

AN EXTENSION THEOREM FOR CLOSING MAPS OF SHIFTS OF FINITE TYPE

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ABSTRACT. If there exists some right-closing factor map $\pi: \Sigma_A \rightarrow \Sigma_B$ between aperiodic shifts of finite type, then any right-closing map $\varphi: X \rightarrow \Sigma_B$ from any shift of finite type X contained in Σ_A can be extended to a right-closing factor map from all of Σ_A onto Σ_B . We prove this and give some consequences.

1. INTRODUCTION

We prove the following extension theorem for closing maps between shifts of finite type. In its statement, Σ_A is the shift of finite type defined by the 0-1 matrix A , and (G_A, \hat{A}) is the dimension group pair associated to Σ_A by Krieger [Kr1].

(1.1) **Theorem.** *Let A and B be aperiodic 0-1 matrices satisfying:*

(i) *A and B have the same spectral radius.*

(ii) *The dimension group pair (G_B, \hat{B}) is a quotient of the dimension group pair (G_A, \hat{A}) .*

(iii) *For each periodic point p of Σ_A there is a periodic point of Σ_B whose period divides the period of p . (We denote this by $\Sigma_A \xrightarrow{\text{per}} \Sigma_B$.)*

If $X \subset \Sigma_A$ is a subshift of finite type and $\tilde{\varphi}: X \rightarrow \Sigma_B$ is a right-closing map, then $\tilde{\varphi}$ can be extended to a 1-to-1 almost everywhere right-closing map $\varphi: \Sigma_A \rightarrow \Sigma_B$.

Theorem (1.1) generalizes Theorem (3.15) of [A2] which is the special case where $X \subseteq \Sigma_A$ is a finite subshift. Our proof of Theorem (1.1) is a more intricate reworking of the proof of the case where $X \subseteq \Sigma_A$ is finite.

Theorem (1.1) also generalizes Theorem 6 of [KM] which is the special case where Σ_B is the full n -shift. Following the proof of the main result of [KM], we prove the following corollary of Theorem (1.1).

(1.2) **Theorem.** *Let S be an aperiodic almost finite type sofic system, and let $\pi_R: \Sigma_R \rightarrow S$ be the canonical right-resolving cover of S by a shift of finite type. Let Σ_B be an aperiodic shift of finite type with the same entropy as S . Then the following two conditions are equivalent.*

(i) *The shift S factors onto the shift Σ_B by a 1-to-1 almost everywhere, right-closing almost everywhere map.*

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(ii) The dimension group pair (G_B, \widehat{B}) is a quotient of the dimension group pair (G_R, \widehat{R}) ; and $S \xrightarrow{\text{per}} \Sigma_B$.

Setting $\Sigma_B = \Sigma_R$ we get the following immediate corollary.

(1.3) **Corollary.** *Let S be an aperiodic almost finite type sofic system. Then the following are equivalent.*

- (i) S is weakly isomorphic to its canonical right-resolving covering shift of finite type Σ_R .
- (ii) $S \xrightarrow{\text{per}} \Sigma_R$.

In §2 we give some background; we prove Theorem (1.1) in §3, and prove Theorem (1.2) in §4. In §5 we give an example.

2. BACKGROUND

We assume some familiarity with shifts of finite type and their generalization, the sofic systems. Sections 2 and 3 of [BMT] are our main references for eventual factoring and dimension groups for shifts of finite type. Section 4 of [BMT] is our main reference for almost finite type sofic systems.

Given a 0-1 matrix A indexed by a finite set \mathcal{A} , we define the shift of finite type Σ_A by

$$\Sigma_A = \{s \in \mathcal{A}^{\mathbb{Z}} : A_{s_i s_{i+1}} = 1 \text{ for } i \in \mathbb{Z}\}$$

and topologize Σ_A by the product of the discrete topologies on its coordinate spaces.

The 0-1 matrix A also defines a directed graph \mathcal{G}_A with states or symbols \mathcal{A} and with a directed edge from state s to state t iff $A_{st} = 1$.

If $s_1 s_2 \cdots s_k$ is an allowable path of k states in \mathcal{G}_A , we call $s_1 s_2 \cdots s_k$ a k -path of \mathcal{G}_A . Given a k -path $s_1 s_2 \cdots s_k$ of \mathcal{G}_A , we denote

$$n[s_1 s_2 \cdots s_k]_{n+k-1} = \{x \in \Sigma_A : x_{n+i-1} = s_i, 1 \leq i \leq k\}.$$

This set is called a k -block of Σ_A . Given a set \mathcal{S} of k -paths we denote

$$n[\mathcal{S}]_{n+k-1} = \bigcup_{s \in \mathcal{S}} n[s]_{n+k-1}.$$

Given $y \in \Sigma_A$ we denote the finite path $y_i y_{i+1} \cdots y_j$ in \mathcal{G}_A by $i(y)_j$.

A k -block map $\pi: \Sigma_A \rightarrow \Sigma_B$ is a shift-commuting map such that there is some $l > 0$ for which

$$(2.1) \quad (\pi y)_0 = (\pi y')_0 \quad \text{if } l_{-k+1}(y)_l = l_{-k+1}(y')_l.$$

The k -block map π is a *conjugacy* if it is exactly 1-to-1. In this case π^{-1} is an m -block map for some $m > 0$.

In the 1-block case we require merely for notational convenience that $l = 0$ in (2.1). In the 1-block case we have

$$(\pi y)_0 = (\pi y')_0 \quad \text{if } y_0 = y'_0.$$

Thus π is defined by a map from single states of \mathcal{G}_A to single states of \mathcal{G}_B that we again call π .

