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COUNTABLE CLOSED LFC-GROUPS WITH p-TORSION

FELIX LEINEN

ABSTRACT. Let LFC be the class of all locally FC-groups. We study the existentially closed groups in the class LFC_p of all LFC-groups H whose torsion subgroup $T(H)$ is a p-group. Differently from the situation in LFC, every existentially closed LFC_p-group is already closed in LFC_p, and there exist $2^{\aleph_0}$ countable closed LFC_p-groups G. However, in the countable case, $T(G)$ is up to isomorphism always a unique locally finite p-group with similar properties as the unique countable existentially closed locally finite p-group E_p.

1. INTRODUCTION

In the last few years, existentially closed groups were an area of intense and fruitful research. Recall that, for a class $\mathfrak{X}$ of groups, $G \in \mathfrak{X}$ is said to be *existentially closed* (e.c.) in $\mathfrak{X}$, if every finite system of equations and inequalities with coefficients in G, which has a solution in some $H \in \mathfrak{X}$ with $G \leq H$, already has a solution in G itself. Strongly related to this concept is the notion of closedness. An $\mathfrak{X}$-group G is said to be *closed* in $\mathfrak{X}$, if whenever B and C are finitely generated $\mathfrak{X}$-groups and whenever $H \in \mathfrak{X}$ satisfies

$$\begin{array}{ccccc} & & C & & \\ & \subsetneq & & \subsetneq & \\ B & & & & H, \\ & \subsetneq & & \subsetneq & \\ & & G & & \end{array}$$

then id_B can be extended to an embedding of C into G. Obviously, every closed $\mathfrak{X}$-group is e.c. in $\mathfrak{X}$. The converse is true for example in classes of locally finite groups, if the underlying language is finite. However, e.c. groups are not closed in general. In the class $\mathfrak{A}$ of all abelian groups, a group is e.c. if and only if it is divisible of infinite p-rank for all primes p, while closed groups must have infinite torsion-free rank too. In particular, there exists a countable infinity of (pairwise nonisomorphic) countable e.c. $\mathfrak{A}$-groups, but just one countable closed $\mathfrak{A}$-group.

In this paper we consider *FC-groups*, i.e., groups whose elements have only finitely many conjugates. We denote by LFC the class of all locally FC-groups, and by $L\mathfrak{F}$ the class of all locally finite groups. Obviously, $\mathfrak{A}$ and $L\mathfrak{F}$

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are contained in LFC. Elementary properties of FC- and of LFC-groups may be found in [19 and 4]. Note that the torsion elements in any LFC-group G form a subgroup $T(G) \in L\mathfrak{F}$ with $G/T(G) \in \mathfrak{A}$. The torsion-free rank of $G/T(G)$ is usually just called the *rank* $r(G)$ of G.

In [5 and 6], F. Haug has studied e.c. LFC-groups extensively. In the countable case, some of his results are as follows.

(1.1) Let G be a countable e.c. LFC-group. Then
- (a) $G' = T(G)$, and G splits over $T(G)$;
- (b) $T(G)$ is isomorphic to P. Hall's countable universal $L\mathfrak{F}$-group ULF (which is the unique countable e.c. $L\mathfrak{F}$-group (see [3 and 14]));
- (c) $G/T(G)$ is divisible (and hence e.c. in the class of all torsion-free $\mathfrak{A}$-groups).

(1.2) For each $\rho \leq \omega$ there exists precisely one countable e.c. LFC-group G_ρ with rank ρ. Whenever $B \subseteq C$ are finitely generated FC-groups of rank $\leq \rho$ with $B \leq G_\rho$, then id_B can be extended to an embedding of C into G_ρ. In particular, G_ρ is closed in the class LFC^ρ of all LFC-groups of rank $\leq \rho$. (An application of [5, Lemma 1.5] and [4, Corollary 1.3] shows that every e.c. LFC^ρ-group is e.c. in LFC, whence G_ρ is actually the unique countable closed LFC^ρ-group.)

Now, let p be a fixed prime. The theory of e.c. $L\mathfrak{F}$-groups has nice parallels in the class $L\mathfrak{F}_p$ of all locally finite p-groups. In particular, there exists a unique countable e.c. $L\mathfrak{F}_p$-group E_p (see [15]). It is thus a quite natural continuation of Haug's investigations, to consider the class LFC_p of all LFC-groups G with $T(G) \in L\mathfrak{F}_p$. Do the statements corresponding to (1.1) and (1.2) hold for the class LFC_p with E_p in place of ULF? In the present paper we will try to illuminate the structure of countable e.c. LFC_p-groups.

In fact, our results about e.c. LFC_p-groups suggest a more general treatment as in [13]. There it was shown that, for every fixed countable abelian p-group A, there exists a unique countable e.c. group E_A in the class $L\mathfrak{F}_{p,A}$ of all $L\mathfrak{F}_p$-groups G with $A \leq Z(G)$. Since A is the Schur multiplier of E_A/A, this gave $2^{\aleph_0}$ isomorphism types of countable $L\mathfrak{F}_p$-groups E_A/A similar to E_p. For example, every E_A/A is verbally complete and characteristically simple, has a unique chief series (of order type $(\mathbb{Q}, <)$), and normality is transitive in E_A/A.

In this paper, we will consider, for each fixed countable abelian p-group A, the class $LFC_{p,A}$ of all $G \in LFC_p$ with $A \leq Z(G)$. Here, the elements of A are constants in the underlying language. Note also that every $LFC_{p,A}$-group G is contained in a closed $LFC_{p,A}$-group of cardinality $\max\{\aleph_0, |G|\}$ (cf. [16, Proposition 1.2] and the remarks after Theorem 4.2). In contrast to (1.2) we will show that every countable e.c. $LFC_{p,A}$-group is already closed in $LFC_{p,A}$ (and thus has infinite rank), and that there exist $2^{\aleph_0}$ countable closed $LFC_{p,A}$-groups for each A. Corresponding results hold for the classes $LFC^\rho_{p,A}$ of $LFC_{p,A}$-groups of rank $\leq \rho$ $(1 \leq \rho \in \omega)$. Since the structure of closed $LFC^\rho_{p,A}$-groups is less homogeneous, we will not study them in detail. Concerning the countable closed $LFC_{p,A}$-groups G, it will turn out that they all have a unique isomorphism type T_A of torsion subgroup, and that (1.1) holds for $LFC_{p,A}$ with T_A in place of ULF, with the only exception that we are not able to show that G splits over $T(G)$. As with E_A/A, the group T_A/A has Schur multiplier A, is verbally complete and characteristically simple, and normality is transitive in T_A/A. However, T_A is not isomorphic to E_A, since in place of the unique chief series of E_A/A, the group T_A/A has a unique normal series (of order type $(\mathbb{Q}, <)$) with infinite elementary-abelian factors. Every countable closed $LFC_{p,A}$-group acts via conjugation on each of these factors in the same way as $K^\times$ acts on K^+ regularly, where K denotes the algebraic closure of $GF(p)$. We will also describe the automorphism group of T_A similarly as in [11, §6 and 1 and 5, §4].

Our techniques are completely different from F. Haug's ones. He employed the permutational product, and also the Černikov embedding of finitely generated FC-groups into direct products of free abelian and of finite groups. Both constructions cannot be used within the class LFC_p, because they increase the number of primes involved in the orders of torsion elements. Instead we modify wreath product embedding techniques as in [9] such that they can be applied to LFC-groups. Our restriction to countable groups derives from these techniques.

2. AT-SERIES AND CLOSEDNESS

Our notions of *normal series, induced series, order type* and *Dedekind completion* of a series coincide with those introduced in [9, pp. 208–209] and [11, §2]. Normal series of $LFC_{p,A}$-groups G refining the series $1 \leq A \leq T(G) \leq G$ will play an important role in our investigations. Such a series we call an *AT-normal series* in G, if it has $G/T(G)$ and $A/1$ among its factors; the factors of such a series between A and $T(G)$ are called *AT-factors.* An *AT-chief series* is an AT-normal series whose AT-factors are chief factors in G; such factors are called *AT-chief factors.*

Lemma 2.1. *Every AT-chief factor of an $LFC_{p,A}$-group G is elementary-abelian and central in $T(G)$.*

Proof. Let M be a periodic minimal normal subgroup of the LFC_p-group G. From [8, 1.B.3] we know that M is elementary-abelian. Choose $m \in M - 1$ and $g \in T(G)$. Then $F = \langle m, m^g \rangle$ is finite. Choose a finitely generated FC_p-group H such that $\langle F, g \rangle \leq H \leq G$ and $F \leq \langle x^H \rangle$ for all $x \in F - 1$. Then every torsion chief factor K/L in H with $m \in K - L$ satisfies $F \leq K$ and $F \cap L = 1$. The torsion subgroup of H is a finite p-group, and hence $K/L \cap Z(T(H)/L) \neq 1$. This immediately yields $K/L \leq Z(T(H)/L)$, and hence $m^g \in mL$. But now $F \cap L = 1$ implies $m^g = m$. This shows that $M \leq Z(T(G))$. □

Lemma 2.2. *Let M/N be an AT-chief factor of the $LFC_{p,A}$-group G. Put $\overline{\overline{G}} = G/C_G(M/N)$ and $\mathscr{k} = GF(p)$. Regard M/N as a $\mathscr{k}\overline{\overline{G}}$-module via conjugation. Then $K = \mathscr{k}\overline{\overline{G}}/An(M/N)$ is a locally finite field, and for any $m_0 \in M - N$ a K-isomorphism $\varepsilon\colon K^+ \to M/N$ is given by $x\varepsilon = (m_0N)^x$ for all $x \in K$.*

Proof. From Lemma 2.1 we know that $\overline{\overline{G}}$ is abelian. Hence there exists a canonical embedding of the ring K into $\mathrm{Hom}_K(M/N, M/N)$, and the latter is a skew field by Schur's Lemma. Now let $G_0 \leq G$ be finitely generated with $m_0 \in G_0$. Since $\overline{\overline{G}}$ is abelian, we have $C_G(M/N) = C_G(m_0N)$ and

$\overline{\overline{G}}_0 \cong G_0/C_{G_0}(m_0N)$. So $G_0 \in FC$ implies that $\overline{\overline{G}}_0$ is finite. Thus $\mathscr{k}\overline{\overline{G}}_0 + An(M/N)/An(M/N)$ is finite, commutative and multiplicatively closed; hence it must be a finite subfield of K. This shows that K is a locally finite field. Moreover ε is surjective because M/N is a simple K-module. Since ε is nontrivial and $\operatorname{Ker}\varepsilon$ is an ideal in the field K, we obtain $\operatorname{Ker}\varepsilon = \{0\}$. □

Because locally finite fields are countable, Lemma 2.2 yields that every AT-chief factor of an $LFC_{p,A}$-group is countable.

Chief factors of $L\mathfrak{F}_p$-groups are central and cyclic of order p. Lemma 2.2 gives a first hint, that this is no longer true for AT-chief factors of $LFC_{p,A}$-groups. We can make this more precise by using semidirect product constructions. If $\theta: G \to H$ is a group homomorphism, and if H acts on a group N, we let

$$N \rtimes_\theta G = \{(g, n) | g \in G, n \in N\} \quad \text{with group multiplication}$$
$$(g_1, n_1)(g_2, n_2) = (g_1g_2, n_1^{g_2\theta}n_2) \quad \text{for all } g_1, g_2 \in G, \; n_1, n_2 \in N.$$

Usually we identify N and G canonically with the corresponding subgroups of $N \rtimes_\theta G$. We suppress θ whenever $\theta = \mathrm{id}$.

Lemma 2.3. *Let K be a locally finite field of characteristic p. Regard K^+ as regular $K^\times$-module. Suppose that G is an $LFC_{p,A}$-group and that there exists a homomorphism $\theta: G \to K^\times$ with $A \leq \operatorname{Ker}\theta$. Then $H = K^+ \rtimes_\theta G$ is an $LFC_{p,A}$-group.*

Proof. Let $H_0 \leq H$ be finitely generated. Since K^+ and $K^\times$ are locally finite, we have $K^+ \rtimes K^\times \in L\mathfrak{F}$. Therefore $H_0 \cap K^+$ is a finite p-group. Clearly, $H_0/H_0 \cap K^+ \cong H_0K^+/K^+$ is an FC_p-group. Now, extensions of finite groups by FC-groups are again FC-groups. Thus $H_0 \in FC_p$. This shows that $H \in LFC_p$. Moreover, $A \leq \operatorname{Ker}\theta$ implies $[A, K^+] = 1$, whence $A \leq Z(H)$. □

Corollary 2.4. *AT-chief factors are in general not central and not cyclic.*

Proof. Let $K \neq GF(p)$ be a locally finite field of characteristic p. Since $K^\times$ is periodic and locally cyclic, there exists a subgroup G of $A \times \mathbb{Q}^+$ with $A \leq G$, and an epimorphism $\theta: G \to K^\times$ with $A \leq \operatorname{Ker}\theta$. Because $K^\times$ acts transitively on $K^+ - \{0\}$, we see that K^+A/A is a noncyclic noncentral AT-chief factor in the $LFC_{p,A}$-group $K^+ \rtimes_\theta G$. □

Corollary 2.4 shows that it is impossible to embed finitely generated FC_p-groups into direct products of free abelian and of finite p-groups, and that finitely generated FC_p-groups need not be residually finite p-groups.

In $LFC_{p,A}$-groups, only elements of infinite order can induce p'-automorphisms on AT-chief factors. Now, for given $g \in G \in LFC_{p,A}$ with $o(g) = \infty$, Lemma 2.3 allows us to construct an $LFC_{p,A}$-supergroup H of G such that g acts as a p'-automorphism on some AT-chief factor of H. Thus we can express by a finite system of equations and inequalities in H that g has infinite order. This provides the key for our

Theorem 2.5. *Every countable e.c. $LFC_{p,A}$-group is closed in $LFC_{p,A}$. If A has finite p-rank, then every e.c. $LFC_{p,A}$-group is closed in $LFC_{p,A}$.*

We denote the class of all finitely generated $LFC_{p,A}$-groups by $FC^0_{p,A}$. Note that every $FC^0_{p,A}$-group F has the form $F = \langle A, f_1, \dots, f_r\rangle$ for some

$f_1, \dots, f_r \in F$, since the elements of A are constants in the underlying language.

