

GL(4, \mathbb{R})-WHITTAKER FUNCTIONS AND ${}_4F_3(1)$ HYPERGEOMETRIC SERIES

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ABSTRACT. In this paper we consider spaces of GL(4, \mathbb{R})-Whittaker functions, which are special functions that arise in the study of GL(4, \mathbb{R}) automorphic forms. Our main result is to determine explicitly the series expansion for a GL(4, \mathbb{R})-Whittaker function that is “fundamental,” in that it may be used to generate a basis for the space of all GL(4, \mathbb{R})-Whittaker functions of fixed eigenvalues.

The series that we find in the case of GL(4, \mathbb{R}) is particularly interesting in that its coefficients are not merely ratios of Gamma functions, as they are in the lower-rank cases. Rather, these coefficients are themselves certain series—namely, they are finite hypergeometric series of unit argument. We suspect that this is a fair indication of what will happen in the general case of GL(n , \mathbb{R}).

1. PRELIMINARIES; OVERVIEW OF THE PROBLEM

We begin by discussing Whittaker functions, and their relation to automorphic forms, in the setting of GL(n , \mathbb{R}). Our results in this paper will concern primarily the group GL(4, \mathbb{R}), yet we desire sufficient generality to be able to compare these results with the known theory in cases of smaller rank.

Let $X \subset \text{GL}(n, \mathbb{R})$ be the group of upper triangular, unipotent matrices: if $x \in X$, we write $x = (x_{ij})$. Also let $Y \subset \text{GL}(n, \mathbb{R})$ be the group of diagonal matrices y of the form

$$y = \{\text{diag}(y_1 y_2 \cdots y_{n-1}, y_2 y_3 \cdots y_{n-1}, \dots, y_{n-1}, 1) \mid y_i > 0 \text{ for all } i\}.$$

Now consider the “generalized upper half-plane”

$$\mathcal{H}^n = \text{GL}(n, \mathbb{R}) / (\text{O}(n, \mathbb{R}) \cdot \mathbb{R}^\times)$$

($\text{O}(n, \mathbb{R})$ is the orthogonal group) as a left homogeneous space for GL(n , \mathbb{R}). By the Iwasawa decomposition, every $z \in \mathcal{H}^n$ has a unique representation $z \equiv xy \pmod{\text{O}(n, \mathbb{R}) \cdot \mathbb{R}^\times}$ with $x \in X$, $y \in Y$.

In this paper we will be concerned with eigenfunctions of the algebra D of GL(n , \mathbb{R})-invariant differential operators on \mathcal{H}^n . One such eigenfunction is given (cf. [11]), for $\nu = (\nu_1, \nu_2, \dots, \nu_{n-1}) \in \mathbb{C}^{n-1}$, by

$$H_\nu(z) = \prod_{i=1}^{n-1} \prod_{j=1}^{n-1} y_i^{b_{ij} \nu_j},$$

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where $b_{ij} = \min\{ij, (n-i)(n-j)\}$. We denote the eigenvalues of H_ν by $\lambda_\nu(d)$ —that is, $dH_\nu = \lambda_\nu(d)H_\nu$ for all $d \in D$ —and have the following. Let \mathscr{W} be the Weyl group (identified with the group of $n \times n$ permutation matrices) of $\mathrm{GL}(n, \mathbb{R})$. Also, if $\rho = (\rho_1, \rho_2, \dots, \rho_{n-1}) \in \mathbb{C}^{n-1}$, let $\rho - \frac{1}{n}$ denote the element $(\rho_1 - \frac{1}{n}, \rho_2 - \frac{1}{n}, \dots, \rho_{n-1} - \frac{1}{n})$. If we define an action $\omega(\nu)$ of \mathscr{W} on \mathbb{C}^{n-1} by requiring, for each $\omega \in \mathscr{W}$, that

$$H_{\nu - \frac{1}{n}}(y) = H_{\omega(\nu) - \frac{1}{n}}(\omega y)$$

for all $y \in Y$, then the eigenvalues $\lambda_\nu(d)$ are found to be *invariant* under this action. That is, we have $\lambda_{\omega(\nu)}(d) = \lambda_\nu(d)$ for all $\omega \in \mathscr{W}$, $d \in D$.

A less simple type of eigenfunction of D may be defined as follows. Let Θ be the character of X given by

$$\Theta(x) = e(x_{1,2} + x_{2,3} + \dots + x_{n-1,n}),$$

where $e(t)$ denotes $e^{2\pi it}$. We then make the following

Definition 1.1. A $\mathrm{GL}(n, \mathbb{R})$ -Whittaker function is a function $f_\nu(z)$, smooth on \mathscr{X}^n and meromorphic on \mathbb{C}^{n-1} (with polar divisors that are independent of z), such that

- (a) $df_\nu = \lambda_\nu(d)f_\nu \forall d \in D$;
- (b) $f_\nu(x_1 z) = \Theta(x_1)f_\nu(z) \forall x_1 \in X, z \in \mathscr{X}^n$

(of course, (a) should hold for all ν in the domain of analyticity of f_ν). The theory of Whittaker functions (for general reductive groups) was initiated by Jacquet [5].

