is an indecomposable nodal cycle iff it is a fundamental cycle of some rational double point.

Since nodal Enriques surfaces have no general moduli it follows that, for a general Enriques surface, there is no embedding in G as a Fano model. Because of this remark, after the classification of $\operatorname{Tris}(X)$, we came on the following kinds of questions:

Let L be a polarization on X satisfying (1.3), $X_L = \phi_L(X)$, G_r the Plücker embedding of the grassmannian of lines of $\mathbf{P}^r$; when do we have

$$X_L \subset \mathbf{P}^N \cap G_r, \qquad r << N?$$

$(N = \dim |L|)$, and what is the special feature of a projective model X_L if X is nodal?

To partially answer them we generalize the construction of Reye congruences to projective models of higher degree $L^2 = 4t - 2 \geq 10$ (§§4 and 5):

Assume the set

$\mathscr{E}_L(t) = \{E \in \mathrm{Div}(X)/E =$ isolated elliptic curve, $E{\cdot}L = t,\ h^1(\mathscr{O}_X(L-2E)) = 0\}$

is not empty, then we show

(1.6) $$X_L \subset G_t \Leftrightarrow L \sim 2E + R + K_X\,;$$

where $E \in \mathscr{E}_L(t)$, $R =$ indecomposable nodal cycle, $E \cdot R = t$;

(1.7) as a surface in G_t, X has rational equivalence class $(t, 3t-2)$;

there exists a linear system W of quadrics of $\mathbf{P}^t$ such that

(1.8) $$\dim(W) = \binom{t}{2} \quad \text{and} \quad X_L = \{l \in G_t \cap \mathbf{P}^N / \operatorname{codim}(W_l, W) = 2\}$$

with $W_l = \{Q \in W/l \subset Q\}$, $\mathbf{P}^N =$ linear span of X in the Plücker space of G_t.

The case $t = 3$ gives Reye congruences and their classical construction by a web of quadrics of $\mathbf{P}^3$. In view of this we define the projective model appearing in (1.6) as a Reye congruence of index t. Then, in §6, we show that Reye congruences of index t exist for each $t \geq 3$.

Of course an embedding of X in G_t defines a rank 2 vector bundle on X: the restriction to X_L of the universal bundle of G_t. Let $\mathscr{Q}_t$ be the dual of such a bundle; by (1.6), (1.7) $c_1(\mathscr{Q}_t)^2 = 4t-2$, $c_2(\mathscr{Q}_t) = t$; we can define $\mathscr{Q}_t$ as a *Reye bundle* of index t. This relates our results to those obtained by I. Dolgachev and I. Reider in [11]: there they study the case $t = 3$ and show that $\mathscr{Q}_3$ is stable, (actually the unique stable rank 2 vector bundle on X with Chern class $c_1^2 = 10$, $c_2 = 3$), and extremal i.e. without moduli. Moreover they are interested in the following problem: to find other examples of rank 2 vector bundles $\mathscr{E}$ on an Enriques surface which are stable and extremal.

In this case, computing the dimension of the moduli space, $\mathscr{E}$ must satisfy $c_1(\mathscr{E})^2 = 4t-2$, $c_2(\mathscr{E}) = t$; therefore our Reye bundle $\mathscr{Q}_t$ seems to be a natural candidate for further examples. During the completion of this paper we learned that stable-extremal rank 2 vector bundles on X were described by Hoil Kim in his Ph.D. thesis [15]. In particular he produces examples $\mathscr{F}_t$ of them with $c_2(\mathscr{F}_t) = t$, any $t \geq 0$. We mention that, applying his results to our situation, it is possible to show that $\mathscr{Q}_t$ is stable-extremal too. Also, we have to mention that the description of $\mathrm{Tris}(X)$ for a Fano model X is independently obtained in [11] as a consequence of the study of global sections of $\mathscr{Q}_3$.

Finally we wish to thank the referee for his help, especially in §6 where our previous degeneration arguments were considerably simplified by his suggestion of using Cremona transformations of $\mathbf{P}^5$ and generic nodal Enriques surfaces.

2. Trisecants to Fano models

Let $V \subset \mathbf{P}^n$ be a smooth projective variety, L a line intersecting V; in the following we will say that L is a *trisecant line* to V if the scheme $V \cdot L$ is zero dimensional of length ≥ 3; (hence, in particular, L is not in V).

Consider a smooth hyperplane section $C = X \cap \mathbf{P}^4$ of a Fano model X; first we want to study the family $\mathrm{Tris}(C)$ of trisecant lines to C. By Berzolari formulae [12] the expected number of trisecant lines to a smooth, irreducible curve of genus g and degree d in $\mathbf{P}^4$ is $t = \frac{1}{6}(d-4)(d-3)(d-2) - g(d-4)$, which gives $t = 20$ in our case. Note that $\mathcal{O}_C(1) \cong \omega_C \otimes \eta$, with $\eta = \mathcal{O}_C(K_X) =$ nontrivial order 2 element of $\mathrm{Pic}(C)$. On the other hand, for a general curve D in $\mathbf{P}^4$ of degree 10 and genus 6, $\mathcal{O}_D(1) \cong \omega_D \otimes \eta$, with $\eta =$ any degree zero line bundle.

So C is a special element of its Hilbert scheme and, at least for this reason, we need to analyze more in detail $\mathrm{Tris}(C)$. For this consider any curve C of genus 6 embedded in $\mathbf{P}^4$ by a very ample linear system $\omega_C \otimes \eta$ with $\eta =$ nontrivial order 2 element of $\mathrm{Pic}(C)$; let

$$\begin{aligned} &C(3) = \text{3-symmetric product of } C, \\ &a\colon C(3) \to \mathrm{Pic}^0(C) \text{ the Abel map}, \\ &W = a(C(3)), \\ &W_\eta = \text{ translation of } W \text{ by } \eta; \text{ then} \end{aligned} \tag{2.1}$$

(2.2) **Proposition.** *Let $d \in C(3)$. There exists a trisecant line containing d if and only if $h^0(\eta(d)) = 1$.*

Proof. The condition d is contained in a trisecant line is equivalent to $h^0(\omega_C \otimes \eta(-d)) = 3$. Since η is isomorphic to its dual $h^0(\omega_C \otimes \eta(-d)) = h^1(\eta(d))$, (Serre duality). Finally, $h^1(\eta(d)) = 3 \Leftrightarrow h^0(\eta(d)) = 1$.

(2.3) **Corollary.** *d is contained in a trisecant line if and only if $a(d) \in W \cap W_\eta$.*

Computing $W \cdot W_\eta$ we obtain by Poincaré formulae $W \cdot W_\eta = (\Theta/3!)^2 = 20$, ($\Theta =$ theta divisor in $\mathrm{Pic}^0(C)$). At this point we need a transversality condition for W and W_η in $\mathrm{Pic}^0(C)$:

(2.4) **Proposition.** *W and W_η are transversal at $a(d)$ if and only if $h^0(\eta(2d)) = 1$.*

Proof. Since $h^0(\eta(d)) = 1$ there exists a unique $d' \in C(3)$ such that $\eta \cong \mathcal{O}_C(d' - d)$. By (2.2), (2.3) $a(d') \in W \cap W_\eta$ and $h^0(\eta(d')) = h^0(\mathcal{O}_C(d)) = 1$. Moreover, since $\omega_C \otimes \eta$ is very ample, $\mathrm{Supp}(d) \cap \mathrm{Supp}(d') = \varnothing$ [10, 0.6]. Writing the derivative of Abel map as in [1, 4.1] one obtains the standard identifications

$$H^0(\omega_C) \cong T_{\mathrm{Pic}^0(C), a(d)}, \qquad H^0(\omega_C(-d)) \cong T_{W, a(d)} \tag{2.5}$$

since the derivative of the translation by η is the identity, one has

$$T_{W_\eta, a(d)} = T_{W, a(d')} = H^0(\omega_C(-d')). \tag{2.6}$$

Since $\mathrm{Supp}(d) \cap \mathrm{Supp}(d') = \varnothing$ we have $H^0(\omega_C(-d - d')) = H^0(\omega_C(-d)) \cap H^0(\omega_C(-d'))$. Therefore, by Riemann-Roch and Serre duality,

$$h^0(\omega_C(-d - d')) = 0 \Leftrightarrow h^1(\eta(2d)) = 0 \Leftrightarrow h^0(\eta(2d)) = 1 \Leftrightarrow W, W_\eta$$

are transversal at $a(d)$ (and also at $(a(d'))$.

Let us apply Proposition (2.4) to a smooth hyperplane section $C = \mathbf{P}^4 \cap X$ of a smooth Fano model. Of course the 10 pairs E_i, E_i' $(i = 1 \cdots 10)$ of plane cubic curves define 10 pairs L_i, L_i' of trisecant lines to C and the divisors

$$d_i = E_i \cdot C = L_i \cdot C, \qquad d_i' = E_i' \cdot C = L_i' \cdot C$$

on C. Note that $\mathcal{O}_C(d_i'-d_i) \cong \mathcal{O}_C(K_X) \cong \eta$, where $\omega_C \otimes \eta = \mathcal{O}_C(1)$. Therefore we have

$$\eta(2d_i) \cong \mathcal{O}_C(E_i + E_i') \cong \eta(2d_i')$$

and the exact sequence

(2.8) $$0 \to \mathcal{O}_X(-C + E_i + E_i') \to \mathcal{O}_X(E_i + E_i') \to \eta(2d_i) \to 0.$$

Since $h^1(\mathcal{O}_X(E_i + E_i')) = 0$ this gives the exact sequence

(2.9) $$0 \to H^0(\mathcal{O}_X(E_i + E_i')) \to H^0(\eta(2d_i)) \to H^1(\mathcal{O}_X(-C + E_i + E_i')) \to 0$$

with $\dim H^0(\mathcal{O}_X(E_i + E_i')) = 1$. Therefore we have shown

(2.10) **Proposition.** *W and W_η are transversal at $a(d_i)$, $a(d_i')$ if and only if $H^1(\mathcal{O}_X(-C + E_i + E_i')) = 0$.*

The main point is now the following known result:

(2.11) **Proposition.** *Let X be a smooth Fano model polarized by C, then $h^1(\mathcal{O}_X(-C + E_i + E_i')) = 0$ $\forall i = 1 \cdots 10$ unless $C + K_X$ is a Reye polarization.*

Proof. Cf. [8] and the next section.