Proof of Theorem 2.5. Let G be e.c. in $LFC_{p,A}$. Suppose that $B, C \in FC^0_{p,A}$ and $H \in LFC_{p,A}$ satisfy $B \leq C \leq H$ and $B \leq G \leq H$. Without loss we may assume that $H = \langle G, C\rangle$. Denote epimorphic images modulo torsion subgroups by bars. Choose $c_1, \dots, c_r \in C$ such that $\overline{C} = \langle \overline{c}_1\rangle \times \cdots \times \langle \overline{c}_r\rangle$. Since $\overline{G}$ is divisible by Theorem 5.2, we may assume that $\overline{c}_1, \dots, \overline{c}_\nu \in \overline{G}$ and $\overline{H} = \overline{G} \times \langle \overline{c}_{\nu+1}\rangle \times \cdots \times \langle \overline{c}_r\rangle$ for some $\nu \leq r$. Let K be the algebraic closure of $GF(p)$. In the case when $p \neq 2$ put $q = p$, otherwise $q = 4$.

We construct a chain $H = H_0 \leq H_1 \leq \cdots \leq H_r$ in $LFC_{p,A}$ with $\overline{H}_i = \overline{H}$ for $1 \leq i \leq r$ as follows. Suppose that H_{i-1} has been found. The pure subgroup $\overline{C}_i$ of $\overline{H}_{i-1}$ generated by $\{\overline{c}_i\}$ has a direct complement $\overline{D}_i$ in $\overline{H}_{i-1}$ which contains the pure subgroup of $\overline{H}_{i-1}$ generated by $\{\overline{c}_j | j \neq i\}$. Since $\overline{C}_i/\langle \overline{c}_i^{q-1}\rangle$ is periodic and locally cyclic, there exists a homomorphism $\varphi_i \colon \overline{C}_i \to K^\times$ with $GF(q)^\times = \langle \overline{c}_i\varphi_i\rangle$. Define $\theta \colon H_{i-1} \to K^\times$ via $h\theta = \overline{h\varphi_i}$ for all $h \in H_{i-1}$, where $\overline{\varphi}_i \colon \overline{H}_{i-1} \to K^\times$ is determined by $\overline{\varphi}_i|\overline{C}_i = \varphi_i$ and $\overline{\varphi}_i|\overline{D}_i \equiv 1$. Put $H_i = K^+ \rtimes_\theta H_{i-1}$ and $K_i = GF(q)^+ \leq K^+ \leq H_i$. Then K_i is a finite p-group, and $c_j \in C(K_i)$ for all $j \neq i$, while c_i induces an automorphism of order $q-1$ on K_i, which permutes the nontrivial elements in K_i transitively. Also, $H_i \in LFC_{p,A}$ with $\overline{H}_i = \overline{H}_{i-1}$ by Lemma 2.3.

Choose a transversal T of A in $T(C)$ with $1 \in T$. Then T is finite, since TA/A is the torsion subgroup of $C/A \in FC^0_p$. Consider the following finite systems of equations and inequalities with unknowns $(x_t | t \in T)$, $(y_1, \dots, y_r)$, and $(z_k | k \in K_i - 1,\ 1 \leq i \leq r)$.

$$(2.1)\quad \begin{array}{ll} x_{t_1}x_{t_2} = x_{t_3}a & \text{whenever } t_1, t_2, t_3 \in T \text{ and } a \in A \text{ satisfy } t_1t_2 = t_3a; \\ y_i^{-1}x_{t_1}y_i = x_{t_2}a & \text{for } 1 \leq i \leq r \text{ whenever } t_1, t_2 \in T \text{ and } a \in A \text{ satisfy} \\ & c_i^{-1}t_1c_i = t_2a; \\ [y_1, y_2] = x_ta & \text{for } 1 \leq i, j \leq r \text{ whenever } t \in T \text{ and } a \in A \\ & \text{satisfy } [c_i, c_j] = ta. \end{array}$$

$$(2.2)\quad \begin{array}{ll} y_i^{-1}z_{k_1}y_i = z_{k_2} & \text{for } 1 \leq i \leq r \text{ for all } k_1, k_2 \in K_i - 1 \text{ with } c_i^{-1}k_1c_i = k_2; \\ [y_i, z_k] = 1 & \text{for } 1 \leq i, j \leq r \text{ with } i \neq j \text{ and all } k \in K_j - 1; \\ \multicolumn{2}{l}{\text{the group table } K_i \text{ in the unknowns } (z_k | k \in K_i - 1) \text{ for } 1 \leq i \leq r} \\ \multicolumn{2}{l}{\text{(including the inequalities } z_k \neq 1).} \end{array}$$

Suppose that G contains a simultaneous solution $(u_t | t \in T)$, $(v_1, \dots, v_r)$, and $(w_k | k \in K_i - 1,\ 1 \leq i \leq r)$ to (2.1) and (2.2). Let $G_0 = \langle A, v_1, \dots, v_r, u_t | t \in T\rangle$. Define $\psi \colon C \to G_0$ via

$$\psi(c_1^{n_1} \cdots c_r^{n_r} \cdot ta) = v_1^{n_1} \cdots v_r^{n_r} \cdot u_ta \quad \text{for all } n_1, \dots, n_r \in \mathbb{Z},\ t \in T,\ a \in A.$$

Then ψ is a homomorphism by virtue of (2.1).

Put $L_i = \langle w_k | k \in K_i - 1\rangle$. From (2.2) we know that $L_i \cong K_i$, and that v_i induces an automorphism of order $q-1$ on L_i which permutes the elements from $L_i - 1$ transitively. In particular, $o(v_i) = \infty$, and there exists an AT-chief factor M_i/N_i in G such that $L_i \leq M_i$ and $N_i \cap L_i = 1$. Clearly $v_i \notin G_i =$

$C_{G_0}(M_i/N_i)$. Since $v_j \in C_{G_0}(L_i)$ for $j \neq i$, Lemma 2.2 yields that $v_j \in G_i$. Thus the canonical embedding

$$G_0/\bigcap\{G_i|1 \leq i \leq r\} \to \prod\{G_0/G_i|1 \leq i \leq r\} = \prod\{\langle v_iG_i\rangle|1 \leq i \leq r\}$$

is an isomorphism, and therefore $G_0/\bigcap\{G_i|1 \leq i \leq r\}$ has Prüfer rank r. (Note that $o(v_iG_i) = q-1$ for all i.) This shows that $\overline{G}_0 = \langle\overline{v}_1\rangle \times \cdots \times \langle\overline{v}_r\rangle$.

If $x = c_1^{n_1}\cdots c_r^{n_r}\cdot ta \in \operatorname{Ker}\psi$, then $\overline{v}_1^{n_1}\cdots\overline{v}_r^{n_r} = 0$, and so $n_1 = \cdots = n_r = 0$. It follows that $u_t = a^{-1} \in \{u_t|t \in T\} \cap A$, and $\operatorname{Ker}\psi$ will be trivial if we can ensure that $\{u_t|t \in T\} \cap A = 1$. If A has finite p-rank, then $A[p^n] = \{a \in A|o(a)|p^n\}$ is finite for all $n \in \omega$. In this case we supplement (2.1) and (2.2) by

$$\text{(2.3)} \qquad x_t \neq a \quad \text{for all } t \in T-1 \text{ and all } a \in A[o(t)].$$

Since ψ is a homomorphism, $o(u_t)$ divides $o(t)$. Thus (2.3) yields that $u_t \notin A$ for every $t \in T-1$. In the other case, G is countable by assumption. Then we may assume without loss that H is countable, whence H_r is contained in a countable closed $LFC_{p,A}$-group H_{r+1}. But Theorem 5.1 yields that $Z(H_{r+1}) = A$. In this case we supplement (2.1) and (2.2) by

$$\text{(2.3)} \qquad [x_t, \xi_t] \neq 1 \quad \text{for all } t \in T-1,$$

where $(\xi_t|t \in T-1)$ are additional unknowns. This yields $u_t \notin Z(G) \geq A$ for all $t \in T-1$.

Finally, let $B = \langle A, b_1, \ldots, b_s\rangle$. Because of $B \leq C$ there exist words w_j in elements from $\{c_1, \ldots, c_r\} \cup T$ such that

$$b_j = w_j a_j \quad \text{for } 1 \leq j \leq s \text{ and suitable } a_j \in A.$$

Let w_j^* be the word obtained from w_j by replacing c_i by y_i and t by x_t. Supplement (2.1)–(2.3) by

$$\text{(2.4)} \qquad b_j = w_j^* a_j \quad \text{for } 1 \leq j \leq s.$$

Since the $LFC_{p,A}$-supergroup H_r resp. H_{r+1} of the e.c. $LFC_{p,A}$-group G contains a simultaneous solution to the systems (2.1)–(2.4), there does already exist a solution in G. As before, (2.1) gives rise to a homomorphism $\psi: C \to G$, which is injective because of (2.2) and (2.3), and which satisfies $\psi|B = \mathrm{id}_B$ by virtue of (2.4). □

By Theorem 5.1 there exists an uncountable closed $LFC_{p,A}$-group with centre greater than A. Therefore, the method of proof of Theorem 2.5 cannot be applied to uncountable e.c. $LFC_{p,A}$-groups G when A has infinite rank, since we cannot ensure that for every $t \in T-1$ there exists an $LFC_{p,A}$-supergroup H of G with $[t, h] \neq 1$ for some $h \in H$.

Question 2.6. *Do there exist uncountable e.c. $LFC_{p,A}$-groups which are not closed in $LFC_{p,A}$ (when A has infinite p-rank)?*

Clearly, every closed $LFC_{p,A}$-group contains a copy of every $FC^0_{p,A}$-group. Together with Theorem 2.5 this immediately yields

Corollary 2.7. *Every countable e.c. $LFC_{p,A}$-group has infinite rank. If A has finite p-rank, then every e.c. $LFC_{p,A}$-group has infinite rank.*

This result sharply contrasts with the fact, that e.c. LFC-groups can have any prescribed rank [5, Lemma 2.2 and 6, §9].

3. Modified standard embeddings

For the investigation of an e.c. group G in some class $\mathfrak{X}$ it is necessary to construct $\mathfrak{X}$-supergroups of G. In LFC and $L\mathfrak{F}_p$ quite elegant constructions were made by means of the permutational product [4 and 11]. By a synthesis of both cases one would expect that it be possible to use the permutational product also within LFC_p. However, there are the following two difficulties. Firstly, the permutational product of torsion-free $\mathfrak{A}$-groups is their central product over the amalgamated subgroup; therefore its torsion subgroup need not be a p-group. Secondly, the $L\mathfrak{F}_p$-case suggests a choice of transversals with respect to AT-chief series in order to apply certain theorems of Gregorac [2]; however, these theorems require the centrality of chief factors in the *whole* group, whereas AT-chief factors are in general just central in the torsion subgroup. And a suitable generalization of Gregoracs theorems could not be found. Therefore, we try to modify the technique of standard embeddings into wreath products (see [7]), which has already been used advantageously for a large variety of group classes (see [9]), and which has turned out to be even more powerful than the permutational product in the study of *countable* e.c. $\mathfrak{X}$-groups.

Let A be any group, and let B be a permutation group on a set Ω. The *unrestricted wreath product* $A \operatorname{Wr}_\Omega B$ is the set $\{(b, f) | b \in B, f: \Omega \to A\}$ with group multiplication

$$(b_1, f_1) \cdot (b_2, f_2) = (b_1 b_2, f_1^{b_2} f_2)$$

where

$$(f_1^{b_2} f_2)(\omega) = f_1(\omega b_2^{-1}) \cdot f_2(\omega) \quad \text{for all } \omega \in \Omega .$$

$A \operatorname{Wr}_\Omega B$ is a split extension of its *base group* $\{(1, f) | f: \Omega \to A\} \cong A^\Omega$ by its *top group* $\{(b, 1) | b \in B\} \cong B$. In the case when B is an arbitrary group, we can choose $\Omega = B$ and define a permutation action of B on Ω via $(\omega)b^{-1} = b \cdot \omega$ for all $\omega \in \Omega$, $b \in B$. In this case we write $A \operatorname{Wr} B$ instead of $A \operatorname{Wr}_\Omega B$. We usually identify canonically B with the top group and A with the 1-*component* $\{(1, f) | f(b) = 1 \text{ for all } b \in B - 1\}$ of $A \operatorname{Wr} B$. The canonical embedding of A onto the *diagonal subgroup* $\{(1, f) | f \text{ constant}\}$ of $A \operatorname{Wr} B$ is called the *diagonal embedding.*

Now, let $\theta: G \to H$ be a group homomorphism with $N = \operatorname{Ker} \theta$. A *countermap* to θ is a map $\theta^*: H \to G$ satisfying

$$(g\theta h)\theta^*\theta = g\theta \cdot h\theta^*\theta \quad \text{for all } g \in G, \; h \in H .$$

See [7] for the existence of countermaps. Every countermap θ^* to θ gives rise to a *standard embedding* $\sigma: G \to W = N \operatorname{Wr} H$ via

$$g\sigma = (g\theta, f_g) \quad \text{for all } g \in G,$$

where

$$f_g(h) = [(g\theta h)\theta^*]^{-1} \cdot g \cdot h\theta^* \quad \text{for all } h \in H .$$

Similar embeddings have been used in [9, Construction 4.1] to obtain $L\mathfrak{X}$-supergroups of countable e.c. $L\mathfrak{X}$-groups for certain classes $\mathfrak{X}$. In these applications it was required that wreath products of finitely generated $\mathfrak{X}$-groups by $L\mathfrak{X}$-groups were $L\mathfrak{X}$-groups again. This is however not true for FC-groups. For example, C_p has infinitely many conjugates in the finitely generated subgroup $\langle C_p, C_\infty\rangle$ of $C_p \operatorname{Wr} C_\infty$. Therefore, we need some further specialization of standard embeddings. Our idea is as follows. In the case when G and H are finitely generated FC-groups (FC^0-groups) and $N \leq T(G)$, choose the countermap $\theta^*\colon H \to G$ in such a way that, for some subgroup H_0 of finite index in H, each f_g is constant on the left cosets of H_0 in H. Then $\operatorname{Im}\sigma$ is contained in

$$W_0 = \{(h, f)|h \in H, f \text{ is constant on each } \tilde{h}H_0(\tilde{h} \in H)\} \leq W.$$

W_0 is a split extension of a direct product of finitely many copies of N by H. Since N is finite, it follows that $W_0 \in FC^0$.

