TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 336, Number 1, March 1993

# CONFORMAL METRICS WITH PRESCRIBED GAUSSIAN CURVATURE ON $S^2$

KUO-SHUNG CHENG AND JOEL A. SMOLLER

ABSTRACT. We consider on $S^2$ the problem of which functions $K$ can be the scalar curvature of a metric conformal to the standard metric on $S^2$. We assume that $K$ is a function of one variable, and we obtain a necessary and sufficient condition for the problem to be solvable. We also obtain several new sufficient conditions on $k$ (which are easy to check), in order that the problem be solvable.

## 1. INTRODUCTION

Consider the two-sphere $S^2 = \{(x_1, x_2, x_3) \in \mathbf{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$, with the standard metric $ds_0^2 = dx_1^2 + dx_2^2 + dx_1^2$. If we make the conformal change of metric, $ds^2 = e^{2u} ds_0^2$, then the Gaussian curvature $K(x)$ of the new metric is

$$K(x) = (1 - \Delta)e^{-2u(x)}, \qquad x \in S^2, \tag{1.1}$$

where $\Delta$ denotes the Laplacian relative to the standard metric $ds_0^2$. L. Nirenberg has raised the following inverse problem: which functions $K$ can be the Gaussian curvature of a metric $ds^2$ which is conformal to the standard metric $ds_0^2$? This problem is equivalent to the question: Which functions $K$ on $S^2$ can be prescribed so that (1.1) has a solution $u$ on $S^2$?

If $d\mu$ denotes the standard surface measure on $S^2$, then rewriting (1.1) as

$$\Delta u + K(x)e^{2u} = 1, \qquad x \in S^2, \tag{1.2}$$

and integrating over the sphere gives

$$\int_{S^2} K e^{2u} \, d\mu = 4\pi. \tag{1.3}$$

Thus, an obvious necessary condition is that $K$ be positive somewhere. Kazdan and Warner [6] found another necessary condition via integration by parts.

---

Received by the editors March 1, 1990 and, in revised form, June 25, 1990 and December 5, 1990.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 53A10.

The first author's research was supported in part by the National Science Council of the Republic of China.

The second author's research was supported in part by the NSF under Grant No. MCS-830123, and in part by the ONR under Grant No. N0014-88-C-0082.


0002-9947/93 \$1.00 + \$.25 per page

Thus, for each eigenfunction $\phi_j$ of $\Delta$ on $S^2$ satisfying $\Delta\phi_j + 2\phi_j = 0$ ($j = 1, 2, 3$), the Kazdan-Warner condition is

$$\int_{S^2} \langle \nabla K, \nabla \phi_j \rangle e^{2u} d\mu = 0, \qquad j = 1, 2, 3. \tag{1.4}$$

Moser [7] proved that if $K$ is an even function on $S^2$, i.e., $K(x) = K(-x)$, then (1.2) has a solution. Recently, several other sufficient conditions were found (see Chang and Yang [1, 2], Chen and Ding [3], and Hong [5]).

In this paper we consider the case of a rotationally symmetric $K$, i.e., $K$ is a function of $x_3$ alone. In [5], Hong considered this case and established some existence theorems using a variational approach. Since $K$ is a function of $x_3$, if we seek solutions $u$ depending only on $x_3$, then (1.2) becomes an ordinary differential equation, and it is the purpose of this paper to study (1.2) by using the techniques of ordinary differential equations.

Thus, assume that $K = K(x_3)$ and that we seek solutions $u$ of (1.2) which depend only on $x_3$. Denoting $x_3$ by $z$, (1.2) becomes

$$\frac{d}{dz}\left[(1-z^2)\frac{du}{dz}\right] + K(z)e^{2u} = 1, \qquad z \in [-1, 1]. \tag{1.5}$$

Let $z = (r^2-1)/(r^2+1)$; then $0 \le r^2 < \infty$. Now define $K_1(r) = K(z)$, i.e.

$$K_1(r) = K\left(\frac{r^2-1}{r^2+1}\right),$$

and consider the following initial-value problem:

$$\begin{aligned} &v''(r) + \frac{1}{r}v'(r) + K_1(r)e^{2v(r)} = 0, \qquad r > 0, \\ &v(0) = \alpha, \qquad v'(0) = 0. \end{aligned} \tag{1.6}$$

We define the set $I$ by $I = \{\alpha \in \mathbf{R}\colon$ (1.6) has a unique solution $v(r, \alpha)$ defined for all $r > 0\}$. Our first result is the following theorem.

**Theorem A.** *Assume that $K$ is a smooth function on $(-1, 1)$ and Hölder continuous on $[-1, 1]$. Then (1.5) has a regular solution $u(z)$ on $[-1, 1]$ if and only if there exists an $\alpha \in I$ satisfying both*

$$\int_0^\infty K_1'(r) r^2 e^{2v(r,\alpha)}\, dr = 0 \tag{1.7}$$

*and*

$$\int_0^\infty K_1(r) r e^{2v(r,\alpha)}\, dr > 0. \tag{1.8}$$

Note that if $K_1'$ is of one sign and not identically zero, then (1.7) cannot hold so that Theorem A implies that (1.5) has no solution; an easy calculation shows that (1.7) is actually the Kazdan-Warner condition (1.4).

Now it is not easy to verify either (1.7) or (1.8). To overcome this difficulty, we define a function $K_2$ by

$$K_2(r) = K\left(\frac{1-r^2}{1+r^2}\right).$$

Our second result is:

**Theorem B.** *Let $K$ be smooth on $(-1, 1)$ and Hölder continuous on $[-1, 1]$. Assume that $K_1$ and $K_2$ are smooth and have only finite-order zeros. Then* (1.5) *has a regular solution $u(z)$ on $[-1, 1]$ if any one of the following statements hold*:

(i) $K_1(r) = K_2(r)$ *for* $0 \leq r \leq 1$ *and* $K_1$ *is positive somewhere* (*cf.* *Moser* [7]).
(ii) $K_1(0) > 0$, $K_2(0) > 0$, *and* $K_1'(0)K_2'(0) > 0$.
(iii) $K_1(0) > 0$, $K_2(0) > 0$, $K_1'(0) = 0$, *and* $K_1''(0)K_2'(0) > 0$.
(iv) $K_1(0) > 0$, $K_2(0) > 0$, $K_1'(0) = K_2'(0) = 0$, *and* $K_1''(0)K_2''(0) > 0$.
(v) $K_1(0) > 0$, $K_1'(0) > 0$, *and* $K_2(0) \leq 0$.
(vi) $K_1(0) > 0$, $K_1'(0) > 0$, $K_1''(0) > 0$, *and* $K_2(0) \leq 0$.
(vii) *Interchange* $K_1$ *and* $K_2$ *in* (iii), (v), *or* (vi).
(viii) $\max[K_1(0), K_2(0)] \leq 0$ *and* $K$ *is positive somewhere* (*cf.* *Hong* [5]).

We remark that if $K$ is smooth, say $C^1$ on $[-1, 1]$, then since

$$K_1'(r) = K'\left(\frac{r^2-1}{r^2+1}\right) \cdot \frac{4r}{(r^2+1)^2},$$

it follows that $K_1'(0) = 0 = K_2'(0)$. However, if $K$ is only of class $C^\alpha$ on $[-1, 1]$ (e.g., $K(z) = a + b\sqrt{1-z^2}$, so $K_1(r) = K((r^2-1)/(r^2+1)) = a + 2r/(r^2+1)$, then $K_1'(0)K_2'(0)$ need not be zero.

The basic idea in this paper is to break the study of (1.5) into two problems:

$$(\mathrm{P}_1) \qquad \frac{d}{dz}\left[(1-z^2)\frac{d\tilde{v}}{dz}\right] + K(z)e^{2\tilde{v}(z)} = 1, \qquad z \in [-1, 0],\ \tilde{v}(-1) = \tilde{\alpha},$$

and

$$(\mathrm{P}_2) \qquad \frac{d}{dz}\left[(1-z^2)\frac{d\tilde{w}}{dz}\right] + K(z)e^{2\tilde{w}(z)} = 1, \qquad z \in [0, 1],\ \tilde{w}(+1) = \tilde{\beta}.$$

Let $\tilde{I}_1$ denote those $\tilde{\alpha}$ for which $(\mathrm{P}_1)$ has a solution $\tilde{v}(z, \tilde{\alpha})$ defined on $[-1, 0]$, and then set

$$A_1 = \{(\tilde{v}(0, \tilde{\alpha}), \tilde{v}'(0, \tilde{\alpha})) : \tilde{\alpha} \in \tilde{I}_1\};$$

$\tilde{I}_2$, $\tilde{w}(z, \beta)$, and $A_2$ are defined similarly. Thus, (1.5) has a solution if and only if the "curves" $A_1$ and $A_2$ meet. To investigate this possibility, we make the change of variables

$$z = \frac{r^2-1}{r^2+1} \quad \text{and} \quad v(r) = \tilde{v}(z) - \log\left(\frac{1+r^2}{2}\right).$$

Then $\tilde{v}$ satisfies $(\mathrm{P}_1)$ iff $v$ satisfies

$$v''(r) + \frac{1}{r}v'(r) + K_1(r)e^{2v(r)} = 0, \qquad 0 \leq r \leq 1,$$
$$v(0) = \alpha \equiv \tilde{\alpha} + \log 2, \quad v'(0) = 0, \qquad \alpha \in I_1 \equiv \tilde{I}_1 + \log 2.$$

Moreover,

$$A_1 = \{(v(1, \alpha), v'(1, \alpha) + 1) \colon \alpha \in I_1\}.$$

Similarly, if $z = (1-t^2)/(1+t^2)$ and $w(t) = \tilde{w}(z) - \log((1+t^2)/2)$, then $\tilde{w}$ satisfies $(P_2)$ iff $w$ satisfies

$$w''(t) + \frac{1}{t}w'(t) + K_2(t)e^{2w(t)} = 0, \qquad t \in [0, 1],$$
$$w(0) = \beta = \tilde{\beta} + \log 2, \quad w'(0) = 0, \quad \beta \in I_2 = \tilde{I}_2 + \log 2.$$

Moreover, $A_2 = \{(w(1, \beta), -(w'(1, \beta)+1)) : \beta \in I_2\}$. To show that $A_1 \cap A_2 \neq \varnothing$, under the hypotheses of Theorem B we show that $A_1$ and $A_2$ actually can be considered as continuous curves in $\mathbf{R}^2$ having no self-intersections, and that their "limiting ends" (for large and small values of $\alpha$ and $\beta$) can be controlled. This is done by obtaining careful estimates on the quantities $v(1, \alpha)$, $v'(1, \alpha)$, and $w(1, \beta)$, $w'(1, \beta)$, and then studying their asymptotic behaviors.

The paper is organized as follows. In §2 we make a detailed study of (1.6); not only is this needed for the proofs of Theorems A and B, but it seems to be of independent interest. The proofs of Theorems A and B are given in §§3 and 4, respectively.

## 2. Preliminaries

We consider the initial-value problem

$$v'' + \frac{1}{r}v' + K(r)e^{2v(r)} = 0, \qquad r > 0,$$
$$v(0) = \alpha, \qquad v'(0) = 0, \tag{2.1}$$

where $K$ is a smooth function on $[0, \infty)$. Let $I$ denote the set of real $\alpha$ such that (2.1) has a (unique) solution $v(r, \alpha)$ on $[0, \infty)$. Note that $I$ may be empty; for example, if $K(r) \le 0$ for all $r \ge 0$, and $K(r) \le -C/r^2$ for large $r$, where $C > 0$, then $I = \varnothing$ (cf. Sattinger [9], Ni [8], or Cheng and Lin [4]).

In what follows we will use the notation "$f \sim g$ at $\infty$," to denote that "there exist positive constants $C_1$, $C_2$ such that $C_1 g \ge f \ge C_2 g$ at $\infty$." We begin with a simple lemma whose standard proof is omitted.

**Lemma 2.1.** *$v$ solves* (2.1) *if and only if $v$ is continuous and satisfies*

$$\begin{aligned} v(r) &= \alpha - \int_0^r \frac{1}{t}\int_0^t sK(s)e^{2v(s)}\,ds\,dt \\ &= \alpha - \int_0^r s\,\log\left(\frac{r}{s}\right)K(s)e^{2v(s)}\,ds, \qquad r > 0. \end{aligned} \tag{2.2}$$

**Lemma 2.2.** *Assume that $K(r) > 0$ for $r \ge r_0$ and $K(r) \sim r^p$ at $\infty$ for some real number $p$. Then, for every $\alpha \in I$,*

$$\frac{p+2}{2} < \int_0^\infty sK(s)e^{2v(s,\alpha)}\,ds < \infty.$$

*Proof*. Suppose first that $\int_0^\infty sK(s)e^{2v(s,\alpha)}\,ds = \infty$. Then there is an $r_1 \ge r_0$ such that for all $r \ge r_1$,

$$\int_0^r sK(s)e^{2v(s,\alpha)}\,ds \ge \max\{2, 2+p\} \equiv a.$$

Using (2.2), we have

$$v'(r, \alpha) = -\frac{1}{r}\int_0^r sK(s)e^{2v(s,\alpha)}\,ds \le -\frac{a}{r}, \qquad r \ge r_1.$$

Thus, for $r \geq r_1$,

$$v(r, \alpha) \leq v(r_1, \alpha) + a \log r_1 - a \log r.$$

Hence

$$\begin{aligned}\int_0^\infty sK(s)e^{2v(s, \alpha)}\,ds &= \int_0^{r_1} sK(s)e^{2v(s, \alpha)}\,ds + \int_{r_1}^\infty sK(s)e^{2v(s, \alpha)}\,ds \\ &\leq \int_0^{r_1} sK(s)e^{2v(s, \alpha)}\,ds + e^{2v(r_1, \alpha)} r_1^{2a} \int_{r_1}^\infty C s^{1+p-2a}\,ds < \infty,\end{aligned}$$

since $1+p-2a < -1$. This contradiction shows that $\int_0^\infty sK(s)e^{2v(s, \alpha)}\,ds < \infty$.

Next, suppose that $\int_0^\infty sK(s)e^{2v(s, \alpha)}\,ds \leq (p+2)/2$. Then as $K(r) > 0$ for $r \geq r_0$, there is an $r_2 \geq r_0$ for which

$$\int_0^r sK(s)e^{2v(s, \alpha)}\,ds \leq \frac{p+2}{2}, \qquad r \geq r_2.$$

Using (2.2) again gives, for $r \geq r_2$,

$$v'(r, \alpha) = -\frac{1}{r} \int_0^r sK(s)e^{2v(s, \alpha)}\,ds \geq -\frac{p+2}{2r}.$$

Thus, for $r \geq r_2$,

$$v(r, \alpha) \geq v(r_2, \alpha) + \frac{p+2}{2} \log r_2 - \frac{p+2}{2} \log r$$

and

$$\begin{aligned}\frac{p+2}{2} &\geq \int_0^\infty sK(s)e^{2v(s, \alpha)}\,ds \\ &= \int_0^{r_2} sK(s)e^{2v(s, \alpha)}\,ds + \int_{r_2}^\infty sK(s)e^{2v(s, \alpha)}\,ds \\ &\geq \int_0^{r_2} sK(s)e^{2v(s, \alpha)}\,ds + C \int_{r_2}^\infty s^{1+p-(p+2)}\,ds = \infty.\end{aligned}$$

This contradiction completes the proof. □

**Lemma 2.3.** *Suppose that $K(r) \leq 0$ for $r \geq r_0$, and $K(r) \sim -r^p$ at $\infty$ for some real number $p$. Then*

$$\int_0^\infty sK(s)e^{2v(s, \alpha)}\,ds \geq \frac{p+2}{2}$$

*for every $\alpha \in I$. In particular, if $K(r) \leq 0$ for all $r > 0$ (and $K(r) \sim -r^p$ at $\infty$), then*

$$0 > \int_0^\infty sK(s)e^{2v(s, \alpha)}\,ds \geq \frac{p+2}{2}.$$

*Remark.* Note that Lemma 2.3 is an a priori estimate. Thus, if $p+2 \geq 0$ and $K(r) \leq 0$ for all $r \geq 0$ and $K \not\equiv 0$, then the lemma implies that $I = \varnothing$, i.e., every solution $v(r, \alpha)$ blows up at some finite $r$ (cf. Sattinger [9]).