A 1-block map $\pi: \Sigma_A \rightarrow \Sigma_B$ is *right resolving* if for every path $t_1 t_2$ of length 2 in \mathcal{G}_B and for every state s_1 of \mathcal{G}_A with $\pi(s_1) = t_1$, there is at most one state s_2 such that $s_1 s_2$ is an edge of \mathcal{G}_A and $\pi(s_2) = t_2$.

A k -block map $\pi: \Sigma_A \rightarrow \Sigma_B$ is *right closing* if it never identifies two distinct left asymptotic points: if $s, s' \in \Sigma_A$ have an $l_0 \in \mathbb{Z}$ such that $s_l = s'_l$ for all $l \leq l_0$ and $\pi(s) = \pi(s')$, then $s = s'$. Notice that a right-resolving map is right closing, and that the composition of two right-closing maps is again right closing.

A *factor map* is an onto block map, and a *factor* of a shift space is its image under a factor map.

A *sofic system* S is a shift space (closed shift invariant subset of $\mathcal{B}^{\mathbb{Z}}$ for some finite alphabet \mathcal{B}) that is a factor of a shift of finite type [We].

A point x in a shift space is *left transitive* if $\{\sigma^n x : n \leq 0\}$ is a dense subset of the space. A factor map $\varphi: X \rightarrow Y$ is *right closing almost everywhere* if it never identifies two left-asymptotic, left-transitive points [BMT].

We define (following [BMT]) the dimension group of integral matrix A and collect here its properties that we use in this paper. If A is an $\alpha \times \alpha$ integral matrix, let V_A be the eventual range of A regarded as a map $A: \mathbb{Q}^\alpha \rightarrow \mathbb{Q}^\alpha$ acting on row vectors. Thus $V_A = A^\alpha(\mathbb{Q}^\alpha)$. Define the dimension group

$$G_A = \{ \vec{q} \in V_A : \vec{q} A^k \in \mathbb{Z}^\alpha \text{ for sufficiently large } k \in \mathbb{N} \},$$

and the automorphism \hat{A} of G_A by $\hat{A} = A|_{G_A}$. The dimension group (G_A, \hat{A}) associated to the shift of finite type Σ_A is a conjugacy invariant.

(2.1) **Proposition** [BMT, Proposition 2.12]. *Let A and B be integral eventually positive matrices with the same spectral radius, λ . Then the following are equivalent.*

- (1) (G_B, \hat{B}) is a quotient of (G_A, \hat{A}) .
- (2) There exists $L \geq 0$ and nonnegative integral matrices S, R such that $AS = SB$ and $RS = B^L$.
- (3) There exists $L \geq 0$ and positive integral matrices S, R such that $AS = SB$ and $RS = B^L$.

Let A be an α -by- α 0-1 matrix and B be a β -by- β 0-1 matrix. A right-resolving 1-block factor map $\pi: \Sigma_A \rightarrow \Sigma_B$ defines an α -by- β 0-1 matrix S as follows:

$$S_{ij} = \begin{cases} 1 & \text{if } \pi(i) = j, \\ 0 & \text{otherwise.} \end{cases}$$

Because π is right resolving, $AS = SB$. Notice that each row of S has a single 1. Since π is surjective, at least one 1 appears in each column of S , so there is a β -by- α matrix R such that RS is the β -by- β identity matrix. Thus by Proposition 1.2 (G_B, \hat{B}) is a quotient of (G_A, \hat{A}) .

Conversely if we are given R, S such that $AS = SB$ and $RS = B^L$ and each row of S is an elementary vector (all 0's except one 1), then we immediately get a right-resolving factor map $\pi: \Sigma_A \rightarrow \Sigma_B$ defined by $\pi(s) = t$ iff $S_{st} = 1$.

It is more difficult to construct a factor map $\varphi: \Sigma_A \rightarrow \Sigma_B$ when the rows of S are not all elementary vectors. It is still more difficult to construct a map φ that extends a given right-closing map $\tilde{\varphi}: X \rightarrow \Sigma_B$ whose domain is a given shift of finite type contained in Σ_A . The construction uses a state-splitting algorithm that at each stage i preserves the relation

$$A^{(i)} S^{(i)} = S^{(i)} B, \quad 0 \leq i \leq n,$$

for a sequence of matrices with $\Sigma_{A^{(i)}} \cong \Sigma_A$, where finally, $S^{(n)}$ has elementary rows. Moreover the right-resolving 1-block map $\mu: \Sigma_{A^{(n)}} \rightarrow \Sigma_B$ defined by $S^{(n)}$, when composed with the conjugacy $\Sigma_A \cong \Sigma_{A^{(n)}}$, gives an extension of the given map $\tilde{\varphi}: X \rightarrow \Sigma_B$.

State splitting. Here we describe the state-splitting construction that we will use to conjugate the shift Σ_A to a new shift $\Sigma_{A'}$ moving closer to our goal.

Given a 0-1 α -by- α matrix A and its associated directed graph \mathcal{G}_A , for each state s in \mathcal{G}_A denote by $\mathcal{F}(s)$ the set of states t such that st is an edge of \mathcal{G}_A .