In the following constructions we will make extensive use of the fact, that in FC^0-groups the centre has finite index. The rank $r(G)$ of an FC^0-group G is equal to its Hirsch number and thus also to the torsion-free rank of $Z(G)$. By an *independent system* of an $\mathfrak{A}$-group we always mean an independent system of torsion-free elements. For any periodic $\mathfrak{A}$-group A, we denote by LFC_A the class of all LFC-groups with $A \leq Z(G)$.

Construction 3.1. *Suppose that we are given a homomorphism $\theta\colon G \to H$ of countable LFC-groups with $A \leq Z(G)$ and $A \leq N = \operatorname{Ker}\theta \leq T(G)$, and that $\{H_n|n \in \omega\}$ is an ascending chain of FC^0-groups with union H. Then there exists an ascending chain of FC^0-groups $\{G_n|n \in \omega\}$ with union G such that $H_n \cap G\theta = G_n\theta$ for all n.*

Proof. Let $N = \{v_m|m \in \omega\}$ and choose elements $g_m \in G$ such that $H_n \cap G\theta = \langle g_0\theta, \dots, g_{k_n}\theta\rangle$ for suitable k_n. Put $G_n = \langle v_0, \dots, v_n, g_0, \dots, g_{k_n}\rangle$. □

Choose recursively a torsion-free $Z_n \leq Z(G_n)$ with $Z_n\theta \leq Z(H_n)$, such that $r(Z_0) = r(G_0)$, and also $r(Z_{n+1}) = r(G_{n+1}) - r(G_n)$ and $Z_{n+1} \cap G_n = 1$.

Proof. Let S be a subset of $Z(G_{n+1})$ which supplements a maximal independent system of $Z_n \cap Z(G_{n+1})$ to one of $Z(G_{n+1})$. Then Z_{n+1} may be generated by a suitable power of S. □

It follows that $Z_{n+1}\theta \cap H_n = Z_{n+1}\theta \cap H_n \cap G\theta = Z_{n+1}\theta \cap G_n\theta = 1$.

Proof. Let $z \in Z_{n+1}$ and $g \in G_n$ with $z\theta = g\theta$. Then $g = zv$ for some $v \in N \cap G_{n+1} \leq T(G_{n+1})$. Thus, $g^{o(v)} = z^{o(v)} \in Z_{n+1} \cap G_n = 1$. But Z_{n+1} is torsion-free, whence $z = 1$. □

Choose recursively a torsion-free $\widehat{Z}_n \leq Z(H_n)$ with $Z_n\theta \leq \widehat{Z}_n$, such that $r(\widehat{Z}_0) = r(H_0)$, and also $r(\widehat{Z}_{n+1}) = r(H_{n+1}) - r(H_n)$ and $\widehat{Z}_{n+1} \cap H_n = 1$.

Proof. Supplement a maximal independent system of $(Z_{n+1}\theta \times \widehat{Z}_n) \cap Z(H_{n+1})$. □

Clearly, $Z_n \cdots Z_0$ and $\widehat{Z}_n \cdots \widehat{Z}_0$ have finite index in G_n resp. H_n. Note also that $\theta|Z_n \cdots Z_0$ is injective as $Z_n \cdots Z_0$ is torsion-free and $N \leq T(G)$. Choose

$C_n \leq Z_n \cdots Z_0$ with $C_n\theta = Z(H_n) \cap (Z_n \cdots Z_0 \cap Z(G_n))\theta$. Then

$$\begin{aligned} C_{n+1}\theta &= Z(H_{n+1}) \cap [Z_{n+1}(Z_n \cdots Z_0 \cap Z(G_{n+1}))]\theta \\ &= Z_{n+1}\theta \cdot (Z(H_{n+1}) \cap [Z_n \cdots Z_0 \cap Z(G_{n+1})]\theta) \\ &\leq Z_{n+1}\theta \cdot (Z(H_n) \cap [Z_n \cdots Z_0 \cap Z(G_n)]\theta) = (Z_{n+1}C_n)\theta, \end{aligned}$$

whence

$$Z_{n+1} \leq C_{n+1} \leq Z_{n+1}C_n. \tag{3.1}$$

Choose a torsion-free $X_0 \leq \widehat{Z}_0$ with $X_0 \cap C_0\theta = 1$ and $r(X_0) = r(\widehat{Z}_0) - r(C_0\theta)$, and then recursively a torsion-free $X_{n+1} \leq \widehat{Z}_{n+1}X_n \cap Z(H_{n+1})$ such that $X_{n+1} \cap C_{n+1}\theta = 1$ and $r(X_{n+1}) = r(\widehat{Z}_{n+1}D_n) - r(C_{n+1}\theta)$, where $D_n = C_n\theta \times X_n$.

Proof. Note that $C_{n+1}\theta \leq (Z_{n+1}C_n)\theta \leq \widehat{Z}_{n+1}D_n = \widehat{Z}_{n+1} \times X_n \times C_n\theta$. In order to find X_{n+1}, supplement a maximal independent system of $C_{n+1}\theta$ to one of $\widehat{Z}_{n+1}D_n$. From (3.1) and $r(C_{n+1}) = r(Z_{n+1}) + r(C_n)$ we know that $C_{n+1}\theta$ contains a maximal independent system of $(Z_{n+1}C_n)\theta$. Therefore we can even get $X_{n+1} \leq \widehat{Z}_{n+1} \times X_n$. □

An easy induction shows that $r(D_n) = r(H_n)$. Moreover,

$$D_n \cap G\theta = C_n\theta \quad \text{and} \quad D_{n+1} \leq \widehat{Z}_{n+1}D_n. \tag{3.2}$$

Proof. $|X_n \cap G_n\theta| = |X_n \cap G_n\theta : X_n \cap C_n\theta| \leq |G_n\theta : C_n\theta| < \infty$, since $r(G_n\theta) = r(G_n) = r(C_n) = r(C_n\theta)$. But X_n is torsion-free. So $X_n \cap G_n\theta = 1$ and $D_n \cap G_n\theta = C_n\theta$. Furthermore, $D_{n+1} = C_{n+1}\theta \cdot X_{n+1} \leq Z_{n+1}\theta \cdot C_n\theta \cdot \widehat{Z}_{n+1} \cdot X_n \leq \widehat{Z}_{n+1}D_n$. □

Now, let $W = (T(G) \rtimes G)\mathrm{Wr}\, H$ where G acts on $T(G)$ via conjugation. We will specify a certain $LFC_{A\delta}$-subgroup $\widehat{W}$ of W, where $\delta: A \to W$ is given by $a\delta = (1, f_a)$ for all $a \in A$ and $f_a \equiv (1, a)$. To this end, put $\widehat{Z}^n = \langle \widehat{Z}_k | k \geq n+1\rangle$, and fix a right transversal T_n of $\widehat{Z}^n H_n$ in $\widehat{Z}^{n+1}H_{n+1}$ with $T_n \subseteq H_{n+1}$. Then $T^n = \bigcup\{T_n \cdots T_k | k \geq n\}$ is a right transversal of $\widehat{Z}^n H_n$ in H. Clearly,

$$h \cdot \widehat{Z}^{n+1}X_{n+1}T^{n+1} \subseteq \tilde{h} \cdot \widehat{Z}^n X_n T^n \text{ and } h \cdot \widehat{Z}^{n+1}D_{n+1}T^{n+1} \subseteq \tilde{h} \cdot \widehat{Z}^n D_n T^n \tag{3.3}$$

$$\text{whenever } h \in H_{n+1} \text{ satisfies } h = z\tilde{h}t \text{ with } z \in \widehat{Z}^n,\ \tilde{h} \in H_n,\ t \in T_n.$$

Let

$$\begin{aligned} W_n = \{(h, f) \in W | h \in H_n, \mathrm{Im}\, f \subseteq T(G_n) \rtimes G_n, f \text{ is constant on each} \\ \text{of the sets } \tilde{h} \cdot \widehat{Z}^n D_n T^n\ (\tilde{h} \in H_n), \text{ the } G_n\text{-component of } f \\ \text{is constant modulo } T(G_n)\}. \end{aligned}$$

Then W_n is a subgroup of W. Moreover, $W_n \leq W_{n+1}$ by (3.3). *And $\widehat{W} = \bigcup\{W_n | n \in \omega\} \in LFC_{A\delta}$.*

Proof. Clearly, $A\delta \leq Z(\widehat{W})$. It remains to show that $W_n \in FC$. Let

$$\Omega_n = \{(h, f) \in W_n | \mathrm{Im}\, f \subseteq T(G_n) \rtimes T(G_n) \text{ and } h \in T(H_n)\}.$$

Since for every $(h, f) \in W_n$ the G_n-component of f is constant modulo $T(G_n)$, we obtain that $\Omega_n \trianglelefteq W_n$. If $(h, f) \in \Omega_n$, then f is constant on each of the finitely many sets $\tilde{h} \cdot \widehat{Z}^n D_n T^n (\tilde{h} \in H_n)$. Hence Ω_n is a finite normal subgroup of W_n. Moreover, $W_n/\Omega_n \cong H_n/T(H_n) \times G_n/T(G_n)$ is a torsion-free $\mathfrak{A}$-group, whence $W_n \in FC$. □

Finally note that $\widehat{W} \in LFC_{p,A\delta}$ *whenever* $G \in LFC_{p,A}$ *and* $H \in LFC_p$.

Construction 3.2. *Adopt the notation introduced in Construction* 3.1, *and suppose in addition that* $G_0 \leq T(G)$. *For each* $n \in \omega$, *choose a right transveral* R_n *of* $G_n\theta \cdot X_n$ *in* H_n, *and let* S_n *be a transversal of* $C_n \cdot (N \cap G_n)$ *in* G_n. *Then* $S_n\theta$ *is a transversal of* $C_n\theta$ *in* $G_n\theta$. *Define* $\tilde{\theta}_n^*: H \to G_n$ *via*

$$(zxc\theta s\theta rt)\tilde{\theta}_n^* = cs \quad \text{for all } z \in \widehat{Z}^n,\ x \in X_n,\ c \in C_n,\ s \in S_n,\ r \in R_n,\ t \in T^n.$$

$\tilde{\theta}_n^*$ *is a countermap to* $\theta|G_n$ *because* $\widehat{Z}^n X_n R_n T^n$ *is a right transversal of* $G_n\theta$ *in* H (*cf.* [7]). *Moreover, we have that*

(3.4) $\tilde{\theta}_n^*$ *is constant on each of the sets* $h \cdot \widehat{Z}^n X_n T^n$ $(h \in H_n)$, *and*

(3.5) $$(c\theta h)\tilde{\theta}_n^* = c \cdot h\tilde{\theta}_n^* \quad \text{for all } c \in C_n,\ h \in H.$$

Define $\theta_n^*: H \to G_n$ *recursively via*

$$\theta_0^* = \tilde{\theta}_0^* \quad \text{and} \quad (g\theta y)\theta_{n+1}^* = (g\theta y)\theta_n^* \cdot y\theta_n^{*-1} \cdot y\tilde{\theta}_{n+1}^* \quad \text{for all } g \in G_n,\ y \in \widehat{Z}^n X_n R_n T^n.$$

Straightforward calculations yield that θ_n^* *is also a countermap to* $\theta|G_n$, *and that the map* $\omega_n: H \to G_{n+1}$ *given by* $h\omega_n = h\theta_n^{*-1} \cdot h\theta_{n+1}^*$ *is constant on each coset* $G_n\theta \cdot h$ $(h \in H)$. *It follows from* (3.3) *that* θ_n^* *satisfies* (3.4) *too. Furthermore,* θ_n^* *satisfies* (3.5).

Proof. Let $c \in C_{n+1}$ and $h \in H$. Then (3.1) yields $c = \tilde{z}\tilde{c}$ for some $\tilde{z} \in Z_{n+1}$, $\tilde{c} \in C_n$. Moreover, $h = g\theta \cdot y$ for some $g \in G_n$, $y \in \widehat{Z}^n X_n R_n T^n$. But then

$$\begin{aligned}(c\theta h)\theta_{n+1}^* &= ((\tilde{z}\tilde{c}g)\theta y)\theta_{n+1}^* = ((\tilde{c}g\tilde{z})\theta y)\theta_{n+1}^* \\ &= (\tilde{c}\theta g\theta\tilde{z}\theta y)\theta_n^* \cdot (\tilde{z}\theta y)\theta_n^{*-1} \cdot (\tilde{z}\theta y)\tilde{\theta}_{n+1}^* \\ &= \tilde{c} \cdot (g\theta\tilde{z}\theta y)\theta_n^* \cdot (\tilde{z}\theta y)\theta_n^{*-1} \cdot (\tilde{z}\theta y)\tilde{\theta}_{n+1}^* \quad \text{by induction} \\ &= \tilde{c} \cdot (g\theta y)\theta_n^* \cdot y\theta_n^{*-1} \cdot \tilde{z} \cdot y\tilde{\theta}_{n+1}^* \quad \text{by (3.4) and (3.1)/(3.5)} \\ &= \tilde{z}\tilde{c} \cdot (g\theta y)\theta_{n+1}^* = c \cdot h\theta_{n+1}^*. \quad \square\end{aligned}$$

Now, the standard embedding $G_0 \to W$ *determined by* θ_0^* *can be extended to an embedding* $\sigma: T \to W$, *given by*

$g\sigma = (g\theta, f_g)$ *for all* $g \in G_n$, *where*

$$f_g(h) = ((g\theta h)\theta_0^{*-1} \cdot (g\theta h)\theta_n^* \cdot h\theta_n^{*-1} \cdot h\theta_0^*,\ h\theta_0^{*-1} \cdot h\theta_n^* \cdot (g\theta h)\theta_n^{*-1} \cdot g \cdot h\theta_0^*) \in (N \cap G_n) \rtimes G_n \text{ for all } h \in H.$$

(*This is the embedding* σ *of* [9, Construction 4.1].) *We list some properties of* σ.