The proof is similar to that of Lemma 2.2 and is omitted.

Next, for $\alpha \in I$, we define $\Phi_\alpha(r)$ by

$$\Phi_\alpha(r) = (1 + rv'(r; \alpha))^2 + K(r) r^2 e^{2v(r, \alpha)}. \tag{2.3}$$

Concerning this function, we have the following lemma.

**Lemma 2.4.** *Suppose that* $K(r) \sim r^p$ *or* $K(r) \sim -r^p$ *at* $\infty$ *for some real number* $p$. *Then for each* $\alpha \in I$, *the following hold*:

(i) $\Phi'_\alpha(r) = K'(r)r^2 e^{2v(r,\alpha)}$, and

(ii)

$$\lim_{r\to\infty} \Phi_\alpha(r) = \left(1 - \int_0^\infty sK(s)e^{2v(s,\alpha)}\,ds\right)^2 = 1 + \int_0^\infty K'(s)s^2 e^{2v(s,\alpha)}\,ds.$$

*Proof*. By direct calculation,

$$\Phi'_\alpha(r) = 2(1+rv')(rv')' + 2(1+rv')Kre^{2v} + K'r^2e^{2v}$$

and since (2.1) implies $(rv')' + rKe^{2v} = 0$, (i) follows. To prove (ii), we first note that from the last two lemmas, the quantity $\int_0^\infty sK(s)e^{2v(s,\alpha)}\,ds$ is finite. Suppose now that $K(r) \sim r^p$ at $\infty$. Then $K(r)$ is positive for large $r$. We claim that $v(r,\alpha)$ is eventually monotone. If not, we could find a sequence $r_i \to \infty$ such that $v(r,\alpha)$ assumes a local minimum at each $r_i$, $i = 1, 2, \ldots$, i.e., $v'(r_i,\alpha) = 0$, $v''(r_i,\alpha) \geq 0$. But then (2.1) shows that

$$v''(r_i,\alpha) = -K(r_i)e^{2v(r_i,\alpha)} < 0$$

for large $i$. This proves our claim. (Similarly, if $K(r) \sim -r^p$ at $\infty$, $v(r,\alpha)$ is also eventually monotone.)

We next claim that

$$\limsup_{r\to\infty} K(r)r^2e^{2v(r,\alpha)} \leq 0.$$

For if not, we can find $\varepsilon > 0$ and a sequence $r_i \to \infty$ such that $K(r_i)r_i^2e^{2v(r_i,\alpha)} > \varepsilon$ for $i = 1, 2, \ldots$. Since $v(r,\alpha)$ is eventually monotone, and $K(r) \sim r^p$ at $\infty$, there exists $\varepsilon' > 0$ such that, for large $r$, $K(r)r^2e^{2v(r,\alpha)} \geq \varepsilon'$. This however contradicts the fact that $\int_0^\infty sK(s)e^{2v(s,\alpha)}\,ds$ is finite. Thus our second claim holds.

In a similar way, we can show that

$$\liminf_{r\to\infty} K(r)r^2e^{2v(r,\alpha)} \geq 0;$$

hence we may conclude that

$$\lim_{r\to\infty} K(r)r^2e^{2v(r,\alpha)} = 0.$$

Thus,

$$\lim_{r\to\infty}\Phi_\alpha(r) = \lim_{r\to\infty}(1 + rv'(r,\alpha))^2 = \lim_{r\to\infty}\left(1 - \int_0^r sK(s)e^{2v(s,\alpha)}\,ds\right)^2$$
$$= \left(1 - \int_0^\infty sK(s)e^{2v(s,\alpha)}\,ds\right)^2.$$

On the other hand if we integrate (i) from 0 to $\infty$, we get

$$\lim_{r\to\infty}\Phi_\alpha(r) = \Phi_\alpha(0) + \int_0^\infty K'(s)s^2e^{2v(s,\alpha)}\,ds = 1 + \int_0^\infty K'(s)s^2e^{2v(s,\alpha)}\,ds,$$

and this completes the proof of the lemma. □

Now consider the problem

$$v'' + \frac{1}{r}v' + K(r)e^{2v} = 0, \qquad r \in [0, a], \tag{2.4}$$
$$v(0) = \alpha, \qquad v'(0) = 0,$$

for fixed $a > 0$. We define the set $J_a$ by $J_a = \{\alpha \in \mathbf{R}\colon$ (2.4) has a unique solution $v(r, \alpha)$ defined on $[0, a]\}$. Concerning this set we have the following theorem.

**Theorem 2.5.** *Assume that $K$ is continuous on $[0, a]$. Then there is an $\alpha_0$ such that $(-\infty, \alpha_0) \subset J_a$.*

This theorem is not difficult to prove using an iteration method; we omit the details.

The next result is quite important in our development.

**Theorem 2.6.** *Assume that $K(r) \geq 0$ for all $r \in [0, a]$, and that $K(0) > 0$, $K'(0) \neq 0$. Then $J_a = \mathbf{R}$ and there exists an $\alpha_0 > 0$ such that, for $\alpha \leq -\alpha_0$,*

$$\begin{aligned} v(a, \alpha) &= \alpha + O(e^{2\alpha}), \\ v'(a, \alpha) &= -\frac{1}{a}e^{2\alpha}\int_0^a sK(s)\,ds + O(e^{4\alpha}), \end{aligned} \tag{2.5}$$

*while for $\alpha \geq \alpha_0$ there is a constant $C > 0$ such that*

$$\begin{aligned} v(a, \alpha) &= -\alpha + O(1), \\ v'(a, \alpha) &= -\frac{2}{a} - \frac{K'(0)}{a}Ce^{-\alpha} + O(e^{-2\alpha}\alpha^2). \end{aligned} \tag{2.6}$$

The proof of this important theorem requires several lemmas.

**Lemma 2.7.** *Assume that $K$ is continuous and $K(r) \geq 0$ on $[0, a]$; then $J_a = \mathbf{R}$.*

*Proof.* Let $\alpha \in \mathbf{R}$, and define a sequence $\{v_n\}$ by

$$v_0(r) \equiv \alpha,$$

$$v_{n+1}(r) = \alpha - \int_0^r s\,\log\left(\frac{s}{r}\right) K(s)e^{2v_n(s)}\,ds, \qquad n = 0, 1, 2, \ldots, \; r \in [0, a].$$

Since $K(r) \geq 0$ for all $r \in [0, a]$, it is easy to see that $\{v_n\}$ is well defined and satisfies

$$v_0(r) \geq v_2(r) \geq \cdots \geq v_{2n}(r) \geq v_{2n+1}(r) \geq \cdots \geq v_3(r) \geq v_1(r).$$

Now it is easy to show that on $0 \leq r \leq a$

$$|v_1(r) - v_0(r)| \leq 2M_1M_2e^{2\alpha}r$$

and

$$|v_{n+1}(r) - v_n(r)| \leq (2M_1M_2e^{2\alpha})^{n+1}\frac{r^{n+1}}{(n+1)!}, \qquad n \geq 1,$$

where $M_1$ and $M_2$ are defined by $M_1 = \sup\{s\,\log(r/s)\colon 0 \leq s \leq r,\ 0 \leq r \leq a\}$ and $M_2 = \sup\{|K(r)|\colon 0 \leq r \leq a\}$. Hence $\sum_{n=0}^{\infty}[v_{n+1}(r) - v_n(r)]$ converges uniformly on $[0, a]$. But as

$$v_n(r) = v_0(r) + (v_1(r) - v_0(r)) + \cdots + (v_n(r) - v_{n-1}(r)),$$

we see that $\{v_n\}$ converges uniformly on $[0, a]$ to a function $v$ which solves (2.4); thus, $\alpha \in J_a$. □

As a consequence of this result, we have

**Corollary 2.8.** *If $K$ is continuous, and $K(r) \geq 0$ on $r \geq 0$, then $I = \mathbf{R}$, i.e., the problem* (1.6) *has a unique solution defined for all $r > 0$.*

**Lemma 2.9.** *Let $u$ and $v$ be nonnegative continuous functions on $[0, \infty]$ satisfying*

$$u(t) \leq a + b \int_0^t s \log\left(\frac{t}{s}\right) \frac{u(s)}{(1+s^2/4)^2} \, ds,$$
$$v(t) = a + b \int_0^t s \log\left(\frac{t}{s}\right) \frac{v(s)}{(1+s^2/4)^2} \, ds,$$

*where $a$ is a positive constant and $b = 21/10$. Then the following hold:*

(i) *$u(t) \leq v(t)$ for all $t \geq 0$, and*
(ii) *there exist positive constants $C_1$ and $C_2$ such that*

$$v(t) \leq a(C_1 + C_2 \log(1+t)) \quad \text{for all } t \geq 0.$$

*Proof.* Define

$$\phi(t) = a + b \int_0^t s \log\left(\frac{t}{s}\right) \frac{u(s)}{(1+s^2/4)^2} \, ds,$$

and set $w(t) = v(t) - \phi(t)$. Then $w(0) = 0$, $w'(0) = 0$, and

$$w''(t) + \frac{1}{t} w'(t) = \left[v''(t) + \frac{1}{t} v'(t)\right] - \left[\phi''(t) + \frac{1}{t}\phi'(t)\right]$$
$$= \frac{bv(t)}{(1+t^2/4)^2} - \frac{bu(s)}{(1+t^2/4)^2} \geq \frac{bw(t)}{(1+t^2/4)^2}.$$

Thus, $w$ cannot be negative on an interval $0 < t < \varepsilon$. Hence $w(t) \geq 0$ on such an interval, so $w(t) \geq 0$ if $t > 0$. It follows that $v(t) \geq \phi(t) \geq u(t)$ for $t \geq 0$, and this proves (i).

To prove (ii) we let $X$ denote the locally convex space of continuous functions on $(0, \infty)$ with the compact-open topology, and consider the set $Y \subset X$ defined by

$$Y = \{w \in X : a \leq w(t) \leq g(t), \ t \geq 0\},$$

where $g(t) = a(1 + Bt^{2-\beta})$; here $B$ and $\beta$ are positive constants to be determined later. It is easy to see that $Y$ is a closed convex subset of $X$. Now let $T$ be the mapping on $Y$ defined by

$$(Tw)(t) = a + b \int_0^t s \log\left(\frac{t}{s}\right) \frac{w(s)}{(1+s^2/4)^2} \, ds.$$

We shall show that $B$ and $\beta$ can be chosen in order to make $TY \subset Y$ and $TY$ to be relatively compact in $Y$.

To do this, we choose $B$ so large and $\beta < 2$ so small as to make the following hold:

$$(2.7) \qquad (2-\beta)^2 \left(\tfrac{9}{40} + \tfrac{1}{2}\right) \geq \tfrac{21}{10} = b.$$

$$(2.8) \qquad B(2-\beta)^2 \tfrac{1}{10} \geq bt^\beta \quad \text{for } 0 \leq t \leq 2.$$

$$(2.9) \qquad B(2-\beta)^2 \tfrac{1}{40} \geq b.$$

Now suppose $w \in Y$; then it is easy to see from (2.6) that $Tw$ is continuous and $a \le (Tw)(t)$ for all $t \ge 0$. Let $h(t) = g(t) - (Tg)(t)$; then

$$h(0) = 0, \qquad h'(0) = 0, \tag{2.10}$$

and

(2.11)
$$\begin{aligned} h''(t) + \frac{1}{t}h'(t) &= a\left[\frac{(2-\beta)^2 B}{t^\beta} - \frac{b(1+Bt^{2-\beta})}{(1+t^2/4)^2}\right] \\ &= \frac{a}{t^\beta(1+t^2/4)^2}\left(B(2-\beta)^2\left(1+\frac{t^2}{2}+\frac{t^4}{16}\right) - b(t^\beta + Bt^2)\right). \end{aligned}$$

If $0 \le t \le 2$, then

$$\begin{aligned} &B(2-\beta)^2\left(1+\frac{t^2}{2}+\frac{t^4}{16}\right) - b(t^\beta + Bt^2) \\ &\quad \ge \left[B(2-\beta)^2\frac{1}{10} - bt^\beta\right] + B(2-\beta)^2\left(\frac{9}{10}+\frac{t^2}{2}\right) - bBt^2 \\ &\quad \ge \left[B(2-\beta)^2\frac{1}{10} - bt^\beta\right] + Bt^2\left[(2-\beta)^2\left(\frac{9}{40}+\frac{1}{2}\right) - b\right] \ge 0 \end{aligned}$$

in view of (2.7) and (2.8). On the other hand, if $t \ge 2$,

$$\begin{aligned} B(2-\beta)^2\left(1+\frac{t^2}{2}+\frac{t^4}{16}\right) - b(t^\beta + Bt^2) &\ge B(2-\beta)^2\left(\frac{t^2}{2}+\frac{t^2}{4}\right) - b(t^2+Bt^2) \\ = \left[B(2-\beta)^2\frac{1}{40} - b\right]t^2 + Bt^2\left[(2-\beta)^2\left(\frac{1}{2}+\frac{9}{40}\right) - b\right] &\ge 0 \end{aligned}$$

in view of (2.7) and (2.9). Thus, for all $t \ge 0$, (2.11) implies $h''(t) + \frac{1}{t}h'(t) \ge 0$, so that (2.10) shows that $h(t) \ge 0$. This proves that $g(t) \ge (Tg)(t)$ for all $t \ge 0$. Thus, if $w \in Y$, then $w \le g$, $Tw \le Tg \le g$, so that $TY \subset Y$.