For each state s_i , $1 \leq i \leq \alpha$, of \mathcal{G}_A fix a partition $\mathcal{S}_i^{(1)}, \dots, \mathcal{S}_i^{(r_i)}$ of the followers $\mathcal{F}(s_i)$ of s_i . We define a new graph $\mathcal{G}_{A'}$ with states

$$0[s_i]_0 \cap 1[\mathcal{S}_i^{(l)}]_1 = 0[s_i \mathcal{S}_i^{(l)}]_1, \quad 1 \leq i \leq \alpha, \quad 1 \leq l \leq r_i,$$

and transitions (edges)

$$0[s_i \mathcal{S}_i^{(l)}]_1 \rightarrow 0[s_j \mathcal{S}_j^{(m)}]_1$$

if and only if $s_j \in \mathcal{S}_i^{(l)}$. We say that $\mathcal{G}_{A'}$ is obtained from \mathcal{G}_A by 1 round of state splitting.

The 1-block map $\alpha: \Sigma_{A'} \rightarrow \Sigma_A$ given by $\alpha(0[s_i \mathcal{S}_i^{(l)}]_1) = s_i$ is a conjugacy (an elementary conjugacy in the sense of [W]). In fact, the image under α of the 1 block $0[0[s_i \mathcal{S}_i^{(l)}]_1]_0$ in $\Sigma_{A'}$ is the union of 2 blocks $0[s_i \mathcal{S}_i^{(l)}]_1$ in Σ_A .

One can prove by induction that if $\mathcal{G}_{A'}$ is obtained from \mathcal{G}_A by m rounds of state splitting then the states of $\mathcal{G}_{A'}$ can be identified with elements of a partition $\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_l$ of the $(m+1)$ -paths of \mathcal{G}_A such that

- (1) the states of $\mathcal{G}_{A'}$ are $0[\mathcal{P}_i]_m$, $1 \leq i \leq l$,
- (2) for each $1 \leq i \leq l$, there is a state $s(i)$ of \mathcal{G}_A such that $0[\mathcal{P}_i]_m \subseteq 0[s(i)]_0$,
- (3) there is a transition $0[\mathcal{P}_i]_m \rightarrow 0[\mathcal{P}_j]_m$ iff $0[\mathcal{P}_i]_m \cap 1[\mathcal{P}_j]_{m+1} \neq \emptyset$,
- (4) if $0[\mathcal{P}_i]_m \cap 1[\mathcal{P}_j]_{m+1} \neq \emptyset$ then $1[\mathcal{P}_j]_{m+1} \subseteq 1[s(i)^{-1}\mathcal{P}_i]_m$, where $s(i)^{-1}\mathcal{P}_i$ is the set of m -paths $\{s_{j_2}s_{j_3} \cdots s_{j_{m+1}} : s(i)s_{j_2}s_{j_3} \cdots s_{j_{m+1}} \in \mathcal{P}_i\}$.

The 1-block map $\alpha: \Sigma_{A'} \rightarrow \Sigma_A$ defined by $\alpha(0[\mathcal{P}_i]_m) = s(i)$ is a conjugacy we call the *ancestor map* after [KM]. The state $0[\mathcal{P}_i]_m$ is a *descendant* of the state $s(i)$.

Legal splitting. Our starting point in constructing $\pi: \Sigma_A \rightarrow \Sigma_B$ is the set of equations $AS = SB$ and $RS = B^L$. In the sequel we denote the i th row of any matrix M by $M_{i,*}$ where we regard $M_{i,*}$ as a row vector. We think of the α -by- β matrix S as assigning mass vectors $w(s)$ to the states s of \mathcal{G}_A by $w(s) = S_{s,*}$. Restating the equation $AS = SB$ in terms of mass vectors, we have for each state s of \mathcal{G}_A

$$(2.2) \quad \sum_{t \in \mathcal{F}(s)} w(t) = w(s)B.$$

We define a *legal* round of state splitting (legal with respect to S) of the graph \mathcal{G}_A to be a round of state splitting in which for each state $s_i \in \mathcal{G}_A$, the followers $\mathcal{F}(s_i)$ of s_i are partitioned into subsets $\mathcal{S}_i^{(1)}, \mathcal{S}_i^{(2)}, \dots, \mathcal{S}_i^{(r_i)}$ in such a way that there are mass vectors $\vec{w}_i^{(1)}, \vec{w}_i^{(2)}, \dots, \vec{w}_i^{(r_i)}$ with $\sum_{j=1}^{r_i} \vec{w}_i^{(j)} = w(s_i)$ and $\sum_{t \in \mathcal{S}_i^{(j)}} w(t) = \vec{w}_i^{(j)} B$.

If the states of the graph $\mathcal{G}_{A'}$ ($\mathcal{G}_{A'}$ being the graph obtained from \mathcal{G}_A by splitting according to these partitions) are assigned mass vectors by

$$w([s_i \mathcal{S}_i^{(j)}]_1) = \vec{w}_i^{(j)},$$

then condition (2.2) will remain true in $\mathcal{G}_{A'}$; moreover some of the entries of S' (whose rows are the mass vectors $\vec{w}_i^{(j)}$) will be reduced in size from those of S .

Finally, a little more terminology. Given a graph \mathcal{G}_A with state masses given by S , we call a state s of \mathcal{G}_A *fat* if $w(s)$ is not an elementary vector, and *lean* if it is. A *fat path* is a path in \mathcal{G}_A having only fat states along it, and a *fat cycle* is a cycle having only fat states along it. An *isolated fat cycle* is a fat cycle $s_0 s_1 \cdots s_{p-1}$ such that each state s_i on the cycle has only one fat successor: state s_{i+1} .