(3.6) *If* $g \in N$, *then* $g\sigma = (1, f_g)$ *where* $f_g(h) = (1, g^{h\theta_0^*})$ *for all* $h \in H$. *In particular,* $\sigma|A = \delta$. *And if we choose* $G_0 \leq N$ *and* $S_0 = \{1\}$, *then* $\sigma|N$ *is the diagonal embedding.*

(3.7) *If* $g \in G_n$, *then* f_g *is constant on each of the sets* $h \cdot \widehat{Z}^n D_n T^n$ $(h \in H_n)$, *and the* G_n*-component of* f_g *constantly equals* g *modulo* $T(G)$. *In particular,* $\operatorname{Im}\sigma \leq \widehat{W}$.

Proof. Because of (3.3)/(3.4) the $T(G_n)$-component f_g^r of f_g is constant on each $h \cdot \widehat{Z}^n X_n T^n$. Now, let $c \in C_n$. Because of $G_0 \le T(G)$ we have $C_0 = 1$, whence $c\theta \in C_n\theta \le \widehat{Z}^0$ by (3.1). Thus (3.5) and (3.4) yield

$$\begin{aligned} f_g^r(c\theta h) &= ((c\theta h)\theta_n^* \cdot (g\theta c\theta h)\theta_n^{*-1} \cdot g)^{(c\theta h)\theta_0^*} \\ &= (c \cdot h\theta_n^* \cdot (g\theta h)\theta_n^{*-1} \cdot c^{-1} \cdot g)^{h\theta_0^*} = f_g^r(h). \end{aligned}$$

By the same arguments, the G_n-component f_g^1 of f_g is constant on each $h \cdot \widehat{Z}^n D_n T^n$ too. Furthermore, $\operatorname{Im}\theta_0^* \subseteq G_0 \le T(G)$. So $f_g^1(h) \cdot T(G) = (g\theta h)\theta_n^* \cdot h\theta_n^{*-1} \cdot T(G) = g \cdot T(G_n)$ by definition of a countermap. □

Construction 3.3. *Adopt the notation introduced in Constructions* 3.1 *and* 3.2. *Suppose in addition that* $N = \operatorname{Ker}\theta$ *is a minimal normal subgroup of* G. *Then, by Lemma* 2.1, *we have that* $T(G) \le C_G(N)$. *This allows it to modify the embedding* $\sigma: G \to \widehat{W}$ *as follows.*

Denote epimorphic images modulo $T(G)$ *by bars. Let* $W^0 = (N \rtimes \overline{G})\mathrm{Wr}\, H$ *and*

$$\begin{aligned} W_n^0 = \{(h, f) \in W^0 | h \in H_n,\ & \operatorname{Im} f \subseteq (N \cap G_n) \rtimes \overline{G}_n, \\ & f \text{ is constant on each of the sets } \tilde{h} \cdot \widehat{Z}^n D_n T^n\ (\tilde{h} \in H_n), \\ & \text{the } \overline{G}_n\text{-component of } f \text{ is constant}\}. \end{aligned}$$

Then $\widehat{W}^0 = \bigcup\{W_n^0 | n \in \omega\} \in LFC$ (*resp.* $\widehat{W}^0 \in LFC_p$ *whenever* $G, H \in LFC_p$). *Define* $\sigma^0: G \to \widehat{W}^0$ *by*

$$g\sigma^0 = (g\theta, f_g^0) \quad \text{for all } g \in G,$$

where

$$f_g^0(h) = (\overline{g}, h\theta_n^* \cdot (g\theta h)\theta_n^{*-1} \cdot g) \quad \text{for all } h \in H.$$

Then σ^0 *is also an embedding with the properties that* $\sigma|N$ *is the diagonal embedding, and that* f_g^0 *is constant on each of the sets* $h \cdot \widehat{Z}^n D_n T^n$ $(h \in H_n)$ *whenever* $g \in G_n$. *Usually we suppress the* 0*'s.*

Embeddings σ as in Constructions 3.2 and 3.3 we call *modified standard embeddings* (*ms-embeddings*). We will need a technical lemma concerning ms-embeddings which corresponds to [9, Corollary 4.4].

Lemma 3.4. *Let* $(h, f) \in W_n$ *with* $h \in H_n - D_n$. *If* B_n *denotes the base group of* W_n, *then* $T(B_n)' \le [[(h, f), T(B_n)], T(B_n)] \le \langle (h, f)^{T(B_n)} \rangle$.

Proof. Let $\widehat{R}_n$ be a right transversal of $D_n\langle h \rangle$ in H_n. Follow the proof of [9, Lemma 4.3(b)] with $\widehat{Z}^n D_n \widehat{R}_n T^n$ in place of T. (Note that $[[(b, f), x_{1j}], x_{2j}] = [x_{1j}, x_{2j}]$ actually holds for *all* j in the proof of [9, Lemma 4.3(b)], whence the assumption $f(b') = 1$ for all $b' \in B - T$ is superfluous). □

4. Many countable closed structures

In this section we will show that there exist $2^{\aleph_0}$ countable closed $LFC_{p,A}$-groups. We need a preparatory lemma similar to [11, Lemma 2.2]. Note that for $U \le G \in LFC_{p,A}$, every AT-normal series with elementary-abelian AT-factors in G induces an AT-normal series with elementary-abelian AT-factors in U in the sense of [11, p. 163].

Lemma 4.1. *Let Σ_U be an AT-chief series in the $FC^0_{p,A}$-group U. Then there exists an $FC^0_{p,A}$-supergroup V of U such that every AT-normal series with elementary-abelian AT-factors in V induces Σ_U in U.*

Proof. Since every $FC^0_{p,A}$-group is a central product of A with some FC^0_p-group over a finite subgroup of A, we may assume without loss that A is finite. Let $1 \leq A = U_n < U_{n-1} < \cdots < U_0 = T(U) \leq U$ be the series Σ_U. Recursively we will construct for $0 \leq l \leq n$ an FC^0_p-supergroup V_l of U/U_l such that

$$hU_l \in \langle gU_l^{V_l}\rangle' \quad \text{for all } h \in U_k,\ g \in U_0 - U_k \text{ and } 1 \leq k \leq l-1. \tag{4.1}$$

To this end put $V_0 = U/U_0$ and assume that V_{l-1} has been found for some $l \in \{1, \ldots, n\}$. Let $\widehat{U} = U/U_l$, and denote the canonical epimorphism $\widehat{U} \to U/U_{l-1} \leq V_{l-1}$ by θ. Identify $\widehat{U}_{l-1}$ with the diagonal subgroup of $H = (\widehat{U}_{l-1} \operatorname{Wr} C_p)\operatorname{Wr} C_{p^2}$. Because of $\exp(\widehat{U}_{l-1}) = p$ we have $Z(H) = \widehat{U}_{l-1} \leq H''$ (see [17, Theorem 4.1]).

Now, as in the 0th step of Construction 3.2, there exists a torsion-free central subgroup D of finite index in V_{l-1} and a countermap $\theta^*\colon V_{l-1} \to \widehat{U}$ to θ such that the corresponding standard embedding

$$\sigma\colon \widehat{U} \to \widehat{U}_{l-1} \operatorname{Wr} V_{l-1} \leq H \operatorname{Wr} V_{l-1}$$

maps $\widehat{U}$ into the FC^0_p-group

$$V_l = \{(v, f) \in H \operatorname{Wr} V_{l-1} | \operatorname{Im} f \text{ is constant on each coset } \tilde{v} \cdot D\ (\tilde{v} \in V_{l-1})\}.$$

Let Ω be the base group of V_l. Fix $h \in U_k$ and $g \in U_0 - U_k$ for some $k \in \{1, \ldots, l-1\}$. By our recursion, (4.1) yields that $\hat{h}\sigma \in \langle \hat{g}\sigma^{V_{l-1}}\rangle' \cdot w$ for some $w \in \Omega''$. Moreover, $w \in \langle \hat{g}\sigma^{V_l}\rangle'$ by Lemma 3.4. Hence $\hat{h}\sigma \in \langle \hat{g}\sigma^{V_l}\rangle'$. Suppress σ. This completes the recursion.

A further application of the above argument, with A in place of $\widehat{U}_{l-1}$, and with $(A \operatorname{Wr} C_q)\operatorname{Wr} C_{q^2}$ in place of H where $q = \exp(A)$, yields an $FC^0_{p,A}$-supergroup V of U such that

$$h \in \langle g^V\rangle' \quad \text{for all } h \in U_k,\ g \in U_0 - U_k \quad \text{and} \quad 1 \leq k \leq n.$$

Now, let M/N be an elementary-abelian AT-factor in V. Choose k minimal with respect to $U_k \leq N$. If $M \cap U = U_k$, then $N \cap U = U_k$, and we are done. Suppose that $M \cap U > U_k$. Every $g \in (M \cap U) - U_k$ satisfies $g \in U_{j-1} - U_j$ for some $j \leq k$. On the other hand $j \geq k$, since (4.1) yields $U_j \leq \langle g^V\rangle' \leq M' \leq N$. Therefore, $g \in U_{k-1} - U_k$, and $U_k \leq N \cap U < M \cap U \leq U_{k-1}$. Since U_{k-1}/U_k is a chief factor in U, we obtain $U_k = N \cap U < M \cap U = U_{k-1}$. □

In order to show the main result of this section we will use a result of B. Maier [16]. This makes it necessary to introduce the notion of a controller. Let $\mathfrak{X}$ be a class of groups. An $\mathfrak{X}$-supergroup V of $U \in \mathfrak{X}$ is an *$\mathfrak{X}$-controller* for U, if the following holds.

(4.2) Whenever G and H are $\mathfrak{X}$-supergroups of V, then there exist an $\mathfrak{X}$-group W and embeddings $\sigma\colon G \to W$ and $\tau\colon H \to W$ such that $\sigma|U = \tau|U$.

Theorem 4.2. *Let $U \in FC^0_{p,A}$ with $U \neq T(U) \neq A$. Then U has no $FC^0_{p,A}$-controller.*

Now, every FC^0-group is embeddable into the direct product of a free abelian group of finite rank and a finite group (see [19, Theorem 1.7]). Since each of these products has only countably many finitely generated subgroups, we obtain that there exist only countably many FC^0-groups. Every $FC^0_{p,A}$-group is the central product of A with some FC^0-group over a finite subgroup of A, whence there exist only countably many $FC^0_{p,A}$-groups. Therefore, [16, Theorem 4.1] applies, and so Theorem 4.2 immediately yields

Corollary 4.3. *There exist $2^{\aleph_0}$ pairwise nonisomorphic countable closed $LFC_{p,A}$-groups for every fixed countable abelian p-group A.*

Proof of Theorem 4.2. Assume that there exists an $FC^0_{p,A}$-controller V for U. Let Σ_V be an AT-chief series in V. Then $V_1 \cap U > V_2 \cap U = A$ for some AT-chief factor V_1/V_2 of Σ_V. Denote epimorphic images modulo torsion subgroups by bars. Fix $u \in U - T(U)$, and let $\overline{V} = \langle \overline{v}_1 \rangle \times \cdots \times \langle \overline{v}_r \rangle$. By Rado's Lemma [18] we may assume that $\overline{u} = \overline{v}_1^m$ for some $m \in \omega$. Fix $u_0 \in (U \cap V_1) - V_2$, put $\mathscr{k} = GF(p)$, and let $\overline{\overline{V}} = \overline{V}/C_{\overline{V}}(V_1/V_2)$. Lemma 2.2 yields that $K = \mathscr{k}\overline{\overline{V}}/An(V_1/V_2)$ is a finite field, and that a K-isomorphism $\varepsilon\colon K^+ \to V_1/V_2$ is given by $x\varepsilon = (u_0 V_2)^x$ for all $x \in K$. Choose a prime $q \neq p$ not dividing $|K^\times|$. Let $\widehat{V} = \langle \overline{v}_0 \rangle \times \langle \overline{v}_2 \rangle \times \cdots \times \langle \overline{v}_r \rangle \geq \overline{V}$ with $\overline{v}_0^q = \overline{v}_1$.

We will construct $FC^0_{p,A}$-supergroups G and H of V with the following properties.

(4.3) $$\overline{V} \leq \widehat{V} \leq \overline{G} \cap \overline{H}.$$

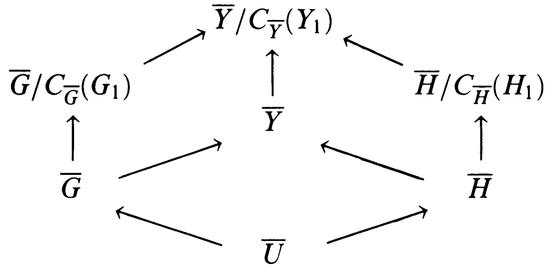
(4.4) There exist AT-chief series Σ_G in G and Σ_H in H which induce Σ_V in V.

(4.5) If V_1/V_2 is induced from the AT-chief factors G_1/G_2 of Σ_G and H_1/H_2 of Σ_H, then
$$o(\overline{v}_0 \cdot C_{\overline{G}}(G_1/G_2)) = q \cdot o(\overline{v}_0 \cdot C_{\overline{H}}(H_1/H_2)).$$

Let us first show that the existence of G and H leads to a contradiction. From Lemma 4.1 we obtain $G \leq \widehat{G} \in FC^0_{p,A}$ and $H \leq \widehat{H} \in FC^0_{p,A}$ such that every AT-normal series with elementary-abelian AT-factors in $\widehat{G}$ resp. $\widehat{H}$ induces Σ_G in G resp. Σ_H in H. Since V is an $FC^0_{p,A}$-controller for U, there exist $Y \in FC^0_{p,A}$ and embeddings $\sigma\colon \widehat{G} \to Y$ and $\tau\colon \widehat{H} \to Y$ with $\sigma|U = \tau|U$. Suppress σ and τ.