It follows from (unpublished) work of Casselman and Zuckerman, and independent work of Kostant [6], that the space S_ν of $\mathrm{GL}(n, \mathbb{R})$ -Whittaker functions with fixed eigenvalues $\lambda_\nu(d)$ has, for almost all values of ν , dimension $n!$. Moreover, it is shown by Hashizume [4] that S_ν is spanned by translates of a certain “fundamental” Whittaker function $M_\nu(z)$. Specifically, if $K = (k_1, k_2, \dots, k_{n-1})$, then there exist complex coefficients $G_K(\nu)$ and a $\mathrm{GL}(n, \mathbb{R})$ -Whittaker function

$$(1.1) \quad M_\nu(z) = \Theta(x)H_\nu(z) \sum_{k_1, k_2, \dots, k_{n-1}=0}^{\infty} G_K(\nu) (\pi y_1)^{2k_1} (\pi y_2)^{2k_2} \dots (\pi y_{n-1})^{2k_{n-1}}$$

such that the set $\{M_{\omega(\nu)}(z) \mid \omega \in \mathscr{W}\}$ spans S_ν . (Actually, Hashizume’s paper treats the more general situation of Whittaker functions on a certain class of semisimple Lie groups.) Hashizume also gives (for each n) an explicit difference equation, in the indices k_1, k_2, \dots, k_{n-1} , that the $G_K(\nu)$ ’s solve uniquely (up to constants). The determination of $M_\nu(z)$, and hence S_ν , then requires only the solution of this equation.

Before discussing what is already known and what we wish to show regarding the $G_K(\nu)$ ’s, we recall briefly the relevance of S_ν and $M_\nu(z)$ to the study of automorphic forms. First, let φ be an automorphic function for $\Gamma = \mathrm{GL}(n, \mathbb{Z})$; that is, $\varphi \in C^0(\mathscr{X}^n)$ and $\varphi(z)$ is invariant under the left action of Γ on \mathscr{X}^n . Then φ has a “Fourier expansion” (cf. [10, 12]). If also φ is an eigenfunction of type ν —that is, we have $d\varphi = \lambda_\nu(d)\varphi$ for all $d \in D$ —then we find that each Fourier coefficient of φ is given by some Whittaker function in S_ν (actually

the expansion of φ may also contain some “degenerate” Fourier coefficients, which involve eigenfunctions of D that do not satisfy condition (b) of Definition 1.1). If in addition φ is at most polynomial in each y_i (in which case we say that φ is an automorphic form of type ν), each Whittaker function so arising must in fact be a constant multiple of a fixed Whittaker function $W_\nu(z) \in \mathcal{S}_\nu$. (The uniqueness of $W_\nu(z)$ follows from a local multiplicity-one theorem of Shalika [12].) $W_\nu(z)$ is often called a “class one principal series” Whittaker function, and corresponds to a certain principal series representation of $\mathrm{GL}(n, \mathbb{R})$ induced from the subgroup XY (cf. [3]).

The automorphic forms whose expansions contain *no* degenerate terms are called *cuspidal forms*. If $\mathcal{L}^2(\Gamma \backslash \mathcal{H}^n)$ denotes the space of automorphic functions that are square-integrable (with respect to the $\mathrm{GL}(n, \mathbb{R})$ -invariant measure) over a fundamental domain for Γ in \mathcal{H}^n , then the eigenvalues of the cusp forms constitute the discrete spectrum of the algebra D acting on $\mathcal{L}^2(\Gamma \backslash \mathcal{H}^n)$. Thus these forms (and consequently the function $W_\nu(z)$) are central to the spectral theory of $\mathcal{L}^2(\Gamma \backslash \mathcal{H}^n)$.

Much explicit information regarding $W_\nu(z)$ is already known (cf., for example, [7] for the case $n = 2$, [1, 16] for the case $n = 3$, and [5, 15] for the general case), and has been obtained without specific knowledge of $M_\nu(z)$. On the other hand, our understanding of the former Whittaker function is far from complete, and can only be aided by better comprehension of the latter.