(2.12) **Theorem.** *Let X be a smooth Fano model polarized by C, π_i, π_i' be the planes containing E_i, E_i'; π_i^*, $\pi_i'^*$ their dual planes. Assume $C + K_X$ is not a Reye polarization then*

$$\operatorname{Tris}(X) = \bigcup_{i=1}^{10} (\pi_i^* \cup \pi_i'^*).$$

Proof. Let $L \in \operatorname{Tris}(X)$, by definition L is not in X so that there exists a $\mathbf{P}^4$ transversal to X and containing L. Let $C = X \cap \mathbf{P}^4$, l a degree 3 divisor contained in the 0-cycle $L \cdot C$. Then $a(l) \in W \cap W_\eta \subset \operatorname{Pic}^0(C)$. By (2.4), (2.11) the scheme $W \cdot W_\eta$ contains the points $a(d_i)$, $a(d_i')$, ($i = 1 \cdots 10$), as isolated components. Let Z be possibly some different excess intersection component of $W \cdot W_\eta$. Let us show directly that $Z = 0$: consider in $C(3) \times \operatorname{Pic}^0(C)$ the incidence correspondence $I = \{(d, \xi)/h^0(\xi(d)) \geq 1\}$. The projection $p_1 \colon I \to C(3)$ has fibre isomorphic to W, ($\forall d \in C(3)$). Hence I is irreducible, $\dim(I) = 6$. The projection $p_2 \colon I \to \operatorname{Pic}^0(C)$ has fibre $W \cdot W_\xi$ over ξ ($W_\xi = W$ translated by ξ); moreover p_2 is surjective and $\deg(p_2) = W^2 = 20$. Applying Stein factorization to p_i it follows that the number of connected components of $p_2^{-1}(\xi) = W \cap W_\xi$ cannot exceed 20. Hence $W \cdot W_\eta = \sum(a(d_i) + a(d_i'))$, $l = d_i$ or d_i' for some i, $L \in \pi_i^*$ or $\pi_i'^*$.

(2.13) *Remark.* Take a smooth Fano model X satisfying the assumption of Theorem 2.12, then a general point p of X is not contained in a trisecant line to X. Hence projecting X from p one obtains a smooth Enriques surface $S \subset \mathbf{P}^4$ of degree 9 and sectional genus 6 which has been blown up in one point. The existence of such an S was previously conjectured. A proof has been recently given in [13] by using a different method.

3. Trisecants to adjoints to Reye models

To complete the results of the previous section we need to describe $\operatorname{Tris}(X_C)$ in the case $C + K_X =$ very ample Reye polarization. With this purpose we give an explicit projective construction for X_C which essentially comes from the results of [8, 10]. Nevertheless we present it as a special case of the more general situation to be discussed in §4. Let

(3.1) $V =$ vector space, $\dim(V) = t+1$, $\mathbf{P}^t = \mathbf{P}(V)$.

Consider the Segre product

(3.2) $$\mathbf{P}^t \times \mathbf{P}^t \subset \mathbf{P}^{\otimes} = \mathbf{P}(V \otimes V)$$

(i.e. the projectivized set of indecomposable vectors of $V \otimes V$) and the involution

(3.3) $$I_t \colon \mathbf{P}^t \times \mathbf{P}^t \to \mathbf{P}^t \times \mathbf{P}^t$$

induced by the linear map $v_1 \otimes v_2 \to v_2 \otimes v_1$. Let Q^+, Q^- be its eigenspaces, they are generated respectively by vectors $(v_1 \otimes v_2 + v_2 \otimes v_1)$ and $(v_1 \otimes v_2 - v_2 \otimes v_1)$. Hence, as usual, we identify Q^+, Q^- to $\mathrm{Sym}^2 V$, $\bigwedge^2 V$ by the isomorphisms $(v_1 \otimes v_2 + v_2 \otimes v_1) \to v_1 v_2$ and $(v_1 \otimes v_2 - v_2 \otimes v_1) \to v_1 \wedge v_2$. Correspondingly we have in $\mathbf{P}^{\otimes}$ the projectivized eigenspaces

(3.4) $$\mathbf{P}^+ = \mathbf{P}(\mathrm{Sym}^2 V), \qquad \mathbf{P}^- = \mathbf{P}\left(\bigwedge^2 V\right)$$

which are the set of fixed points of I_t. We are interested in the linear projections

(3.5) $$p_+ \colon \mathbf{P}^{\otimes} \to \mathbf{P}^+, \qquad p_- \colon \mathbf{P}^{\otimes} \to \mathbf{P}^-$$

respectively of centers $\mathbf{P}^-$, $\mathbf{P}^+$; restricting these projections to $\mathbf{P}^t \times \mathbf{P}^t$ we get

(3.6) $$\mathbf{P}^- \overset{p_-}{\longleftarrow} \mathbf{P}^t \times \mathbf{P}^t \overset{p_+}{\to} \mathbf{P}^+$$

with $p_+(v_1 \otimes v_2) = v_1 v_2$, $p_-(v_1 \otimes v_2) = v_1 \wedge v_2$. Therefore

(3.7) $$p_-(\mathbf{P}^t \times \mathbf{P}^t) = G_t, \qquad p_+(\mathbf{P}^t \times \mathbf{P}^t) = \Sigma_t$$

with $G_t =$ Plücker embedding of the grassmannian of lines of $\mathbf{P}^t$, $\Sigma_t =$ 2-symmetric product of $\mathbf{P}^t$. Since p_+ is the quotient map $\mathbf{P}^t \times \mathbf{P}^t \to \mathbf{P}^t \times \mathbf{P}^t / \langle I_t \rangle$ it turns out that $\mathrm{Sing}(\Sigma_t) = p_+(\Delta)$, where Δ is the diagonal of $\mathbf{P}^t \times \mathbf{P}^t$; it is known that each point of $\mathrm{Sing}(\Sigma_t)$ occurs with multiplicity $2t-2$. Finally $\deg \Sigma_t = \frac{1}{2}\binom{2t}{t}$.

Consider the case $t = 3$, then $\mathbf{P}^- = \mathbf{P}^5$, $\mathbf{P}^+ = \mathbf{P}^9$, $\mathbf{P}^{\otimes} = \mathbf{P}^{15}$. Let $\Sigma = \Sigma_3$, $G = G_3$; fix any 5-dimensional projective space

(3.8) $$\Lambda \subset \mathbf{P}^+$$

intersecting properly Σ along the reduced, irreducible surface of degree 10

(3.9) $$X = \Lambda \cdot \Sigma.$$

Assume X is normal with at most rational double points, then

(3.10) **Proposition.** *X is a Fano model. Let C be its polarization, then $C + K_X$ is a Reye polarization.*

Proof. Since X has no point of multiplicity 4, $\mathrm{Sing}(\Sigma) \cap X = \varnothing$. Let $\widetilde{X} = p_+^{-1}(X)$, then $p_+ \colon \widetilde{X} \to X$ is an étale double covering and $i = I_t / \widetilde{X}$ its associated fixed points free involution; moreover $\widetilde{X}$ is normal with at most rational double points. Note that $\widetilde{X}$ is a complete intersection in $\mathbf{P}^3 \times \mathbf{P}^3$ of four hyperplane sections; by Bertini's theorem this implies that $\widetilde{X}$ is connected and, since it is normal, irreducible. Finally, by adjunction formula, $\widetilde{X}$ is a $K3$-surface. Since i is fixed-point-free and $X = \widetilde{X}/\langle i \rangle$, X is an Enriques surface and, by (1.4), a Fano model.

Now consider the other projection $p_- : \widetilde{X} \to \mathbf{P}^-$; $p_-(\widetilde{X})$ is a Fano model of X which is contained in G, hence a Reye congruence. Let C' be its polarization: clearly $C \not\sim C'$; on the other hand the pull back of $\mathscr{O}_X(C' - C)$ to $\widetilde{X}$ is trivial. Since, as an étale double covering of X, $\widetilde{X}$ is defined by $\mathscr{O}_X(K_X)$, it follows $C - C' \sim K_X$.

(3.11) **Lemma.** *Let X_C be a smooth Fano model;*

$$(u, v) = (h^1(\mathscr{O}_X(E_i + E_i' - C)),\ h^1(\mathscr{O}_X(C' - 2E_i))).$$

Then (u, v) must be one of the pairs $(0, 0), (1, 0), (0, 1)$.

Proof. Let for instance $u > 0$; applying Riemann Roch it follows that $C' - E_i - E_i' \sim R$ with R effective, $p_a(R) = 0$. Consider the exact sequence

$$0 \to H^0(\mathscr{O}_X(-R)) \to H^0(\mathscr{O}_X) \to H^0(\mathscr{O}_R) \to H^1(\mathscr{O}_X(-R)) \to 0,$$

by Serre duality $v = h^1(\mathscr{O}_X(-R))$. Since R is a curve of degree 4 in the smooth model X_C it is easy to check that $v > 0 \Leftrightarrow R$ is not connected. Assume $v > 0$ then $R = A + B$ with A, B effective, $A \cap B = \varnothing$. Since $E_i \cdot (A + B) = 3$ we have $E_i \cdot A$ or $E_i \cdot B \le 1$. Hence the linear system $|E_i + A + B| = |C' - E_i'|$ is not numerically 2-connected. On the other hand, $|C' - E_i'|$ is base-point-free [14] hence numerically 2-connected [10, 4.3.4, 4.4.1], contradiction. Therefore it must be $v = 0$. Since R is connected and $p_a(R) = 0$, $h^0(\mathscr{O}_X(R)) = 1$. Hence $h^0(\mathscr{O}_X(R)) = h^0(\mathscr{O}_X(C' - E_i - E_i')) = h^1(\mathscr{O}_X(E_i + E_i' - C)) = u = 1$. A completely similar argument works if we assume $v > 0$.

(3.12) **Lemma.** *Assume $C' = C + K_X$ is a very ample Reye polarization, then* $h^1(\mathscr{O}_X(C' - 2E_i)) = 0$, $i = 1 \cdots 10$.

Proof. Let π_i, π_i' be the supporting planes of the cubic curves E_i, E_i' in the smooth Reye model $X_{C'}$; it is very well known that $\pi_i \cap \pi_i' \neq \varnothing$. Hence $h^0(\mathscr{O}_X(C' - E_i - E_i')) = h^1(\mathscr{O}_X(E_i + E_i' - C)) > 0$ and, by Lemma 3.11, $h^1(\mathscr{O}_X(C' - 2E_i)) = 0$.

(3.13) **Proposition.** *Let X be a smooth Fano model, then the following conditions are equivalent*:

(1) $h^1(\mathscr{O}_X(E_i + E_i' - C)) = 1$ *for some* $i = 1 \cdots 10$,
(2) $C \sim 2E + R$, $E =$ *isolated elliptic curve,* $R =$ *indecomposable nodal cycle,* $E \cdot R = 3$,
(3) $C' = C + K_X$ *is a Reye polarization,*
(4) $h^1(\mathscr{O}_X(E_i + E_i' - C)) = 1$ *for each* $i = 1 \cdots 10$,
(5) $X_C = \Lambda \cdot \Sigma$ *as in* (3.9).

Proof. Consider as in (4.11) $\mathscr{E}_C = \{E \in \mathrm{Div}(X)/E =$ isolated elliptic curve, $E \cdot C = 3$, $h^1(\mathscr{O}_X(C' - 2E)) = 0\}$. By Lemmas (3.11), (3.12) the cubic curves E_i, E_i' belong to $\mathscr{E}_C(3)$. Then the equivalence of (1), (2), (3), (4) is just the case $t = 3$ of Theorem (4.12). Finally (3) $\Rightarrow$ (5) by Lemma 5.6 and (5) $\Rightarrow$ (3) by Proposition 3.10.