Consider an AT-chief series Σ_Y in Y. Clearly $\Sigma_Y \cap G = \Sigma_G$ and $\Sigma_Y \cap H = \Sigma_H$, whence $\Sigma_Y \cap V = \Sigma_V$ by (4.4). Let Y_1/Y_2 be the AT-chief factor in Σ_Y inducing V_1/V_2 in V. Then Y_1/Y_2 induces G_1/G_2 in G and H_1/H_2 in H. Considering everything modulo Y_2 we may assume that $Y_2 = G_2 = H_2 = V_2 = 1$. Clearly, $C_{\overline{X}}(X_1) = C_{\overline{X}}(u_0)$ holds for G, H, and Y in place of X, because X_1 is a minimal normal subgroup in X, and because $\overline{X}/C_{\overline{X}}(X_1)$ is abelian.

Thus, the diagram of canonical homomorphisms



commutes. Now, $\overline{v}_0$ is the unique qmth root of $\overline{u}$ in the torsion-free abelian group $\overline{Y}$. Hence $\overline{v}_0 \cdot C_{\overline{G}}(G_1) \equiv \overline{v}_0 \cdot C_{\overline{Y}}(Y_1) \equiv \overline{v}_0 \cdot C_{\overline{H}}(H_1)$, which contradicts (4.5).

It remains to construct G and H. As in the proof of Lemma 4.1 we may assume that A is finite. Then V_2 is finite. Suppose that we can find FC_p^0-supergroups $\widetilde{G}$ and $\widetilde{H}$ of V/V_2 satisfying (4.3)–(4.5) correspondingly. Then, by the 0th step of Construction 3.2 there exist, for the canonical homomorphisms $V \to V/V_2 \leq \widetilde{G}$ resp. $V \to V/V_2 \leq \widetilde{H}$, standard embeddings of V into $FC_{p,A}^0$-subgroups G of $V_2 \operatorname{Wr} \widetilde{G}$ resp. H of $V_2 \operatorname{Wr} \widetilde{H}$. Since every $v \in V_2$ is mapped onto $(1, f_v) \in V_2 \operatorname{Wr} \widetilde{G}$, where $\operatorname{Im} f_v$ is contained in the V-conjugacy class of v, there exists an AT-chief series in $V_2 \operatorname{Wr} \widetilde{G}$ and hence in G, which induces Σ_V in V such that G inherits the properties (4.3)–(4.5) from $\widetilde{G}$. Similarly, H satisfies (4.3)–(4.5). Therefore we may assume from now on that $V_2 = 1$.

By the 0th step of Construction 3.2 there exists, for the canonical homomorphism $V/V_1 \to \overline{V} \leq \widehat{V}$, a standard embedding μ of V/V_1 into an FC_p^0-subgroup W of $T(V)/V_1 \operatorname{Wr} \widehat{V}$ with $\overline{W} = \widehat{V}$. As before, there is a T-chief series in W which induces the series $(\Sigma_V/V_1)\mu$ in $(V/V_1)\mu$.

Let $L = K(\alpha)$ for some primitive qth root of unity α. Since q does not divide $|K^\times|$, there exists $z \in K$ such that $z^q = \overline{\overline{v}}_1 + An(V_1)$. Extend the canonical homomorphism $\theta \colon \overline{V} \to \overline{\overline{V}} \leq K^\times$ to homomorphisms $\theta_1 \colon \widehat{V} \to K^\times$ and $\theta_2 \colon \widehat{V} \to L^\times$ via

$$\overline{v}_0\theta_1 = z \quad \text{and} \quad \overline{v}_0\theta_2 = z \cdot \alpha.$$

Put $P = K^+ \rtimes_{\theta_1} \widehat{V}$ and $Q = L^+ \rtimes_{\theta_2} \widehat{V}$. Since K^+ is an irreducible $K^\times$-module, K^+ is a minimal normal subgroup of P. Similarly, because L is the smallest subring of L containing $\operatorname{Im} \theta_2$, we see that L^+ is a minimal normal subgroup of Q.

By Construction 3.3 there is a countermap μ^* to the composition $\overline{\mu}$ of the canonical epimorphism $V \to V/V_1$ and of μ such that, for some torsion-free central subgroup D of finite index in W, the ms-embedding $\sigma \colon V \to P \operatorname{Wr} W$ given by $v\sigma = (v\overline{\mu}, f_v)$ where $f_v(w) = (\overline{v}, (w\mu^* \cdot [(v\overline{\mu} \cdot w)\mu^*]^{-1} \cdot v)\varepsilon^{-1})$, satisfies

$$V\sigma \leq G = \{(w, f) \in P \operatorname{Wr} W \mid f \text{ is constant on each of the cosets } \tilde{w} \cdot D \\ (\tilde{w} \in W),\ \text{and the } \widehat{V}\text{-component of } f \text{ is constant}\} \in FC_p^0.$$

G inherits the property (4.3) from W: we may identify $\widehat{V}$ canonically with $\{(\overline{v}, 1)\cdot\overline{v}\delta\cdot T(G)|\overline{v}\in\widehat{V}\}$, where $\delta\colon P\to W$ denotes the diagonal embedding. (Observe that

$$T(G)=\{(w, f)\in G|w\in T(W),\ \text{and the } \widehat{V}\text{-component of } f \text{ is trivial}\}$$

and that $\overline{v\sigma}=(\overline{v}, 1)\cdot\overline{v}\delta\cdot T(G)$ for all $v\in V$.)

Now $\sigma|V_1=\varepsilon^{-1}\delta$, and $G_1=K^+\delta$ is a minimal normal subgroup in G. Therefore, the normal series $1\le G_1\le T(G)\le G$ can be refined to a T-normal series in G which induces $\Sigma_V\sigma$ in $V\sigma$. Put $x=(\overline{v}_0, 1)\cdot\overline{v}_0\delta$. Then $\overline{x}=\overline{v}_0$ in $\overline{G}$, and

$$|\{u_0\sigma^{(x^k)}|k\in\mathbb{Z}\}|=|\{(u_0\varepsilon^{-1})^{\overline{v}_0^k}|k\in\mathbb{Z}\}\delta|=|\{u_0\varepsilon^{-1}\cdot z^k|k\in\mathbb{Z}\}|=o(z),$$

whence $o(\overline{v}_0\cdot C_{\overline{G}}(G_1))=o(z)$ by Lemma 2.2. Similarly, we can find an ms-embedding $\tau\colon V\to H\le Q\,\mathrm{Wr}\,W$ where H is an FC_p^0-group with the properties (4.3) and (4.4) such that $o(\overline{v}_0\cdot C_{\overline{H}}(H_1))=o(z\cdot\alpha)=o(z)\cdot o(\alpha)=q\cdot o(z)$. □

Observe, that the groups G and H in the proof of Theorem 4.2 will have the same rank as V, if we replace G by

$$\{(w, f)\in P\,\mathrm{Wr}\,W|f \text{ is constant on each of the cosets } \tilde{w}\cdot D\ (\tilde{w}\in W),$$
$$\text{and the } \widehat{V}\text{-component of } f \text{ constantly equals } \overline{w}\},$$

and similarly H. Therefore we can even show that there exist $2^{\aleph_0}$ countable closed groups in the class of all $LFC_{p,A}$-groups of rank $\le\rho$ (for each fixed $\rho\in\omega-\{0\}$).

5. Some properties

In this section we will collect properties of countable closed $LFC_{p,A}$-groups G which correspond to those which hold for countable e.c. LFC-groups [5, §§1–2] resp. the countable e.c. $L\mathfrak{F}_{p,A}$-group E_A [13 and 11], and we will determine the action of G on its AT-chief factors.

Theorem 5.1. (a) *Let G be a countable closed $LFC_{p,A}$-group. Then $C_G(T(G))=A$. In particular, $Z(G)=A=Z(T(G))$.*

(b) *Let $\widehat{A}$ be an abelian p-group with $A\le\widehat{A}$. Then there exists a closed $LFC_{p,A}$-group G of cardinality* $\max\{|\widehat{A}|, \aleph_1\}$ *with $\widehat{A}\le Z(G)$.*

Proof. (a) Assume that there exists $x\in C_G(T(G))-A$. Put $\widehat{G}=G\times C_p$ and $C_p=\langle c\rangle$. Let $\{G_n|n\in\omega\}$ be an ascending chain of FC^0-groups with union $\widehat{G}$. Following Construction 3.1 with the canonical epimorphism $\widehat{G}\to\widetilde{G}=\widehat{G}/(A\times C_p)$ in place of $\theta\colon G\to H$ we obtain for each $n\in\omega$ a partition $\widetilde{G}=\Omega_{n,0}\dot\cup\cdots\dot\cup\Omega_{n,k_n}$ such that $\widetilde{G}_n$ permutes the $\Omega_{n,i}$ regularly by left multiplication, and such that the groups

$$W_n=\{(\tilde{g}, f)\in(A\times C_p)\cap G_n\,\mathrm{Wr}\,\widetilde{G}|g\in G_n,\ \text{and } f \text{ is constant on each } \Omega_{n,i}\}$$

form an ascending chain of FC_p-subgroups of $(A\times C_p)\mathrm{Wr}\,\widetilde{G}$. Thus, Construction 3.2 yields an ms-embedding σ of $\widehat{G}$ into the $LFC_{p,A\sigma}$-group $W=\bigcup\{W_n|n\in\omega\}\le(A\times C_p)\mathrm{Wr}\,\widetilde{G}$. (Because of $A\times C_p\le Z(\widehat{G})$ we can replace $(A\times C_p)\rtimes\widehat{G}$ by $A\times C_p$ in the base group.)

Without loss we may assume that $\tilde{x} \in \widetilde{G}_0$ permutes the $\Omega_{0,i}$ nontrivially (see Construction 3.1). Let $y = (1, f) \in W_0$ where

$$f(\tilde{g}) = \begin{cases} c & \text{if } \tilde{g} \in \Omega_{0,0}, \\ 1 & \text{else.} \end{cases}$$

Then $[x\sigma, y] \neq 1$ and $y^p = 1$. Since G is e.c. in $LFC_{p,A}$, there does already exist some $z \in G$ such that $[x, z] \neq 1$ and $z^p = 1$, in contradiction to the choice of x.

(b) Follow the construction in the proof of [4, Theorem 2.2] with $\widehat{A}$ in place of A, and p^n in place of $n!$. Then $\overline{G} = B \rtimes E \in LFC_{p,A}$. Since every infinite $LFC_{p,A}$-group is contained in a closed $LFC_{p,A}$-group of the same cardinality, it suffices to show that $\overline{G}$ satisfies $\widehat{A} \leq A(\overline{G})$ (see proof of [4, Theorem 2.2]).

Fix $a \in \widehat{A}$ and $g\colon \overline{G} \to \omega - \{0\}$. Then there exists $m \in \omega - \{0\}$ and $\xi < \omega_1$ such that $\{\beta < \xi | g(d(\beta)) = m\}$ is infinite. Put $t = g(e(\xi, a))$. Let $tm = p^k l$ with $(p, l) = 1$. Then there exists $\beta < \xi$ with $g(d(\beta)) = m$ and $s = \varphi_\xi(\beta) \geq k$. Put $u = p^{s-k}$. Then

$$[d(\beta)^{g(d(\beta))}, e(\xi, a)^{g(e(\xi,a))}]^u = (z(s, a)\cdot c(s, a))^{tmu} = (z(s, a)\cdot c(s, a))^{p^s l} = a^l.$$

It follows that $a^l \in A_g(\overline{G})$. But now $(p, l) = 1$ yields $a \in \langle a^l \rangle \leq A_g(\overline{G})$. □

In the following, G will always denote a countable closed $LFC_{p,A}$-group. The proof below of the divisibility of $G/T(G)$ however only requires that G is e.c. in $LFC_{p,A}$ (this was used in the proof of Theorem 2.5 above).

Theorem 5.2. *$T(G)$ is verbally complete, and $G/T(G)$ is divisible. In particular, $G' = T(G)$, and $G/T(G) \cong \mathbb{Q}^{(\omega)}$.*

Proof. Fix $h \in T(G)$ and any word $w(x_1, \dots, x_\nu) \neq 1$. Let $F = \langle f_1, \dots, f_\nu \rangle$ be a finite p-group such that $w = w(f_1, \dots, f_\nu) \in Z(F)$ and $o(w) = o(h)$ (see [3, Lemma 7]). Identify w with h. Let $\delta\colon G \to G \operatorname{Wr} F/\langle h \rangle$ be the diagonal embedding. Put $V = T(G) \operatorname{Wr} F/\langle h \rangle$. Because of $h \in Z(F)$, any Krasner-Kaloujnine-embedding $\sigma\colon F \to V$ extends $\delta|\langle h \rangle$. Let $\widehat{V} = V \rtimes G$, where $g \in G$ acts on V via conjugation by $g\delta$. Suppress δ and σ. Applying Construction 3.1 to the canonical epimorphism $\theta\colon G \to \overline{G} = G/T(G)$ (with $G_0 = 1$) we obtain for each $n \in \omega$ a partition $\overline{G} = \Omega_{n,0} \dot\cup \cdots \dot\cup \Omega_{n,k_n}$ such that $\overline{G}_n$ permutes the $\Omega_{n,i}$ regularly by left multiplication, and such that the groups

$$\begin{aligned} W_n = \{(\overline{g}, f) \in \widehat{V} \operatorname{Wr} \overline{G} | g \in G_n,\ &\operatorname{Im} f \subseteq (T(G_n) \operatorname{Wr} F/\langle h \rangle) \rtimes G_n, \\ &f \text{ is constant on each } \Omega_{n,i}, \\ &\text{and the } G_n\text{-component of } f \text{ is constant modulo } T(G_n)\} \end{aligned}$$

form an ascending chain of FC-subgroups of $\widehat{V} \operatorname{Wr} \overline{G}$. Thus, Construction 3.2 yields an ms-embedding τ of G into the $LFC_{p,A\varepsilon}$-group $W = \bigcup\{W_n | n \in \omega\} \leq \widehat{V} \operatorname{Wr} \overline{G}$, where $\varepsilon\colon V \to \widehat{V} \operatorname{Wr} \overline{G}$ denotes the diagonal embedding. Because of $G_0 = 1$ we have $\tau|T(G) = \varepsilon|T(G)$. In particular, $f_1\varepsilon, \dots, f_v\varepsilon$ is a solution to $w(x_1, \dots, x_v) = h\tau$ in $T(W)$. Since G is e.c. in $LFC_{p,A}$, this shows that the equation $w(x_1, \dots, x_v) = h$ has a solution in $T(G)$. Thus, $T(G)$ is verbally complete.