There exists also a more direct relation between $M_\nu(z)$ and automorphic forms: namely, consider the Poincaré series

$$P(z; \nu) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} M_\nu(\gamma z),$$

where $\Gamma_\infty = \Gamma \cap X$. This series has been studied when $n = 2$ (cf. [8, 9]) and $n = 3$ (cf. [14]) and is itself intimately related to the spectral theory of $\mathcal{L}^2(\Gamma \backslash \mathcal{H}^n)$. (When $n = 3$, the series does not actually converge, but makes sense as a linear functional on spaces of cusp forms.) Specifically, there is found in either case a correspondence between the poles of $P(z; \nu)$ and the eigenvalues of cusp forms in $\mathcal{L}^2(\Gamma \backslash \mathcal{H}^n)$ (there remain many open questions regarding the location of these eigenvalues; such questions have motivated much work in the field of automorphic forms). Moreover, a certain linear combination (over $\omega \in \mathcal{W}$) of the $P(z; \omega(\nu))$ ’s yields an “Eisenstein series”; the latter is an automorphic form (of type ν) whose eigenvalues belong to the *continuous* spectrum of D acting on $\mathcal{L}^2(\Gamma \backslash \mathcal{H}^n)$.

In developing the above theory of $P(z; \nu)$, when $n = 2$ or $n = 3$, one utilizes explicit expressions for the coefficients $G_K(\nu)$ of $M_\nu(z)$. The difference equations defining these coefficients have been solved by Whittaker and Watson [17] (when $n = 2$) and by Bump [1] (when $n = 3$); we briefly recall these results. If, for $e \in \mathbb{C}$, we define

$$(1.2) \quad (e)_k = (k + e - 1)(k + e - 2) \cdots (1 + e)e \quad (k \in \mathbb{Z}^+); \quad (e)_0 = 1,$$

then we have

$$G_K(\nu) = \frac{1}{k_1! (\nu_1 + \frac{1}{2})_{k_1}}$$

when $n = 2$ and

$$G_K(\nu) = \frac{\left(\frac{3\nu_1+3\nu_2}{2}\right)_{k_1+k_2}}{k_1! k_2! \left(\frac{3\nu_2+1}{2}\right)_{k_1} \left(\frac{3\nu_1+3\nu_2}{2}\right)_{k_1} \left(\frac{3\nu_1+1}{2}\right)_{k_2} \left(\frac{3\nu_1+3\nu_2}{2}\right)_{k_2}}$$

when $n = 3$. Note that $G_K(\nu)$ may, in the above two cases, be interpreted as a ratio of terms involving Euler's Gamma function $\Gamma(s)$ (cf. [17]). This is so because, from the functional equation $\Gamma(s+1) = s\Gamma(s)$, we find that

$$(1.3) \quad (e)_k = \frac{\Gamma(e+k)}{\Gamma(e)}.$$

(The Gamma function actually has simple poles at the nonpositive integers; if e is such a number then the right-hand side of (1.3) is equal either to 0 or to the appropriate residue.) We also have $k! = \Gamma(k+1)$.

In this paper we show that, for $n = 4$, the coefficient $G_K(\nu)$ is *not* a ratio of Gamma functions, but is instead a terminating hypergeometric series of unit argument (see §2 for the definitions and basic properties of hypergeometric series, and §3 for the precise statement and proof of our result). This generalizes the behavior of $G_K(\nu)$ in the lower-rank cases, in that such a hypergeometric series may (as is clear from the definitions in §2) be regarded as a *finite sum* of ratios of Gamma functions. It seems likely to us that, for $n > 4$, the coefficients $G_K(\nu)$ will also involve terminating hypergeometric series that will be more complicated than those arising for $n = 4$. In particular, preliminary evidence in the case $n = 5$ suggests that these series may involve several indices of summation. Such “multiple” hypergeometric series are often called “Lauricella functions”; we will not discuss these functions in this paper). In any case, the determination of the $G_K(\nu)$'s for general values of n should be relevant to the theory of Poincaré series as described above, as well as to more general questions in the theory of automorphic forms.

For the remainder of this paper we restrict our attention to the case $n = 4$.

2. $\mathrm{GL}(4, \mathbb{R})$: THE DIFFERENCE EQUATION FOR THE COEFFICIENTS $G_K(\nu)$

We begin by writing the difference equation that defines, in the case $n = 4$, the coefficients $G_K(\nu)$ of $M_\nu(z)$. This appears as formula (4.1) in [4], and is given in terms of the simple roots of the Lie algebra of $\mathrm{GL}(4, \mathbb{R})$. Actually, the formula is stated there in much greater generality, but we may easily specialize to the situation at hand. The equation that we get is

$$(2.1a) \quad P_K(\nu) G_K(\nu) = G_{K(1)}(\nu) + G_{K(2)}(\nu) + G_{K(3)}(\nu),$$

where $K = (k_1, k_2, k_3) \in (\mathbb{Z}^+)^3$, $K(1) = (k_1 - 1, k_2, k_3)$, $K(2) = (k_1, k_2 - 1, k_3)$, $K(3) = (k_1, k_2, k_3 - 1)$, and

$$(2.1b) \quad \begin{aligned} P_K(\nu) = & k_1^2 + k_2^2 + k_3^2 - 2k_1k_2 - 2k_2k_3 + (2\nu_3 - \tfrac{1}{2})k_1 \\ & + (2\nu_2 - \tfrac{1}{2})k_2 + (2\nu_1 - \tfrac{1}{2})k_3. \end{aligned}$$

Here we are assuming that $G_{K(i)}(\nu) = 0$ if $k_i = 0$; we then see that (2.1) determines the coefficients $G_K(\nu)$ uniquely up to a constant (that depends on the choice of $G_{(0,0,0)}(\nu)$).