3. PROOF OF THE EXTENSION THEOREM

In an attempt to motivate the following thicket of technical lemmas, we first outline the proof of Theorem (1.1).

First we show in Lemma (3.1) that we can assume without loss of generality that $\mathcal{G}_A, \mathcal{G}_B, X \subseteq \Sigma_A$, and $\tilde{\varphi}: X \rightarrow \Sigma_B$ satisfy some hypotheses convenient to the construction of $\varphi: \Sigma_A \rightarrow \Sigma_B$. These are

- (i) $X \subseteq \Sigma_A$ is presented as a subgraph $\mathcal{G}_{\tilde{A}} \subseteq \mathcal{G}_A$,
- (ii) $\tilde{\varphi}: \mathcal{G}_{\tilde{A}} \rightarrow \Sigma_B$ is a right-resolving 1-block map,
- (iii) the symbols of the sofic system $\tilde{\varphi}(X)$ comprise the states of a proper subgraph $\mathcal{G}_{\tilde{B}} \subseteq \mathcal{G}_B$,
- (iv) each state s of $\mathcal{G}_{\tilde{A}}$ is followed in \mathcal{G}_A by a state t not in $\mathcal{G}_{\tilde{A}}$; similarly each state s of $\mathcal{G}_{\tilde{B}}$ is followed in \mathcal{G}_B by a state t not in $\mathcal{G}_{\tilde{B}}$. We say that $\mathcal{G}_{\tilde{A}} \subseteq \mathcal{G}_A$ and $\mathcal{G}_{\tilde{B}} \subseteq \mathcal{G}_B$ satisfy the *escape hypothesis*.

The extension $\varphi: \Sigma_A \rightarrow \Sigma_B$ of $\tilde{\varphi}: \Sigma_{\tilde{A}} \rightarrow \Sigma_{\tilde{B}}$ will be constructed by applying rounds of legal splitting to the graph \mathcal{G}_A with state masses initially given by SB^N for some $N > 0$, to produce a graph $\mathcal{G}_{A'}$ with all lean states. The map $\varphi: \Sigma_A \rightarrow \Sigma_B$ will then be the composition of the inverse ancestor map $\alpha^{-1}: \Sigma_A \rightarrow \Sigma_{A'}$ with the 1-block mass map $\mu: \Sigma_{A'} \rightarrow \Sigma_B$ defined by $\mu(s) = t$ where $w(s) = e_t$.

In order to ensure that φ is an extension of $\tilde{\varphi}$, the splittings leading from \mathcal{G}_A to $\mathcal{G}_{A'}$ are controlled in such a way that the graph $\mathcal{G}_{A'}$ contains as a subgraph, a graph isomorphic to the $(L+1)$ -block graph $\mathcal{G}_{\tilde{A}}^{[L+1]}$, for some $L > 0$. The state s in $\mathcal{G}_{A'}$ identified with state $0[s_0 s_1 \cdots s_L]_L$ of $\mathcal{G}_{\tilde{A}}^{[L+1]}$ is such that

$$s \cap \Sigma_{\tilde{A}} = 0[s_0 s_1 \cdots s_L]_L \cap \Sigma_{\tilde{A}}.$$

Here, s is regarded as a subset of Σ_A , and $s_0 s_1 \cdots s_L$ is an $(L+1)$ path in $\mathcal{G}_{\tilde{A}}$ (and in \mathcal{G}_A). The legal splittings leading to $\mathcal{G}_{A'}$ are arranged so that $w(s) = e_{\tilde{\varphi}(s_L)}$ in $\mathcal{G}_{A'}$. Now s has ancestor s_0 in \mathcal{G}_A , so $(\mu \circ \alpha^{-1})|_{\Sigma_{\tilde{A}}} = \tilde{\varphi} \circ \sigma^L$. Thus, strictly speaking, it is $\varphi \circ \sigma^{-L}$ rather than φ that is the extension of $\tilde{\varphi}$.

We divide the rounds of splitting leading from \mathcal{G}_A to $\mathcal{G}_{A'}$ into two phases. The first phase, described in the proof of the Marker Splitting Lemma (3.4), eliminates all but a finite number of fat cycles. The second phase, described in the Isolated Cycle Lemma (3.6), eliminates the remaining fat cycles.

(3.1) **Lemma.** *By replacing Σ_A and Σ_B by conjugate shifts of finite type, we can assume without loss of generality that*

- (i) *the shift of finite type $X \subseteq \Sigma_A$ is presented by a subgraph $\mathcal{G}_{\tilde{A}}$ of the graph \mathcal{G}_A with the property that an edge st of \mathcal{G}_A is an edge of $\mathcal{G}_{\tilde{A}}$ whenever s and t are both states of $\mathcal{G}_{\tilde{A}}$;*
- (ii) *the map $\tilde{\varphi}: \Sigma_{\tilde{A}} \rightarrow \Sigma_B$ is a right-resolving 1-block map;*
- (iii) *$\tilde{\varphi}(\Sigma_{\tilde{A}}) \subseteq \Sigma_{\tilde{B}}$ where $\mathcal{G}_{\tilde{B}}$ is a proper subgraph of \mathcal{G}_B ;*
- (iv) *$\mathcal{G}_{\tilde{A}} \subseteq \mathcal{G}_A$ and $\mathcal{G}_{\tilde{B}} \subseteq \mathcal{G}_B$ satisfy the escape hypothesis.*

Proof. By the proof of Theorem 5 of [KM], we can assume (i) and (ii) without loss of generality. By replacing \mathcal{G}_A with a higher order edge graph, we can assume that there is a cycle in \mathcal{G}_A no state of which is in $\mathcal{G}_{\tilde{A}}$.