We now show that $TY$ is relatively compact in $Y$. Thus let $\{w_n\} \subset Y$, $w_n \to w \in Y$ in the space $X$. Then $\{w_n\}$ converges to $w$ uniformly on any compact interval on $[0, \infty)$. Since

$$|(Tw_m)(t) - (Tw)(t)| \le \int_0^t s \log\left(\frac{t}{s}\right) \frac{1}{(1+s^2/4)^2}|w_m(s) - w(s)|\,ds, \tag{2.12}$$

we see that $Tw_m$ converges to $Tw$ on any compact interval of $[0, \infty)$; thus $T$ is a continuous mapping. Furthermore

$$|(Tw_m)'(t)| \le \int_0^t \frac{s}{t}\frac{|w_m(s)|}{(1+s^2/4)^2}\,ds \le \int_0^t \frac{s}{t}\frac{g(s)}{(1+s^2/4)^2}\,ds \le \int_0^\infty \frac{g(s)}{(1+s^2/4)^2}\,ds.$$

Thus, $\{(Tw_m)'\}$ is uniformly bounded, $\{Tw_m\}$ is equicontinuous, and $TY$ is relatively compact in $Y$. We may now apply the Schauder fixed-point theorem to conclude that $T$ has a fixed-point $w_0$ in $Y$, i.e.,

$$w_0(t) = a + b\int_0^t s \log\left(\frac{s}{t}\right)\frac{w_0(s)}{(1+s^2/4)^2}\,ds$$

or, equivalently,

$$\begin{aligned} w_0'' + \frac{1}{t}w_0' &= \frac{bw_0}{(1+t^2/4)^2}, \\ w_0(0) = a, \qquad w_0'(0) &= 0. \end{aligned} \tag{2.13}$$

From the uniqueness of solutions of (2.13), we conclude that $w_0 \equiv v$. Hence $v \in Y$, $v \le g$, and

$$\begin{aligned} v(t) &= (Tv)(t) \le (Tg)(t) \\ &= a + b\int_0^t s\ \log\left(\frac{t}{s}\right)\frac{a(1+Bs^{2-\beta})}{(1+s^2/4)^2}\,ds \\ &= a\left[1 + b\int_0^t s\ \log\left(\frac{t}{s}\right)\frac{(1+Bs^{2-\beta})}{(1+s^2/4)^2}\,ds\right]. \end{aligned}$$

Now for $t > 1$, we have

$$\int_0^t s\ \log\left(\frac{t}{s}\right)\frac{(1+Bs^{2-\beta})}{(1+s^2/4)^2}\,ds = \int_0^1 + \int_1^t .$$

But as

$$\begin{aligned} &\int_0^1 s\ \log\left(\frac{t}{s}\right)\frac{(1+Bs^{2-\beta})}{(1+s^2/4)^2}\,ds \\ &\quad = \log t\int_0^1 s\frac{(1+Bs^{2-\beta})}{(1+s^2/4)^2}\,ds - \int_0^1 s\ \log(s)\frac{(1+Bs^{2-\beta})}{(1+s^2/4)^2}\,ds \\ &\quad = K_1\log t + K_2 \end{aligned}$$

and

$$\begin{aligned} \int_1^t s\ \log\left(\frac{t}{s}\right)\frac{(1+Bs^{2-\beta})}{(1+s^2/4)^2}\,ds &\le \log t\int_1^t s\frac{(1+Bs^{2-\beta})}{(1+s^2/4)^2}\,ds \\ &\le \log t\int_1^\infty s\frac{(1+Bs^{2-\beta})}{(1+s^2/4)^2}\,ds = K_3\log t, \end{aligned}$$

we have

$$v(t) \le a(C_1 + C_2\log t) \quad \text{for all } t \ge 1.$$

Thus, by choosing $C_1$ and $C_2$ appropriately, we have

$$v(t) \le a(C_1 + C_2\log(1+t)) \quad \text{for all } t \ge 0.$$

This proves (ii) and completes the proof of Lemma 2.9. □

We can now give the proof of Theorem 2.6.

*Proof of Theorem* 2.6. From Lemma 2.7, we know that $J_a = \mathbf{R}$. For any $\alpha$, from the proof of Lemma 2.7 we have $v(a, \alpha) = \lim_{n\to\infty} v_n(a, \alpha)$, where

$$v_n(a, \alpha) = v_0(a, \alpha) + (v_1 - v_0)(a, \alpha) + \cdots + (v_n - v_{n-1})(a, \alpha).$$

But, for $\alpha$ near $-\infty$, $2M_1M_2e^{2\alpha} < 1$ and as

$$|(v_K - v_{K-1})(a, \alpha)| \le \frac{2M_1M_2a}{K+1}e^{2\alpha}[2M_1M_2e^{2\alpha}a]^K\frac{1}{K!},$$

we have

$$|(v_K - v_{K-1})(a, \alpha)| \le M_1M_2ae^{2\alpha}\frac{a^K}{K!},$$

from which it follows that $|v_n(a;\alpha)-\alpha| \le Ce^{2\alpha}$, so $v(a;\alpha)=\alpha+O(e^{2\alpha})$. Similarly, $v'(a,\alpha)=\lim_{n\to\infty} v_n'(a,\alpha)$, where

$$\begin{aligned} v_n'(a,\alpha) &= v_1'(a,\alpha)+(v_2'-v_1')(a,\alpha)+\cdots+(v_n'-v_{n-1}')(a,\alpha) \\ &= -\frac{1}{a}\int_0^1 sK(s)e^{2\alpha}\,ds+\sum_{K=1}^{n}(v_K'-v_{K+1}')(a,\alpha) \end{aligned}$$

and

$$(v_K'-v_{K-1}')(a,\alpha) = -\frac{1}{a}\int_0^a sK(s)[e^{2v_{K-1}}-e^{2v_{K-2}}]\,ds.$$

But, from the proof of Lemma 2.7, we have that $|v_n'(a,\alpha)-v_1'(a,\alpha)| \le Ce^{4\alpha}$, and thus we see that (2.5) is valid.

Now in order to prove (2.6) we first make the change of variables $t = e^{\alpha}\sqrt{K(0)}r$ and $y(t)=v(r)-\alpha$. Then $y$ satisfies

$$\begin{gathered} y''(t)+\frac{1}{t}y'(t)+\frac{1}{K(0)}K\left(\frac{e^{-\alpha}t}{\sqrt{K(0)}}\right)e^{2y(t)}=0, \\ y(0)=0, \qquad y'(0)=0. \end{gathered} \tag{2.14}$$

Choose $\delta>0$ so small that

$$|K(r)-K(0)| \le Cr, \qquad K'(r)=K'(0)+O(1)r$$

for $0\le r\le\delta$. Let $z(t)=y(t)+\log(1+t^2/4)$; then $z$ satisfies

$$\begin{gathered} z''+\frac{1}{t}z'(t)+\frac{1}{K(0)}K\left(\frac{e^{-\alpha}t}{\sqrt{K(0)}}\right)\frac{1}{(1+t^2/4)^2}e^{2z(t)} \\ -\frac{1}{(1+t^2/4)^2}=0, \qquad z(0)=0, \qquad z'(0)=0. \end{gathered} \tag{2.15}$$

Using (2.15), we have (cf. (2.2)),

$$\begin{aligned} z(t) &= -\int_0^t s\,\log\left(\frac{r}{s}\right)\frac{1}{(1+s^2/4)^2}\left[\frac{1}{K(0)}K\left(\frac{e^{-\alpha}s}{\sqrt{K(0)}}\right)e^{2z(s)}-1\right]ds \\ &= -\int_0^t s\,\log\left(\frac{r}{s}\right)\frac{1}{(1+s^2/4)^2}\left[e^{2z(s)}-1\right]ds \\ &\quad -\int_0^t s\,\log\left(\frac{r}{s}\right)\frac{1}{(1+s^2/4)^2}\left[\frac{1}{K(0)}K\left(\frac{e^{-\alpha}t}{\sqrt{K(0)}}\right)-1\right]e^{2z(s)}\,ds. \end{aligned}$$

Let $u(t)=|z(t)|$; then

$$\begin{aligned} u(t) \le &\int_0^t s\,\log\left(\frac{t}{s}\right)\frac{1}{(1+s^2/4)^2}\left|\frac{1}{K(0)}K\left(\frac{e^{-\alpha}s}{\sqrt{K(0)}}\right)-1\right|e^{2\mu(s)}\,ds \\ &+\int_0^t s\,\log\left(\frac{t}{s}\right)\frac{1}{(1+s^2/4)^2}\left|\frac{e^{2z(s)}-1}{z(s)}\right|u(s)\,ds. \end{aligned} \tag{2.16}$$

For $0\le s\le e^{\alpha}\sqrt{K(0)}\delta$, we have

$$\left|\frac{1}{K(0)}K\left(\frac{e^{-\alpha}s}{\sqrt{K(0)}}\right)-1\right| \le C\frac{e^{-\alpha}s}{\sqrt{K(0)}}=Ce^{-\alpha}s. \tag{2.17}$$

(We shall use the same $C$ to denote different positive constants.)

Suppose that for $0 \le t \le e^{\alpha}\sqrt{K(0)}\delta$, we have

$$u(t) \le \tfrac{1}{2}\log(\tfrac{21}{20}). \tag{2.18}$$

Then, using (2.16)–(2.18), we get

$$\begin{aligned} u(t) &\le Ce^{-\alpha}\int_0^t s\,\log\left(\frac{t}{s}\right)\frac{s}{(1+s^2/4)^2}\,ds \\ &\quad + \int_0^t s\,\log\left(\frac{t}{s}\right)\frac{1}{(1+s^2/4)^2}2e^{2u(s)}u(s)\,ds \\ &\le Ce^{-\alpha}\alpha + \frac{21}{10}\int_0^t s\,\log\left(\frac{t}{s}\right)\frac{1}{(1+s^2/4)^2}u(s)\,ds \end{aligned} \tag{2.19}$$

for $0 \le t \le e^{\alpha}\sqrt{K(0)}\delta$ and for large $\alpha$. Using Lemma 2.9 and (2.19), we conclude that, for $0 \le t \le e^{\alpha}\sqrt{K(0)}\delta$,

$$u(t) \le Ce^{-\alpha}\alpha(C_1 + C_2\log(1+t)), \qquad C \text{ independent of } \alpha.$$

Thus, for $\alpha$ large, we have

$$u(t) \le Ce^{-\alpha}\alpha^2, \qquad 0 \le t \le e^{\alpha}\sqrt{K(0)}\delta.$$

(Note that this confirms (2.18); *Proof*. since $u(0) = 0 = u'(0)$, $u(t) < \frac{1}{2}\log(\frac{21}{20})$ for small $t$. Let $t_0 \le e^{\alpha}\sqrt{K(0)}\delta$ be the first $t_0$ for which $u(t_0) = \frac{1}{2}\log(\frac{21}{20})$. Now on $[0, t_0]$, (2.16)–(2.19) hold so $u(t) \le Ce^{-\alpha}\alpha^2$, so if $\alpha$ is chosen so large that $Ce^{-\alpha}\alpha^2 < \frac{1}{2}\log(\frac{21}{20})$, we see $u(t_0) < \frac{1}{2}\log(\frac{21}{20})$, a contradiction.) Therefore, we see that there exists a constant $\alpha_0 > 0$ such that, for $\alpha \ge \alpha_0$ and $0 \le r \le \delta$,

$$v(r, \alpha) = \alpha - \log\left[1 + \left(\frac{e^{\alpha}\sqrt{K(0)}}{2}r\right)^2\right] + O(e^{-\alpha}\alpha^2). \tag{2.20}$$

From Lemma 2.4(i),

$$\begin{aligned} (1+\delta v'(\delta, \alpha))^2 + K(\delta)\delta^2 e^{2v(\delta,\alpha)} &= 1 + \int_0^{\delta} K'(r)r^2 e^{2v(r,\alpha)}\,dr \\ &= 1 + K'(0)\int_0^{\delta} r^2 e^{2v(r,\alpha)}\,dr + O(1)\int_0^{\delta} r^3 e^{2v(r,\alpha)}\,dr. \end{aligned} \tag{2.21}$$

Using (2.20) in (2.21) gives, for large $\alpha$ (after a little algebra),

$$\begin{aligned} (1+\delta v'(\delta, \alpha))^2 &= 1 + K'(0)K(0)^{-3/2}e^{-\alpha}C\int_0^{e^{\alpha}\sqrt{K(0)}\delta}\frac{s^2\,ds}{(1+s^2/4)^2} \\ &\quad + O(1)e^{-2\alpha}\alpha^2. \end{aligned} \tag{2.22}$$

Now as

$$v'(\delta, \alpha) = -\frac{1}{\delta}\int_0^{\delta} sK(s)e^{2v(s,\alpha)}\,ds$$

and $\int_0^{\delta} sK(s)e^{2v(s,\alpha)}\,ds > 1$ for large $\alpha$ (cf. 2.20), we have $1 + \delta v'(\delta, \alpha) < 0$. Thus, from (2.22), we have, for large $\alpha$,

$$\delta v'(\delta, \alpha) = -2 - CK'(0)e^{-\alpha} + O(1)e^{-2\alpha}\alpha^2 \tag{2.23}$$

for some $C > 0$. For large $\alpha$, say $\alpha \geq \alpha_0$, (2.20) gives

$$v(\delta, \alpha) = -\alpha + O(1). \tag{2.24}$$

Now for $r \geq \delta$, we have

$$v(r, \alpha) = v(\delta, \alpha) + \delta v'(\delta, \alpha) \log\left(\frac{r}{\delta}\right) - \int_\delta^t s \log\left(\frac{r}{s}\right) K(s) e^{2v(s, \alpha)}\, ds.$$

and

$$v'(r, \alpha) = \frac{\delta v'(\delta, \alpha)}{r} - \frac{1}{r}\int_\delta^r sK(s)e^{2v(s, \alpha)}\, ds.$$

Hence, for $\alpha \geq \alpha_0$, we have

$$v(a, \alpha) = -\alpha + O(1) \tag{2.25}$$

and

$$v'(a, \alpha) = -\frac{2}{a} - \frac{CK'(0)}{a}e^{-\alpha} + O(e^{-2\alpha}\alpha^2)$$

for some $C > 0$. The proof of Theorem 2.6 is complete.

Next, we shall obtain some results analogous to those in Theorem 2.6 for various hypotheses on the function $K$.