The solution of (2.1) will be the concern of the next section. There we will see that $G_K(\nu)$ is essentially equal to a “terminating, Saalschutzhian hypergeometric

series of type ${}_4F_3(1)$." Such a series arises as a Taylor coefficient of a product of "Gauss functions," and in fact we will use known identities for Gauss functions to show that our choice of $G_K(\nu)$ does solve (2.1). We will need some basic facts, then, concerning the entities just mentioned; unless otherwise stated, all of the results in the remainder of this section are from Slater [13].

If $e_1, e_2, \dots, e_n, f_1, f_2, \dots, f_{n-1}, z \in C$, we define

$${}_nF_{n-1}(e_1, e_2, \dots, e_n; f_1, f_2, \dots, f_{n-1}; z) = \sum_{k=0}^{\infty} \frac{(e_1)_k (e_2)_k \cdots (e_n)_k}{k! (f_1)_k (f_2)_k \cdots (f_{n-1})_k} z^k,$$

where $(e)_k$ is given by (1.2). If we are to avoid poles we need to assume that no f_i is a negative integer or zero; on the other hand, if one of the e_i 's is a nonpositive integer, then the series *terminates* (since then $(e_i)_k$ will be zero for $k > -e_i$).

The above series is called a "hypergeometric series of type ${}_nF_{n-1}$ "; if $z = 1$ it is said to be "of type ${}_nF_{n-1}(1)$." We have absolute convergence for $|z| < 1$, or for $|z| = 1$ and $\operatorname{Re}(\sum f_i - \sum e_i) > 0$; if $\operatorname{Re}(\sum f_i - \sum e_i) = 1$ then the series is said to be "Saalschutzian."

We will be concerned with series of type ${}_4F_3$ and ${}_2F_1$. The latter is often called a "Gauss function"; we will follow the standard practice of writing F for ${}_2F_1$. Of particular importance to us will be the relation between ${}_4F_3$ and the product of two Gauss functions: namely,

$$\begin{aligned} F(a, b; c; z)F(a', b'; c'; z) \\ (2.2) \quad = \sum_{k_2=0}^{\infty} \frac{(a)_{k_2}(b)_{k_2}}{k_2!(c)_{k_2}} z^{k_2} {}_4F_3(a', b', 1 - c - k_2, -k_2; \\ c', 1 - a - k_2, 1 - b - k_2; 1) \end{aligned}$$

for $|z| < 1$ (cf. [2, p. 187]). This identity may be proved by comparing coefficients of z^{k_2} on either side. Note that the ${}_4F_3(1)$ series on the right is Saalschutzian if and only if $a + b - c = c' - a' - b'$.

3. SOLUTION OF THE DIFFERENCE EQUATION FOR THE $G_K(\nu)$'s

In this section we wish to solve the difference equation (2.1). We will find a solution $G_K(\nu)$ that is, up to some factors to be described below, a terminating, Saalschutzian hypergeometric series of type ${}_4F_3(1)$. The arguments of this series will involve the parameter ν as well as the index K .

Specifically, let us put $\mu = 2\nu_1 + 2\nu_2 + 2\nu_3 - 3/2$. For the remainder of this paper, we will let

$$(3.1) \quad \begin{aligned} a &= -k_1 - 2\nu_3 + 1/2; & b &= -k_3 - 2\nu_1 + 1/2; & c &= 2\nu_2 + 1/2; \\ a' &= k_1 + \mu + 1; & b' &= k_3 + \mu + 1; & c' &= \mu + 1; \end{aligned}$$

note that $a + b - c = c' - a' - b'$. We also introduce, for these values of $a, b, c; a', b', c'$, the notation

$$(3.2) \quad {}_4F_3^*[K] = \frac{(a)_{k_2}(b)_{k_2}}{k_2!(c)_{k_2}} {}_4F_3(a', b', 1 - c - k_2, -k_2; c', 1 - a - k_2, 1 - b - k_2; 1)$$

for $k_2 = 0, 1, 2, \dots$; if $k_2 = -1$, we let ${}_4F_3^*[K] = 0$. We then have

Theorem 3.1. *The coefficient*

$$(3.3) \quad G_K(\nu) = \frac{(-1)^{k_1+k_3}}{k_1! k_3! (a)_{k_1} (b)_{k_3} (a+a')_{k_2} (b+b')_{k_2}} {}_4F_3^*[K]$$

is, up to a constant, the unique solution to (2.1).