We define a shift of finite type Γ with $\Sigma_{\tilde{A}} \subseteq \Gamma \subseteq \Sigma_A$ and $\Gamma \neq \Sigma_A$ as follows. For each pair of states (s, t) in $\mathcal{G}_{\tilde{A}}$, choose a path $u(s, t)$ in \mathcal{G}_A such that $su(s, t)t$ is a path in \mathcal{G}_A . For one pair of states s, t in $\mathcal{G}_{\tilde{A}}$ choose paths $u(s, t)$ and v so that $su(s, t)t$ and svt are both paths in \mathcal{G}_A and $|v| = 1 + |u(s, t)|$. Let Γ be the subshift of Σ_A where $\Sigma_{\tilde{A}} \subseteq \Gamma$ and where the only allowed excursions possibly outside of $\mathcal{G}_{\tilde{A}}$ are those along paths of the form $su(s, t)t$ or svt .

Now Γ is a shift of finite type since one can determine whether or not x is in Γ by examining each n -word of x , where n is the maximum length of the paths $su(s, t)t$. We have $\Gamma \neq \Sigma_A$ since \mathcal{G}_A has a cycle no state of which is in $\mathcal{G}_{\tilde{A}}$. Clearly Γ is irreducible. We threw in the path svt to ensure that Γ has positive entropy and is aperiodic.

By replacing \mathcal{G}_A by a still higher order edge graph, we can assume that Γ is presented by a subgraph \mathcal{G}_C with $\mathcal{G}_{\tilde{A}} \subseteq \mathcal{G}_C \subseteq \mathcal{G}_A$. We can also assume that if s, t are both states of \mathcal{G}_C and st is an edge of \mathcal{G}_A , then st is an edge of \mathcal{G}_C . Let γ be the number of states in \mathcal{G}_C , (so that C is a γ -by- γ matrix). Let \vec{d} be the column vector with

$$d_s = \# \text{ of followers of state } s \text{ of } \mathcal{G}_C \text{ that are not themselves} \\ \text{in } \mathcal{G}_C.$$

We define a splitting of \mathcal{G}_A as follows:

- (i) If state s is not in \mathcal{G}_C , do not split s at all;
- (ii) If state s is in \mathcal{G}_C , split s according to the partition $\mathcal{P}_0, \mathcal{P}_1, \dots, \mathcal{P}_{d_s}$ where \mathcal{P}_0 is the set of all followers of s that are in \mathcal{G}_C , and each of $\mathcal{P}_1, \dots, \mathcal{P}_{d_s}$ is a singleton set containing a distinct follower of s not in \mathcal{G}_C .

If \mathcal{G}_A is split according to this partition, then the resulting graph $\mathcal{G}^{(1)}$ will contain a subgraph isomorphic in the natural way to \mathcal{G}_C (that we identify with \mathcal{G}_C). If we define the column vector $\vec{d}^{(1)}$ for $\mathcal{G}_C \subseteq \mathcal{G}^{(1)}$ as \vec{d} was for $\mathcal{G}_C \subseteq \mathcal{G}_A$, then

$$\vec{d}^{(1)} = C \vec{d}.$$

If we repeatedly split \mathcal{G}_A in this way, obtaining successive graphs $\mathcal{G}^{(1)}, \mathcal{G}^{(2)},$

..., we obtain successive vectors $\vec{d}^{(i)}$

$$\vec{d}_s^{(i)} = \# \text{ of followers of state } s \text{ in } \mathcal{G}_C \text{ that are in } \mathcal{G}^{(i)} \text{ but not in } \mathcal{G}_C,$$

and $\vec{d}^{(i)} = C^i \vec{d}$. Since Γ is aperiodic, $C^i > 0$ for large i . Thus \mathcal{G}_C , and hence \mathcal{G}_A , satisfies the escape hypothesis in $\mathcal{G}^{(i)}$ for large i .

Similarly, by splitting \mathcal{G}_B we can assume $\mathcal{G}_B \subseteq \mathcal{G}_B$ satisfies the escape hypothesis. \square

Following [Kr2] denote by $\mathcal{N}[J, K]$, $K \geq J$, the set of all K -paths $a_1 a_2 \cdots a_k$ of \mathcal{G}_A such that for every j , $0 < j < J$, there is at least one k , $0 < k \leq K - j$, such that $a_k \neq a_{k+j}$.

(3.2) **Lemma** [Kr2, Lemma 2]. *For all $J, K \in \mathbb{N}$, $K \geq J$, there is an $N \in \mathbb{N}$ and a finite union of N -blocks $F \subseteq \Sigma_A$ such that*

$$F \cap \sigma^j F = \emptyset, \quad 0 < j < J,$$

and

$$\bigcup_{-J < j < J} \sigma^j F \supseteq \bigcup_{a \in \mathcal{N}[J, K]} o[a]_{k-1}.$$

An easy corollary is

(3.3) **Lemma**. *For all $J \in \mathbb{N}$, there is $N \in \mathbb{N}$ and a set of N -paths \mathcal{B} such that*

(1) *No element of \mathcal{B} nontrivially overlaps any other by more than $N - J$ states.*

(2) *The shift of finite type $\Sigma \subseteq \Sigma_A$ defined by forbidding occurrences of N -paths in \mathcal{B} ,*

$$\Sigma = \bigcap_{j=-\infty}^{\infty} (\Sigma_A \setminus j[\mathcal{B}]_{N+j-1})$$

consists of a finite number of periodic points, and these are exactly the periodic points of Σ_A having period at most $J - 1$.