(3.14) **Proposition.** *Let X be a smooth Fano model polarized by C. Assume $C + K_X$ is Reye then*

(1) $\mathrm{Tris}(X)$ *is the blowing up of* $\mathbf{P}^3$ *in* 20 *points*;
(2) *the union of trisecant lines to* X *is a determinantal quartic hypersurface* $\Delta = \{\det(d_{ij}) = 0\}$, *where* (d_{ij}) *is a* 4×4 *symmetric matrix of linear forms.*

Proof. By (3.11), $C + K_X =$ Reye polarization $\Leftrightarrow X = \Lambda \cdot \Sigma \subset \mathbf{P}(\mathrm{Sym}^2 V)$ as in (3.9). $\mathbf{P}(\mathrm{Sym}^2 V)$ is the parameter space for quadrics Q in $\mathbf{P}(V^*) = \mathbf{P}^3$ and Σ parametrizes rank 2 quadrics. Hence Λ is a 5-dimensional linear system of quadrics of $\mathbf{P}^{3*}$ and

$$X = \Lambda \cdot \Sigma = \{Q \in \Lambda / \mathrm{rank}(Q) = 2\}.$$

Let $p \in \mathbf{P}^{3*}$, $\Lambda_p = \{Q \in \Lambda / p \in \mathrm{Sing}(Q),\ \dim(\Lambda_p) = d\}$. If $d = 1$ then, counting properly multiplicities, Λ_p contains three rank 2 quadrics and it is a trisecant line to X. In the same way $d = 2 \Rightarrow \Lambda_p \cdot X =$ plane cubic: we know that there are exactly 20 points o_i $(i = 1 \cdots 20)$ such that $d = 2$. If $d = 3$ then Λ contains a rank 1 quadric, which is a point of multiplicity 4 for $\Lambda \cdot \Sigma = X$: against our assumptions. Let P be a 3-secant line to X, Δ the quartic hypersurface parametrizing singular quadrics of Λ. Since $\mathrm{Sing}(\Delta) = \Lambda \cdot \Sigma = X$ it follows that $P \subset \Delta$. Since, by definition P is not in X, all the members of P have rank 3 but for three of them having rank 2. Looking at the projective classification of pencil of quadrics one can deduce that, in our case, P has a (unique) singular base point p. This yields a morphism f: $\mathrm{Tris}(X) \to \mathbf{P}^3$ sending P to p. f blows up $\mathbf{P}^3$ in the points o_i. The union of trisecant lines to X is the quartic hypersurface considered above. This completes the proof: for brevity we omitted some easy details.

4. Enriques surfaces in grassmannians

In this section we construct projective models X of Enriques surfaces which are contained in the intersection of G_t with a $(2t-1)$-space: $X \subset \mathbf{P}^{2t-1} \cap G_t$. The degree of X in $\mathbf{P}^{2t-1}$ is $4t-2$, the rational equivalence class of X in G_t is $(t, 3t-2)$. These models are the natural generalization of classical Reye congruences.

Let $\mathscr{H}_t$, $t \geq 3$, be the Hilbert scheme of normal Enriques surfaces of degree $4t-2$, $\mathscr{R}_t$ the Hilbert scheme of normal Enriques surfaces in G_t having rational equivalence class $(t, 3t-2)$, $f: \mathscr{R}_t \to \mathscr{H}_t$ the obvious morphism between these two Hilbert schemes. As a consequence of our generalization we would like to conjecture the following:

(4.1) $f(\mathscr{R}_t)$ is a closed codimension one subset of $\mathscr{H}_t$ and its points correspond to all nodal Enriques surfaces.

For Fano models $(t = 3)$ this is generically true, [8, 10]. Let L be an irreducible polarization on X such that

(4.2) $L^2 = 4t - 2$, $\quad \phi_L$ is a degree 1 morphism, $\quad X_L$ is normal.

(4.3) **Definition.** We say that the pair (X, L) fits in a Reye diagram of index t if the following conditions are satisfied

(1) there exists a commutative diagram

$$(4.4)\qquad \begin{array}{ccccc} \mathbf{P}^- & \xrightarrow{p_-} & \mathbf{P}^\otimes & \xrightarrow{p_+} & \mathbf{P}^+ \\ \cup & & \cup & & \cup \\ G_t & \xrightarrow{p_-} & \mathbf{P}^t \times \mathbf{P}^t & \xrightarrow{p_+} & \Sigma_t \\ \cup & & \cup & & \cup \\ X_{L'} & \xrightarrow{p_-/\widetilde{X}} & \widetilde{X} & \xrightarrow{p_+/\widetilde{X}} & X_L \end{array}$$

where $\widetilde{X}$ is the $K3$ cover of X', $L' = K_X + L$ and for the top arrow we use the same notations of (3.1)–(3.6).

(2) The projections of $\widetilde{X}$ on the two factors of $\mathbf{P}^t \times \mathbf{P}^t$ are not contained in a hyperplane.

(4.5) **Proposition.** *Assume $L = 2E + R$, with $E =$ isolated elliptic curve, $R =$ indecomposable nodal cycle, $E \cdot L = t$. Then the pair (X, L) fits in a Reye diagram of index t.*

Proof. $L^2 = 4t - 2$; let $\pi: \widetilde{X} \to X$ be the $K3$ étale double cover of X, $i: \widetilde{X} \to \widetilde{X}$ the induced involution, $\widetilde{L} = \pi^* L$. Since R is connected $\pi^* R$ splits

$$\pi^* R = R_1 + R_2,$$

$\pi/R_j: R_j \to R =$ isomorphism $(j = 1, 2)$. This yields two polarizations of degree $2t - 2$ on $\widetilde{X}$:

$$L_1 = R_1 + \widetilde{E}, \qquad L_2 = R_2 + \widetilde{E}$$

with $|\widetilde{E}| = |\pi^* E| =$ elliptic pencil. Since $h^1(\mathscr{O}_{\widetilde{X}}(R_j)) = 0$ the sequence

$$(4.6)\qquad 0 \to H^0(\mathscr{O}_{\widetilde{X}}(R_j)) \to H^0(\mathscr{O}_{\widetilde{X}}(L_j)) \to H^0(\mathscr{O}_{\widetilde{E}}(L_j)) \to 0$$

is exact and $\dim |L_j| = t$. Let $\phi_j: \widetilde{X} \to \mathbf{P}^t$ be the map defined by $|L_j|$, $X_j = \phi_j(\widetilde{X})$: by (4.6) and the very ampleness of $\mathscr{O}_{\widetilde{E}}(L_j)$, $\phi_j/\widetilde{E}$ is an embedding. This easily implies $\deg(\phi_j) = 1$. Then, applying standard properties of linear systems on a $K3$ surface, it follows that ϕ_j is a morphism and X_j is normal with at most rational double points. Since $|L_1 + L_2| = |\pi^* L|$ the map $\phi_{\widetilde{L}}$ has the same properties; since π is étale and $\pi^* L \sim \widetilde{L} \sim \pi^* L'$ the same is still true for $|L|$, $|L'|$ on X. Consider

$$\widetilde{X} \xrightarrow{\Psi} \mathbf{P}^t \times \mathbf{P}^t \xrightarrow{\sigma} \mathbf{P}^\otimes$$

($\sigma =$ Segre embedding as in (3.2); $\Psi = (\phi_1 \times \phi_2)$). $\sigma \cdot \Psi$ is the map defined by the vector space $\mathrm{Im}(\mu) \subset H^0(\mathscr{O}_{\widetilde{X}}(\widetilde{L}))$ where

$$\mu: H^0(\mathscr{O}_{\widetilde{X}}(L_1)) \otimes H^0(\mathscr{O}_{\widetilde{X}}(L_2)) \to H^0(\mathscr{O}_{\widetilde{X}}(\widetilde{L}))$$

is the multiplication map. First we want to show that μ is surjective. Observe that $|\widetilde{L} - \widetilde{E}|$ has no fixed components (it contains $R_1 + |L_2|$ and $|L_1| + R_2$) and positive self-intersection. Therefore $h^1(\mathscr{O}_{\widetilde{X}}(\widetilde{L} - \widetilde{E})) = 0$ and the sequence

$$(4.7)\qquad 0 \to H^0(\mathscr{O}_{\widetilde{X}}(\widetilde{L} - \widetilde{E})) \to H^0(\mathscr{O}_{\widetilde{X}}(\widetilde{L})) \to H^0(\mathscr{O}_{\widetilde{E}}(\widetilde{L})) \to 0$$

is exact. Let $T = H^0(\mathcal{O}_{\widetilde{X}}(L_1)) \otimes H^0(\mathcal{O}_{\widetilde{X}}(L_2))$, $T_{\widetilde{E}} = H^0(\mathcal{O}_{\widetilde{E}}(L_1)) \otimes H^0(\mathcal{O}_{\widetilde{E}}(L_2))$; using (4.6) we get a homomorphism of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & T & \longrightarrow & T_{\widetilde{E}} & \longrightarrow & 0 \\ & & {\scriptstyle\mu_1}\downarrow & & {\scriptstyle\mu}\downarrow & & {\scriptstyle\mu_2}\downarrow & & \\ 0 & \longrightarrow & H^0(\mathcal{O}_{\widetilde{X}}(\widetilde{L} - \widetilde{E})) & \longrightarrow & H^0(\mathcal{O}_{\widetilde{X}}(\widetilde{L})) & \longrightarrow & H^0(\mathcal{O}_{\widetilde{E}}(\widetilde{L})) & \longrightarrow & 0 \end{array}$$

where μ_1, μ_2 are the induced multiplication maps. The surjectivity of μ_2 for two very *ample* line bundles $\mathcal{O}_{\widetilde{E}}(L_j)$ on an elliptic curve is standard. To finish we must show that μ_1 is surjective. Observe that dim $H^0(\mathcal{O}_{\widetilde{X}}(\widetilde{L} - \widetilde{E})) = 2t$, dim $K = 2t + 1$. Fix $h_j \in H^0(\mathcal{O}_{\widetilde{X}}(L_j))$, $h_j \neq 0$ and vanishing on $\widetilde{E}$. Then K is spanned by the vector spaces

$$K_1 = \langle h_1 \rangle \otimes H^0(\mathcal{O}_{\widetilde{X}}(L_2)), \qquad K_2 = H^0(\mathcal{O}_{\widetilde{X}}(L_1)) \otimes \langle h_2 \rangle.$$

Therefore: $w \in \operatorname{Ker}(\mu_1) \Rightarrow (w = h_1 \otimes s_2 + s_1 \otimes h_2$ and $\operatorname{div}(h_1 s_2) = \operatorname{div}(h_2 s_1)) \Rightarrow \widetilde{E} + \operatorname{div}(s_2) = \widetilde{E} + \operatorname{div}(s_1)$. Since R_1, R_2 are disjoint and we can choose $\widetilde{E}$ irreducible, it follows $\operatorname{div}(s_j) = E_j + R_j$; $(E_j \in |\widetilde{E}|)$. Hence $\operatorname{Ker}(\mu_1)$ is contained in $H^0(\mathcal{O}_{\widetilde{X}}(L_1 - R_1)) \otimes H^0(\mathcal{O}_{\widetilde{X}}(L_2 - R_2)) \cong H^0(\mathcal{O}_{\widetilde{X}}(\widetilde{E})) \otimes H^0(\mathcal{O}_{\widetilde{X}}(\widetilde{E}))$. The multiplication map on the latter vector space has 1-dimensional kernel. Hence $\dim \operatorname{Ker}(\mu_1) = 1$ and μ, μ_1 are surjective. Now it is quite easy to reconstruct diagram (4.4) for the pair (X, L): since μ is surjective $\operatorname{Ker}\mu$ has codimension $h^0(\mathcal{O}_{\widetilde{X}}(\widetilde{L})) = 4t$ in $V^* \otimes V^* = H^0(\mathcal{O}_{\widetilde{X}}(L_1)) \otimes H^0(\mathcal{O}_{\widetilde{X}}(L_2))$.