Remark. The right-hand side of (3.3) is clearly meromorphic in ν_1 , ν_2 , and ν_3 . We will assume throughout, then, that these variables are chosen to avoid any poles.

Proof of Theorem 3.1. Note that, according to (3.2), equation (2.2) reads

$$(3.4) \quad F(a, b; c; z) F(a', b'; c'; z) = \sum_{k_2=0}^{\infty} z^{k_2} {}_4F_3^*[K].$$

This equation will allow us to prove Theorem 3.1 by proving an analogous identity for a product of two Gauss functions. Namely, our theorem is clearly implied by the following two propositions.

Proposition 3.1. *Let D denote the differential operator $z \frac{d}{dz}$, and*

$$(3.5a) \quad \begin{aligned} \Delta = & [a(c' - a') + b(c' - b') - z(a + a')(b + b')] \\ & + [(c' + a + b - 1) - z(a + a' + b + b')]D + [1 - z]D^2. \end{aligned}$$

The coefficient $G_K(\nu)$ above solves (2.1) if and only if

$$(3.5b) \quad \begin{aligned} \Delta \{ & F(a, b; c; z) F(a', b'; c'; z) \} \\ = & a(c' - a')F(a + 1, b; c; z)F(a' - 1, b'; c'; z) \\ & + b(c' - b')F(a, b + 1; c; z)F(a', b' - 1; c'; z). \end{aligned}$$

Proof. We begin by noting that

$$(e + 1)_{k-1} = \frac{(e)_k}{e} \quad \text{and} \quad (e)_{k-1} = \frac{(e)_k}{k + e - 1};$$

these identities follow readily from the definitions. From equation (3.3), then, we see that

$$\begin{aligned} G_{K(1)}(\nu) &= \frac{(-1)^{(k_1-1)+k_3}}{(k_1-1)! k_3! (a+1)_{k_1-1} (b)_{k_3} (a+a')_{k_2} (b+b')_{k_2}} {}_4F_3^*[K(1)] \\ &= \frac{-ak_1 (-1)^{k_1+k_3}}{k_1! k_3! (a)_{k_1} (b)_{k_3} (a+a')_{k_2} (b+b')_{k_2}} {}_4F_3^*[K(1)]; \\ G_{K(2)}(\nu) &= \frac{(-1)^{k_1+k_3}}{k_1! k_3! (a)_{k_1} (b)_{k_3} (a+a')_{k_2-1} (b+b')_{k_2-1}} {}_4F_3^*[K(2)] \\ &= \frac{(k_2+a+a'-1)(k_2+b+b'-1)(-1)^{k_1+k_3}}{k_1! k_3! (a)_{k_1} (b)_{k_3} (a+a')_{k_2} (b+b')_{k_2}} {}_4F_3^*[K(2)]; \\ G_{K(3)}(\nu) &= \frac{(-1)^{k_1+(k_3-1)}}{k_1! (k_3-1)! (a)_{k_1} (b+1)_{k_3-1} (a+a')_{k_2} (b+b')_{k_2}} {}_4F_3^*[K(3)] \\ &= \frac{-bk_3 (-1)^{k_1+k_3}}{k_1! k_3! (a)_{k_1} (b)_{k_3} (a+a')_{k_2} (b+b')_{k_2}} {}_4F_3^*[K(3)]. \end{aligned}$$

We also find readily that

$$P_K(\nu) = k_2^2 + k_2(c' + a + b - 1) + a(c' - a') + b(c' - b'),$$

where $P_K(\nu)$ is as in (2.1b). Putting this information, along with the expression (3.3) for $G_K(\nu)$, into the difference equation (2.1a) (and cancelling where possible) yields the equation

$$\begin{aligned} & [k_2^2 + k_2(c' + a + b - 1) + a(c' - a') + b(c' - b')] {}_4F_3^*[K] \\ &= -ak_1 {}_4F_3^*[K(1)] + (k_2 + a + a' - 1)(k_2 + b + b' - 1) \\ (3.6) \quad & \times {}_4F_3^*[K(2)] - bk_3 {}_4F_3^*[K(3)] \\ &= -a(a' - c') {}_4F_3^*[K(1)] + (k_2 + a + a' - 1)(k_2 + b + b' - 1) \\ & \times {}_4F_3^*[K(2)] - b(b' - c') {}_4F_3^*[K(3)] \end{aligned}$$

since $k_1 = a' - c'$ and $k_3 = b' - c'$. (We check the equivalence of (2.1) and (3.6) even if some k_i is zero: if this happens then $G_{K(i)}(\nu) = 0$, so we want the ${}_4F_3^*[K(i)]$ -term on the right-hand side of (3.6) to vanish. But if $k_i = 0$ for $i = 1$ or $i = 3$ then the coefficient of ${}_4F_3^*[K(i)]$ in (3.6) is zero, as desired. If $k_2 = 0$ then ${}_4F_3^*[K(2)]$ is itself zero, by assumption.) We now need to show that (3.5) is equivalent to (3.6).