Proof. Set $K = 2J - 1$ and express the set F of Lemma (3.2) as

$$F = \bigcup_i [b^{(i)}]_{i+N-1}$$

and let \mathcal{B} be the set of N -paths $b^{(i)}$. Then

$$\Sigma = \bigcap_{j=-\infty}^{\infty} \Sigma_A \setminus j[\mathcal{B}]_{j+N-1} \subseteq \bigcap_{j=-\infty}^{\infty} \bigcap_{a \in \mathcal{N}[J, 2J-1]} \Sigma_A \setminus j[a]_{j+2J-2}.$$

Thus if $x \in \Sigma$, then for each $(2J - 1)$ -path w of x there is a j , $0 < j < J$, such that w is j -periodic. Now suppose $2J - 1 \leq L$ and that every L -path w of x is j -periodic for some $j < J$, j perhaps depending on w . We show inductively that every $(L + 1)$ -path v of x is j -periodic for some $j < J$, j perhaps depending on the $(L + 1)$ -path v . Let $s_1 s_2 \cdots s_{L+1}$ be an $(L + 1)$ path from x . Now $s_1 s_2 \cdots s_L$ is j periodic for some $j < J$ and $s_2 s_3 \cdots s_{L+1}$ is k periodic for some $k < J$. Now $s_{L+1} = s_{L+1-k}$ because

$s_2 s_3 \cdots s_{L+1}$ is k periodic; $s_{L+1-k} = s_{L+1-k-j}$ because $s_1 s_2 \cdots s_L$ is j periodic and $L+1-k-j \geq 2 > 1$; $s_{L+1-k-j} = s_{L+1-j}$ because $s_2 s_3 \cdots s_{L+1}$ is k periodic and $L+1-k-j \geq 2$. Thus, $s_{L+1} = s_{L+1-j}$ so $s_1 s_2 \cdots s_{L+1}$ is j periodic. It follows by induction that any finite path from x , and hence x itself, is j periodic for some $j < J$. Thus Σ contains only periodic points of period less than J .

On the other hand, since $F \cap \sigma^j F = \emptyset$, $0 < j < J$, none of the N paths in \mathcal{B} is j periodic for any $0 < j < J$. Thus none of the periodic points of period less than J in Σ_A is removed by disallowing the N -paths in \mathcal{B} . Thus Σ consists of exactly the periodic points of Σ_A of period less than J . \square

(3.4) Marker Splitting Lemma. *Let A and B be aperiodic 0-1 matrices and let R, S be positive matrices and $l > 0$ satisfy $AS = SB$ and $RS = B^l$. Let \mathcal{G}_A be a subgraph of \mathcal{G}_A and let \mathcal{G}_B be a subgraph of \mathcal{G}_B , both subgraphs satisfying the escape hypothesis, and let $\tilde{\varphi}: \Sigma_A \rightarrow \Sigma_B$ be any right-resolving 1-block map. Then there is $L > 0$ such that the graph \mathcal{G}_A with mass vectors given by SB^L can be split by a finite number of rounds of legal splittings into a graph \mathcal{G}' having the following properties.*

(1) *If s is a state of \mathcal{G}' and $s \cap \Sigma_A \neq \emptyset$, then $s \cap \Sigma_A$ is a union of $(L+1)$ blocks in Σ_A and*

$$w(s) \geq \Sigma[e_{\tilde{\varphi}(s_L)} : 0[s_0 s_1 \cdots s_L]_L \cap s \cap \Sigma_A \neq \emptyset].$$

(2) *\mathcal{G}' has only a finite number of fat cycles.*

(3) *The states along a fat cycle \mathcal{C} in \mathcal{G}' are $(L+1)$ -blocks $0[s_0 s_1 \cdots s_L]_L \subseteq \Sigma_A$, where the path $s_0 s_1 \cdots s_L$ is j periodic for some $j < J$.*

(4) *The mass of a state $s = 0[s_0 s_1 \cdots s_L]_L$ on a fat cycle \mathcal{C} in \mathcal{G}' is $w(s) = S_{s_L, *}$.*

(5) *Each fat cycle \mathcal{C} in \mathcal{G}' is isolated.*

Proof. In the proof of the Eventual Factors Theorem [BMT], it is shown that the entries of the matrix RA^n grow exponentially in n with growth rate λ , the spectral radius of A (and B); while the magnitude of the entries of the matrix $A^{l+n} - SRA^n$ grow exponentially in n with growth rate that is strictly less than λ . Thus there is $n_0 > 0$ such that for $n \geq n_0$,

$$e_1 RA^n + e_s (A^{l+n} - SRA^n) > 0$$

for each state s in \mathcal{G}_A .