**Theorem 2.10.** *Assume that $K(r) \geq 0$ for $0 \leq r \leq a$, and $K(0) > 0$, $K'(0) = 0$, $K''(0) \neq 0$. Then $J_a = \mathbf{R}$ and there exists an $\alpha_0 > 0$ such that, for $\alpha \leq -\alpha_0$,*

$$v(a, \alpha) = \alpha - O(e^{2\alpha}), \tag{2.26}$$

$$v'(a, \alpha) = -\frac{1}{a}e^{2\alpha}\int_0^a sK(s)\, ds + O(e^{4\alpha}), \tag{2.27}$$

*while for $\alpha \geq \alpha_0$ there is a constant $C > 0$ such that*

$$v(a, \alpha) = -\alpha + O(1), \tag{2.28}$$

$$v'(a, \alpha) = -\frac{2}{a} - \frac{CK''(0)}{a}\alpha e^{-2\alpha} + O(e^{-2\alpha}). \tag{2.29}$$

*Proof.* We need only prove the asymptotic behavior of $v(a, \alpha)$ and $v'(a, \alpha)$ near $\alpha = \infty$ (cf. Theorem 2.6). For this, we choose $\delta > 0$ so small that for $0 \leq r \leq \delta$ the following hold:

$$|K(r) - K(0)| \leq Cr^2, \tag{2.30}$$

$$K'(r) = K''(0)r + O(r^2). \tag{2.31}$$

Now, as in the proof of Theorem 2.6, we set $t = e^\alpha\sqrt{K(0)}r$, $y(t) + \alpha = v(r)$, and $y$ satisfies (2.14). If $z(t) = y(t) + \log(1 + t^2/4)$, then $z$ satisfies (2.15). With $u(t) = |z(t)|$, arguments similar to those in the proof of Theorem 2.6 yield (2.16). Now as

$$\left|\frac{1}{K(0)}K\left(\frac{e^{-\alpha}s}{\sqrt{K(0)}}\right) - 1\right| \leq C\frac{e^{-2\alpha}s^2}{K(0)},$$

we have, instead of (2.19) (assuming (2.18)),

$$u(t) \leq Ce^{-2\alpha}\alpha^2 + \frac{21}{10}\int_0^t s\ \log\left(\frac{t}{s}\right)\frac{1}{(1 + s^2/4)^2}u(s)\, ds$$

for $0 \le t \le e^{\alpha}\sqrt{K(0)}\delta$, for large $\alpha$. It follows (as before) that, for large $\alpha$ and $0 \le t \le e^{\alpha}\sqrt{K(0)}\delta$,

$$u(t) \le Ce^{-2\alpha}\alpha^3 .$$

Thus, we have (as before) that, for some $\alpha_0 > 0$,

$$v(r, \alpha) = \alpha - \log\left[1 + \left(\frac{e^{\alpha}\sqrt{K(0)}}{2} r\right)^2\right] + O(\alpha^3 e^{-2\alpha}) \tag{2.32}$$

for $\alpha \ge \alpha_0$ and $0 \le r \le \delta$. Similarly to (2.21), we have

$$\begin{aligned}(1 + \delta v'(\delta, \alpha))^2 + K(\delta)\delta^2 e^{2v(\delta, \alpha)} &= 1 + \int_0^{\delta} K'(r) r^2 e^{2v(r, \alpha)}\, dr \\ &= 1 + K''(0)\int_0^{\delta} r^3 e^{2v(r, \alpha)}\, dr + O(1)\int_0^{\delta} r^4 e^{2v(r, \alpha)}\, dr \\ &= 1 + CK''(0)\alpha e^{-2\alpha} + O(e^{-2\alpha})\end{aligned}$$

for some $C > 0$. Hence, for some $C > 0$,

$$\delta v'(\delta, \alpha) = -2 - CK''(0)\alpha e^{-2\alpha} + O(e^{-2\alpha}) . \tag{2.33}$$

The rest of the proof now follows as in the proof of Theorem 2.6. □

**Theorem 2.11.** *Assume that $K(r) \ge 0$ for $0 \le r \le a$ and that $K(r) = Ar^m + O(r^{m+1})$ for small $r$, where $m > 0$ and $A > 0$. Then $J_a = \mathbf{R}$, and there exists an $\alpha_0 > 0$ such that, for $\alpha \ge \alpha_0$,*

$$v(a, \alpha) = -\alpha + O(1),$$
$$v'(a, \alpha) = -\frac{m+2}{a} + \frac{1}{a} O(\alpha e^{-2\alpha/(m+2)}) .$$

*Proof.* Choose $\delta > 0$ so small that, for $0 \le r \le \delta$,

$$|K(r) - Ar^m| \le Cr^{m+1}$$

for some constant $C > 0$. Let $t = \beta r^{(m+2)/2}$, $\beta = 2\sqrt{A}e^{\alpha}/(m+2)$, and $y(t) = v(r, \alpha) - \alpha$. Then $y$ satisfies

$$\begin{aligned} &y'' + \frac{1}{t} y' + \frac{K(r)}{Ar^m} e^{2y} = 0, \\ &y(0) = 0, \qquad y'(0) = 0 . \end{aligned} \tag{2.34}$$

Let $z(t) = y(t) + \log(1 + t^2/4)$; then $z$ satisfies

$$\begin{aligned} &z'' + \frac{1}{t} z' + \frac{K(r)}{Ar^m} \frac{e^{2z}}{(1 + t^2/4)^2} - \frac{1}{(1 + t^2/4)^2} = 0, \\ &z(0) = 0, \qquad z'(0) = 0 . \end{aligned}$$

Thus,

$$\begin{aligned} z(t) = &-\int_0^t s \log\left(\frac{t}{s}\right) \frac{1}{(1 + s^2/4)^2} \left[e^{2z(s)} - 1\right] ds \\ &-\int_0^t s \log\left(\frac{t}{s}\right) \frac{1}{(1 + s^2/4)^2} \left[\frac{K(r)}{Ar^m} - 1\right] e^{2z(s)}\, ds . \end{aligned}$$

Using arguments similar to those in the proof of Theorem 2.6, we have, for $0 \le r \le \delta$ and $\alpha \ge \alpha_0 > 0$,

$$v(r, \alpha) = \alpha - \log\left[1 + \left(\frac{\sqrt{A}e^{\alpha}r}{m+2}\right)^2\right] + O\left(e^{-2\alpha/(m+2)}\alpha^2\right). \tag{2.35}$$

Then, by similar arguments as before, we obtain

$$v(a, \alpha) = -\alpha + O(1)$$
$$v'(a, \alpha) = -\frac{m+2}{a} + \frac{1}{a}O\left(e^{-2\alpha/(m+2)}\alpha\right)$$

for $\alpha \ge \alpha_0$; the proof is complete. □

**Theorem 2.12.** *Assume that* $K(r) \le 0$ *for* $0 \le r \le a$ *and that* $K(r) < 0$ *for* $a - \delta \le r < a$. *Then there exists an* $\alpha^* \in \mathbf{R}$ *such that* $J_a = (-\infty, \alpha^*)$. *If in addition* $K(a) < 0$, *then*

$$v(r, \alpha^*) = -\log\left(1 - \frac{r}{a}\right) + C + o(1) \tag{2.36}$$

*for* $r$ *near* $a$. *If on the other hand* $K(r) = -A(1 - r/a)^m + O(1 - r/a)^{m+1}$ *for some* $A > 0$ *and* $m > 0$, *for* $r$ *near* $a$, *then*

$$v(r, \alpha^*) = -\frac{m+2}{2}\log\left(1 - \frac{r}{a}\right) + C + o(1) \tag{2.37}$$

*for* $r$ *near* $a$.

In order to prove this theorem we need two lemmas, whose proofs we omit as they are similar to results of Cheng and Lin [4].

**Lemma 2.13.** *Let* $K$ *and* $\widetilde{K}$ *be two piecewise continuous functions on* $[0, a]$. *Let* $w$ *be a continuous function on* $[0, a]$ *satisfying*

$$w(r) \ge \tilde{\alpha} + \int_0^r s\,\log\left(\frac{r}{s}\right)(-\widetilde{K}(s))e^{2w(s)}\,ds, \qquad r \in [0, a]. \tag{2.38}$$

*Suppose that* $\widetilde{K}(r) \le K(r) \le 0$ *for all* $r \in [0, a]$, *and* $\tilde{\alpha} \ge \alpha$. *Then* (2.4) *has a (unique) solution* $v(r, \alpha)$ *on* $[0, a]$, *i.e.,* $(-\infty, \tilde{\alpha}] \subset J_a$.

**Lemma 2.14.** *Assume that* $K(r) \le 0$ *on* $[0, a]$, *and* $K \not\equiv 0$. *Let* $\alpha_1, \alpha_2 \in J_a$ *with* $\alpha_1 < \alpha_2$. *Then* $v(r, \alpha_1) < v(r, \alpha_2)$ *and* $v'(r, \alpha_1) < v'(r, \alpha_2)$ *for* $0 < r \le a$.

*Proof of Theorem* 2.12. We divide the proof into several steps.
*Step* 1: $(-\alpha)$ sufficiently large implies $\alpha \in J_a$. Let $\inf\{K(r): 0 \le r \le a\} = -e^{2\beta} < 0$, and define $w(r)$ by

$$w(r) = \alpha_0 - \log\left[1 - \tfrac{1}{4}r^2 \exp 2(\alpha_0 + \beta)\right]. \tag{2.39}$$

where $\alpha_0$ will be chosen below. Then $w$ satisfies

$$w(r) = \alpha_0 + \int_0^r s\,\log\left(\frac{r}{s}\right)e^{2\beta}e^{2w(s)}\,ds \tag{2.40}$$

on $0 \le r < 2e^{-(\alpha_0+\beta)}$. Choose $\alpha_0$ so that $2e^{-(\alpha_0+\beta)} > a$. Since $0 \ge K(r) \ge -e^{2\beta}$ for $r \in [0, a]$, if $\alpha \le \alpha_0$, Lemma 2.13 implies that $\alpha \in J_a$. Hence $(-\infty, \alpha_0] \subset J_a$.

*Step* 2: $\alpha \in J_a$ implies $(-\infty, \alpha] \subset J_a$. This is a trivial consequence of Lemma 2.13.

Thus $J_a$ is a nonempty interval of the form $(-\infty, \alpha^*)$.

*Step* 3: $\alpha^* < \infty$. Since $K(r) < 0$ for $r$ near $a$, we can find $a_1 < a_2 < a$ and $\xi$ such that

$$K(r) \le -e^{2\xi} < 0, \qquad a_1 \le r \le a_2.$$

Let

$$\widetilde{K}(r) = \begin{cases} 0 & \text{if } 0 \le r \le a_1, \\ -e^{2\xi} & \text{if } a_1 < r < a_2, \\ 0 & \text{if } a_2 \le r < a, \end{cases}$$

and consider the initial-value problem

$$\begin{aligned} &w'' + \frac{1}{r}w' + \widetilde{K}(r)e^{2w} = 0, \qquad r > 0, \\ &w(0) = \alpha - \xi, \quad w'(0) = 0, \quad a_1^2 e^{2\alpha} > 1. \end{aligned} \tag{2.41}$$

The solution $w(r, \alpha - \xi)$ is (by direct verification)

$$w(r, \alpha - \xi) = \begin{cases} \alpha - \xi \quad \text{if } 0 \le r < a, \\ \alpha - \xi - \log\left(\frac{r}{a_1}\right) + \log\left[\cos\left(\tan^{-1}\frac{1}{A}\right)\right] \\ \quad -\log\left\{\cos\left[A\log\left(\frac{r}{a_1}\right) + \tan^{-1}\left(\frac{1}{A}\right)\right]\right\} \quad \text{if } a_1 \le r < b, \end{cases}$$

where

$$A = [a_1^2 e^{2\alpha} - 1]^{1/2}, \qquad b = a_1 \exp\left\{\frac{1}{A}\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{1}{A}\right)\right]\right\}. \tag{2.42}$$

We claim that if $\alpha$ is so large that $b < a_2$, then $(\alpha - \xi) \notin J_a$. For, if $(\alpha - \xi) \in J_a$, then $v(r, \alpha - \xi)$ satisfies

$$v(r, \alpha - \xi) = (\alpha - \xi) + \int_0^r s \,\log\left(\frac{r}{s}\right)(-K(s))e^{2v(s, \alpha - \xi)}\,ds. \tag{2.43}$$

But since $K(r) \le \widetilde{K}(r) \le 0$ for $r \in [0, a]$, Lemma 2.13 implies that (2.41) has a solution $w(r; \alpha - \xi)$ defined on $[0, a]$. But this is impossible since $w(r; \alpha - \xi)$ exists only on $[0, b]$, and $b < a_2 < a$. Hence $(\alpha - \xi) \notin J_a$ so that $J_a = (-\infty, \alpha^*)$, and $\alpha^* < \infty$. Now since $v'(r, \alpha^*) > 0$ and solutions of (2.1) depend continuously on $\alpha$, we have that $\lim_{r \to a} v(r, \alpha^*) = +\infty$.

*Step* 4. Suppose $K(a) < 0$; then, for $r$ near $a$ (cf. (2.36)),

$$v(r, \alpha^*) = -\log\left(1 - \frac{r}{a}\right) - \frac{1}{2}\log[-a^2 K(a)] + o(1).$$

Let $w(r) = v(r, \alpha^*) + \log(1 - r/a)$; then $w$ satisfies

$$w'' + \frac{1}{r}w' = \frac{1}{(1 - r/a)^2}\left[-K(r)e^{2w} - \frac{1}{a^2} - \frac{1}{ar}\left(1 - \frac{r}{a}\right)\right]. \tag{2.44}$$

Let $v_0 = -\frac{1}{2}\log[-a^2 K(a)]$ and let $\varepsilon > 0$ be given. Choose $\tilde{\gamma} > 0$ and $\delta > 0$ such that, for all $r$, with $a - \tilde{\gamma} \le r \le a$,

$$-K(r)e^{2(v_0 + \varepsilon)} - \frac{1}{a^2} - \frac{1}{ar}\left(1 - \frac{r}{a}\right) > \delta$$

and

$$-K(r)e^{2(v_0-\varepsilon)} - \frac{1}{a^2} - \frac{1}{ar}\left(1 - \frac{r}{a}\right) < -\delta .$$

We claim that there exists a $\gamma$, $0 < \gamma < \tilde{\gamma}$, such that

$$v_0 - \varepsilon < w(r) < v_0 + \varepsilon , \qquad a - \gamma \le r < a . \tag{2.45}$$

In view of the arbitrariness of $\varepsilon$, this will prove (2.36).

Thus, suppose (2.45) does not hold. Then one of the following must hold:

(i) $w(r) \ge v_0 + \varepsilon$ and $w'(r) < 0$ for all $r \in [a - \tilde{\gamma}, a)$.
(ii) $\exists r_1 \in [a - \tilde{\gamma}, a)$ such that $w(r_1) \ge v_0 + \varepsilon$ and $w'(r_1) \ge 0$.
(iii) $w(r) \le v_0 - \varepsilon$ and $w'(r) > 0$ for all $r \in [a - \tilde{\gamma}, a)$.
(iv) $\exists r_2 \in [a - \tilde{\gamma}, a)$ such that $w(r_2) \le v_0 - \varepsilon$ and $w'(r_2) \le 0$.

We shall successively rule out these four possibilities.

Suppose that (i) holds. Then from (2.44), we have (in view of our choice of $\tilde{\gamma}$ and $\delta$)

$$w''(r) + \frac{1}{r}w'(r) > \frac{\delta}{(1 - r/a)^2}, \qquad a - \tilde{\gamma} \le r \le a,$$

so, on this range, $(rw'(r))' > \delta r/(1 - r/a)^2$. Thus, for some $r$ near $a$,

$$w'(r) > \frac{1}{r}(a - \tilde{\gamma})w'(a - \tilde{\gamma}) + \frac{1}{r}\int_{a-\tilde{\gamma}}^{r} \frac{\delta s}{(1 - s/a)^2}\, ds > 0,$$

and this contradicts the fact that $w'(r) < 0$ for all $r \in [a - \tilde{\gamma}, a)$; thus (i) does not hold.