Moreover $\widetilde{X}_{\widetilde{L}}$ spans $\mathbf{P}(K)$, with $K = \operatorname{Ker}(\mu)^{\perp} = \{u \in V \otimes V / h(u) = 0, h \in \operatorname{Ker}(\mu)\}$. Since $i^* L_1 \sim L_2$ we can assume that i is induced by I_t as in (3.3) and that $\mathbf{P}(K)$ is an invariant space of I_t. Let $j \colon K^* \to K^*$ be the involution induced by I_t on K^* then, under the restriction isomorphism $K^* \cong H^0(\mathcal{O}_{\widetilde{X}}(\widetilde{L}))$, the eigenspaces of j are $\pi^* H^0(\mathcal{O}_X(L))$, $\pi^* H^0(\mathcal{O}_X(L'))$. Therefore, with the same notations of (3.4), we have $p_+(\widetilde{X}) = X_L$ or $X_{L'}$, $(p_-(\widetilde{X}) = X_L$ or $X_{L'})$. This implies that the condition (1) of (4.3) is satisfied up to showing $\pi^* H^0(\mathcal{O}_X(L)) = +1$ eigenspace of i: we omit for brevity the proof of this last fact. The surjectivity of μ implies that condition (2) of (4.3) is also satisfied.

(4.8) **Corollary.** $h^1(\mathcal{O}_X(L' - 2E)) = 0$.

Proof. Observe that $h^1(\mathcal{O}_X(L' - 2E)) = h^1(\mathcal{O}_X(-R))$ by Serre duality and that the condition $h^1(\mathcal{O}_X(-R)) = 0$ is equivalent to $h^0(\mathcal{O}_R) = 1$.

Let $CH^*(G_t)$ be the Chow ring of G_t; we recall that $CH^2(G_t)$ is isomorphic to $\mathbf{Z}^2$ and generated by

$\sigma_{11} =$ class of $\{l \in G_t / l \subset H\}$, ($H$ a given hyperplane),

$\sigma_{20} = \{l \in G_t / l \cap M \neq \varnothing\}$, ($M$ a given codimension 3 subspace).

Therefore, by the intersection pairing, $CH^2(G_t) \cong CH^{2t-4}(G_t)$ and the rational equivalence class of a surface S in G_t is $(\sigma_{11} \cdot S, \sigma_{20} \cdot S)$.

(4.9) **Corollary.** *The rational equivalence class of* $X_{L'}$ *is* $(t, 3t - 2)$.

Proof. Let h_1, h_2 be the obvious generators of $CH^*(\mathbf{P}^t \times \mathbf{P}^t)$; an easy exercise shows that, under the homomorphism of Chow rings

$$p_-^*: CH^*(G_t) \to CH^*(\mathbf{P}^t \times \mathbf{P}^t),$$

$p_-^*\sigma_{20} = h_1^2 + h_2^2 + h_1h_2$ and $p_-^*\sigma_{11} = h_1h_2$. On the other hand one computes $\widetilde{X}_{\widetilde{L}} \sim (2t-2)(h_1^{t-2}h_2^t + h_1^t h_2^{t-2}) + 2t(h_1h_2)^{t-1}$. Hence $\sigma_{11}\cdot X_{L'} = \frac{1}{2}p_-^*\sigma_{11}\cdot\widetilde{X}_{\widetilde{L}} = t$, $\sigma_{20}\cdot X_{L'} = \frac{1}{2}p_-^*\sigma_{20}\cdot\widetilde{X}_{\widetilde{L}} = 3t-2$.

Let G_t^* be the dual grassmannian of G_t and $g: G_t^* \to \mathbf{P} = \mathbf{P}H^0(\mathscr{O}_{G_t}(1))$ the map sending $M \in G_t^*$ in the codimension 1 Schubert cycle $\sigma_M = \{l \in G_t / l \cap M \neq \varnothing\}$. g is just the Plücker embedding of G_t^*.

Observe that σ_M is ruled by a pencil of codimension 2 Schubert cycles of class σ_{11}: the elements of this pencil are the grassmannians of lines of the hyperplanes through M.

Assume σ_M does not contain $X_{L'}$: since $\sigma_{11}\cdot X_{L'} = t$ the hyperplane section $L' = \sigma_M \cdot X_{L'}$ is a t-gonal curve. Let $r: H^0(\mathscr{O}_{G_t}(1)) \to H^0(\mathscr{O}_X(L'))$ be the restriction map; of course r induces a linear projection $p: \mathbf{P} \to \mathbf{P}H^0(\mathscr{O}_X(L'))$ of center $|\mathscr{I}_X(1)|$, ($\mathscr{I}_X =$ ideal of $X_{L'}$ in G_t). We expect that, for general $X_{L'}$, p/G_t is a birational morphism. However: assume p/G_t is generically finite, then

(4.10) **Corollary.** *$|\mathscr{O}_X(L')|$ contains a codimension 1 family of t-gonal curves.*

For any polarization L on X let us define

(4.11) $\mathscr{E}_L(s) = \{E/E =$ isolated elliptic curve, $E\cdot L = s$, $h^1(\mathscr{O}_X(L'-2E)) = 0\}$ $(L' = L + K_X)$, then

(4.12) **Theorem.** *Let L be a polarization on X as in (4.2), $L^2 = 4t-2 \geq 10$. Assume $\mathscr{E}_L(t) \neq \varnothing$, then the following conditions are equivalent:*

(1) *The pair (X, L) fits in a Reye diagram of index t,*

(2) *$h^1(\mathscr{O}_X(E+E'-L)) = 1$ for all $E \in \mathscr{E}_L(t)$,*

(3) *$h^1(\mathscr{O}_X(E+E'-L)) = 1$ for some $E \in \mathscr{E}_L(t)$,*

(4) *$L \sim 2E + R$ with $E =$ isolated elliptic curve, $R =$ indecomposable nodal cycle, $E\cdot R = t$.*

Proof. (1) $\Rightarrow$ (2): let $E \in \mathscr{E}_L(t)$, first we show $h^0(\mathscr{O}_X(L'-E-E')) > 0$. By assumption $X_{L'}$ fits in diagram (4.4), therefore, with the same notations used there, we have that $\widetilde{E} = (p_-/\widetilde{X})^*(E)$ is a degree $2t$ elliptic curve which is invariant with respect to the involution $I_t(x, y) = (y, x)$. Let $\widetilde{E}_i$ be the projection of $\widetilde{E}$ on the ith factor of $\mathbf{P}^t \times \mathbf{P}^t$ then $\deg(\widetilde{E}_i) = t$ and $\widetilde{E}_i$ is contained in a hyperplane. This means $\widetilde{E} \subset \{x_0 = y_0 = 0\}$ where $(x_0 : \cdots : x_t) \times (y_0 : \cdots : y_t)$ are suitable coordinates on $\mathbf{P}^t \times \mathbf{P}^t$. The same holds for $\widetilde{E}' = (p_-/\widetilde{X})^*(E')$. Assume we have again $\widetilde{E}' \subset \{x_0 = y_0 = 0\}$. Let $H = \{x_0y_0 = 0\}$; note that, by (4.3)(2), $\widetilde{X}$ is not in H and that H is $+1$-invariant with respect to I_t. Therefore $H\cdot\widetilde{X}$ would be the pull back of an element of $|\mathscr{O}_X(L)|$ and $h^0(\mathscr{O}_X(L-E-E')) > 0$. Since $h^0(\mathscr{O}_X(L-E-E')) = h^1(\mathscr{O}_X(L'-2E)) = 0$ this is impossible. Hence, up to changing coordinates, we can assume $\widetilde{E} \subset \{x_0 = y_0 = 0\}$, $\widetilde{E}' \subset \{x_1 = y_1 = 0\}$. Finally consider $\tilde{\sigma} = \{x_0y_1 - x_1y_0 = 0\}$ and assume $\widetilde{X} \subset \tilde{\sigma}$. Then the rational function on $\widetilde{X}$: $\tilde{f} = x_1/x_0 = y_1/y_0$

is $+1$-invariant with respect to the fixed-point-free involution of $\widetilde{X}$. On the other hand $\tilde{f} = (p_-/\widetilde{X})^* f$ where $\operatorname{div}(f) = E - E'$. As is well known the pull back of f is -1-invariant, hence $\widetilde{X}$ is not in $\tilde{\sigma}$. Note that $\tilde{\sigma} = p_-^*(\sigma)$, where σ is a hyperplane section of G_t (more precisely: a codimension 1 Schubert cycle). Hence $X \cdot \sigma = R + E + E'$, $h^0(\mathscr{O}_X(L' - E - E')) > 0$. To show $h^1(\mathscr{O}_X(E + E' - L)) = 1$ first observe that $h^0(\mathscr{O}_X(R)) = h^1(\mathscr{O}_X(E + E' - L))$. Moreover $h^1(\mathscr{O}_X(-R)) = h^1(\mathscr{O}_X(L' - 2E)) = 0 \Rightarrow h^0(\mathscr{O}_R) = 1 \Rightarrow R$ is connected. Then, since $p_a(R) = 0$, $h^0(\mathscr{O}_X(R)) = 1$.

(2) $\Rightarrow$ (3): obvious.

(3) $\Rightarrow$ (4): $h^1(\mathscr{O}_X(E + E' - L)) = 1 \Rightarrow L' - E - E' \sim R$, with R effective, $p_a(R) = 0$, $E \cdot R = t$. The condition $h^0(\mathscr{O}_R) = 1$ can be shown as above. (4) $\Rightarrow$ (1) is Proposition 4.5.

(4.13) **Definition.** Let L be a polarization on X as in (4.2). Assume (i) $\mathscr{E}_L(t) \neq \varnothing$, (ii) the equivalent conditions of Theorem (4.13) are satisfied. Then we say that $L' = L + K_X$ is a Reye polarization of index t and $X_{L'}$ a Reye congruence of index t.