So let us assume that (3.5) is true. Since the replacement of a by $a + 1$ and a' by $a' - 1$ is equivalent to the substitution $k_1 \rightarrow k_1 - 1$, we get from (3.4)

$$F(a + 1, b; c; z) F(a' - 1, b'; c'; z) = \sum_{k_2=0}^{\infty} z^{k_2} {}_4F_3^*[K(1)].$$

Similarly,

$$F(a, b + 1; c; z) F(a', b' - 1; c'; z) = \sum_{k_2=0}^{\infty} z^{k_2} {}_4F_3^*[K(3)].$$

Then (3.5b) may be rewritten as

$$\begin{aligned} (3.7) \quad & \left\{ [a(c' - a') + b(c' - b') - z(a + a')(b + b')] \right. \\ & \left. + [(c' + a + b - 1) - z(a + a' + b + b')]D + [1 - z]D^2 \right\} \sum_{k_2=0}^{\infty} z^{k_2} {}_4F_3^*[K] \\ &= a(c' - a') \sum_{k_2=0}^{\infty} z^{k_2} {}_4F_3^*[K(1)] + b(c' - b') \sum_{k_2=0}^{\infty} z^{k_2} {}_4F_3^*[K(3)]. \end{aligned}$$

It is clear that

$$D^n \left(\sum_{k_2=0}^{\infty} z^{k_2} {}_4F_3^*[K] \right) = \sum_{k_2=0}^{\infty} k_2^n z^{k_2} {}_4F_3^*[K]$$

for a positive integer n , so (3.7) becomes

$$\begin{aligned}
 (3.8) \quad & \sum_{k_2=0}^{\infty} \left\{ [a(c' - a') + b(c' - b') - z(a + a')(b + b')] \right. \\
 & \quad \left. + [(c' + a + b - 1) - z(a + a' + b + b')]k_2 + [1 - z]k_2^2 \right\} z^{k_2} {}_4F_3^*[K] \\
 & = a(c' - a') \sum_{k_2=0}^{\infty} z^{k_2} {}_4F_3^*[K(1)] + b(c' - b') \sum_{k_2=0}^{\infty} z^{k_2} {}_4F_3^*[K(3)].
 \end{aligned}$$

Since, for any function $H(k_2)$ such that $H(-1) = 0$, we have

$$\sum_{k_2=0}^{\infty} z^{k_2+1} H(k_2) = \sum_{k_2=0}^{\infty} z^{k_2} H(k_2 - 1),$$

we may rewrite (3.8) by collecting terms with an extra power of z :

$$\begin{aligned}
 (3.9) \quad & \sum_{k_2=0}^{\infty} [a(c' - a') + b(c' - b') + (c' + a + b - 1)k_2 + k_2^2] z^{k_2} {}_4F_3^*[K] \\
 & + \sum_{k_2=0}^{\infty} [-(a + a')(b + b') - (a + a' + b + b') \\
 & \quad \times (k_2 - 1) - (k_2 - 1)^2] z^{k_2} {}_4F_3^*[K(2)] \\
 & = a(c' - a') \sum_{k_2=0}^{\infty} z^{k_2} {}_4F_3^*[K(1)] + b(c' - b') \sum_{k_2=0}^{\infty} z^{k_2} {}_4F_3^*[K(3)].
 \end{aligned}$$

Moving the second summation in (3.9) from the left to the right side of the equality and then comparing coefficients of z^{k_2} on either side, we get exactly equation (3.6).

The above sequence of equalities may easily be rearranged to show that (3.6) implies (3.5) as well. \square

To complete the proof of Theorem 3.1, we need only to show that equation (3.5) is true. We will prove this in Proposition 3.2; before proceeding, we derive some basic differential properties of $F(a, b; c; z)F(a', b'; c'; z)$.

First of all, we find from [13, equations (1.4.1.1) and (1.4.16) respectively], that

$$\begin{aligned}
 \frac{d}{dz} F(e, f; g; z) &= \frac{ef}{g} F(e + 1, f + 1; g + 1; z); \\
 (g - e)(g - f) F(e, f; g + 1; z) &= g(g - e - f) F(e, f; g; z) \\
 &\quad + ef(1 - z) F(e + 1, f + 1; g + 1; z)
 \end{aligned}$$

for arbitrary e, f, g . Combining these, we find that

$$\begin{aligned}
 \frac{d}{dz} F(e, f; g; z) &= \frac{1}{g(1 - z)} [(g - e)(g - f) F(e, f; g + 1; z) - g(g - e - f) F(e, f; g; z)].
 \end{aligned}$$