Suppose now that (ii) holds. Then (2.44) at $r = r_1$ gives

$$w''(r) + \frac{1}{r}w'(r) > \frac{\delta}{(1 - r/a)^2} > 0. \tag{2.46}$$

Since $w'(r_1) \ge 0$, we conclude that $w$ is strictly increasing for $r > r_1$, $r$ near $r_1$. By continuity, (2.46) holds for $r > r_1$, and $r$ near $r_1$; say on $[r_1, \bar{r})$, $a - \tilde{\delta} < r_1 < \bar{r} < a$. Now as

$$\bar{r}w'(\bar{r}) - r_1 w'(r_1) > \delta \int_{r_1}^{\bar{r}} \frac{s}{(1 - s/a)^2}\, ds > 0$$

we see $w'(\bar{r}) > 0$; thus, $w(\bar{r}) > v_0 + \varepsilon$. If

$$w''(\bar{r}) + \frac{1}{\bar{r}}w'(\bar{r}) = \frac{\delta}{(1 - \bar{r}/a)^2},$$

then from (2.44) we get the contradiction

$$\delta = \left[-K(\bar{r})e^{2w(\bar{r})} - \frac{1}{a^2} - \frac{1}{a\bar{r}}\left(1 - \frac{\bar{r}}{a}\right)\right] > \delta .$$

Thus, (2.46) holds for all $r \ge r_1$. If we set $z(t) = w(e^t)$, then (2.46) shows that $z$ satisfies $z''(t) \ge \delta/(1 - e^t/a)^2$ and, as $z' > 0$, we see that this implies $z(t) \to +\infty$ as $t \to \ln a$, i.e., $\lim_{r\to a} w(r) = +\infty$. Thus, there exists an $r_3$, $r_1 < r_3 < a$, such that

$$-K(r)e^{2w(r)} - \frac{1}{a^2} - \frac{1}{ar}\left(1 - \frac{r}{a}\right) \ge Ce^{w(r)} \tag{2.47}$$

for some constant $C > 0$ and for $r_3 \leq r < a$. From (2.44) and (2.47), we have

$$w'' + \frac{1}{r}w' \geq \frac{Ce^{w(r)}}{(1-r/a)^2}, \qquad r_3 \leq r < a, \tag{2.48}$$
$$w'(r_3) > 0.$$

Now set $t = (1 - r/a)^{-1}$ and $g(t) = w(r)$; then (2.48) becomes

$$g'' + \left(\frac{1}{t} + \frac{1}{t-1}\right)g' \geq a^2Ct^{-2}e^{g(t)}, \qquad t \geq t_3, \tag{2.49}$$
$$g(t_3) = w(r_3), \qquad g'(t_3) > 0, \qquad t_3 = \left(1 - \frac{r_3}{a}\right)^{-1}$$

But using similar arguments as in Cheng and Lin [4], we can prove that no $g$ satisfies (2.49); this shows that (ii) cannot hold.

Suppose that (iii) holds. Then (2.44) gives

$$w''(r) + \frac{1}{r}w'(r) \leq -\frac{\delta}{(1-r/a)^2}, \qquad a - \tilde{\gamma} \leq r \leq a.$$

Hence

$$w'(r) \leq \frac{1}{r}(a-\tilde{\gamma})w'(a-\tilde{\gamma}) - \frac{1}{r}\int_{a-\delta}^{r} \frac{\delta s}{(1-s/a)^2}\,ds < 0$$

for $r$ near $a$; this contradicts the fact that $w'(r) > 0$ for all $r \in [a - \tilde{\gamma}, a)$; thus (iii) cannot hold.

Finally, suppose that (iv) holds. Then, at $r = r_2$,

$$w''(r) + \frac{1}{r}w'(r) < -\frac{\delta}{(1-r/a)^2} < 0. \tag{2.50}$$

As in case (ii), we conclude that (2.50) is true for $r \in [r_2, a)$, $w' < 0$ on this interval, and

$$\lim_{r\to a} w(r) = -\infty. \tag{2.51}$$

Hence, for $r$ near $a$, we have (cf. (2.44))

$$w''(r) + \frac{1}{r}w'(r) = \frac{1}{(1-r/a)^2}\frac{1}{a^2} + \text{(higher-order terms)}. \tag{2.52}$$

Thus (as above, setting $z(t) = w(e^t)$), we have, for $r$ near $a$,

$$w(r) = \log(1 - r/a) + O(1). \tag{2.53}$$

Hence $v(r, \alpha^*) = w(r) - \log(1 - r/a) + O(1)$ for $r$ near $a$. But this contradicts the fact that $v(r, \alpha^*) \to \infty$ as $r \to a$. Thus (iv) cannot hold, and this proves (2.45) and also (2.36).

To prove (2.37), we let

$$v(r, \alpha^*) = -\frac{m+2}{2}\log\left(1 - \frac{r}{a}\right) + w(r); \tag{2.54}$$

then $w$ satisfies

(2.55)
$$w'' + \frac{1}{r}w' = \frac{1}{(1-r/a)^2}\left[\frac{-K(r)}{(1-r/a)^m}e^{2w} - \frac{(m+2)}{2}\frac{1}{a^2} - \frac{(m+2)}{2}\frac{1}{ar}\left(1 - \frac{r}{a}\right)\right].$$

The proof now follows along the same lines as in the proof of (2.36)—we omit the details. The proof of Theorem 2.12 is considered complete. □

Now we consider the problem

$$v'' + \frac{1}{r}v' + K(r)e^{2v} = 0, \qquad a \le r \le b, \; a > 0, \tag{2.56}$$
$$v(a) = \alpha, \qquad v'(a) = \beta$$

Denote by $v(r, \alpha, \beta)$ the solution of (2.56).

**Theorem 2.15.** *Assume that* $K(r) \le 0$ *for* $r \in [a, b]$, *and that* $K(r) < 0$ *for* $r$ *near* $b$. *Then given* $\alpha \in \mathbb{R}$, *there is a unique* $\beta^* = \beta^*(\alpha)$ *such that* (2.56) *has a* (*unique*) *solution* $v(r, \alpha, \beta)$ *on* $[a, b]$ *for all* $\beta < \beta^*$. *Furthermore,* $v(r, \alpha, \beta^*)$ *is defined on* $[a, b)$ *and satisfies*

$$v(r, \alpha, \beta^*(\alpha)) = -\log\left(1 - \frac{r}{b}\right) + C + o(1)$$

*for* $r$ *near* $b$ *if* $K(b) < 0$, *and*

$$v(r, \alpha, \beta^*(\alpha)) = -\frac{m+2}{2}\log\left(1 - \frac{r}{b}\right) + C + o(1)$$

*for* $r$ *near* $b$ *if* $K(r) = -A(1 - r/b)^m +$ (*higher-order terms*), *for some* $m > 0$ *and* $A > 0$. *Moreover* $\beta^*(\alpha)$ *is a strictly decreasing function which satisfies*

$$\lim_{\alpha\to\infty} \beta^*(\alpha) = -\infty \quad \text{and} \quad \lim_{\alpha\to-\infty} \beta^*(\alpha) = \infty.$$

We will outline the proof of this theorem since it is similar to the proof of Theorem 2.12. First we need two lemmas whose proofs are similar to the proofs of Lemmas 2.13 and 2.14, and will thus be omitted.

**Lemma 2.16.** *Let* $K$ *and* $\widetilde{K}$ *be two piecewise continuous functions on* $[a, b]$, *and let* $w$ *be a continuous function on* $[a, b]$ *satisfying*

$$w(r) \ge \tilde{\alpha} + a\tilde{\beta}\log\left(\frac{r}{a}\right) + \int_a^r s \, \log\left(\frac{r}{s}\right)(-\widetilde{K}(s))e^{2w(s)}\,ds. \tag{2.57}$$

*If* $\widetilde{K}(r) \le K(r) \le 0$ *for* $r \in [a, b]$, $\tilde{\alpha} \ge \alpha$, $\tilde{\beta} \ge \beta$, *then* (2.56) *has a* (*unique*) *solution* $v(r, \alpha, \beta)$ *defined on* $[a, b]$.

**Lemma 2.17.** *Assume that* $K(r) \le 0$ *for* $r \in [a, b]$, *and* $K \not\equiv 0$. *Let* $v(r, \alpha_1, \beta_1)$ *and* $v(r, \alpha_2, \beta_2)$ *be two solutions of* (2.56) *corresponding to* $(\alpha, \beta) = (\alpha_i, \beta_i)$, $i = 1, 2$. *If* $\alpha_1 < \alpha_2$, $\beta_2 \le \beta_2$ *or* $\alpha_1 \le \alpha_2$, $\beta_1 < \beta_2$, *then, for* $r \in (a, b]$, $v(r, \alpha_1, \beta_1) < v(r, \alpha_2, \beta_2)$ *and* $v'(r, \alpha_1, \beta_1) \le v'(r, \alpha_2, \beta_2)$.

*Proof of Theorem* 2.15. The proof will again be divided into several steps.

*Step* 1. Given $\alpha \in \mathbf{R}$, if $\beta < 0$ and $(-\beta)$ is sufficiently large, (2.56) has a solution $v(r, \alpha, \beta)$ defined on $[a, b]$. (The arguments are similar to the proof of Theorem 2.12.)

*Step* 2. Given $\alpha \in \mathbf{R}$, if, for $\beta = \beta_1$, (2.56) has a solution $v(r, \alpha, \beta_1)$ on $[a, b]$, then, again by Lemma 2.16, (2.56) has a solution $v(r, \alpha, \beta)$ for all $\beta \le \beta_1$.

*Step* 3. Given $a \in \mathbf{R}$, (2.56) has no solution defined on $[a, b]$ if $\beta$ is sufficiently large. The idea of the proof is similar to Step 3 in the proof of Theorem 2.12, and the details will be omitted.

*Step* 4. Using Steps 1, 2, and 3, we conclude that, for a given $\alpha \in \mathbf{R}$, there exists a unique $\beta^* = \beta^*(\alpha)$ such that (2.56) has a solution $v(r, \alpha, \beta)$ on $[a, b]$ for all $\beta < \beta^*(\alpha)$. It is easy to see that at $\beta^*$, the solution $v(r, \alpha, \beta^*)$ of (2.56) satisfies

$$\lim_{r \to b} v(r, \alpha, \beta^*(\alpha)) = \infty .$$

Similar arguments as in the proof of Theorem 2.12 shows the corresponding asymptotic behavior of $v(r, \alpha, \beta^*)$ as described in the statement of Theorem 2.15.

*Step* 5. $\lim_{\alpha\to\infty} \beta^*(\alpha) = -\infty$ and $\lim_{\alpha\to-\infty} \beta^*(\alpha) = +\infty$. Thus, suppose that $\lim_{\alpha\to\infty} \beta^*(\alpha) = B > -\infty$. Let $\alpha_0$ be chosen such that the following problem does not have a solution defined in $[a, b)$ (cf. Theorem 2.12):

$$\begin{aligned} &v'' + \frac{1}{r} v' + K(r) e^{2v} = 0, \qquad r \in [a, b), \\ &v(a) = \alpha_0, \qquad v'(a) = 0. \end{aligned}$$

Let $\alpha = 2\alpha_0 - aB \log(b/a)$ and let $v(r, \alpha)$ be the solution of

$$\begin{aligned} &v'' + \frac{1}{r} v' + K(r) e^{2v} = 0, \qquad r \in [a, b), \\ &v(a) = \alpha, \qquad v'(a) = \beta^*(\alpha). \end{aligned}$$

Then

$$\begin{aligned} v(r, \alpha) &= \alpha + a\beta^*(\alpha) \log\left(\frac{r}{a}\right) + \int_a^r s \ \log\left(\frac{r}{s}\right) [-K(s)] e^{2v(s, \alpha)}\, ds \\ &= 2\alpha_0 - aB \log\left(\frac{b}{a}\right) + a\beta^*(\alpha) \log\left(\frac{r}{a}\right) \\ &\quad + \int_a^r s \ \log\left(\frac{r}{s}\right) [-K(s)] e^{2v(s, \alpha)}\, ds \\ &\geq 2\alpha_0 + \int_a^r s \ \log\left(\frac{r}{s}\right) [-K(s)] e^{2v(s, \alpha)}\, ds . \end{aligned}$$

But Lemma 2.16 shows that this is not possible; hence $B = -\infty$. Similarly, $\lim_{\alpha\to-\infty} \beta^*(\alpha) = +\infty$. This completes the outline of the proof of Theorem 2.15.

## 3. Proof of Theorem A

Suppose first that (1.5) has a regular solution $u(z)$ defined on $[-1, 1]$. Let $z = (r^2 - 1)/(r^2 + 1)$, $K_1(r) = K((r^2 - 1)/(r^2 + 1)) = K(z)$, and $v(r) = u(z) - \log((1 + R^2)/2)$. Then $v$ satisfies

$$\begin{aligned} &v''(r) + \frac{1}{r} v'(r) + K_1(r) e^{2v(r)} = 0, \qquad r \in [0, \infty), \\ &v'(0) = 0, \quad v(0) = u(-1), \quad v(r) = -2\log r + C + o(1) \quad \text{at } r = \infty . \end{aligned} \tag{3.1}$$

Now evaluating (2.3) at $r = \infty$ gives $\lim_{r\to\infty} \Phi_\alpha(r) = 1$ (where $\alpha = u(-1)$). Thus from Lemma 2.4(ii), we have

$$1 = 1 + \int_0^\infty K_1'(s) s^2 e^{2v(s, \alpha)}\, ds ,$$

and so

$$\int_0^\infty K_1'(s)s^2e^{2v(s,\alpha)}\,ds = 0. \tag{3.2}$$

Now again using (ii) in Lemma 2.4, we have

$$1 = \left(1 - \int_0^\infty sK_1(s)e^{2v(s,\alpha)}\,ds\right)^2,$$

so $\int_0^\infty sK_1(s)e^{2v(s,\alpha)}\,ds = 2 > 0$, and $\alpha = u(-1) \in I$.

Conversely, if there exists an $\alpha \in I$ such that (3.2) holds and

$$\int_0^\infty sK_1(s)e^{2v(s,\alpha)}\,ds > 0,$$

then from Lemmas 2.2, 2.3, and 2.4 we have

$$\int_0^\infty sK_1(s)e^{2v(s,\alpha)}\,ds = 2,$$

and since $\alpha \in I$, if $r > 0$, we have

$$rv'(r,\alpha) + \int_0^r sK_1(s)e^{2v(s,\alpha)}\,ds = 0.$$

Hence $rv'(r,\alpha) \to -2$ as $r \to \infty$. It follows that for large $r$

$$v(r,\alpha) = -2\log r + C + o(1).$$

Now set $z = (r^2-1)/(r^2+1)$, and $u(z) = v(r,\alpha) + \log((1+r^2)/2)$. Then $u(z)$ is a regular solution of (1.5), since

$$u(1) = \lim_{r\to\infty}\left[v(r,\alpha) + \log\left(\frac{1+r^2}{2}\right)\right]$$

exists. This completes the proof of Theorem A.

## 4. Proof of Theorem B

We begin by considering the following problem:

$$\frac{d}{dz}\left[(1-z^2)\frac{d\tilde{v}}{dz}\right] + K(z)e^{2\tilde{v}} = 1, \qquad z \in [-1,0],\ \tilde{v}(-1) = \tilde{\alpha}. \tag{4.1}$$

We define sets $\tilde{I}_1$ and $A_1$ by $\tilde{I}_1 = \{\tilde{\alpha} \in \mathbf{R}\colon (4.1)$ has a unique solution $v(z,\tilde{\alpha})$ on $[-1,0]\}$ and $A_1 = \{(\tilde{v}(0,\tilde{\alpha}), \tilde{v}'(0,\alpha)) \in \mathbf{R}^2\colon \tilde{\alpha} \in \tilde{I}_1\}$. Similarly, we consider the problem

$$\frac{d}{dz}\left[(1-z^2)\frac{d\tilde{w}}{dz}\right] + K(z)e^{2\tilde{w}} = 1, \qquad z \in [0,1],\ \tilde{w}(1) = \tilde{\beta}, \tag{4.2}$$

and we define sets $\tilde{I}_2$ and $A_2$ by $\tilde{I}_2 = \{\tilde{\beta} \in \mathbf{R}\colon (4.2)$ has a unique solution $w(z,\tilde{\beta})$ on $[0,1]\}$ and $A_2 = \{(\tilde{w}(0,\beta), \tilde{w}'(0,\beta)) \in \mathbf{R}^2\colon \tilde{\beta} \in \tilde{I}_2\}$. We then have the following theorem whose simple proof is omitted.