5. Reye constructions

The original construction of Reye congruences in G_3 was given as follows:

(5.1) Let W be a general 3-dimensional linear system of quadrics of $\mathbf{P}^3$ and

$$X = \{l \in G_3 / l \text{ is included in two quadrics of } W\}$$

then X is a Reye congruence and, conversely, all Reye congruences arise in this way. At the end we want to point out that (5.1) generalizes again to Reye congruences in G_t:

(5.2) **Theorem.** *Let X be a Reye congruence of index t. Then there exists a linear system W of quadrics of $\mathbf{P}^t$ such that*

(i) $\dim W = \binom{t}{2}$;

(ii) $X = \{l \in G_t \cap \mathbf{P}^{2t-1} / \operatorname{codim}(W_l, W) = 2\}$;

where $W_l = \{Q \in W / l \subset Q\}$, $\mathbf{P}^{2t-1} =$ linear span of X in the Plücker space of G_t.

To show (5.2) we first consider the following situation: $E =$ elliptic curve; $\mathscr{O}_E(L_1)$, $\mathscr{O}_E(L_2) =$ line bundles of degree $t \geq 3$ on E. Assume $L_1 \not\sim L_2$ and consider

$$E \xrightarrow{\gamma_1 \times \gamma_2} \mathbf{P}^{t-1} \times \mathbf{P}^{t-1} \xrightarrow{\sigma} \mathbf{P}^{t^2-1}$$

with $\sigma =$ Segre inclusion and $\gamma_j =$ map associated to $|L_j|$. Let $E' = \sigma \cdot (\gamma_1 \times \gamma_2)(E)$, then

(5.3) **Lemma.** *Let Λ be the $(2t-1)$-space spanned by E' in the ambient space of $\mathbf{P}^{t-1} \times \mathbf{P}^{t-1}$ then $E' = \Lambda \cap \mathbf{P}^{t-1} \times \mathbf{P}^{t-1}$.*

Proof. For brevity we will leave some details as an exercise. Consider the multiplication map $\mu \colon H^0(\mathscr{O}_E(L_1)) \otimes H^0(\mathscr{O}_E(L_2)) \to H^0(\mathscr{O}_E(L_1 + L_2))$. Since μ is surjective $\dim \operatorname{Ker}(\mu) = t^2 - 2t$ and E' is contained in $t^2 - 2t$ linearly independent hyperplanes of $\mathbf{P}^{t^2-1}$. Let Λ be the intersection of them and

$$Z = \Lambda \cap \mathbf{P}^{t-1} \times \mathbf{P}^{t-1}.$$

We have to show that $E' = Z$: this is done by induction on t.

$t = 3$: in this case $\Lambda = \mathbf{P}^5$, $E' =$ sextic curve. Note that the Segre variety $\mathbf{P}^2 \times \mathbf{P}^2$ is a 4-fold of degree 6 in $\mathbf{P}^8$. Assume $E' \subset Z$, then, by Bézout theorem and its corollaries, $\dim Z \geq 2$ and there exists an irreducible component Y of Z which contains properly E'. Let h_j $(j = 1, 2)$ be the obvious generators of $CH^*(\mathbf{P}^2 \times \mathbf{P}^2)$, $\pi_j\colon \mathbf{P}^2 \times \mathbf{P}^2 \to \mathbf{P}^2$ the canonical projections. If $\dim Y = 2$ then $\deg(Y) \leq 5$; assume $\dim(Y) = 2$, $\deg(Y) = 5$ or $\dim(Y) = 3$ then $Y \sim h_j^2 + 2h_1h_2$ or $Y \sim h_j$. In both cases one can check that $\pi_j(Y)$ is a line. Since $\pi_j(E') \subset \pi_j(Y)$ it follows that $\pi_j(E')$ does not span $\mathbf{P}^2$: a contradiction. Hence $\dim(Y) = 2$, $\deg(Y) = 4$; this implies $Y \sim h_1^2 + h_2^2 + h_1h_2$, $Y =$ Veronese surface. Hence Y is the graph of a projective isomorphism $f\colon \mathbf{P}^2 \to \mathbf{P}^2$ such that $f(\pi_1(E')) = \pi_2(E')$. Since $L_1 \not\sim L_2$ this is again a contradiction. Therefore $E' = \Lambda \cap \mathbf{P}^2 \times \mathbf{P}^2$.

$t > 3$: let $p = (p_1, p_2) \in E'$, $\phi_i\colon \mathbf{P}^{t-1} \to \mathbf{P}^{t-2}$ the projection from p_i and

$$\phi = \phi_1 \times \phi_2\colon \mathbf{P}^{t-1} \times \mathbf{P}^{t-1} \to \mathbf{P}^{t-2} \times \mathbf{P}^{t-2}.$$

The fundamental locus of ϕ is $\{p_1\} \times \mathbf{P}^{t-1} \cup \mathbf{P}^{t-1} \times \{p_2\}$. Let Γ be the linear span of this set,

$$\gamma\colon \mathbf{P}^{t^2-1} \to \mathbf{P}^{(t-1)^2-1}$$

the projection from Γ; then $\sigma \cdot \phi = \gamma/\mathbf{P}^{t-1} \times \mathbf{P}^{t-1}$ (with $\sigma =$ Segre embedding of $\mathbf{P}^{t-2} \times \mathbf{P}^{t-2}$). Now observe the following:

(i) $\phi/E' = \psi_1 \times \psi_2$ with $\psi_i =$ morphism associated to $|L_i - p|$;
(ii) γ/Λ is the projection from $l =$ tangent line to E' at p and $\Gamma \cap \Lambda = l$.

Let $\Lambda' = \gamma(\Lambda)$, $E'' = \phi(E')$; then Λ' is the linear span of E'' and, by (i) and the induction

$$E'' = \Lambda' \cap \mathbf{P}^{t-2} \times \mathbf{P}^{t-2}.$$

Therefore, to complete the proof of our statement, it remains only to show that

$$\forall q \in E' \qquad \overline{(\gamma/\Lambda)^*(\phi(q))} \cdot \overline{\phi^*(\phi(q))} = p + q \tag{5.4}$$

(the overline denoting Zariski closure). Now $\overline{(\gamma/\Lambda)^*(\phi(q))} = \alpha_q =$ plane containing l and q; while $\overline{\phi^*(\phi(q))} = l_1 \times l_2$, with $l_i =$ line through p_i. The Segre map embeds $l_1 \times l_2$ in $\mathbf{P}^{t^2-1}$ as a smooth quadric surface S_q. Let P_q be the 3-space of S_q, l_q the line joining p to q; then $l_q \subseteq \alpha_q \cap S_q$. Clearly (5.4) holds if and only if: (1) l_q is not in S_q, (2) α_q is not in P_q (i.e. $l_q = \alpha_q \cap P_q$).

Assume $l_q \subset S_q$ then l_q is $\{p_1\} \times l_1$ or $l_1 \times \{p_2\}$ and $\pi_1(l_q)$ or $\pi_2(l_q)$ is a point; since l_q is bisecant to E' and $\pi_i\colon E' \to \mathbf{P}^{t-1}$ is an embedding we get a contradiction. Hence (1) holds.

Assume $l \subset P_q$; by (ii) $l \subset \Gamma \cap P_q$ which is just the plane of the lines $\{p_1\} \times l_2, l_1 \times \{p_2\}$ (i.e. the tangent plane to S_q at p). Hence the image of l by the tangent map $d(\pi)_p$ is l_i. This implies $l_i =$ tangent line to $\pi_i(E')$ at p_i and $p = q$. Therefore $p \neq q \Rightarrow l$ not in $P_q \Rightarrow \alpha_q$ not in P_q and (2) holds for $p \neq q$. After some tedious remarks one can show (2) also in the case $p = q$.

(5.5) *Remark.* With some more effort one can show the same for any divisor of canonical type.

Now let X be an Enriques surface, $L' = L + K_X$ a Reye polarization of index t on it; by (4.4) the pair (X, L) defines the Reye diagram

$$\begin{array}{ccccc}
\mathbf{P}^- & \xrightarrow{p_-} & \mathbf{P}^\otimes & \xrightarrow{p_+} & \mathbf{P}^+ \\
\cup & & \cup & & \cup \\
G_t & \xleftarrow{p_-} & \mathbf{P}^t \times \mathbf{P}^t & \xrightarrow{p_+} & \Sigma_t \\
\cup & & \cup & & \cup \\
X_{L'} & \xrightarrow{p_-/\widetilde{X}} & \widetilde{X} & \xrightarrow{p_+/\widetilde{X}} & X_L
\end{array}$$

we have

(5.6) **Lemma.** *Let $\widetilde{\Lambda}$ be the linear span of $\widetilde{X}$ in $\mathbf{P}^\otimes$, then $\widetilde{X} = \widetilde{\Lambda} \cap \mathbf{P}^t \times \mathbf{P}^t$.*

Proof. Let $\pi_j : \mathbf{P}^t \times \mathbf{P}^t \to \mathbf{P}^t$ be the canonical projections $(j = 1, 2)$, $X_j = \pi_j(\widetilde{X})$. By (4.12) $L \sim 2E + R$, with $E =$ isolated elliptic curve, $R =$ indecomposable nodal cycle and $E \cdot R = t$. As in the proof of (4.5) consider $|\widetilde{E}| = |\pi^* E|$, $R_1 + R_2 = \pi^* R$: we know that, for the fixed points free involution i on $\widetilde{X}$, $i = I_t/\widetilde{X}$, where I_t is the involution on $\mathbf{P}^t \times \mathbf{P}^t$ sending (x, y) in (y, x); moreover $\mathscr{O}_{X_j}(1) \cong \mathscr{O}_{\widetilde{X}}(\widetilde{E} + R_j)$. Therefore we can fix projective coordinates $(x_0 \cdots x_t) \times (y_0 \cdots y_t)$ on $\mathbf{P}^t \times \mathbf{P}^t$ such that $i = I_t/\widetilde{X}$ and the divisor associated to the rational function x_0/x_1 on X_1 is $F_0 - F_1$, $(F_0, F_1 \in |\widetilde{E}|)$. Since x_0/x_1 is anti-invariant we have $-x_0/x_1 = i^*(x_0/x_1) = y_0/y_1$, so that $\widetilde{X} \subset H$ where the equation of H in $\mathbf{P}^t \times \mathbf{P}^t$ is $x_0 y_1 + x_1 y_0 = 0$. H contains the pencil of divisors

$$H_z = \{z_0 x_0 + z_1 x_1 = z_0 y_0 - z_1 y_1 = 0\}$$

with $z = (z_0, z_1) \in \mathbf{P}^1$. Observe that $H_z \cap \widetilde{X} \supset \widetilde{E}_z$ for some $\widetilde{E}_z \in |\widetilde{E}|$ and that $H_z = \mathbf{P}^{t-1} \times \mathbf{P}^{t-1}$. Then consider $\mathscr{O}_{\widetilde{E}_z}(L_j) = \mathscr{O}_{\widetilde{E}_z}(\widetilde{E} + R_j)$: the embedding of $\widetilde{E}_z$ in H_z is exactly the one described above in Lemma (5.3); moreover, applying standard exact sequences, one easily shows $\mathscr{O}_{\widetilde{E}_z}(L_1) \neq \mathscr{O}_{\widetilde{E}_z}(L_2)$. Therefore, by Lemma (5.4), we obtain

$$\widetilde{E}_z = \Lambda_z \cap H_z,$$

Λ_z being the linear span of $\widetilde{E}_z$. Finally $\widetilde{E}_z = \widetilde{\Lambda} \cap H_z \ (\forall z) \Rightarrow \widetilde{X} = \widetilde{\Lambda} \cap H \Rightarrow \widetilde{X} = \widetilde{\Lambda} \cap \mathbf{P}^t \times \mathbf{P}^t$ (because H is a hyperplane section of $\mathbf{P}^t \times \mathbf{P}^t$).