Assume that there exists a prime q and some $\overline{z} \in \overline{G}$ which has no qth root in $\overline{G}$. Put $C_q = \langle c \rangle$ and $W = G \operatorname{Wr} C_q$. Let $V = \langle G\delta, cz \rangle$ where $\delta\colon G \to W$

denotes the diagonal embedding. Then $A\delta \le Z(V)$ and $(cz)^q = z\delta$. Since G is e.c. in $LFC_{p,A}$, this will be a contradiction if we can prove that $V \in LFC_p$.

Let $V_0 = \langle G_0\delta, cz\rangle$ for some $G_0 \le G$ with $z \in G_0 \in FC_p^0$. Then $(G_0\delta)' \le (T(G_0))\delta \le \Omega = \{(1, f) \in W | \mathrm{Im} f \subseteq T(G_0)\}$ and $[g\delta, cz] = [g\delta, z] \in \Omega$ for all $g \in G_0$. Hence, $V_0' \le \langle \Omega^{V_0}\rangle = \Omega$. Since Ω is a finite p-group, it remains to show that V_0/Ω is torsion-free abelian (then $V_0 \in FC_p$). Clearly, V_0/Ω is abelian. Thus, V_0/Ω is the central product of the torsion-free abelian groups $G_0\delta\cdot\Omega/\Omega \cong G_0/T(G_0)$ and $\langle cz\rangle\Omega/\Omega$ over $\langle z\delta\rangle\Omega/\Omega$. Since $G_0\delta\cdot\Omega/\Omega$ contains no qth root of $z\delta\cdot\Omega$, it follows that V_0/Ω is also torsion-free. □

In view of F. Haug's results one is tempted to conjecture that G is a split extension of $T(G)$ by $\mathbb{Q}^{(\omega)}$. However, we did not even find a way to answer

Question 5.3. *Can every embedding $\mathbb{Z} \to G$ be extended to an embedding $\mathbb{Q} \to G$?*

Theorem 5.4. *Let M/N be an AT-chief factor in G.* (a) *If V is a nontrivial set of words, then $N \le V(\langle g^{T(G)}\rangle)$ and $M = \langle g^G\rangle$ for every $g \in M - N$. In particular, G has a unique AT-chief series.*

(b) *Every $K \trianglelefteq T(G)$ satisfies $K \le A$ or $A \le K$, and every $K \trianglelefteq G$ satisfies $K \le T(G)$ or $T(G) \le K$.*

(c) *If $g \in M - N$, then gN contains elements of order p, and any such two are conjugate in $T(G)$.*

(d) *The normal series induced in $T(G)/A$ by the unique AT-chief series in G has order-type $(\mathbb{Q}, <)$.*

Proof. (a) Use §3 and follow the proof of [9, Theorem 4.7].

(b) If $h \in T(G)$ and $g \in G - T(G)$, then an application of Construction 3.2 and Lemma 3.4 similar as in (a), with $\theta: G \to G/T(G)$, $G_0 = 1$, $G_1 = \langle g\rangle$ and $\widehat{Z}_1 = Z_1\theta = C_1\theta = D_1 = \langle g\theta^2\rangle \lneqq \langle g\theta\rangle$, yields that $h \in \langle g^G\rangle$.

(c) Use §3 and follow the proof of [9, Theorem 4.10(b)].

(d) Follow the proof of [9, Theorem 4.11(a)/(b)] to show that the order-type in question is dense without a maximal element. Now, suppose that $T(G)/A$ has a minimal normal subgroup M/A. The group $G \times C_p$ is contained in a closed $LFC_{p,A\times C_p}$-group H, and $C_p \le \langle g^H\rangle''$ for every $g \in M - A$. Since G is closed in $LFC_{p,A}$, we obtain that $C_p \le \langle g^G\rangle'' \le M'' = 1$, a contradiction. □

Lemma 5.5. *If U is a finite subgroup of G, then every conjugacy class of elements in $T(G)$ contains an element which centralizes U.*

Proof. Extend the conjugacy action of G on $T(G)$ to an action of G on $T(G)\mathrm{Wr}\, C_p$ via $(c, f)^g = (c, f^g)$ where $f^g(d) = f(d)^g$ for all $d \in C_p$. Use Construction 3.2 to find an embedding σ of G into an $LFC_{p,A\sigma}$-subgroup W of $((T(G)\mathrm{Wr}\, C_p) \rtimes G)\mathrm{Wr}\, G/T(G)$ which contains the diagonal subgroup, and such that $\sigma | T(G)$ is the diagonal embedding. Then the diagonal subgroup contains the required elements (see [10, Lemma 2.2]). □

Theorem 5.6. (a) *Let M/N be an AT-chief factor in G. If $k = GF(p)$ and $\overline{\overline{G}} = G/C_G(M/N)$, then $\overline{\overline{G}}$ is isomorphic to the multiplicative group of the algebraic closure of k via the canonical embedding $\overline{\overline{G}} \to k\overline{\overline{G}}/An(M/N)$. In particular, M/N is infinite, $\overline{\overline{G}}$ acts transitively on $M/N - \{1\}$, and any two elements of order p in $M - N$ are conjugate in G.*

(b) *If* M_1/N_1 *and* M_2/N_2 *are* AT*-chief factors in* G, *if* $g \in G - T(G)$, *and if* n *is any* p'*-number, then there exists an* AT*-chief factor* M/N *between* M_1/N_1 *and* M_2/N_2 *such that* g *acts as an automorphism of order* n *on* M/N. *In particular,* $T(G)$ *is the intersection of all* $C_G(M/N)$ *where* M/N *is an* AT*-chief factor in* G.

(c) *Let* $M_1/N_1, \dots, M_r/N_r$ *be* AT*-chief factors in* G. *Let* K *be the algebraic closure of* $GF(p)$. *Because of* (a) *we can identify* $K^\times$ *with* $G/C_G(M_i/N_i)$ *for each* i. *Fix* K*-isomorphisms* $\varepsilon_i\colon K^+ \to M_i/N_i$ *as in Lemma* 2.2. *Then, for any* $k_1, \dots, k_r \in K^\times$, *there exists* $g \in G$ *such that* $g \cdot C_G(M_i/N_i) = k_i$.

Proof. (a) Since $\overline{\overline{G}}$ is divisible by Theorem 5.2, it suffices to find for every prime $q \neq p$ an element of order q in $\overline{\overline{G}}$. Adopt the notation introduced in Lemma 2.2. Let L be the algebraic closure of K. Then the p'-component of $\mathbb{Q}^+/\mathbb{Z}$ is isomorphic to $L^\times$, and this gives rise to an epimorphism $\mu\colon \mathbb{Q}^+ \to L^\times$. Choose $x \in \mathbb{Q}$ with $o(x\mu) = q$. Let $\nu\colon \overline{G} = G/T(G) \to L^\times$ be the composition of the canonical homomorphisms $\overline{G} \to \overline{\overline{G}}$ and $\overline{\overline{G}} \to K^\times \leq L^\times$. Define $\varphi\colon \mathbb{Q}\times\overline{G} \to L^\times$ via $(z, \overline{g})\varphi = z\mu\cdot\overline{g}\nu$. Consider L^+ as regular $L^\times$-module. Identify K^+ with M/N via ε and suppress ε.

Let $\{G_n/N | n \in \omega\}$ be an ascending chain of FC_p^0-groups with union G/N, where $G_0 = N$. Let $\{L_n | n \in \omega\}$ be an ascending chain of finite subfields of L such that $x\mu \in L_0$, $|L_n| \geq p^n$, $(M \cap G_n)/N \leq L_n^+$, and such that the preimage U_n of $L_n^\times$ under φ contains $\overline{G}_n$. Applying Construction 3.1 to the canonical epimorphism $\theta\colon G/N \to G/M$ we obtain for each $n \in \omega$ a partition $G/M = \Omega_{n,0}\dot\cup\cdots\dot\cup\Omega_{n,k_n}$ such that G_nM/M permutes the $\Omega_{n,i}$ regularly by left multiplication, and such that the groups

$$\begin{aligned} W_n = \{(gM, f) \in (L^+ \rtimes_\varphi (\mathbb{Q}\times\overline{G}))\mathrm{Wr}\, G/M | g \in G_n, \operatorname{Im} f \subseteq L_n^+ \rtimes_\varphi U_n,\\ f \text{ is constant on each } \Omega_{n,i},\\ \text{and the } (\mathbb{Q}\times\overline{G})\text{-component of } f \text{ is constant}\} \end{aligned}$$

form an ascending chain of FC-subgroups of $(L^+ \rtimes_\varphi (\mathbb{Q}\times\overline{G}))\mathrm{Wr}\, G/M$. Thus, Construction 3.3 yields an ms-embedding σ of G/N into the LFC_p-group $W = \bigcup\{W_n | n \in \omega\} \leq (L^+ \rtimes_\varphi (\mathbb{Q}\times\overline{G}))\mathrm{Wr}\, G/M$. Identify $L^+ \rtimes_\varphi (\mathbb{Q}\times\overline{G})$ canonically with the diagonal subgroup of $(L^+ \rtimes_\varphi (\mathbb{Q}\times\overline{G}))\mathrm{Wr}\, G/M$, and let $\Omega = L_0^+ \rtimes_\varphi U_0 \leq W$. Because of $G_0 = N$ we have that $\sigma|M/N$ is the diagonal embedding.

Now, let η be the composition of the canonical epimorphism $G \to G/N$ and σ. Apply Construction 3.2 to $\eta\colon G \to W$ to obtain an embedding τ of G into an $LFC_{p,A\tau}$-subgroup V of $(T(G) \rtimes G)\mathrm{Wr}\, W$. By Theorem 5.2 and Lemma 3.4 we have $y = (m_0\eta, 1) \in m_0\tau \cdot \langle m_0\tau^V\rangle'$. Let F be an $FC_{p,A\tau}^0$-subgroup of V such that $\langle m_0\tau, (\omega, 1) | \omega \in \Omega\rangle \leq F$ and $y \in m_0\tau \cdot \langle m_0\tau^F\rangle'$. Since G is closed in $LFC_{p,A}$, there exists an embedding $\lambda\colon F \to G$ with $\tau\lambda|\langle A, m_0\rangle = \mathrm{id}$. Let $J = \langle (\omega, 1) | \omega \in \Omega\rangle\lambda$. Then $J_0 = \langle (\omega, 1) | \omega \in L_0^+\rangle\lambda$ is a minimal normal subgroup in J, and $y\lambda \in m_0N$ enforces that $J_0 \cap N = 1$ and $J_0 \leq M$. Thus, $(x, 1)\lambda$ induces an automorphism of order q on J_0N/N and hence on M/N.

(b) Let L be the algebraic closure of $GF(p)$. Since $\overline{G} = G/T(G)$ is divisible, there exists an epimorphism $\varphi\colon \overline{G} \to L^\times$ such that $o(\overline{g}\varphi) = n$. Consider L^+ as regular $L^\times$-module, and put $\widehat{L} = L^+ \rtimes_\varphi \overline{G}$. Identify L^+ canonically with the diagonal subgroup of $L^+ \operatorname{Wr} C_p$, and extend the action of $L^\times$ on L^+ to an action of $L^\times$ on $L^+ \operatorname{Wr} C_p$ as in the proof of Lemma 5.5. Then $\widehat{L} \le \widetilde{L} = (L^+ \operatorname{Wr} C_p) \rtimes_\varphi \overline{G}$.

Suppose that $M_2 \le N_1$. By Theorem 5.4(d) there exist AT-chief factors M_i^*/N_i^* such that $M_2 \le N_2^* \le M_2^* \le N_1^* \le M_1^* \le N_1$. Choose $m_i \in M_i^* - N_i^*$ with $o(m_i) = p$ (Theorem 5.4(c)). Use Construction 3.3 with $\mathrm{id}\colon G/N_1^* \to G/N_1^*$ in place of $\theta\colon G \to H$ to find an embedding σ of G/N_1^* into an LFC_p-subgroup V of $\widetilde{L} \operatorname{Wr} G/N_1^*$, which contains the diagonal subgroup, and such that $gN_1^*\sigma = (gN_1^*, f_{gN_1^*})$ where $f_{gN_1^*} \equiv (\overline{g}, 1)$. Identify $\widetilde{L}$ canonically with the diagonal subgroup of $\widetilde{L} \operatorname{Wr} G/N_1^*$. Fix $l_0 = 1 \in L^+$. Since $L^+ \le (L^+ \operatorname{Wr} C_p)'$, Lemma 3.4 yields that $l_0 \in \langle m_1 N_1^* \sigma^V \rangle$. Let L_0 be a finite subfield of L containing $\overline{g}\varphi$, and let $\overline{U}_0$ be the preimage of $L_0^\times$ under φ. Put $\Omega = L_0^+ \rtimes_\varphi \overline{U}_0 \le \widetilde{L} \le V$.

Now, apply Construction 3.2 to extend the standard embedding $\tau_0\colon \langle m_1 \rangle \to 1 \operatorname{Wr} V$ given by $m_1\tau_0 = (m_1 N_1^* \sigma, 1)$ to an embedding τ of G into an $LFC_{p,A\tau}$-subgroup W of $(T(G) \rtimes G)\operatorname{Wr} V$. Then Lemma 3.4 yields that $m_2\tau \in \langle (l_0, 1)^W \rangle$ and $(gN_1^*\sigma, f) \in g\tau \cdot \langle m_1 \tau^W \rangle$ where $f \equiv (g, 1)$.