Still following [BMT], it follows from a matrix calculation that for any $n > n_0$, the set of J paths (where $J = l + n$) in \mathcal{G}_A with initial state a follower of state s can be partitioned into subsets $\mathcal{S}(s; i, j)$, $1 \leq i \leq \beta$, $1 \leq j \leq S_{s, i}$, such that

$$\Sigma[e_{d_j} : d_1 d_2 \cdots d_J \in \mathcal{S}(s; i, j)] = e_i RA^n \quad \text{if } i \neq 1 \text{ or } j \neq 1$$

and

$$\Sigma[e_{d_j} : d_1 d_2 \cdots d_J \in \mathcal{S}(s; 1, 1)] = e_1 RA^n + e_s (A^J - SRA^n).$$

It follows from $(A^J - SRA^n)S = 0$ that

$$\Sigma[S_{d_j, *} : d_1 d_2 \cdots d_J \in \mathcal{S}(s; i, j)] = e_i B^J.$$

Here, we further require that if s is a state of \mathcal{G}_A , then the set of J paths $\mathcal{S}(s; \tilde{\varphi}(s), 1)$ contains all of the J paths in \mathcal{G}_A with initial state a follower of

state s . We show that this is possible for sufficiently large J as follows. The spectral radius of \tilde{A} is less than that of A , so for sufficiently large n

$$e_s \tilde{A}^{l+n} \leq \begin{cases} e_{\tilde{\varphi}(s)} R A^n & \text{if } \tilde{\varphi}(s) \neq 1, \\ e_1 R A^n + e_s (A^{l+n} - S R A^n) & \text{if } \tilde{\varphi}(s) = 1, \end{cases}$$

for each state s of $\mathcal{G}_{\tilde{A}}$. Now

$$\Sigma[e_{d_j} : s d_1 d_2 \cdots d_J \text{ is a path in } \mathcal{G}_{\tilde{A}}] = e_s \tilde{A}^J = e_s \tilde{A}^{l+n},$$

so for sufficiently large J , we can choose $\mathcal{S}(s; \tilde{\varphi}(s), 1)$ as described.

Fix such a J and such partitions. For this J , fix a set of N paths \mathcal{B} as in Lemma (3.3).

We will specify $N + 2J - 2$ rounds of legal splitting applied to the graph \mathcal{G}_A with initial mass vectors given by $S B^{N+J-2}$. But first we will present the graph \mathcal{H} resulting from these splittings. The graph \mathcal{H} satisfies properties (1)–(4), but not necessarily (5); some fat cycles of \mathcal{H} might not be isolated. Later, we will further split \mathcal{H} to isolate the fat cycles while preserving properties (1)–(4).

States of \mathcal{H} :

$$(1) \quad 0[c]_{N+J-2},$$

where c is an $(N + J - 1)$ -path of \mathcal{G}_A such that no N -path $b \in \mathcal{B}$ occurs in c .

$$(2) \quad 0[b\mathcal{S}(b_N; i, j)]_{N+J-1},$$

where $b \in \mathcal{B}$ and $1 \leq i \leq \beta$, $1 \leq j \leq S_{b_N, i}$.

$$(3) \quad 0[db\mathcal{S}(b_N; i, j)]_{N+J-1+|d|},$$

where $b \in \mathcal{B}$ and db is a $(|d| + N)$ path in \mathcal{G}_A with $|d| \leq J - 1$, and i, j are as in (2).

Transitions of \mathcal{H} :

$$(1) \quad 0[c]_{N+J-2} \rightarrow 0[d]_{N+J-2},$$

if $d_1 d_2 \cdots d_{N+J-2} = c_2 c_3 \cdots c_{N+J-1}$ and $d_J d_{J+1} \cdots d_{J+N-1} \notin \mathcal{B}$.

$$(1') \quad 0[c]_{N+J-2} \rightarrow 0[db\mathcal{S}(b_N; i, j)]_{N+2J-2},$$

if $c_2 c_3 \cdots c_{N+J-1} = db_1 b_2 \cdots b_{N-1}$ and $b = b_1 b_2 \cdots b_N \in \mathcal{B}$.

$$(2) \quad 0[b\mathcal{S}(b_N; i, j)]_{N+J-1} \rightarrow 0[c]_{N+J-2},$$

if c is an $(N + J - 1)$ path in \mathcal{G}_A such that $b_1 c \in b\mathcal{S}(b_N; i, j)$ and no element of \mathcal{B} occurs as a suffix of c (which is the earliest possible such occurrence).

$$(2') \quad 0[b\mathcal{S}(b_N; i, j)]_{N+J-1} \rightarrow 0[db'\mathcal{S}(b'_N; i', j')]_{N+2J-2},$$

if db' is an $(N + J - 1)$ path such that $b_1 db' \in b\mathcal{S}(b_N; i, j)$ and $b' \in \mathcal{B}$.

$$(3) \quad 0[db\mathcal{S}(b_N; i, j)]_{N+J-1+|d|} \rightarrow 0[d'b\mathcal{S}(b_N; i, j)]_{N+J-2+|d|},$$

where $d = d_1 d'$ (d' may be the empty string).

Masses of states of \mathcal{H} :

$$(1) \quad w(0[c]_{N+J-2}) = S_{C_{N+J-1}, *},$$

$$(2) \quad w(0[b\mathcal{S}(b_N; i, j)]_{N+J-1}) = e_i B^{J-1},$$

$$(3) \quad w(0[db\mathcal{S}(b_N; i, j)]_{N+J-1+|d|}) = e_i B^{J-1-|d|}.$$

Now we verify that \mathcal{H} with its mass vectors can be obtained by legally splitting \mathcal{G}_A with its mass vectors given by BS^{N+J-2} . Set $L = N + J - 2$. The rounds of splitting are divided into four stages.

1. Perform $N - 1$ rounds of *complete* splitting (at each round the set of followers of a state is partitioned into singleton sets), yielding the graph whose states are the N -blocks of states of \mathcal{G}_A . If after each round j , $1 \leq j \leq N - 1$, the state ${}_0[s_1s_2 \cdots s_{j+1}]_j$ is assigned mass vector $S_{s_{j+1}, *B^{L-j}}$ then each of these rounds is legal.