Let $\mathscr{I}_{\widetilde{X}}$ be the ideal of $\widetilde{X}$ in $\mathbf{P}^t \times \mathbf{P}^t$, J the ideal of $\widetilde{E}_z$ in H_z; consider the natural restriction map

$$r : H^0(\mathscr{I}_{\widetilde{X}}(1)) \to H^0(\mathscr{T}(1));$$

By Lemma (5.3), to show $\widetilde{E}_z = \widetilde{\Lambda} \cap H_z$, it suffices to show that r is surjective. For this let $\mathscr{I}_z$ be the ideal of H_z in $\mathbf{P}^t \times \mathbf{P}^t$; observe that $H^1(\mathscr{I}_{\widetilde{X}}(1)) = H^1(\mathscr{I}_z(1)) = 0$ because $\widetilde{X}$ and H_z are linearly normal. For the same reason

$H^1(\mathscr{O}_{\widetilde{X}}(\widetilde{L}-\widetilde{E}))=0$. Then we have a commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & K & \to & H^0(\mathscr{I}_{\widetilde{X}}(1)) & \xrightarrow{r} & H^0(\mathscr{T}(1)) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & H^0(\mathscr{I}_z(1)) & \to & H^0(\mathscr{O}_{\mathbf{P}^t\times\mathbf{P}^t}(1)) & \to & H^0(\mathscr{O}_{H_z}(1)) & \to & 0 \\
 & & r'\downarrow & & \downarrow & & \downarrow & & \\
0 & \to & H^0(O_{\widetilde{X}}(\widetilde{L}-\widetilde{E})) & \to & H^0(\mathscr{O}_{\widetilde{X}}(\widetilde{L})) & \to & H^0(\mathscr{O}_{\widetilde{E}_z}(1)) & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

the second and third rows (columns) of which are exact. By the snake lemma r is surjective iff r' is surjective. One easily computes $h^0(\mathscr{I}_z(1))=2t+1$, $h^0(\mathscr{O}_{\widetilde{X}}(\widetilde{L}-\widetilde{E}))=2t$. Let $s=x_0y_1+x_1y_0$, since $\operatorname{div}(s)=H$, s belongs to K.

Let s' be another element of K: as for any element of $H^0(\mathscr{I}_z(1))$ we can write $s'=(z_0x_0+z_1x_1)A+(z_0y_0-z_1y_1)B$ (with $A(B)=$ linear form in $(y_0\cdots y_t)$ (in $(x_0\cdots x_t)$)). Let $H'=\operatorname{div}(s')$; then H_z moves in a pencil $\{H'_t,\ t\in\mathbf{P}^1\}$ on H' having the same properties of the pencil $\{H_z\}$ of H. In particular, changing t by z, we have $\widetilde{E}_z\subset H'_z\ \forall z$. Let $h_{1,z}, h_{2,z}$ be the equations of H'_z, then, for the two projections of $\widetilde{E}_z$, we have $\pi_1(\widetilde{E}_z)\subset\{h_{1,z}=0\}\cap\{z_0x_0+z_1x_1=0\}$ and $\pi_2(\widetilde{E}_x)\subset\{h_{2,z}=0\}\cap\{z_0y_0-z_1y_1=0\}$. Now there is a unique hyperplane containing $\pi_j(\widetilde{E}_z)$ because

$$h^0(\mathscr{O}_{X_j}(1)\otimes\mathscr{O}_{X_j}(-\widetilde{E}_x))=h^0(\mathscr{O}_{X_j}(R))=1.$$

Therefore $H_z=H'_z\ \forall z$, $H=H'$ and $s'=cs$ for some constant c. Hence $\dim(K)=1$ and the proof is complete.

Finally we can give a

Proof of (5.2). By (5.6) $\widetilde{X}=\widetilde{\Lambda}\cap\mathbf{P}^t\times\mathbf{P}^t$. Let $\Lambda_+=p_+(\widetilde{\Lambda})$, $\Lambda_-=p_-(\widetilde{\Lambda})$. By the Reye diagram $X_{L'}\subset\Lambda_-\cap G_t\subset\mathbf{P}^-$, $X_L\subset\Lambda_+\cap\Sigma_t\subset\mathbf{P}^+$ and $\dim(\Lambda_-)=\dim(\Lambda_+)=2t-1$. We denote by W the linear system of hyperplanes of $\mathbf{P}^+$ containing Λ_+. Clearly $\dim(W)=\dim(\mathbf{P}^+)-\dim(\Lambda_-)-1=\binom{t}{2}$.

Recall, (3.4), that $\mathbf{P}^+=\mathbf{P}(\operatorname{Sym}^2V^*)=\mathbf{P}H^0(\mathscr{O}_{\mathbf{P}(V)}(2))$ so that W is a linear system of quadrics of $\mathbf{P}^t=\mathbf{P}(V)$. Now identify (canonically) $\operatorname{Sym}^2(V^*)$ to the vector space of symmetric bilinear maps on $V\times V$; then denote by $\widehat{W}$ the subvector space whose projectivization is W. Let $x\in\mathbf{P}^t\times\mathbf{P}^t$, x not in the diagonal. x corresponds (up to scalars) to a pair (u_1, u_2) of linearly independent vectors of V; moreover the line $u=p_-(x)$ is the projectivization of the vector space $U=\langle u_1, u_2\rangle$.

Observe that, by Lemma (5.6), $x\in\widetilde{X}$ if and only if $u\in\Lambda_-$ and $q(u_1, u_2)=0$ for all symmetric bilinear maps $q\in\widehat{W}$. Let $\widehat{W}_U=\{q\in\widehat{W}/U$ is isotropic for $q\}$: it is a standard exercise in linear algebra showing that $\operatorname{codim}(\widehat{W}_U, \widehat{W})\leq 2$ if and only if there exist vectors $u_1, u_2\in U/q(u_1, u_2)=0\ \forall q\in\widehat{W}$. Hence $X_{L'}=p_-(\widetilde{X})=\{u\in G_t\cap\Lambda_-/\operatorname{codim}(W_u, W)\leq 2\}$. Assume $\operatorname{codim}(W_u, W)\leq 1$ (for some $u\in X_{L'}$); then W has a based point p on u. Equivalently (p, p) is contained in each hyperplane of $\mathbf{P}^{\otimes}$ which contains $\widetilde{\Lambda}$ and is $+1$-invariant with respect to I_t. On the other hand, since it is in the diagonal of $\mathbf{P}^t \times \mathbf{P}^t$, (p, p) is in all -1 hyperplanes of I_t Hence $(p, p) \in \widetilde{\Lambda}$ and, by Lemma 5.6, $(p, p) \in \widetilde{X}$: this is a contradiction because $I_t/\widetilde{X}$ is base-point-free. Hence $\operatorname{codim}(W_u, W) = 2$ and the proof is complete.

6. Existence of Reye congruences of index $t \geq 3$

Now we want to check for which values of t the Reye construction is possible. Let X be a nodal Enriques surface,

$\mathscr{R}(X) = \{R \subset X/R =$ indecomposable nodal cycle$\}$;

$\mathscr{E}(X) = \{E \subset X/E =$ isolated elliptic curve$\}$, and

(6.1) $$m\colon \mathscr{R}(X) \times \mathscr{E}(X) \to \mathbf{N}$$

the intersection product. By (4.12) X admits a Reye polarization of index t if and only if $t \in \operatorname{Im}(m)$. In this section we show that actually $\operatorname{Im}(m) = \mathbf{N}$ on a general nodal X. To make precise what we mean by general we give the following

(6.2) **Definition** [10, vol. II]. A nodal Enriques surface X is said to be generic if all the elements of $\mathscr{R}(X)$ have the same numerical equivalence class modulo $2L$, $L = \operatorname{Num}(X)$.

(6.3) **Definition.** Let X be a nodal Enriques surface, C a Reye polarization on X, we will say that C is good if $C \cdot R \geq 4$ $\forall R \in \mathscr{R}(X)$.

Note that C is very ample because there is no $R \in \mathscr{R}(X)$ such that $R \cdot C = 0$. Moreover [10, vol. II]:

(6.4) X is generic if and only if X admits a good Reye polarization.

Finally it is known [9, 10] that a general (in sense of moduli) nodal X is generic.

Let C be a good Reye polarization on X; E_i, E_i' the twenty plane cubics in the smooth Reye model X_C; it is well known (cf. proof of (3.11)) that

(6.5) $$C - E_i - E_i' \sim R_i$$

with $R_i =$ indecomposable nodal cycle, $C \cdot R_i = 4$. Actually, since C is good and $h^0(\mathscr{O}_X(C - R_i)) = 1$, R_i is a rational normal quartic curve in $\mathbf{P}^4$. Our program is to fix one of these curves, e.g.

(6.6) $$R = R_1$$

and construct a sequence $\{F_t, t \geq 1\}$ of isolated elliptic curves such that $F_t \cdot R = t$. First we construct such a sequence numerically: in the lattice $L = \operatorname{Num}(X)$ we consider all the sets $\{e_1 \cdots e_{10}\}$ of isotropic vectors satisfying $e_i \cdot e_j = 1 - \delta_{ij}$, $\frac{1}{3}\sum e_i \in L$ as in (1.1). Let $c = \frac{1}{3}\sum e_i$, we will say that c is a *Fano vector.* Let $l_{ijk} = c - e_i - e_j - e_k$, (with $i, j, k =$ distinct elements of $\{1 \cdots 10\}$); we define

(6.7) $$s_{ijk}\colon L \to L$$

by $s_{ijk}(v) = v + (v \cdot l_{ijk}) l_{ijk}$. Since $l_{ijk}^2 = -2$ s_{ijk} is a reflection and $s_{ijk}(s_{ijk}(v)) = v$. Let $c' = s_{ijk}(c)$. Observe that c' has the following properties:

(6.8) $$c' = 2c - e_i - e_j - e_k.$$

Let $e'_m = s_{ijk}(e_m)$, then $\{e'_1 \cdots e'_{10}\}$ is again a set of isotropic vectors satisfying (1.1), moveover

$$e'_m = e_m, \quad m \neq i, j, k, \quad e'_i = c - e_j - e_k, \\ e'_j = c - e_i - e_k, \quad e'_k = c - e_i - e_j. \tag{6.9}$$

In particular $c' = s_{ijk}(c) = \frac{1}{3}\sum e'_i$ is still a Fano vector. Finally, $\forall v \in L$, consider the intersection numbers

$$d = v \cdot c, \quad v_i = v \cdot e_i, \quad d' = v \cdot c', \quad v'_i = e'_i \cdot v;$$

by definition of s_{ijk} we have

$$d' = 2d - v_i - v_j - v_k, \quad v'_i = d - v_j - v_k, \quad v'_j = d - v_i - v_k, \\ v'_k = d - v_i - v_j, \quad v'_m = v_m \quad (m \neq i, j, k). \tag{6.10}$$