Let F be an $FC^0_{p,A\tau}$-subgroup of W such that $\langle g\tau, m_1\tau, m_2\tau, (\omega, 1) | \omega \in \Omega \rangle \le F$ and also $m_2\tau \in \langle (l_0, 1)^F \rangle$, $(l_0, 1) \in \langle m_1\tau^F \rangle$, and $(gN_1^*\sigma, f) \in g\tau \cdot \langle m_1 \tau^F \rangle$. Since G is closed in $LFC_{p,A}$, there exists an embedding $\lambda\colon F \to G$ such that $\tau\lambda | \langle A, m_1, m_2, g \rangle = \mathrm{id}$. Let M/N be the unique AT-chief factor in G with $x = (l_0, 1)\lambda \in M - N$. Because of $m_2 \in \langle x^G \rangle$ and $x \in \langle m_1^G \rangle$ we have that $M_2 \le N_2^* \le N \le M \le M_1^* \le N_1$ (Theorem 5.4(a)). Because of $y = (gN_1^*\sigma, f)\lambda \in g \cdot \langle m_1^G \rangle \subseteq \overline{g}$ we have that g acts on M/N as y does. Let $J = \langle (\omega, 1) | \omega \in \Omega \rangle \lambda$. Then $J_0 = \langle (\omega, 1) | \omega \in L_0^+ \rangle \lambda$ is a minimal normal subgroup in J, and $x \in M - N$ enforces that $J_0 \cap N = 1$ and $J_0 \le M$. Thus, y induces an automorphism of order q on J_0N/N and hence on M/N.

(c) Suppose that $M_{i+1} \le N_i$ for $1 \le i \le r-1$. Proceeding recursively, we may assume that there exists $h \in G$ with $h \cdot C_G(M_i/N_i) = k_i$ for $1 \le i \le r-1$. Choose a finite subfield $K_0 \le K$ such that $k_i \in K_0$ for all i. An iterated application of Theorem 5.4(c) and Lemma 5.5 yields embeddings $\mu_i\colon K_0^+ \to G$ such that $k\mu_i N_i = k\varepsilon_i$ for all $k \in K_0$ and all i, and such that $[K_0\mu_i, K_0\mu_j] = 1$ for all i, j. Note that $\langle K_0\mu_i | 1 \le i \le r-1 \rangle \cap M_r = 1$, and that $\langle K_0\mu_i | 1 \le i \le r \rangle \cap N_r = 1$.

Applying Construction 3.3, $\langle K_0\mu_i | 1 \le i \le r-1 \rangle N_r/N_r \to 1 \operatorname{Wr} G/M_r$, the canonical embedding, can be extended to an embedding σ of G/N_r into an LFC_p-subgroup V of $(M_r/N_r \rtimes \overline{G})\operatorname{Wr} G/M_r$ which contains the diagonal subgroup, and such that $\sigma | M_r/N_r$ is the diagonal embedding. Choose $g_r \in G$ such that $g_r \cdot C_G(M_r/N_r) = k_r$. Put $v = (hM_r, f) \in V$ where $f \equiv (\overline{g}_r, 1)$. Then v acts on $L_i = (K_0\mu_i N_r/N_r)\sigma$ as right multiplication with $k_i\mu_i N_r\sigma$. Put $V_0 = \langle v, L_1, \dots, L_r \rangle$.

Now use Construction 3.2 with $\sigma\colon G/N_r \to V$ in place of $\theta\colon G \to H$ to extend the canonical embedding $\langle K_0\mu_i | 1 \le i \le r \rangle \to 1 \operatorname{Wr} V$ to an embedding τ of G into an $LFC_{p,A\tau}$-subgroup W of $(T(G) \rtimes G)\operatorname{Wr} V$. Let W_0 be an $FC^0_{p,A\tau}$-subgroup of W with $(v_0, 1) \in W_0$ for all $v_0 \in V_0$. Since G is closed in $LFC_{p,A}$, there exists an embedding $\lambda\colon W_0 \to G$ with $\tau\lambda | \langle A, K_0\mu_1, \dots, K_0\mu_r \rangle = \mathrm{id}$. Now $g = (v, 1)\lambda$ is the desired element. □

It follows from Theorem 5.6(a) that $T(G)/A$ is not isomorphic to E_A/A, where E_A denotes the unique countable e.c. $L\mathfrak{F}_{p,A}$-group, since every abelian factor of E_A/A has order p (see [13]).

Theorem 5.7. *Let $K \trianglelefteq G$ with $A < K < T(G)$ and $K \neq \langle g^G \rangle$ for all $g \in G$.*

(a) *For every $FC^0_{p,A}$-subgroup F of $T(G)$ there exists an embedding $\sigma : F \to K$ with $\sigma|K \cap F = \mathrm{id}$.*

(b) *Every normal subgroup of K is normal in $T(G)$, and $T(G)$ induces via conjugation locally inner automorphisms on K.*

(c) *$T(G)$ splits over K, and every finite p-subgroup U of $T(G)$ with $U \cap K = 1$ is contained in a complement to K in $T(G)$.*

Proof. (a) Let F_0 be a finite subgroup of F such that $F = AF_0$. Because of $Z(T(G)) = A$ there exists a finite group $F_0^* \leq T(G)$ such that $F_0 \leq F_0^*$ and $C_{F_0}(F_0^*) = A \cap F_0$. Use §3 and follow the proof of [9, Theorem 4.8(a)] to obtain an embedding $\tau : F_0^* \to K$ with $\tau|F_0^* \cap K = \mathrm{id}$. Then $A \cap F_0\tau \leq C_{F_0\tau}(F_0^*\tau) = (A \cap F_0)\tau = A \cap F_0$. Therefore, $\sigma : F \to K$ given by $(af)\sigma = a \cdot f\tau$ for all $a \in A$, $f \in F_0$ is the desired embedding.

(b) Follow the proof of [9, Theorem 4.8(b)/(c)].

(c) Use §3 and follow the proofs of [10, Theorems 4.1/4.2]. □

Theorem 5.8. *Normality is transitive in $T(G)$, and for every proper subnormal subgroup S of G there exists an AT-chief factor M/N in G such that $N < S < M$.*

Proof. Use §3 and follow the proof of [9, Theorem 4.11(f)] to show that every subnormal subgroup S of $T(G)$ satisfies either $S \leq A$, or $S = K$ for some normal torsion subgroup K in G which does not occur in any AT-chief factor, or $N \leq S \leq M$ for some AT-chief factor M/N in G. Because of $M/N \leq Z(T(G)/N)$ this implies that normality is transitive in $T(G)$.

Now, let $S_2 \trianglelefteq S_1 \trianglelefteq G$. If $S_2 \leq T(G)$, then we are done. Suppose now that S_2 is not contained in $T(G)$. Then $T(G) \leq S_1$ by Theorem 5.4(b). Let $g \in S_2 - T(G)$. Applying Construction 3.2 as in the proof of Theorem 5.4(b), we obtain from Lemma 3.4 that $T(G) \leq [[g, T(G)], T(G)] \subseteq S_2$, whence $S_2 \trianglelefteq G$. □

6. The torsion subgroup

The aim of this section is to show the uniqueness of torsion subgroup in countable closed $LFC_{p,A}$-groups via an algebraic characterization in terms of injectivity. As before, G will always denote a countable closed $LFC_{p,A}$-group.

Theorem 6.1. *Let $\Sigma_{T(G)} = \{(M_q, N_q) | q \in \mathbb{Q} \cup \{-\infty\}\}$ be the series in $T(G)$ induced from the unique AT-chief series in G. Let H be a countable $L\mathfrak{F}_{p,A_0}$-group with an A_0T-normal series $\Sigma_H = \{(K_j, L_j) | j \in J \cup \{-\infty\}\}$, which has central elementary-abelian A_0T-factors. Let $\alpha : J \cup \{-\infty\} \to \mathbb{Q} \cup \{-\infty\}$ be an order-preserving injection, and for every $j \in J$, let $\beta_j : K_j/L_j \to M_{j\alpha}/N_{j\alpha}$ be an embedding. (Note that $\beta_{-\infty} : A_0 \to A$.) If $\sigma_0 : U \to G$ is an embedding of some finite subgroup $U \leq H$ satisfying $(K_j \cap U)\sigma_0 = M_{j\alpha} \cap U\sigma_0$, $(L_j \cap U)\sigma_0 = N_{j\alpha} \cap U\sigma_0$, and inducing β_j on $(K_j \cap U)L_j/L_j$ for every $j \in J$, then σ_0 can be extended to an embedding $\sigma : H \to G$ satisfying $K_j\sigma = M_{j\alpha} \cap H\sigma$, $L_j\sigma = N_{j\alpha} \cap H\sigma$, and inducing β_j on K_j/L_j for every $j \in J$.*

Proof. Use §3 and Lemma 5.5, and follow the proofs of [10, Theorems 3.1/3.2] (see also [1, Satz]). □

Theorem 6.1 shows that $T(G)$ embeds every countable $L\mathfrak{F}_{p,A_0}$-group ($A_0 \le A$) in every possible way. In combination with a back-and-forth argument, Theorem 6.1 immediately yields

Corollary 6.2. *If G and H are countable closed $LFC_{p,A}$-groups, then $T(G) \cong T(H)$.*

We denote the unique isomorphism type of torsion subgroup of the countable closed $LFC_{p,A}$-groups by T_A; we will also write T_p instead of T_1. Note that Theorem 6.1 characterizes T_A, i.e., a countable $L\mathfrak{F}_{p,A}$-group T is isomorphic to T_A if and only if it has an AT-normal series Σ of order type $(\mathbb{Q}\cup\{-\infty\}, <)$ with infinite central elementary-abelian factors such that Theorem 6.1 holds with T in place of $T(G)$ and with Σ in place of $\Sigma_{T(G)}$. In the following, $\Sigma = \{(M_q, N_q) | q \in \mathbb{Q}\cup\{-\infty\}\}$ will always denote the distinguished AT-normal series in T_A.

Theorem 6.3. (a) *An isomorphism $\alpha: U \to V$ between finite subgroups of T_A is induced by conjugation in T_A if and only if α induces the identity on each $(M_q \cap U)N_q/N_q$ $(q \in \mathbb{Q} \cup \{-\infty\})$.*

(b) *Let K be the algebraic closure of $GF(p)$. Because of Theorem* 5.6(a) *we may identify $K^\times$ with each $G/C_G(M_q/N_q)$ $(q \in \mathbb{Q})$. Fix K-isomorphisms $K^+ \to M_q/N_q$ as in Lemma 2.2. Then an isomorphism $\alpha: U \to V$ between finite subgroups of T_A is induced by conjugation in G, if and only if*

(1) *there exist $k_q \in K^\times$ such that α acts on $(U \cap M_q)N_q/N_q$ as k_q does, and*

(2) *α induces the identity on $A \cap U$.*

Proof. (a) If $U \le A$, then we are done. Otherwise, let M/N be the unique AT-chief factor in G with $U \cap N < U \cap M = U$. Choose $U \cap N \le V < U$ with $|U : V| = p$. As in the proof of [11, Theorem 6.1], we may assume that $\alpha|V = \mathrm{id}$. Fix $w \in U - V$.

Apply Construction 3.2 to the canonical epimorphism $\theta: G \to G/N$, with $G_0 = V$ and $G_1 = \langle U, w\alpha\rangle$. This yields an embedding σ of G into an $LFC_{p,A\sigma}$-subgroup W of $(T(G) \rtimes G)\mathrm{Wr}\, G/N$ such that $\sigma: V \to ((V \cap N) \rtimes 1)\mathrm{Wr}\, G/N$. Let $\widehat{R}$ be a right transversal of $U\theta$ in $G_1\theta$. In the notation of Construction 3.2, put $T = \widehat{Z}^1 V\theta \cdot \widehat{R} \cdot T^1$ (observe that $D_1 = 1$). Then T is a right transversal of $\langle w\theta\rangle$ in G/N, and as in the proof of [11, Theorem 6.1] the element $\tilde{g} = (1, s) \in T(W_1)$ given by

$$\begin{aligned} s((w\theta)^r t) &= f_{w^r}(t) \cdot f_{w^r\alpha}(t)^{-1} = (1, (w^r \cdot (w\alpha)^{-r})^{((w\theta)^r t)\theta_0^*}) \\ &= (1, w^r \cdot (w\alpha)^{-r}) \quad \text{for all } t \in T,\ 0 \le r \le p-1, \end{aligned}$$

satisfies $w\sigma^{\tilde{g}} = w\alpha\sigma$.

It remains to show that $[V\sigma, \tilde{g}] = 1$. Let $v\sigma = (v\theta, f_v) \in W_0$ for $v \in V$. Since $U\theta$ is abelian, our choice of T ensures that $[(v\theta, 1), \tilde{g}] = 1$. Moreover, $[(1, f_v), \tilde{g}] = 1$ as in the proof of [11, Theorem 6.1].

(b) Let α be given. From Theorem 5.6(c) we obtain $g \in G$ such that $g \cdot C_G(M_q/N_q) = k_q$ for all $q \in \mathbb{Q}$ with $M_q \cap U > N_q \cap U$. But now, $g^{-1} \cdot \alpha$ induces the identity on each $(M_q \cap U)N_q/N_q$ $(q \in \mathbb{Q} \cup \{-\infty\})$, whence there exists $t \in T(G)$ such that $t = g^{-1} \cdot \alpha$. □

Theorem 6.4. (a) *$N_q \cong T_A$ and $T_A/N_q \cong T_E$ for every $q \in \mathbb{Q}$, where E denotes the countably infinite elementary-abelian p-group.*

(b) *If $K \triangleleft T_A$ with $A < K$ does not satisfy $N_q \leq K \leq M_q$ for some $q \in \mathbb{Q}$, then $K \cong T_A$ and $T_A/K \cong T_p$.*

Proof. Because of Theorem 5.7(c), the groups in question satisfy the characterizations of T_A resp. T_E, T_p given by Theorem 6.1. □

Question 6.5. *In the situation of Theorem* 6.4(b), *is G/K a closed LFC_p-group*?