**Theorem 4.1.** (1.5) *has a regular solution* $u(z)$ *on* $[-1, 1]$ *if and only if* $A_1 \cap A_2 \neq \varnothing$.

Now let $z = (r^2 - 1)/(r^2 + 1)$ and $\tilde{v}(z) = v(r) + \log((1 + r^2)/2)$. Then $\tilde{v}$ satisfies (4.1) if and only if $v$ satisfies

$$(4.3)\qquad \begin{gathered} v''(r) + \frac{1}{r}v'(r) + K_1(r)e^{2v(r)} = 0, \qquad r \in [0, 1], \\ v(0) = \tilde{\alpha} + \log 2 \equiv \alpha, \quad v'(0) = 0, \quad \alpha \in I_1 = \tilde{I}_1 + \log 2, \end{gathered}$$

where $K_1(r) = K\left((r^2 - 1)/(r^2 + 1)\right)$, as before. With this notation we have

$$A_1 = \{(v(1, \alpha), v'(1, \alpha) + 1) \colon \alpha \in I_1\}.$$

Similarly, let $z = (1 - t^2)/(1 + t^2)$ and $\tilde{w}(z) = w(t) + \log\left((1 + t^2)/2\right)$. Then if $\tilde{w}$ satisfies (4.2), $w$ satisfies

$$\begin{gathered} w''(t) + \frac{1}{t}w'(t) + K_2(t)e^{2w(t)} = 0, \qquad t \in [0, 1], \\ w(0) = \tilde{\beta} + \log 2 \equiv \beta, \quad w'(0) = 0, \quad \beta \in I_2 \equiv \tilde{I}_2 + \log 2, \end{gathered}$$

where $K_2(t) = K\left((1 - t^2)/(1 + t^2)\right)$. We then have

$$A_2 = \{(w(1, \beta), -w'(1, \beta) - 1) \colon \beta \in I_2\}.$$

In what follows, we set

$$A_1(\alpha) = (v(1, \alpha), v'(1, \alpha) + 1) \equiv (x_1(\alpha), y_1(\alpha)), \qquad \alpha \in I_1;$$

hence $A_1 = \{A_1(\alpha) \colon \alpha \in I_1\}$. We then have the following lemma.

**Lemma 4.2.** *Assume that* $K_1$ *is smooth, has only finite-order zeros,* $K_1(0) > 0$, *and either* $K_1'(0) \neq 0$ *or, if* $K_1'(0) = 0$, *then* $K_1''(0) \neq 0$. *Then either* $I_1 = (-\infty, \infty)$ *or* $I_1$ *is a finite or countable infinite union of open intervals* $J_1, J_2, J_3, \ldots$, *where*

$$J_1 = (-\infty, a_1^*), \quad J_2 = (a_2, \infty), \quad \text{and} \quad J_i = (a_i, a_i^*) \quad \text{for } i \geq 3.$$

*Furthermore the following four statements hold:*

(i) $x_1(\alpha)$ *and* $y_1(\alpha)$ *are smooth functions on* $I_1$ *and* $A_1(\alpha) \neq A_1(\alpha')$ *if* $\alpha \neq \alpha'$.

(ii) *There exists an* $\alpha_0 > 0$ *such that, for* $\alpha \leq -\alpha_0$, $x_1(\alpha) = \alpha - O(e^{2\alpha})$ *and* $y_1(\alpha) = 1 - e^{2\alpha}\int_0^1 sK_1(s)\,ds + O(e^{4\alpha})$.

(iii) *There exists an* $\alpha_1 > 0$ *such that, for* $\alpha \geq \alpha_1$, $x_1(\alpha) = -\alpha + O(1)$ *and either* $y_1(\alpha) = -1 - CK_1'(0)e^{-\alpha} + O(\alpha^2 e^{-2\alpha})$ *if* $K_1'(0) \neq 0$, *or* $y_1(\alpha) = -1 - CK_1''(0)\alpha e^{-2\alpha} + O(e^{-2\alpha})$ *if* $K_1'(0) = 0$ *and* $K_1''(0) \neq 0$.

(iv) *If* $(a_i^*, a_j) \cap I_1 = \varnothing$ *for some* $i$ *and* $j$, *we have either*

$$x_1(a_i^* - 0) = x_1(a_j + 0) = y_1(a_i^* - 0) = y_1(a_j + 0) = \infty$$

*or*

$$x_1(a_i^* - 0) = x_1(a_j + 0) = y_1(a_i^* - 0) = y_1(a_j + 0) = -\infty.$$

*In either case* $\inf\{\|A_1(\alpha_1) - A_1(\alpha_2)\| \colon \alpha_1 \in [a_i^* - \delta, a_i^*),\ \alpha_2 \in (a_j, a_j + \delta]\} = 0$, *where* $\delta > 0$ *is a sufficiently small number such that* $[a_i^* - \delta, a_i^*) \subset J_i$ *and* $(a_j, a_j + \delta] \subset J_j$, *and where* $\|\cdot\|$ *denotes the Euclidean norm on* $\mathbf{R}^2$.

*Proof*. Using the standard "continuous dependence on initial conditions" theorem, it is easy to check that $I_1$ is an open set and that $x_1$ and $y_1$ are continuous functions on $I_1$. Since solutions of (4.3) are unique, it follows that $A_1(\alpha) \neq A_1(\alpha')$ if $\alpha \neq \alpha'$; this proves (i).

If $K_1(r) \geq 0$ for all $r \in [0, 1]$, then, from Theorems 2.6 and 2.10, we conclude that $I_1 = \mathbf{R}$ and that (ii) and (iii) hold. In summary, if $K_1(r) \geq 0$, then (i)–(iii) hold and $I_1 = \mathbf{R}$; (iv) holds vacuously.

Now assume that $K_1(r) \geq 0$ on $[0, a]$, $K_1(r) \leq 0$ on $[a, 1]$ for some $0 < a < 1$, and $K_1(r) < 0$ for $r$ near 1. We consider the problems

$$v'' + \frac{1}{r}v' + K_1(r)e^{2v} = 0, \qquad r \in [0, a], \qquad (4.5)$$
$$v(0) = \alpha, \qquad v'(0) = 0, \qquad \alpha \in \mathbf{R},$$

and

$$v'' + \frac{1}{r}v' + K_1(r)e^{2v} = 0, \qquad r \in [a, 1], a > 0, \qquad (4.6)$$
$$v(a) = \gamma, \qquad v'(a) = \beta.$$

We denote by $v(r, \alpha)$ and $v(r, \gamma, \beta)$ the solutions of (4.5) and (4.6), respectively, if they exist. Let $\Gamma_\alpha = (v(a, \alpha), v'(a, \alpha))$ and $\Gamma = \{\Gamma_\alpha : \alpha \in \mathbf{R}\}$. Again by Theorem 2.6 and the standard "continuous dependence" theorems, it is easy to see that $\Gamma$ is a smooth curve in $\mathbf{R}^2$. From Theorem 2.15, for each $\gamma \in \mathbf{R}$, there exists a $\beta^* = \beta^*(\gamma)$ such that (4.6) has a solution $v(r, \gamma, \beta)$ on $[a, 1]$ for all $\beta < \beta^*(\gamma)$. Let

$$\Omega = \{(\gamma, \beta) : \beta < \beta^*(\gamma), \ \gamma \in \mathbf{R}\}.$$

Then $\alpha \in I$ if and only if $\Gamma_\alpha \in \Omega$. Now if $\alpha \in \mathbf{R}$, then Theorems 2.6 and 2.10 imply that $\alpha \in J_a$. If now $v'(a, \alpha) < \beta^*(v(a, \alpha))$, we may apply Theorem 2.15 to conclude that $\Gamma_\alpha \in \Omega$. Now if $\alpha$ tends to $-\infty$, we see from (2.5) or (2.27) that $v'(a, \alpha) \to 0$ and $v(a, \alpha) \to -\infty$; thus, $v'(a, \alpha) < \beta^*(v(a, \alpha))$ since (cf. Theorem 2.15) $\beta^*$ is positive somewhere. It follows that $\Gamma_\alpha \in \Omega$ for $\alpha$ near $-\infty$, and similarly $\Gamma_\alpha \in \Omega$ for $\alpha$ near $+\infty$. Thus, if $\Gamma \cap \partial\Omega$ is empty, then $I_1 = \mathbf{R}$. From Theorems 2.6 and 2.10, $v(a, \alpha)$ and $v'(a, \alpha)$ satisfy (2.5) and (2.6), or (2.26), (2.27), (2.28), and (2.29). Then integrating (4.3) from $r = a$ to $r = 1$ yields (ii) and (iii); (iv) holds vacuously. If $\Gamma \cap \partial\Omega \neq \varnothing$, then we may write $I_1$ as the union of disjoint open intervals: $I_1 = J_1 \cup J_2 \cup \cdots$, where each $J_i$ is as described in the statement of the theorem. Then (ii) and (iii) hold as before. From Theorem 2.15 (the asymptotic form of $v(r, \alpha, \beta^*)$), we see that

$$x_1(a_i^* - 0) = x_1(a_j + 0) = y_1(a_i^* - 0) = y_1(a_j^* + 0) = +\infty$$

for all $i \geq 1$ and $j \geq 2$. Furthermore, if $(a_i^*, a_j) \cap I_1 = \varnothing$ for some $i$ and $j$, then, again using Theorem 2.15, we conclude that

$$\inf\{\|A_1(\alpha_1) - A_1(\alpha_2)\| : \alpha_1 \in [a_i^* - \delta, a_i^*), \ \alpha_2 \in (a_j, a_j + \delta]\} = 0.$$

Thus, the lemma is proved in this case.

Next, assume that $K_1(r) \geq 0$ for $r \in [0, a]$, $K_1(r) \leq 0$ for $r \in [a, b]$, and $K_1(r) \geq 0$ for $r \in [b, 1]$. We consider the following three problems:

$$v'' + \frac{1}{r}v' + K_1(r)e^{2v} = 0, \qquad r \in [0, a], \qquad (4.7)$$
$$v(0) = \alpha, \qquad v'(0) = 0, \qquad \alpha \in \mathbf{R},$$

$$(4.8)\qquad \begin{aligned} &v'' + \frac{1}{r}v' + K_1(r)e^{2v} = 0, \qquad r \in [a, b], \\ &v(a) = \gamma, \qquad v'(a) = \beta, \end{aligned}$$

$$(4.9)\qquad \begin{aligned} &v'' + \frac{1}{r}v' + K_1(r)e^{2v} = 0, \qquad r \in [b, 1], \\ &v(b) = \delta, \qquad v'(b) = \xi. \end{aligned}$$

Let the solutions of (4.7), (4.8), and (4.9) (if they exist; cf. Lemma 2.7) be denoted by $v(r, \alpha)$, $v(r, \gamma, \beta)$, and $v(r, \delta, \xi)$, respectively. Using arguments similar to the proof of Lemma 2.7, we conclude that (4.9) has a solution for all $\delta$ and $\xi$. Let

$$\Gamma_\alpha = (v(a, \alpha), v'(a, \alpha)) \quad \text{and} \quad \Gamma = \{\Gamma_\alpha : \alpha \in \mathbf{R}\}.$$

From Theorem 2.15 we know that, for each $\gamma \in \mathbf{R}$, there exists $\beta^* = \beta^*(\gamma)$ such that (4.8) has a solution $v(r, \gamma, \beta)$ for all $\beta < \beta^*(\gamma)$. Let

$$\Omega = \{(\gamma, \beta) : \beta < \beta^*(\gamma), \ \gamma \in \mathbf{R}\}.$$

Then using the same arguments as before, we conclude that $\alpha \in I$ if and only if $\Gamma_\alpha \in \Omega$. Hence, either $I_1 = \mathbf{R}$ or $I_1 = J_1 \cup J_2 \cup \cdots$, where the $J_i$'s are as described in the statement of the theorem. If $I_1 = \mathbf{R}$, then, as before, statements (i)–(iv) hold. If $I_1 \neq \mathbf{R}$, then (ii) and (iii) hold, and we need only prove (iv).

To this end, we first study $x_1(a_1^* - 0)$. Thus, suppose $\alpha < a_1^*$, where $(a_1^* - \alpha)$ is small; then $\Gamma_\alpha$ is near $\partial\Omega$. If $\gamma = v(a, \alpha)$ and $\beta = v'(a, \alpha)$, then Theorem 2.15 implies that both $v(b, \gamma, \beta)$ and $v'(b, \gamma, \beta)$ are large. Now, for $r$ near $b$, we have that $K_1(r) = A(r-b)^m + O(r-b)^{m+1}$ for some odd integer $m \geq 1$ and for some constant $A > 0$. Again using Theorem 2.15 (the asymptotic behavior of $v(r, \gamma, \beta^*(\gamma))$ for $r$ near $b$), we conclude that, for $\alpha$ near $a_1^*$,

$$(4.10)\qquad v(b, \gamma, \beta) = \frac{m+2}{2}\log\left(\frac{1}{\varepsilon}\right) + C + O(\varepsilon)$$

and

$$(4.11)\qquad v'(b, \gamma, \beta) = \frac{m+2}{2}\cdot\frac{1}{\varepsilon} + C + O(\varepsilon),$$

where $\varepsilon$ is a positive number depending only on $\alpha$ such that $\lim_{\alpha\to a_1^*}\varepsilon(\alpha) = 0$ and the $C$'s are constants independent of $\alpha$ for $\alpha$ near $a_1^*$.