Then the product rule gives us, for arbitrary $e, f, g; e', f', g'$,

$$\begin{aligned}
 (3.10) \quad & \frac{d}{dz} F(e, f; g; z) F(e', f'; g'; z) \\
 &= \frac{(g-e)(g-f)}{g(1-z)} F(e, f; g+1; z) F(e', f'; g'; z) \\
 &+ \frac{(g'-e')(g'-f')}{g'(1-z)} F(e, f; g; z) F(e', f'; g'+1; z) \\
 &- \frac{(g-e-f+g'-e'-f')}{(1-z)} F(e, f; g; z) F(e', f'; g'; z).
 \end{aligned}$$

The completion of our proof will follow from (3.10) with little difficulty; however, some of the formulas involved will be lengthy. To minimize this, we introduce some simplifying notation: we write F for $F(a, b; c; z)$ and F' for $F(a', b'; c'; z)$. Moreover, should a parameter of either of these series be incremented, this will be denoted by appending to the symbol for the series the new parameter in square-brackets: for example $F(a, b; c+1; z)$ will be written $F[c+1]$; $F(a', b'-1, c; z)$ will be written $F'[b'-1]$; etc. We may now prove

Lemma 3.1. *If $a + b - c = c' - a' - b'$, then*

$$\frac{d}{dz} F F' = \frac{(c-a)(c-b)}{c(1-z)} F[c+1] F' + \frac{(c'-a')(c'-b')}{c'(1-z)} F F'[c'+1].$$

Proof. This follows immediately from (3.10). \square

Lemma 3.2. *If $a + b - c = c' - a' - b'$, then*

$$\begin{aligned}
 \frac{d^2}{dz^2} F F' = \frac{1}{(1-z)^2} & \left[\frac{(c-a)_2 (c-b)_2}{(c)_2} F[c+2] F' \right. \\
 & + 2 \frac{(c-a)(c-b)(c'-a')(c'-b')}{cc'} F[c+1] F'[c'+1] \\
 & \left. + \frac{(c'-a')_2 (c'-b')_2}{(c')_2} F F'[c'+2] \right].
 \end{aligned}$$

Proof. This follows by applying d/dz to both sides of Lemma 3.1, and using equation (3.10) to differentiate the terms on the right. \square

With the aid of the above two lemmas, we are now ready to prove

Proposition 3.2. *Equation (3.5) is true for any $a, b, c; a', b', c'$ such that $a + b - c = c' - a' - b'$.*

Proof. Note that

$$D^2 = z^2 \frac{d^2}{dz^2} + D.$$

By Lemma 3.2, then, (3.5) is equivalent to :

(3.11)

$$\begin{aligned}
 & \left\{ [a(c' - a') + b(c' - b') - z(a + a')(b + b')] \right. \\
 & \quad \left. + [(c' + a + b) - z(a + a' + b + b' + 1)] D \right\} F F' \\
 & + \frac{z^2}{(1 - z)} \left[\frac{(c - a)_2 (c - b)_2}{(c)_2} F[c + 2] F' \right. \\
 & \quad \left. + 2 \frac{(c - a)(c - b)(c' - a')(c' - b')}{cc'} F[c + 1] F'[c' + 1] \right. \\
 & \quad \left. + \frac{(c' - a')_2 (c' - b')_2}{(c')_2} F F'[c' + 2] \right] \\
 & = a(c' - a') F[a + 1] F'[a' - 1] + b(c' - b') F[b + 1] F'[b' - 1].
 \end{aligned}$$

To the series $F[a + 1]$ and $F'[a' - 1]$ on the right-hand side of (3.11) we apply the contiguity relations

$$\begin{aligned}
 eF(e + 1, f; g; z) &= \frac{e - (g - f)z}{1 - z} F(e, f; g; z) \\
 &+ \frac{(g - e)(g - f)z}{g(1 - z)} F(e, f; g + 1; z); \\
 F(e' - 1, f'; g'; z) &= (1 - z) F(e', f'; g'; z) \\
 &+ \frac{(g' - f')z}{g'} F(e', f'; g' + 1; z)
 \end{aligned}$$

(cf. [13, equations (1.4.4) and (1.4.8) respectively]); these relations may in fact be applied to $F[b + 1]$ and $F'[b' - 1]$ as well, since $F(e, f; g; z) = F(f, e; g; z)$. We find after some simplification that

$$\begin{aligned}
 & a(c' - a') F[a + 1] F'[a' - 1] + b(c' - b') F[b + 1] F'[b' - 1] \\
 & = ((c' - a')[a - (c - b)z] + (c' - b')[b - (c - a)z]) F F' \\
 & \quad + \frac{z(c - a)(c - b)(2c' - a' - b')}{c} F[c + 1] F' \\
 & \quad + \frac{2z^2(c - a)(c - b)(c' - a')(c' - b')}{cc'(1 - z)} F[c + 1] F'[c' + 1] \\
 & \quad + \frac{z(c' - a')(c' - b')}{c'(1 - z)} ((a + b) + z(a + b - 2c)) F F'[c' + 1].
 \end{aligned}$$