As in the case of countable e.c. $L\mathfrak{F}_{p,A}$-groups, we can show that different groups A yield different factors T_A/A:

Theorem 6.6. *$H_n(T_A, \mathbb{Z}) = 0$ for all $n \geq 1$. In particular, T_A/A has Schur multiplier A.*

For the proof of Theorem 6.6 we use a result of K. Varadarajan [20] which can be applied correspondingly to show that every e.c. $L\mathfrak{F}_{p,A}$-group G satisfies $H_n(G, \mathbb{Z}) = 0$ for all $n \geq 1$. To this end we need the following definition. A group G is said to be *pseudo-mitotic*, if for every finitely generated subgroup H of G there exist embeddings $\psi_1, \psi_2 \colon H \to G$ and an element $g \in G$ such that

(a) $$h\psi_2 = h \cdot h\psi_1 \quad \text{for all } h \in H,$$

(b) $$[H, H\psi_1] = 1, \quad \text{and}$$

(c) $$h\psi_2 = (h\psi_1)^g \quad \text{for all } h \in H.$$

Proof of Theorem 6.6. Since every pseudo-mitotic group G satisfies $H_n(G, \mathbb{Z}) = 0$ for all $n \geq 1$ [20], we only need to show that T_A is pseudo-mitotic. Let $H \leq T_A$ be finite, and let G be a countable closed $LFC_{p,A}$-group with $T_A = T(G)$. Without loss we may assume that H is not contained in A. Identify $T(G)$ with the diagonal subgroup of $V = T(G)\mathrm{Wr}\, H$, and extend the conjugacy action of G on $T(G)$ to an action of G on V as in the proof of Lemma 5.5. Denote by $\tau \colon H \to V$ the canonical embedding of H onto the top group of V. Clearly, $[H, H\tau] = 1$, and from Theorem 5.2 and [13, (3.2.3)] we have that $H \leq \langle h\tau^V \rangle'$ for all $h \in H - 1$.

Use Construction 3.2 to find an embedding σ of G into an $LFC_{p,A\sigma}$-subgroup W of $(V \rtimes G)\mathrm{Wr}\, G/T(G)$ which contains the diagonal subgroup, and such that $\sigma|T(G)$ is the diagonal embedding. Since G is e.c. in $LFC_{p,A}$, we obtain an embedding $\psi_1 \colon H \to G$ such that $[H, H\psi_1] = 1$ and

$$H \leq \langle h\psi_1^G \rangle' \quad \text{for all } h \in H - 1. \tag{6.1}$$

Define $\psi_2 \colon H \to G$ via $h\psi_2 = h \cdot h\psi_1$. Since H is not contained in A, (6.1) yields that $A \cap H\psi_1 = 1$, and that $H \leq N$ where M/N denotes the AT-chief factor in G with $1 = N \cap H\psi_1 < M \cap H\psi_1$. Therefore, ψ_2 is an embedding, and $\psi_1^{-1}\psi_2 \colon H\psi_1 \to H\psi_2$ induces the identity on each $(M_q \cap H\psi_1)N_q/N_q$ $(q \in \mathbb{Q} \cup \{-\infty\})$ and is thus induced by conjugation in T_A (Theorem 6.3). □

7. Automorphisms

In this section we will show that the structure of $\mathrm{Aut}(T_A/A)$ is similar to that of $\mathrm{Aut}(E_A/A)$ and $\mathrm{Aut}(ULF)$ (cf. [1 and 5, §4]). As before, G denotes a countable closed $LFC_{p,A}$-group with $T(G) = T_A$.

Theorem 7.1. *Consider the diagram*

$$\begin{array}{ccc} \mathrm{Aut}(G) & \xrightarrow{\ \varphi\ } & \mathrm{Aut}(G/A) \\ {\scriptstyle\eta}\big\downarrow & & {\scriptstyle\psi}\big\downarrow \\ \mathrm{Aut}(T_A) & \xrightarrow{\ \chi\ } & \mathrm{Aut}(T_A/A) \end{array}$$

of canonical homomorphisms. Then χ *and* ψ *are embeddings, and* $\mathrm{Ker}\,\eta \cong \mathrm{Ker}\,\varphi \cong \mathrm{Hom}(\mathbb{Q}^{(\omega)}, D(A))$, *where* $D(A)$ *denotes the divisible radical of* A. *In particular,* η *and* φ *are embeddings if* A *contains no Prüfer p-subgroup.*

Proof. Clearly, $g^{-1} \cdot g\alpha \in A$ for all $g \in G$, $\alpha \in \mathrm{Ker}\,\varphi$. Therefore, straightforward calculations show that an isomorphism $\tau\colon \mathrm{Ker}\,\varphi \to \mathrm{Hom}(G, A) \cong \mathrm{Hom}(\mathbb{Q}^{(\omega)}, D(A))$ is given by $[g](\alpha\tau) = g^{-1} \cdot g\alpha$. (Note that $A \le T_A = G' \le \mathrm{Ker}(\alpha\tau)$.)

Since $C_{G/A}(T_A/A) = 1$ by Theorem 5.4, it follows as in [5, Lemma 4.1] that $\mathrm{Ker}\,\psi = 1$, and that $g^{-1} \cdot g\alpha \in A$ for all $g \in G$, $\alpha \in \mathrm{Ker}\,\eta$. As above, we obtain $\mathrm{Ker}\,\eta \cong \mathrm{Hom}(\mathbb{Q}^{(\omega)}, D(A))$. Correspondingly, $\mathrm{Ker}\,\chi \cong \mathrm{Hom}(T_A, A)$ is trivial as T_A is perfect. □

In the same way, an assertion on [1, p. 202] can be improved: The canonical homomorphism $\mathrm{Aut}(E_A) \to \mathrm{Aut}(E_A/A)$ is an embedding.

Now, put

$$\mathrm{Stab}(\Sigma) = \{\alpha \in \mathrm{Aut}(T_A) | \alpha \text{ centralizes every } M_q/N_q \ (q \in \mathbb{Q} \cup \{-\infty\})\},$$

and let $L\,\mathrm{Inn}(T_A)$ be the group of all locally inner automorphisms of T_A. Denote by $A(\mathbb{Q})$ the group of all order-preserving permutations of $\mathbb{Q}$, and by $GL(\aleph_0, p)$ the group of all automorphisms of the $\aleph_0$-dimensional $GF(p)$-vector space V. Fix isomorphisms $\gamma_q\colon M_q/N_q \to V$ $(q \in \mathbb{Q})$. Then an embedding

$$\phi\colon \mathrm{Aut}(T_A)/\mathrm{Stab}(\Sigma) \to \mathrm{Aut}(A) \times [GL(\aleph_0, p)\mathrm{Wr}_{\mathbb{Q}} A(\mathbb{Q})]$$

is given by

$$(\alpha \cdot \mathrm{Stab}(\Sigma))\phi = (\alpha|_A, (\overline{\alpha}, f_\alpha)),$$

where

$$(M_{q\overline{\alpha}}, N_{q\overline{\alpha}}) = (M_q\alpha, N_q\alpha) \quad \text{and} \quad f_\alpha(q) = \gamma^{-1}_{q\overline{\alpha}^{-1}} \cdot \alpha \cdot \gamma_q \quad \text{for all } q \in \mathbb{Q}.$$

Theorem 7.2. (a) $\mathrm{Stab}(\Sigma) = L\,\mathrm{Inn}(T_A)$, *and* $L\,\mathrm{Inn}(T_A)$ *contains* $2^{\aleph_0}$ *automorphisms of order* m *for every p-number* m, *and for* $m = \infty$.

(b) $\phi\colon \mathrm{Aut}(T_A)/\mathrm{Stab}(\Sigma) \to \mathrm{Aut}(A) \times [GL(\aleph_0, p)\mathrm{Wr}_{\mathbb{Q}} A(\mathbb{Q})]$ *is actually an isomorphism. In particular,* T_A/A *is characteristically simple.*

Proof. (a) follows from Theorem 6.3, and as in the proof of [9, Theorem 5.1].

(b) Follow the lines of proof of [11, Theorem 6.2] or [1, Korollar], using Theorem 6.1. □

Corollary 7.3. *Let* U *and* V *be finite subgroups of* T_A *with*

$$\Sigma \cap U = \{(M_{q_i} \cap U, N_{q_i} \cap U) | 0 \le i \le r\}.$$

Then an isomorphism α *between* U *and* V *is induced by an automorphism of* T_A, *if and only if*

(1) $\alpha|_{A \cap U}$ *can be extended to an automorphism of* A, *and*

(2) *the map* $\lambda\colon \{q_0, \dots, q_r\} \to \mathbb{Q}$ *given by* $(M_{q_i\lambda} \cap V,\ N_{q_i\lambda} \cap V) = ((M_{q_i} \cap U)\alpha,\ (N_{q_i} \cap U)\alpha)$ *can be extended to an order-preserving permutation of* $\mathbb{Q}$.

As in [5, §4], for groups $E \le H$ we put

$$F_E(H) = \{\alpha \in \mathrm{Aut}(H) \mid \alpha \text{ is locally induced by conjugation with an element from } E\}$$

and

$$F\,\mathrm{Inn}(H) = \{\alpha \in \mathrm{Aut}(H) \mid \alpha \in F_E(H) \text{ for some finitely generated subgroup } E \text{ of } H\}.$$

Theorem 7.4. *If* E *is an* $FC^0_{p,A}$*-subgroup of* G, *then* $F_{E/A}(G/A)$ *is isomorphic to the universal profinite completion of* E/A.

Proof. Let $\{G_n \mid n \in \omega\}$ be an ascending chain of $FC^0_{p,A}$-subgroups of G with $G_0 = E$ and union G. As in the proof of [5, Theorem 4.4] it suffices to find for every $m \in \omega$ some $n \in \omega$ such that $C_{E/A}(G_n/A) \le (E/A)^m$. Choose $E_0 \in FC^0$ with $E = E_0A$. Let $\{1 = e_0, \dots, e_r\}$ be a transversal of E_0^m in E_0. Without loss we may assume that E_0^m is a torsion-free subgroup of $Z(E_0)$.

Put $\widetilde{G} = G \times C_{p^2}$ and $C_{p^2} = \langle c\rangle$. Let $\{\widetilde{G}_n \mid n \in \omega\}$ be an ascending chain of FC^0-groups with $\widetilde{G}_0 = 1$, $\widetilde{G}_1 = E_0 \times C_{p^2}$, and union $\widetilde{G}$. Following Construction 3.1 with the canonical epimorphism $\theta\colon \widetilde{G} \to H = \widetilde{G}/(A \times C_{p^2})$ and with $Z_1 = C_1 = E_0^m$, $\widehat{Z}_1 = D_1 = E_0^m\theta$, we obtain for each $n \in \omega$ a partition $H = \Omega_{n,0} \dot\cup \cdots \dot\cup \Omega_{n,k_n}$ such that $\widetilde{G}_n\theta$ permutes the $\Omega_{n,i}$ regularly by left multiplication, and such that the groups

$$W_n = \{(h, f) \in (A \times C_{p^2}) \cap \widetilde{G}_n \,\mathrm{Wr}\, H \mid h \in \widetilde{G}_n\theta, \text{ and } f \text{ is constant on each } \Omega_{n,i}\}$$

form an ascending chain of FC-subgroups of $(A \times C_{p^2})\mathrm{Wr}\,H$. In the case when $n = 1$, the $\Omega_{n,i}$ are precisely the sets $(e_iE_0^m)\theta \cdot T$, where T is a right transversal of $E_0\theta$ in H. Now, Construction 3.2 yields an ms-embedding σ of $\widetilde{G}$ into the $LFC_{p,A\sigma}$-group $W = \bigcup\{W_n \mid n \in \omega\} \le (A \times C_{p^2})\mathrm{Wr}\,H$ with $g\sigma = (g\theta, f_g)$ for all $g \in G$. (Because of $A \times C_{p^2} \le Z(\widetilde{G})$ we can replace $(A \times C_{p^2}) \rtimes \widetilde{G}$ by $A \times C_{p^2}$ in the base group.)

Let $y = (1, f) \in W_1$, where

$$f(h) = \begin{cases} c & \text{if } h \in E_0^m\theta \cdot T, \\ 1 & \text{else.} \end{cases}$$

Then $[y, E_0^m\sigma] = 1$, and $[[y, e_i\sigma], e_i\sigma] \ne 1$ for $1 \le i \le r$. Since G is e.c. in $LFC_{p,A}$ (and since E_0^m is finitely generated), we obtain an element $x \in G$ such that $[x, E_0^m] = 1$ and $[[x, e_i], e_i] \ne 1$ for $1 \le i \le r$. The latter implies that $[x, e_i] \notin Z(G) = A$ (by Theorem 5.1). Choose $n \in \omega$ with $x \in G_n$. Then $C_{E/A}(G_n/A) \le (E/A)^m$. □

Note that the above method also yields an alternative proof of [5, Theorem 4.4].

Theorem 7.5. $\mathbb{R}^+ \lesssim \mathrm{Aut}(T_A/A)$.

Proof. Let G be a countable closed $LFC_{p,A}$-supergroup of $A \times \mathbb{Q}^+$. Then $\mathbb{Q}^+ \lesssim G/A$, and so we may follow the proof of [6, Korollar 5.9] with Theorem

7.4 in place of [6, Satz 5.7], to show that $\mathbb{R}^+ \lesssim \mathrm{Aut}(G/A)$. But $\mathrm{Aut}(G/A) \lesssim \mathrm{Aut}(T_A/A)$ by Theorem 7.1. □

Theorem 7.6. *$F\,\mathrm{Inn}(G/A)$ is an LFC_p-group with*

$$T(F\,\mathrm{Inn}(G/A)) = F\,\mathrm{Inn}(G/A)' = T(\mathrm{Inn}(G/A)) \cong T_A/A.$$

Proof. Copy the proof of [5, Theorem 4.5]. □

Finally, let us note that the following can be proved from Theorems 6.1 and 7.2(b) in the same way as the corresponding result in [12] for E_p.

Theorem 7.7. *Let K be any field. Then the augmentation ideal of KT_p is the unique proper ideal in KT_p which is invariant under the basis transformations of KT_p induced by $\mathrm{Aut}(T_p)$.*

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FACHBEREICH 17-MATHEMATIK, JOHANNES GUTENBERG-UNIVERSITÄT, SAARSTRASSE 21, D-6500 MAINZ, GERMANY

E-mail address: leinen@mzdmza.zdv.uni-mainz.de