Now we consider the following functions on L:

$$f = s_{178} \cdot s_{256} \cdot s_{234}, \qquad g = s_{178} \cdot s_{256} \cdot s_{234} \cdot s_{1910} \tag{6.11}$$

and

$$h_t = g^{t-1} \cdot f, \qquad t \geq 1. \tag{6.12}$$

Then we define in L the elements

$$c_t = h_t(c) = \frac{1}{3}\sum e^t_i, \qquad f_t = e^t_{10}, \qquad r = c - 2e_1 \tag{6.13}$$

($e^t_i = h_t(e_i)$). Let $v_i = r \cdot e_i$, one immediately computes $v_1 = 3$, $v_i = 1$, $i \geq 2$. Then, applying repeatedly the previous formulae (6.10), one computes with some pain:

$$r \cdot e^t_1 = 2t + 2, \qquad r \cdot e^t_2 = 2t + 1, \qquad r \cdot e^t_3 = \cdots = r \cdot e^t_8 = t + 1, \\ r \cdot e^t_9 = r \cdot e^t_{10} = t \quad \text{and} \quad c_t \cdot r = 4t + 3. \tag{6.14}$$

In particular it follows $r \cdot f_t = t$. Since $f_t^2 = 0$, $f_t \cdot e^t_1 = 1$, f_t has the numerical properties of the class of an isolated elliptic curve on X. On the other hand $r^2 = -2$ so that, numerically, r can be the class of an indecomposable nodal cycle. Now we shall show that, for a given (nodal) Enriques surface X, one can construct in $\mathrm{Num}(X)$ elements r, f_t which are represented by curves $R \in \mathscr{R}(X)$, $F_t \in \mathscr{E}(X)$. Let

(6.15) **Proposition.** *Let X_C be a smooth Fano model; E_i, E'_i its twenty plane cubics. Fix three distinct $i, j, k \in \{1 \cdots 10\}$ and consider the linear system*

$$|H| = |2C - E_i - E_j - E_k|.$$

Assume there is no line in X_C, then H is a very ample Fano polarization.

Proof. Let $L = \mathrm{Num}(X)$, $s_{ijk}L \to L$ be the reflection considered in (6.7), c, h the numerical classes of C, H; note that s_{ijk} is an isometry of L and that, by definition, $s_{ijk}(c) = h$. Therefore H is constructed as in (1.2) from ten isotropic vectors satisfying (1.1). By (1.3) $|H|$ irreducible $\Rightarrow \phi_H$ is a birational morphism. Since H has positive self-intersection the irreducibility of $|H|$ is equivalent to $h^1(\mathscr{O}_X(H)) = 0$ [10]. Consider the exact sequence

$$0 \to \mathscr{O}_X(H - C) \to \mathscr{O}_X(H) \to \mathscr{O}_C(H) \to 0. \tag{6.16}$$

Since $C \cdot (H - C) = 1$ and X_C does not contain lines, it follows that $h^0(\mathcal{O}_X(H - C)) = 0$. Since $C \cdot (C - H) = -1$, it follows that

$$h^0(\mathcal{O}_X(K_X + C - H)) = h^2(\mathcal{O}_X(H - C)) = 0.$$

Then, by Riemann-Roch, $h^1(\mathcal{O}_X(H - C)) = 0$. Furthermore $\mathcal{O}_C(H)$ is a degree 11 line bundle on a genus 6 curve so that $h^1(\mathcal{O}_C(H)) = 0$. Hence, passing to the long exact sequence associated to (6.16), we obtain $h^1(\mathcal{O}_X(H)) = 0$ and $|H|$ irreducible. By (1.3) X_H has at most rational double points.

Let $l, m \in \{1 \cdots 10\}$, $l \neq m$, $F_{lm} = C - E_l - E_m$, and consider the exact sequence

$$(6.17) \qquad 0 \to \mathcal{O}_X(F_{lm} - E_n) \to \mathcal{O}_X(F_{lm}) \to \mathcal{O}_{E_n}(F_{lm}) \to 0$$

$(n \neq l, m)$. Observe that, with exactly the same proof used for $H - C$, $h^0(\mathcal{O}_X(F_{lm} - E_n)) = h^1(\mathcal{O}_X(F_{lm} - E_n)) = 0$. Then, since $h^0(\mathcal{O}_{E_n}(F_{lm})) = 1$, it follows that F_{lm} is an isolated curve in X_C. The degree of F_{lm} is 4 $(C \cdot F_{lm} = 4)$ and the arithmetic genus 1 $(F_{lm}^2 = 0)$. We can show that F_{lm} is a nef divisor: since it is isolated F_{lm} is not nef if and only if $D \cdot F_{lm} < 0$ for some component $D \subset F_{lm}$. Now F_{lm} is a quartic curve and X_C does not contain lines; therefore either F_{lm} is irreducible and nef because $F_{lm}^2 = 0$ or $F_{lm} = D_1 + D_2$ with $D_i =$ smooth conic. In this latter case $D_i^2 = -2$ so that $(D_1 + D_2)^2 = 0 \Rightarrow D_1 \cdot D_2 = 2 \Rightarrow D_i \cdot F_{lm} = 0$; hence F_{lm} is nef. Now assume H is not very ample, then X_H has a rational double point and $H \cdot R = 0$ for some indecomposable nodal cycle R. Observe that

$$(6.18) \qquad H \sim F_{xy} + F_{zs} + E_s, \qquad s \neq z,$$

where (x, y, z) is any permutation of (i, j, k); then $R \cdot H = 0 \Rightarrow R \cdot (F_{xy} + F_{zs} + E_s) = 0$ and, since E_s is also nef, $F_{xy} \cdot R = F_{zs} \cdot R = E_s \cdot R = 0$. In particular, $E_s \cdot R = 0$ $\forall s = 1 \cdots 10$: on the other hand $C \sim \frac{1}{3} \sum E_s$ so that $C \cdot R = 0$: a contradiction because C is very ample. Hence H is very ample too.

Now let us fix

$$(6.19) \qquad \begin{aligned} X &= \text{good nodal Enriques surface,} \\ C &= \text{good Reye polarization on } X, \\ E_i, E_i' &= \text{the } 20 \text{ plane cubics in } X_C. \end{aligned}$$

As above let $e_i =$ numerical class of E_i, $c = \frac{1}{3} \sum e_i =$ class of C, $r = c - 2e_1 =$ class of the rational quartic $R \subset X_C$ as in (6.6). Consider as in (6.13) the Fano vectors $c_t = \frac{1}{3} \sum e_i^t = h_t(c)$ and $f_t = e_{10}^t$, then

(6.20) **Proposition.** (i) *c_t is the numerical equivalence class of a very ample Fano polarization C_t;*

(ii) *X_{C_t} does not contain lines.*

Proof. By induction on t $(t = 1)$. By definition

$$h_1 = f = s_{178} \cdot s_{256} \cdot s_{234}.$$

Let $c = \frac{1}{3} \sum e_i$ be as above, we construct from c the Fano vectors

$$a = s_{234}(c) = \frac{1}{3} \sum e_i^a, \quad b = s_{256}(a) = \frac{1}{3} \sum e_i^b,$$
$$c_1 = s_{178}(b) = \frac{1}{3} \sum e_i^1 = h_1(c).$$

Let $r = c - 2e_1$, and consider the intersection numbers

$$x_i = r \cdot e_i, \quad a_i = r \cdot e_i^a, \quad b_i = r \cdot e_i^b, \quad y_i = r \cdot e_i^1$$

and recall that $x_1 = 3$, $x_i = 1$, $i \geq 2$ (cf. (6.13)). Then, applying the formulae (6.10) to $s_{234}, s_{256}, s_{178}$, it is easy to compute

$$\begin{aligned} (a_1 \cdots a_{10}) &= (3, 2, 2, 2, 1, 1, 1, 1, 1, 1), & a \cdot r &= 5, \\ (b_1 \cdots b_{10}) &= (3, 3, 2, 2, 2, 2, 1, 1, 1, 1), & b \cdot r &= 6, \\ (y_1 \cdots y_{10}) &= (4, 3, 2, 2, 2, 2, 2, 2, 1, 1), & c_1 \cdot r &= 7. \end{aligned} \tag{6.21}$$

a is the class of $A = 2C - E_2 - E_3 - E_4$. Since C is good A is very ample by Proposition (6.15). Let $L \subset X_A$ be a line, l its numerical class. Since X is good $l - r$ is divisible by 2 in $\mathrm{Num}(X)$; then $l \cdot e_i^a = a_i \bmod 2$ and, in particular, $l \cdot e_i^a$ is odd for $i \neq 2, 3, 4$. On the other hand $l \cdot e_i^a \geq 0$ because e_i^a represents a nef divisor, (since A is very ample $2e_i^a$ corresponds to an elliptic pencil, cf. (1.4)). Then we compute $2 < \frac{1}{3} \sum e_i^a \cdot l = a \cdot l = 1$. Hence X_A cannot contain lines. Now consider b: b is the class of $B = 2A - E_2^a - E_5^a - E_6^a$ which is again a very ample Fano polarization by Proposition (6.15). Applying exactly the same arguments used for X_A one shows that X_B does not contain lines. So, by (6.15) again, c_1 is the class of the very ample Fano polarization $C_1 = 2B - E_1^b - E_7^b - E_8^b$.

To complete the first step of induction we must show that X_{C_1} does not contain lines: assume L is a line (of numerical class l) in X_{C_1}, let $l_i = e_i^1 \cdot l$. Since X is good $l_i = y_i \bmod 2$. Then, using (6.21) and $l \cdot c_1 = 1$, one obtains

$$(l_1 \cdots l_{10}) = (0, 1, 0, 0, 0, 0, 0, 0, 1, 1).$$

Note that $s_{178}^2 = \mathrm{id}$ so that $s_{178}(e_i^1) = e_i^b$. Let $l_i' - e_i^b \cdot l$; then, applying formulae (6.10), it is not difficult to compute $(l_1' \cdots l_{10}') = (1, 1, 0, 0, 0, 0, 1, 1, 1, 1)$ so that $b \cdot l = 3$: since $b \cdot r = 6$ and X is good this is a contradiction. Hence there is no line in X_{C_1}.