We put this back into (3.11), and notice: all multiples of $F[c + 1] F'[c' + 1]$, as well as all constant (i.e., z -independent) multiples of $F F'$, cancel. So we get

$$\begin{aligned}
(3.12) \quad & \left\{ [-z(a+a')(b+b')] + [(c'+a+b) - z(a+a'+b+b'+1)]D \right\} F F' \\
& + \frac{z^2}{(1-z)} \left[\frac{(c-a)_2(c-b)_2}{(c)_2} F[c+2] F' \right. \\
& \quad \left. + \frac{(c'-a')_2(c'-b')_2}{(c')_2} F F'[c'+2] \right] \\
& = z(-(c'-a')(c-b) - (c'-b')(c-a)) F F' \\
& \quad + \frac{z(c-a)(c-b)(2c'-a'-b')}{c} F[c+1] F' \\
& \quad + \frac{z(c'-a')(c'-b')}{c'(1-z)} ((a+b) + z(a+b-2c)) F F'[c'+1].
\end{aligned}$$

We compute readily (since $a+b-c = c'-a'-b'$) that

$$-(a+a')(b+b') + (c'-a')(c-b) + (c'-b')(c-a) = -(c-a)(c-b) - (c'-a')(c'-b').$$

Then, by virtue of Lemma 3.1, (3.12) becomes

$$\begin{aligned}
(3.13) \quad & z[-(c-a)(c-b) - (c'-a')(c'-b')] F F' \\
& + \frac{z^2}{(1-z)} \left[\frac{(c-a)_2(c-b)_2}{(c)_2} F[c+2] F' + \frac{(c'-a')_2(c'-b')_2}{(c')_2} F F'[c'+2] \right] \\
& = \frac{z(c-a)(c-b)}{c} \left((2c'-a'-b') \right. \\
& \quad \left. - \frac{(c'+a+b) - z(a+a'+b+b'+1)}{1-z} \right) F[c+1] F' \\
& \quad + \frac{z(c'-a')(c'-b')}{c'(1-z)} \left((a+b) + z(a+b-2c) \right. \\
& \quad \left. - (c'+a+b) + z(a+a'+b+b'+1) \right) F F'[c'+1].
\end{aligned}$$

But the identity $a+b-c = c'-a'-b'$ also gives us

$$(2c'-a'-b') - \frac{(c'+a+b) - z(a+a'+b+b'+1)}{1-z} = -\frac{(c+z(a+b-2c-1))}{1-z};$$

$$(a+b) + z(a+b-2c) - (c'+a+b) + z(a+a'+b+b'+1) = -(c'+z(a'+b'-2c'-1)),$$

so that we may rewrite (3.13) as

$$\begin{aligned}
& z[-(c-a)(c-b) - (c'-a')(c'-b')] F F' \\
& + \frac{z^2}{(1-z)} \left[\frac{(c-a)_2(c-b)_2}{(c)_2} F[c+2] F' + \frac{(c'-a')_2(c'-b')_2}{(c')_2} F F'[c'+2] \right] \\
& = -\frac{z(c-a)(c-b)}{c(1-z)} (c + z(a+b-2c-1)) F[c+1] F' \\
& \quad - \frac{z(c'-a')(c'-b')}{c'(1-z)} (c' + z(a'+b'-2c'-1)) F F'[c'+1]
\end{aligned}$$

or

$$\begin{aligned}
 & -z(c-a)(c-b) \left[F - \frac{(c+z(a+b-2c-1))}{c(1-z)} F[c+1] \right. \\
 & \quad \left. - z \frac{(c+1-a)(c+1-b)}{(c)_2(1-z)} F[c+2] \right] F' \\
 (3.14) \quad & = z(c'-a')(c'-b') \left[F' - \frac{(c'+z(a'+b'-2c'-1))}{c'(1-z)} F'[c'+1] \right. \\
 & \quad \left. - z \frac{(c'+1-a')(c'+1-b')}{(c')_2(1-z)} F'[c'+2] \right] F.
 \end{aligned}$$

To prove equation (3.14) we merely note that, in fact, both sides *vanish*; this follows immediately from the contiguity relation

$$\begin{aligned}
 & g(g+1)(1-z) F(e, f; g; z) \\
 & = (g+1)(c+z(e+f-2g-1)) F(e, f; g+1; z) \\
 & \quad + z(g+1-e)(g+1-f) F(e, f; g+2; z)
 \end{aligned}$$

(cf. [13, equation (1.4.15)]. This completes the proof of Proposition 3.2. \square

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