Now consider (4.9), where $\delta = v(b, \gamma, \beta)$ and $\xi = v'(b, \gamma, \beta)$; here $v(b, \gamma, \beta)$ and $v'(b, \gamma, \beta)$ are as in (4.10) and (4.11), respectively. From (4.9) (cf. (2.2)) we have

$$(4.12)\qquad v'(r, \delta, \xi) = \frac{1}{r}\left(b\xi - \int_b^r sK_1(s)e^{2v(s, \delta, \xi)}\,ds\right).$$

We claim that for $\varepsilon$ sufficiently small there exists an $r_0 > b$ such that $v'(r, \delta, \xi) > 0$ for all $b \leq r < r_0$ and $v'(r_0, \delta, \xi) = 0$. For suppose not; then, for all $r \geq b$, $v'(r, \delta, \xi) > 0$. Thus, $v(r, \delta, \xi) > v(b, \delta, \xi)$ for all $r > b$. Set $r =$

$b+k\varepsilon$ for some $k>0$, $k\varepsilon$ small. From (4.11) and (4.12)
(4.13)

$$\begin{aligned} v'(b+k\varepsilon, \delta, \xi) &= \frac{1}{(b+k\varepsilon)}\left[bv'(b, \gamma, \beta) - \int_b^{b+k\varepsilon} sK_1(s)e^{2v(s,\delta,\xi)}\,ds\right] \\ &\le \frac{1}{(b+k\varepsilon)}\left[b\frac{m+2}{2}\frac{1}{\varepsilon} + bC + O(\varepsilon) - \int_b^{b+k\varepsilon} sK_1(s)e^{2C}\left(\frac{1}{\varepsilon}\right)^{m+2} ds\right]. \end{aligned}$$

Now choose $k$ so large that

$$\frac{m+2}{2} - \frac{A}{m+1}k^{m+1}e^{2C} < 0.$$

Then, for small $\varepsilon$,
(4.14)

$$\begin{aligned} v'(b+k\varepsilon, \delta, \xi) &\le \frac{1}{(b+k\varepsilon)}\left[b\cdot\frac{m+2}{2}\cdot\frac{1}{\varepsilon} + bC + O(\varepsilon)\right. \\ &\qquad \left. - \int_b^{b+k\varepsilon} sA(s-b)^m e^{2c}\left(\frac{1}{\varepsilon}\right)^{m+2} ds + O(\varepsilon)\right] \\ &= \frac{1}{(b+k\varepsilon)}\left[b\cdot\frac{1}{\varepsilon}\left(\frac{m+2}{2} - \frac{A}{m+1}k^{m+1}e^{2C}\right) + bC + O(C)\right] \\ &< 0. \end{aligned}$$

This contradiction proves that our claim is valid. Furthermore, the above estimates allow us to conclude that

$$r_0 = b + k_0\varepsilon + O(\varepsilon^2),$$

where $k_0 > 0$ is a constant independent of $\varepsilon$. Moreover, since $v'(b, \delta, \xi) = \xi$, we have

$$\begin{aligned} v(r_0, \delta, \xi) &= v(r_0, \delta, \xi) - v(b, \delta, \xi) + v(b, \delta, \xi) \\ &= v'(b, \delta, \xi)(r_0 - b) + O(r_0 - b) + v(b, \delta, \xi) \\ &= \xi(r_0 - b) + O(r_0 - b) + v(b, \delta, \xi) = v(b, \delta, \xi) + C + o(1). \end{aligned}$$

Thus, from (4.10), we see that

$$v(r_0, \delta, \xi) = \frac{m+2}{2}\log\left(\frac{1}{\varepsilon}\right) + C + O(\varepsilon).$$

Now we consider the following problem:

$$\begin{aligned} &v'' + \frac{1}{r}v' + K_1(r)e^{2v} = 0, \qquad r_0 \le r \le 1, \\ &v(r_0) = v(r_0, \delta, \xi), \qquad v'(r_0) = 0. \end{aligned} \tag{4.15}$$

Let $v(r, \varepsilon)$ be the solution of (4.15); we are interested in the asymptotic behavior of $v(1, \varepsilon)$ and $v'(1, \varepsilon)$ as $\varepsilon \to 0$. To this end, we choose $\Delta > 0$ so small that, for $b \le r \le b + \Delta$,

$$K_1(r) = A(r-b)^m + O((r-b)^{m+1}). \tag{4.16}$$

Let $r = b + (k(\varepsilon) + \eta)\varepsilon$, where $\eta$ is a new independent variable, $0 \le \eta \le \Delta/\varepsilon - k(\varepsilon)$. Now set $w(\eta) = v(r, \varepsilon) - v(r_0)$. Then $w(\eta)$ satisfies the following:
(4.17)

$$w'' + \frac{\varepsilon}{b + (\eta + k(\varepsilon))\varepsilon} w' + K(b + (k(\varepsilon) + \eta)\varepsilon)\left(\frac{1}{\varepsilon}\right)^m e^{2C} e^{20(\varepsilon)} e^{2w(\eta)} = 0,$$
$$w(0) = 0, \qquad w'(0) = 0.$$

Let $w_0(\eta)$ be the unique solution of

$$\text{(4.18)} \qquad \begin{gathered} w_0'' + \left[A(\eta + k(\varepsilon))^m e^{2C}\right] e^{2W_0} = 0, \qquad \eta > 0, \\ w_0(0) = 0, \qquad w_0'(0) = 0. \end{gathered}$$

We shall prove in Lemma 4.3 below that, for large $\eta$,

$$\text{(4.19)} \qquad w_0(\eta) = -C\eta + C + o(1)$$

and

$$\text{(4.20)} \qquad w_0'(\eta) = -C + o(1).$$

Now let

$$w(\eta) = w_0(\eta) + z(\eta);$$

then $z$ satisfies the equation

(4.21)

$$z''(\eta) + A(\eta + k(e))^m e^{2C} e^{2w_0(\eta)} \frac{e^{2z(\eta)} - 1}{z(\eta) - 0} z(\eta)$$
$$= -\frac{\varepsilon}{b + (\eta + k(\varepsilon))\varepsilon} w_0'(\eta) - \frac{\varepsilon}{b + (\eta + k(\varepsilon))\varepsilon} z'(\eta)$$
$$- e^{2C} e^{2w_0(\eta)} e^{2z(\eta)} \left[K(b + \{\eta + k(\varepsilon)\}\varepsilon)\left(\frac{1}{\varepsilon}\right)^m e^{20(\varepsilon)} - A(\eta + k(\varepsilon))^m\right],$$
$$z(0) = 0, \qquad z'(0) = 0.$$

For small $\eta$, $z$ and $z'$ are small; thus, suppose that for $0 \le \eta \le C\varepsilon^{-1/4}$ we have $|z(\eta)| \le C$ and $|z'(\eta)| \le C$. Then the right-hand side of (4.21) is $O(\varepsilon)$ for $0 \le \eta \le C\varepsilon^{-1/4}$. If we integrate (4.21), we obtain

$$z(\eta) = -\int_0^\eta (\eta - s)B(s)z(s)\,ds + \int_0^\eta O(\varepsilon)(\eta - s)\,ds,$$

where $O(\varepsilon)$ denotes the right-hand side of (4.21), and

$$B(\eta) = A(\eta + k(\varepsilon))^m e^{2w_0(\eta)} C.$$

Hence

$$\text{(4.22)} \qquad |z(\eta)| \le C\varepsilon \frac{1}{2}\eta^2 + \int_0^\eta (\eta - s)B(s)|z(s)|\,ds,$$

so that, for $0 \le \eta \le C\varepsilon^{-1/4}$, we have

$$\text{(4.23)} \qquad |z(\eta)| \le C\varepsilon^{1/2} + \int_0^\eta (\eta - s)B(s)|z(s)|\,ds.$$

Using a similar argument as in the proof of Lemma 2.9, it is not very difficult to see that for large $\eta$,

$$|z(\eta)| \le \varepsilon^{1/2}(C + C\eta) \quad \text{and} \quad |z'(\eta)| \le \varepsilon^{1/2} C. \tag{4.24}$$

Finally, for $0 \le r - r_0 \le C\varepsilon^{3/4}$, $\eta = (r - r_0)/\varepsilon \le C\varepsilon^{-1/4}$, so from (4.19) and our above estimates

$$\begin{aligned} v(r, \varepsilon) &= v(r_0) + w(\eta) = v(r_0) + w_0(\eta) + z(\eta) \\ &= \frac{m+2}{2}\log\left(\frac{1}{\varepsilon}\right) + C + O(\varepsilon) - C\eta - C + O(\varepsilon^{1/4}) \\ &= \frac{m+2}{2}\log\left(\frac{1}{\varepsilon}\right) + C + O(\varepsilon) - C\left(\frac{r - r_0}{\varepsilon}\right) - C + O(\varepsilon^{1/4}). \end{aligned}$$

Thus,

$$\begin{aligned} v(r_0 + C\varepsilon^{3/4}, \varepsilon) &= \frac{m+2}{2}\log\left(\frac{1}{\varepsilon}\right) + C + O(\varepsilon) - C\varepsilon^{-1/4} + O(\varepsilon^{1/4}) \\ &= -C\varepsilon^{-1/4}\left[1 - \frac{m+2}{2}\frac{1}{C}\varepsilon^{1/4}\log\left(\frac{1}{\varepsilon}\right) + O(\varepsilon^{1/2})\right]. \end{aligned} \tag{4.25}$$

Also, for $r \ge r_0$, since $v(r, \varepsilon) = v(r_0) + w_0(\eta) + z(\eta)$, we have

$$v'(r, \varepsilon) = w_0'(\eta) + z'(\eta).$$

Thus, (4.20) and (4.24) imply that

$$v'(r_0 + C\varepsilon^{3/4}, \varepsilon) = -\frac{1}{\varepsilon}C + O(\varepsilon^{-1/2}). \tag{4.26}$$

Then integrating (4.15) from $r = r_0 + C\varepsilon^{3/4}$ to $r = 1$ gives

$$v'(1, \varepsilon) = v'(r_0 + C\varepsilon^{3/4}, \varepsilon)(r_0 + C\varepsilon^{3/4}) - \int_{r_0 + C\varepsilon^{3/4}}^{1} rK_1(r)e^{2v(r, \varepsilon)}\,dr$$

and so

$$\begin{aligned} v(1, \varepsilon) = {} & v(r_0 + C\varepsilon^{3/4}, \varepsilon) + v'(r_0 + C\varepsilon^{3/4}, \varepsilon)(r_0 + C\varepsilon^{3/4})\log\left(\frac{1}{r_0 + C\varepsilon^{3/4}}\right) \\ & - \int_{r_0 + C\varepsilon^{3/4}}^{1} s\,\log\left(\frac{1}{s}\right) K_1(s)e^{2v(s, \varepsilon)}\,ds. \end{aligned}$$

Using (4.25), (4.26), and the fact that $v'(r, \varepsilon) < 0$ for $r \ge r_0$, we have

$$v'(1, \varepsilon) \le -\frac{1}{\varepsilon}C + O(\varepsilon^{1/2}) \quad \text{and} \quad v(1, \varepsilon) \le -C\varepsilon^{-1/4} - C\varepsilon^{-1}.$$

This implies that

$$\lim_{\varepsilon \to 0} v'(1, \varepsilon) = -\infty = \lim_{\varepsilon \to 0} v(1, \varepsilon);$$

hence $x_1(a_1^* - 0) = -\infty = y_1(a_1^* - 0)$. Similar arguments can be used to prove that

$$x_1(a_i^* - 0) = -\infty = y_1(a_i^* - 0) \quad \text{for all } i \ge 1,$$

and

$$x_1(a_j + 0) = -\infty = y_1(a_j + 0) \quad \text{for all } j \ge 2.$$

Furthermore, from the above analysis and the asymptotic behavior of $v(r, \gamma, \beta^*(\gamma))$ (cf. Theorem 2.15 and the fact that the constant $C$ in the asymptotic expansion of $v(r, \gamma, \beta^*(\gamma))$ is independent of $\gamma$), we can easily deduce that (iv) of Lemma 4.2 holds.

Since $K_1$ is smooth and has only finite-order zeros, $K_1$ has at most a finite number of sign changes. We can use arguments similar to those above to prove the lemma in this more general case; we omit the details. The proof of Lemma 4.2 is considered complete. □

**Lemma 4.3.** *Let $w_0$ be the solution of* (4.18). *Then there exists a constant $C > 0$ such that* (4.19) *and* (4.20) *hold for sufficiently large $\eta$.*

*Proof*. The existence and uniqueness of the solution $w_0$ of (4.18) is well known. Using (4.18), we see that $w_0$ satisfies

$$w_0(\eta) = -\eta \int_0^\eta A(s+k(\varepsilon))^m e^{2C} e^{2w_0(s)}\, ds + \int_0^\eta sA(s+k(\varepsilon))^m e^{2C} e^{2w_0(s)}\, ds. \tag{4.27}$$

We claim that

$$0 < \int_0^\infty A(s+k(\varepsilon))^m e^{2C} e^{2w_0(s)}\, ds < \infty. \tag{4.28}$$

For, suppose (4.28) is false; then

$$\int_0^\infty A(s+k(\varepsilon))^m e^{2C} e^{2w_0(s)}\, ds = \infty.$$

Thus, given any (large) $N > 0$, there is an $\eta_0$ such that

$$\int_0^\eta A(s+k(\varepsilon))^m e^{2C} e^{2w_0(s)}\, ds \geq N \quad \text{for all } \eta \geq \eta_0.$$

But then, for all $\eta \geq \eta_0$,

$$w_0'(\eta) = -\int_0^\eta A(s+k(\varepsilon))^m e^{2C} e^{2w_0(s)}\, ds \leq -N.$$

Hence

$$w_0(\eta) \leq w_0(\eta_0) - N(\eta - \eta_0), \qquad \eta \geq \eta_0,$$

and

$$\begin{aligned} \infty &= \int_0^\infty A(s+k(\varepsilon))^m e^{2C} e^{2w_0(s)}\, ds \\ &\leq \int_0^{\eta_0} A(s+k(\varepsilon))^m e^{2C} e^{2w_0(s)}\, ds \\ &\quad + \int_{\eta_0}^\infty A(s+k(\varepsilon))^m e^{2C} e^{2w_0(\eta_0)} e^{-N(s-\eta_0)}\, ds \\ &< \infty. \end{aligned}$$

This contradiction proves our claim (4.28). From (4.27) and (4.28) we see at once that (4.19) and (4.20) hold. This completes the proof of Lemma 4.3. □

**Lemma 4.4.** *Assume that $K_1$ is smooth, has only finite-order zeros, and, for $r$ near zero, $K_1(r) = Ar^m + O(r^{m+1})$ for some $m > 0$ and $A > 0$. Then either $I_1 = \mathbf{R}$ or $I_1$ is a finite or countably infinite union of open intervals $J_1, J_2, J_3, \dots$, where $J_1 = (-\infty, a_1^*)$, $J_2 = (a_2, \infty)$, and $J_i = (a_i, a_i^*)$ for $i \geq 3$ (if needed). Furthermore, statements* (i), (ii), *and* (iv) *of Lemma* 4.2 *are valid, but* (iii) *of Lemma* 4.2 *is replaced by*

(iii)′ *For* $\alpha \to \infty$,

$$x_1(\alpha) = -\alpha + O(1) \quad \text{and} \quad y_1(\alpha) = -(m+1) + O(e^{-2\alpha/(m+2)}\alpha).$$

*Proof.* The proof of this lemma is almost the same as the proof of Lemma 4.2. The only thing that must be changed is that in order to obtain (iii)′ we must use Theorem 2.11. We omit the details. □

**Lemma 4.5.** *Assume that $K_1$ is smooth, has only finite-order zeros, and $K_1(r) \leq 0$ for $r$ near zero. Then either $I_1 = (-\infty, a^*)$ or $I_1$ is a finite or countably infinite union of open intervals $J_1, J_2, J_3, \dots$, where $J_1 = (-\infty, a_1^*)$, $J_2 = (a_2, a_2^*)$, and $J_i = (a_i, a_i^*)$ for $i \geq 3$ (if needed), where $a_i^* < a_2$ for all $i \geq 3$. Furthermore, statements* (i) *and* (ii) *of Lemma* 4.2 *still hold, but* (iii) *and* (iv) *of Lemma* 4.2 *are replaced by the following:*

(iii)″ *If $I_1 = (-\infty, a^*)$, then either*

$$x_1(a^* - 0) = \infty = y_1(a^* - 0)$$

*or*

$$x_1(a^* - 0) = -\infty = y_1(a^* - 0).$$

(iv)″ *If $I_1 = J_1 \cup J_2 \cup \cdots$, then statement* (iv) *of Lemma* 4.2 *holds except for $a_2^*$, and*

$$x_1(a_2^* - 0) = -\infty = y_1(a_2^* - 0).$$

*Proof.* Since $K_1(r) \leq 0$ for $r$ near 0, we must use Theorem 2.12 to begin the argument. If $K_1(r) \leq 0$ for all $r$ in $[0, 1]$, then Theorem 2.12 implies that $I_1 = (-\infty, a^*)$, and (iii)″ holds with $x_1(a^* - 0) = \infty = y_1(a^* - 0)$. If $K_1(r) \leq 0$ for $r \in [0, a]$, and $K_1(r) \geq 0$ for $r \in [a, 1]$, then we also have $I_1 = (-\infty, a^*)$, and (iii)″ holds with $x_1(a^* - 0) = -\infty = y_1(a^* - 0)$; this last statement can be proved exactly as in the proof of Lemma 4.2. If $K_1$ changes sign twice (or more) in $[0, 1]$, then either (iii)″ or (iv)″ holds. The proof of this is almost the same as in the proof of Lemma 4.2; we omit the details. □