($t > 1$) This time we start with $c_t = \frac{1}{3} \sum e_i^t$ and we assume $c_t =$ class of C_t, where X_{C_t} is a smooth Fano model not containing lines. Recall that $c_{t+1} = h_t(c) = g(c_t)$ with $g = s_{178} \cdot s_{256} \cdot s_{243} \cdot s_{1910}$ (cf. (6.13)). Then consider the Fano vectors:

$$\begin{aligned} d &= \tfrac{1}{3} \sum e_i^d = s_{1910}(c_t), & m &= \tfrac{1}{3} \sum e_i^m = s_{234}(d), \\ n &= \tfrac{1}{3} \sum e_i^n = s_{256}(m), & c_{t+1} &= \tfrac{1}{3} \sum e_i^{t+1} = s_{178}(n), \end{aligned}$$

and the intersection numbers

$$u_i = e_i^t \cdot r, \quad d_i = e_i^d \cdot r, \quad m_i = e_i^m \cdot r, \quad n_i = e_i^n \cdot r, \quad v_i = e_i^{t+1} \cdot r$$

(r as above). From (6.14) we know that

$$u_1 = 2t + 2, \qquad u_2 = 2t + 1, \qquad u_3 = \cdots = u_8 = t + 1, \qquad u_9 = u_{10} = t$$

and that $c_t \cdot r = 4t + 3$. Therefore, using the formulae (6.10), we can explicitly compute

$$
\begin{aligned}
&(d_1 \cdots d_{10}) = (2t+3\,,\, 2t+1\,,\, t+1\,,\, \dots\,,\, t+1)\,,\\
&(m_1 \cdots m_{10}) = (2t+3\,,\, 2t+2\,,\, t+2\,,\, t+2\,,\, t+1\,,\, \dots\,,\, t+1)\,,\\
(6.22)\quad &(n_1 \cdots n_{10}) = (2t+3\,,\, 2t+3\,,\, t+2\,,\, t+2\,,\\
&\qquad\qquad t+2\,,\, t+2\,,\, t+1\,,\, \dots\,,\, t+1)\,,\\
&(v_1 \cdots v_{10}) = (2t+4\,,\, 2t+3\,,\, t+2\,,\, \dots\,,\, t+2\,,\, t+1\,,\, t+1).
\end{aligned}
$$

Now the proof goes as in the case $t = 1$: let L be any element of $\mathscr{R}(X)$, l its class in $\mathrm{Num}(X)$. By (6.15) $D = 2C_t - E_1^t - E_9^t - E_{10}^t$ is a very ample Fano polarization of class d. Let $l_i^d = l \cdot e_i^d$, then $l_i^d \geq 0$ because e_i^d represents a nef divisor. Since X is good one computes from (6.22): t even $\Rightarrow (l_1^d\,,\, \dots\,,\, l_{10}^d) = (1\,,\, \dots\,,\, 1) \bmod 2 \Rightarrow l \cdot d \geq 3$; t odd $\Rightarrow (l_1^d\,,\, \dots\,,\, l_{10}^d) = (1\,,\, 1\,,\, 0\,,\, \dots\,,\, 0) \bmod 2 \Rightarrow l \cdot d \geq 2$. This implies that there is no line in X_D and that $M = 2D - E_2^d - E_3^d - E_4^d$ is a very ample Fano polarization representing m. Now we just go on in the same way: let $l_i^m = l \cdot e_i^m$; for the same reasons as above we have $l_i^m \geq 0$ and

$$
\begin{aligned}
(l_1^m \cdots l_{10}^m) &= (1\,,\, 0\,,\, 0\,,\, 0\,,\, 1\,,\, 1\,,\, 1\,,\, 1\,,\, 1\,,\, 1) \bmod 2, \quad (t \text{ even}),\\
(l_1^m \cdots l_{10}^m) &= (1\,,\, 0\,,\, 1\,,\, 1\,,\, 0\,,\, 0\,,\, 0\,,\, 0\,,\, 0\,,\, 0) \bmod 2, \quad (t \text{ odd}).
\end{aligned}
$$

This implies that $l \cdot m = 1$ if and only if t is odd and

$$(l_1^m \cdots l_{10}^m) = (1\,,\, 0\,,\, 1\,,\, 1\,,\, 0\,,\, 0\,,\, 0\,,\, 0\,,\, 0\,,\, 0).$$

But now, if such an l exists, we have $l \cdot s_{234}(e_2^m) = l \cdot (m - e_3^m - e_4^m) = -1$; impossible because $s_{234}(e_2^m) = e_2^d =$ class of plane cubic in $X_D =$ class of a nef divisor. Therefore $N = 2M - E_2^m - E_5^m - E_6^m$ is a very ample Fano polarization representing n. Let $l_i^n = l \cdot e_i^n$ then $l_i^n \geq 0$ and we have this time

$$
\begin{aligned}
(l_1^n \cdots l_{10}^n) &= (1\,,\, 1\,,\, 0\,,\, 0\,,\, 0\,,\, 0\,,\, 1\,,\, 1\,,\, 1\,,\, 1) \bmod 2, \quad (t \text{ even}),\\
(l_1^n \cdots l_{10}^n) &= (1\,,\, 1\,,\, 1\,,\, 1\,,\, 1\,,\, 1\,,\, 0\,,\, 0\,,\, 0\,,\, 0) \bmod 2, \quad (t \text{ odd}).
\end{aligned}
$$

Clearly $l \cdot n \geq 2$ so that X_N does not contain lines. Finally we obtain from this the very ample Fano polarization $C_{t+1} = 2N - E_1^n - E_7^n - E_8^n$ of class c_{t+1}. Let $l_i = c_{t+1} \cdot l$, then

$$
\begin{aligned}
(l_1 \cdots l_{10}) &= (0\,,\, 1\,,\, 0\,,\, 0\,,\, 0\,,\, 0\,,\, 0\,,\, 0\,,\, 1\,,\, 1) \bmod 2, \quad (t \text{ even}),\\
(l_1 \cdots l_{10}) &= (1\,,\, 1\,,\, 1\,,\, 1\,,\, 1\,,\, 1\,,\, 0\,,\, 0\,,\, 0\,,\, 0) \bmod 2, \quad (t \text{ odd}).
\end{aligned}
$$

If t is odd $l \cdot c_{t+1} \geq 2$ and $X_{C_{t+1}}$ does not contain lines; if t is even it is completely clear how to complete the proof using the previous arguments.

Finally we can show

(6.22) **Theorem.** *Let X be a good nodal Enriques surface, $m\colon \mathscr{E}(X) \times \mathscr{R}(X) \to$ $\mathbf{N}$ the intersection map considered in* (6.1). *Then*

(1) *m is surjective*;

(2) *X admits a Reye polarization of index t for each $t \geq 3$.*

Proof. Fix on X a good Reye polarization C and its numerical class $c = \frac{1}{3}\sum e_i$. Then, as in (6.13), reconstruct from c the Fano vectors

$$c_t = \tfrac{1}{3}\sum e_i^t \qquad (t \geq 1)$$

and consider also $r = c - 2e_1$, $f_t = e_{10}^t$. By (6.6) r is represented by an element of $\mathscr{R}(X)$ (a rational quartic curve R in X_C). By (6.14) $r \cdot f_t = t$. By Proposition (6.20) c_t is the class of a very ample Fano polarization C_t. Hence, by (1.4), f_t is represented by $F_t =$ plane cubic curve in $X_{C_t} =$ isolated elliptic curve on X. Therefore $F_t \in \mathscr{E}(X)$ and $m(F_t, R) = t$, $t \geq 1$. On the other hand, it is a standard fact that for a given $R \in \mathscr{R}(X)$ there exists $E \in \mathscr{E}(X)$ such that $E \cdot R = 0$. Therefore m is surjective. Finally, by (4.12), the surjectivity of m implies (2).

(6.23) *Remark* (Cremona transformations of $\mathbf{P}^5$). We want to explain without proofs the true geometric construction underlying the numerical arguments used in the section. Fix three distinct planes π_i, π_j, π_k in $\mathbf{P}^5$ such that: (i) there is no hyperplane containing all of them, (ii) any two of them intersect exactly in one point. Then consider the linear system Σ of the quadrics containing $\pi_i \cup \pi_j \cup \pi_k$: Σ is 5-dimensional and defines a birational transformation $s_{ijk}: \mathbf{P}^5 \to \mathbf{P}^5$, this is called a standard Cremona transformation of $\mathbf{P}^5$. Under s_{ijk} each π_m $(m = i, j, k)$ is blown up to a hyperplane, while the hyperplanes containing any two of π_i, π_j, π_k are contracted to three new planes satisfying (i), (ii). Taking them and constructing the corresponding Cremona transformation we obtain the inverse of s_{ijk}. Let $X_C \subset \mathbf{P}^5$ be a smooth Fano model, X_C not contained in a quadric, (note that this is always possible up to replacing C by $C + K_X$). Assume that three plane cubics of X_C, e.g. E_i, E_j, E_k, are contained in π_i, π_j, π_k; then s_{ijk}/X_C is the morphism associated to $|2C - E_i - E_j - E_k|$ and, at least if there is no line in X_C, $s_{ijk}(X_C)$ is a new Fano model by (6.18). Therefore, thinking more geometrically, the reflections s_{ijk}: $\mathrm{Num}(X) \to \mathrm{Num}(X)$ we have used throughout this section could be considered as Cremona transformations of $\mathbf{P}^5$ applied to a suitable Fano model.

References

1. E. Arbarello, M. Cornalba, P. Griffiths, and J. Harris, *Geometry of algebraic curves*. I, Springer-Verlag, Berlin, 1984.
2. I. Shafarevich, *Algebraic surfaces*, Proc. Steklov Inst. Math. **75** (1964).
3. W. Barth, *Lectures on* $K3$ *and Enriques surfaces*, Algebraic Geometry, Sitges 1983, Lecture Notes in Math., vol. 1124, Springer, Berlin, 1985, pp. 21–57.
4. W. Barth, C. Peters, and A. Van de Ven, *Complex algebraic surfaces*, Springer-Verlag, Berlin, 1984.
5. A. Beauville, *Surfaces algébriques complexes*, Astérisque **54**, Soc. Mat. de France, 1980.
6. A. Coble, *The ten nodes of the rational sextic and of the Cayley symmetroid*, Amer. J. Math. **40** (1918), 317–340.
7. F. Cossec, *On the Picard group of Enriques surfaces*, Math. Ann. **271** (1985), 577–600.
8. ——, *Reye congruences*, Trans. Amer. Math. Soc. **280** (1983), 737–751.
9. F. Cossec and I. Dolgachev, *Rational curves on Enriques surfaces*, Math. Ann. **272** (1985), 369–384.
10. ——, *Enriques surfaces*. I, Birkhaüser, Basel, 1989; II (to appear).
11. I. Dolgachev and I. Reider, *On rank* 2 *vector bundles with* $c_1^2 = 10$ *and* $c_2 = 3$ *on Enriques surfaces*, Proc. USA-URSS Conf. on Algebraic Geometry, Chicago, 1989.
12. P. Le Barz, *Formules multisecantes pour les courbes gauches quelconques*, Enumerative Geometry and Classical Algebraic Geometry, Birkhäuser, Basel, 1982, pp. 165–198.

13. E. L. Livorni, *On the existence of some surfaces*, Algebraic Geometry: Proc. Internat. Conf. (L'Aquila, 1988), Lecture Notes in Math., vol. 1417, Springer, Berlin, 1990.
14. A. Verra, *On Enriques surfaces as a fourfold cover of* $\mathbf{P}^2$, Math. Ann. **266** (1983), 241–250.
15. Hoil Kim, *Stable vector bundles on Enriques surfaces*, Ph.D. thesis, Univ. of Michigan, 1990.

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