We now make an important remark; namely that Lemmas 4.2, 4,4, and 4.5 have analogous statements when $K_1$ is replaced by $K_2$. The statements (i) still hold as stated, but e.g. in Lemma 4.2, statements (ii), (iii), and the first part of (iv) are to be replaced by the following:

$(\text{ii})_{4.2}$ There exists a $\beta_0 > 0$ such that, for $\beta \leq -\beta_0$, $x_2(\beta) = \beta - O(e^{2\beta})$ and

$$y_2(\beta) = -1 + e^{2\beta} \int_0^1 s\, K_2(s)\, ds + O(e^{4\beta}).$$

$(\text{iii})_{4.2}$ There exists a $\beta_1 > 0$ such that, for $\beta \geq \beta_1$, $x_2(\beta) = -\beta + O(1)$ and either

$$y_2(\beta) = 1 + CK_2'(0)e^{-\beta} + O(\beta^2 e^{-2\beta}) \quad \text{if } K_1'(0) \neq 0,$$

or

$$y_2(\beta) = 1 + CK_2'(0)\beta e^{-2\beta} + O(e^{-2\beta}) \quad \text{if } K_2'(0) = 0 \text{ and } K_2''(0) \neq 0.$$

$(\text{iv})_{4.2}$ If $(b_i^*, b_j) \cap I_2 = \varnothing$ for some $i$ and $j$, we have either

$$x_2(b_i^* - 0) = \infty = x_2(b_j + 0) \quad \text{and} \quad y_2(b_i^* - 0) = -\infty = y_2(b_j + 0),$$

or

$$x_2(b_i^* - 0) = -\infty = x_2(b_j + 0) \quad \text{and} \quad y_2(b_i^* - 0) = \infty = y_2(b_j + 0).$$

Similar changes are to be made in the analogous statements in Lemmas 4.4 and 4.5. To assist the reader, we shall also write these out explicitly. Thus, in Lemma 4.4, statements (i)′ and (iii)′ are to be replaced by the following:

$(\text{ii})'_{4.4}$ Same as in $(\text{ii})_{4.2}$ (above).

$(\text{iii})'_{4.4}$ For $\beta \to \infty$,

$$x_2(\beta) = -\beta + O(1) \quad \text{and} \quad y_2(\beta) = -(m+2) + O\left(\beta e^{-2/(m+2)}\beta\right).$$

In Lemma 4.5, statements (ii)″, (iii)″, and (iv)″ are to be replaced by the following:

$(\text{ii})''_{4.5}$ Same as in $(\text{ii})_{4.2}$ (above).

$(\text{iii})''_{4.5}$ If $I_2 = (-\infty, b^*)$, then either

$$x_2(b^* - 0) = \infty \quad \text{and} \quad y_2(b^*, 0) = -\infty,$$

or

$$x_2(b^* - 0) = -\infty \quad \text{and} \quad y_2(b^*, 0) = \infty.$$

$(\text{iv})''_{4.5}$ If $I_1 = J_1 \cup J_2 \cup \cdots$, then either

$$x_2(b_i^* - 0) = \infty = x_2(b_j + 0) \quad \text{and} \quad y_2(b_i^* - 0) = -\infty = y_2(b_j^* + 0),$$

or

$$x_2(b_i^* - 0) = -\infty = x_2(b_j + 0) \quad \text{and} \quad y_2(b_i^* - 0) = \infty = y_2(b_j + 0).$$

Moreover, $x_2(b_2^* - 0) = -\infty$ and $y_2(b_2^* - 0) = \infty$, and, in all cases,

$$\inf\{\|A_2(\beta_1) - A_2(\beta_2)\| : \beta_1 \in [b_i^* - \delta, b_i^*), \ \beta_2 \in (b_j, b_j + \delta]\} = 0,$$

where $\delta > 0$ is chosen so that $[b_i^* - \delta, b_i^*) \subset J_i$ and $(b_j, b_j + \delta] \subset J_j$, and $\|\cdot\|$ denotes the Euclidean norm on $\mathbf{R}^2$.

We can now give the proof of Theorem B in §1.

*Proof of Theorem* B. From Lemmas 4.2, 4.4, and 4.5, we know that $A_1$ is the union of nonintersecting smooth curves. We can make $A_1$ into a single "continuous" curve as follows. Suppose that $I_1 = (-\infty, a_1^*) \cup (a_j, a_j^*) \cup \cdots$ for some $j$, and $(a_1^*, a_j) \cap I_1 = \varnothing$. Then from Lemmas 4.2, 4.4, and 4.5, we know that

$$x_1(a_1^* - 0) = y_1(a_1^* - 0) = x_1(a_j + 0) = y_1(a_j + 0) = +\infty \text{ or } -\infty$$

and

$$\inf\{\|A_1(\alpha_1) - A_1(\alpha_2)\| : \alpha_1 \in [a_i^* - \delta, a_i^*), \ \alpha_2 \in (a_j, a_j + \delta]\} = 0,$$

where $\delta$ is as described on the above quoted lemmas. Thus, we can identify the $a_1^*$-end of the curve $\{A_1(\alpha) : \alpha \in (-\infty, a_1^*)\}$ with the $a_j$-end of the curve $\{A_1(\alpha) : \alpha \in (a_j, a_j^*)\}$. After similar identifications of different components of $A_1$, we can consider $A_1$ as a "continuous" curve with two free ends. One is the end corresponding to $\alpha \to -\infty$; the other is the end corresponding to $\alpha \to +\infty$

(if $K_1(r) \geq 0$ for small $r$), or is the end corresponding to $\alpha \to a^*$ or $a_2^*$ (if $K_1(r) \leq 0$ for small $r$, as in Lemma 4.5).

Now assume that condition (i) in the statement of the theorem holds, i.e., that $K_1(r) = K_2(r)$ for all $r \in [0, 1]$. It follows that $A_2$ is the "mirror image" of $A_1$ with respect to the "mirror" $y = 0$. Now $K_1$ is positive somewhere; hence the free end of $A_1$ corresponding to the end $\alpha \to \infty$ (or $\alpha \to a^*$) is below the mirror (cf. statements (ii) and (iv) of Lemmas 4.2, 4.4, and 4.5, respectively; note too the proof of Lemma 4.5). The free end of $A_1$ corresponding to $\alpha \to -\infty$ is above the mirror. Hence there exists a point where $A_1$ crosses the mirror. (The reader should draw a picture!) This point must lie on $A_1 \cap A_2$, so that (i) holds, in view of Theorem 4.1.

(ii) Suppose now that $K_1(0) > 0$, $K_2(0) > 0$, and $K_1'(0)K_2'(0) > 0$. Assume too that $K_1'(0) > 0$ and $K_2'(0) > 0$ (the proof for the other case is similar, and will be omitted). In this case, it follows from (ii) in Lemma 4.2 and its analogue (ii)′ given above (before the beginning of the proof of Theorem B) that the "free end" of the curve $A_1$ corresponding to $\alpha \to -\infty$ is always *below* the "free end" of the curve $A_2$ corresponding to $\beta \to +\infty$. Moreover, both of these "free ends" approach the line $y = 1$ at the $x = -\infty$ end. Similarly, the "free end" of the curve $A_1$ corresponding to $\alpha \to +\infty$ is also *below* the "free end" of the curve $A_2$ corresponding to $\beta \to -\infty$. These two "free ends" both approach the line $y = -1$ at the $x = -\infty$ end. It follows easily from this (draw a picture!) that $A_1 \cap A_2 \neq \varnothing$. Similar arguments can be applied to the case $K_1'(0) < 0$ and $K_2'(0) < 0$. Thus statement (ii) is proved.

Similar arguments can be used to prove statements (iii) and (iv).

(v) Suppose $K_1(0) > 0$, $K_1'(0) > 0$, and $K_2(0) \leq 0$. Then from Lemmas 4.2, 4.4, and 4.5 (and their analogous statements concerning $A_2$) we know that the free end of the curve $A_1$ corresponding to $\alpha \to \infty$ is always *below* the free end of the curve $A_2$ corresponding to $\beta \to -\infty$. Both of these ends approach $y = -1$ as $x \to -\infty$. Moreover, the free end of the curve $A_1$ corresponding to $\alpha \to -\infty$ is *below* the free end of the curve $A_2$ if this free end of the curve $A_2$ also approaches $y = 1$ and $x \to -\infty$ or this free end approaches $(-\infty, \infty)$. In the other case, the free end of curve $A_2$ approaches $(\infty, -\infty)$. In every case, (draw a picture!) $A_1 \cap A_2 \neq \varnothing$; this proves (v).

Similar arguments can be used to prove (vi), and (vii) requires no proof. Finally, we prove (viii). Thus, suppose that $\max\{K_1(0), K_2(0)\} \leq 0$ and that $K$ is positive somewhere. Now from Lemmas 4.4(ii) and 4.5(ii) and their above analogues for $A_2$, we know that we must *always* have

$$\lim_{\alpha \to -\infty} A_1(\alpha) = (-\infty, 1) \quad \textit{and} \quad \lim_{\beta \to -\infty} A_2(\beta) = (-\infty, -1). \tag{4.29}$$

Now we shall obtain the asymptotic behavior of the "other" free end of both $A_1(\alpha)$ and $A_2(\beta)$. For this we introduce the notation

$$A_i^+ = (\overline{x}_i, \overline{y}_i), \qquad i = 1, 2,$$

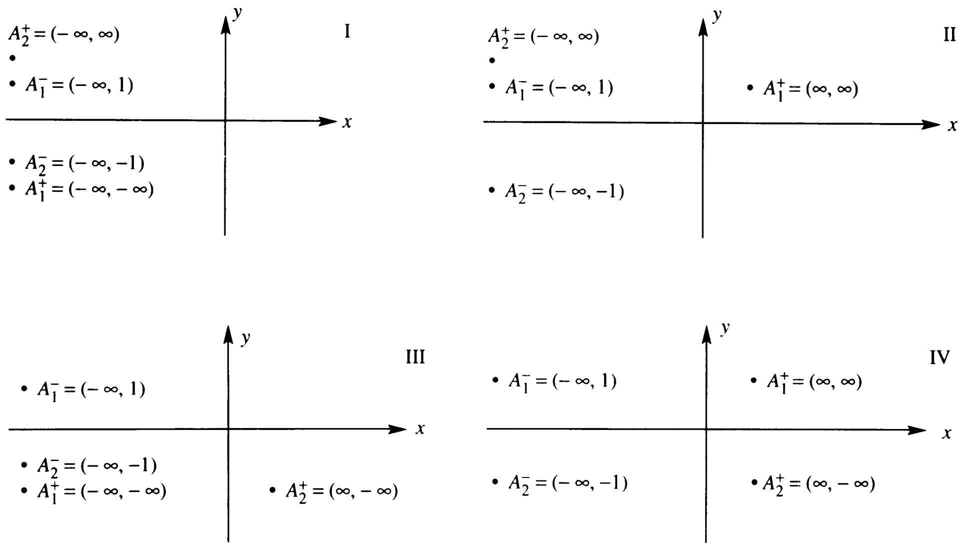
to denote the asymptotic limit of the curve $A_i$ at the "other" free end of $A_i$; that is, $A_i^+ = (\overline{x}_i, \overline{y}_i)$ means that $A_i$ tends to $(\overline{x}_i, \overline{y}_i)$ at the "other" free end; note that $\overline{x}_i$ and $\overline{y}_i$ can take on the values $\pm\infty$! Similarly, we rewrite (4.29) as

$$A_1^- = (-\infty, 1), \qquad A_2^- = (-\infty, -1). \tag{4.30}$$

**Lemma 4.6.** *$A_1^+ = (\infty, \infty)$ and $A_2^+(\infty, -\infty)$ cannot be simultaneously true.*

*Proof.* If $A_1^+ = (\infty, \infty)$, then Lemma 4.5(iii) and (iv) show that $K_1(r) \le 0$ for all $r$ in $[0, 1]$. Similarly if $A_2^+ = (\infty, -\infty)$, (iii)$''_{4.5}$ and (iv)$''_{4.5}$ show that $K_2(r) \le 0$ for all $r$ in $[0, 1]$. Since $K(r) > 0$ somewhere, we must have $K_1(r) > 0$ somewhere, or $K_2(r) > 0$ somewhere, and this completes the proof.

Now using (4.29) together with (iii)$'_{4.4}$, (iii)$''_{4.5}$, and(iv)$''_{4.5}$, we have either $A_2^+ = (-\infty, \infty)$ (so that $A_2^+ > A_1^-$), or $A_2^+ = (\infty, -\infty)$. Thus, there are only four possibilities for the asymptotic behavior of the two free ends of both $A_1(\alpha)$ and $A_2(\beta)$; these are depicted as follows:




However our lemma shows that case IV cannot occur. It is easy to check that in each of the remaining cases we have $A_1 \cap A_2 \neq \varnothing$. This completes the proof of Theorem B.

## Acknowledgment

Most of the results in this paper were completed during the period 1987–1988 when one of the authors (Cheng) was visiting the University of Michigan. He wants to thank the Department of Mathematics for partial support during his stay in the department.

## References

1. S. Y. A. Chang and P. C. Yang, *Prescribing Gaussian curvature on* $S^2$, Acta Math. **159** (1987), 216–259.
2. ——, *Conformal deformation of metrics on* $S^2$, J. Differential Geom. **27** (1988), 259–296.
3. W.-X. Chen and W.-Y. Ding, *Scalar curvature on* $S^2$, Trans. Amer. Math. Soc. **303** (1987), 365–382.
4. K.-S. Cheng and J.-T. Lin, *On the elliptic equations* $\Delta u = K(x)u^\sigma$ *and* $\Delta u = K(x)e^{2u}$, Trans. Amer. Math. Soc. **304** (1987), 639–668.
5. C.-W. Hong, *A best constant and the Gaussian curvature*, Proc. Amer. Math. Soc. **97** (1986), 737–747.

6. J. Kazdan and F. Warner, *Curvature functions for compact* 2*-manifolds*, Ann. of Math. (2) **99** (1974), 14–47.

7. J. Moser, *On a nonlinear problem in differential geometry*, Dynamical Systems (M. Peixoto, ed.), Academic Press, New York, 1973.

8. W.-M. Ni, *On the elliptic equation* $\Delta u + K(x)e^{2u} = 0$ *and conformal metrics with prescribed Gaussian curvatures*, Invent. Math. **66** (1982), 343–352.

9. D. H. Sattinger, *Conformal metrics in* $\mathbf{R}^2$ *with prescribed curvature*, Indiana Univ. Math. J. **22** (1972),1–4.

INSTITUTE OF APPLIED MATHEMATICS, NATIONAL CHUNG CHENG UNIVERSITY, MING-HSUING, CHIA-YI 62117, TAIWAN

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MICHIGAN, ANN ARBOR, MICHIGAN 48109-1003