2. Perform $J - 1$ rounds of splitting, completely splitting every state except the states ${}_0[b]_{N-1}$, $b \in \mathcal{B}$, which are not split at all in any of the $J - 1$ rounds. Call the resulting graph \mathcal{G}_1 . After the j th of these $J - 1$ rounds the states of the graph are the blocks

$$(1) \quad {}_0[b]_{N-1}, \quad b \in \mathcal{B}$$

$$(2) \quad {}_0[cb]_{N-1+i},$$

where $b \in \mathcal{B}$, $0 < i \leq j$, and c is an i path in \mathcal{G}_A such that cb is an $(N + i)$ path in \mathcal{G}_A ,

$$(3) \quad {}_0[c]_{N-1+j},$$

where c is an $(N + j)$ path in \mathcal{G}_A such that no path $b \in \mathcal{B}$ is a subpath of c . If after the k th of these $J - 1$ rounds the states are assigned masses by

$$(1) \quad w({}_0[b]_{N-1}) = S_{b_N, *}B^{J-1}, \quad b \in \mathcal{B},$$

$$(2) \quad w({}_0[cb]_{N-1+i}) = S_{b_N, *}B^{J-1-i}, \quad 1 \leq i \leq k,$$

$$(3) \quad w({}_0[c]_{N-1+k}) = S_{c_{N+k}, *}B^{L-(N-1+k)},$$

then each of these rounds is legal.

3. Perform the following round of splitting. For $b \in \mathcal{B}$, split ${}_0[b]_{N-1}$ into the states ${}_0[b\mathcal{S}(b_N; i, j)]_{N+J-1}$ for $1 \leq i \leq \beta$ and $1 \leq j \leq S_{b_N, i}$. Assign the displayed state the mass vector e_iB^{J-1} .

We show that this splitting is possible and legal. If $c = c_1c_2 \cdots c_{N+J-1}$ is a path in \mathcal{G}_A with $c_1c_2 \cdots c_{N-1} = b_2b_3 \cdots b_N$, then no N path $b' \in \mathcal{B}$ can begin in c before position J , so ${}_0[c]_{N+J-2}$ is a state of \mathcal{G}_1 following state ${}_0[b]_{N-1}$. Hence we can partition the followers of ${}_0[b]_{N-1}$ into the subsets

$$\{{}_0[c]_{N+J-2} : c_Nc_{N+1} \cdots c_{N+J-1} \in \mathcal{S}(b_N; i, j)\}.$$

Now the mass of the state ${}_0[c]_{N+J-2}$ in \mathcal{G}_1 is $S_{c_{N+J-1}, *}$, so the total mass of the elements of the above subset specified by $\mathcal{S}(b_N; i, j)$ is e_iB^J by the choice of the set of J paths $\mathcal{S}(b_N; i, j)$ given by the Eventual Factors Theorem. Thus state ${}_0[b]_{N-1}$ can be legally split as described.

4. Perform the following $J - 1$ rounds of splitting. At the k th of these $J - 1$ rounds, split the states ${}_0[db]_{N-1+k}$, where $b \in \mathcal{B}$ and $|d| = k$ into the states ${}_0[db\mathcal{S}(b_N; i, j)]_{N+J-1+k}$ having mass vectors e_iB^{J-1-k} . That these rounds are possible and legal follows by induction on k .

We show that the graph \mathcal{H} satisfies property (1). Recall that \mathcal{G}_1 is the graph resulting from the splittings in stages 1 and 2. Each state of \mathcal{G}_1 is of the form ${}_0[d]_m$, where d is an $(m + 1)$ path in \mathcal{G}_A and $N - 1 \leq m \leq L$, and $w({}_0[d]_m) = S_{d_{m+1}, *}B^{L-m}$.

If ${}_0[d]_m \cap \Sigma_{\tilde{A}} \neq \emptyset$, then $d = d_1 d_2 \cdots d_{m+1}$ is an $(m+1)$ path in $\mathcal{G}_{\tilde{A}}$, so ${}_0[d]_m \cap \Sigma_{\tilde{A}}$ is the union of $(L+1)$ blocks ${}_0[d\mathcal{S}]_L \cap \Sigma_{\tilde{A}}$, where \mathcal{S} is the set of $(L-m)$ paths in $\mathcal{G}_{\tilde{A}}$ with initial state a follower of state d_{m+1} . Now

$$\begin{aligned} w({}_0[d]_m) &= S_{d_{m+1}, * } B^{L-m} \geq e_{\tilde{\varphi}(d_{m+1})} B^{L-m} \\ &= \Sigma[e_{t_{L-m}} : \tilde{\varphi}(d_{m+1}) t_1 t_2 \cdots t_{L-m} \text{ is a path in } \mathcal{G}_B] \\ &\geq \Sigma[e_{\tilde{\varphi}(s_{L-m})} : d_{m+1} s_1 s_2 \cdots s_{L-m} \text{ is a path in } \mathcal{G}_{\tilde{A}}], \end{aligned}$$

the last inequality follows because $\tilde{\varphi} : \Sigma_{\tilde{A}} \rightarrow \Sigma_B$ is right resolving.

Thus property (1) holds for \mathcal{G}_1 . We now show that property (1) is preserved in the rounds of splitting in stages 3 and 4, thus showing that (1) holds for \mathcal{H} as well.

For convenience, we call a legal splitting that preserves property (1) $\tilde{\varphi}$ -respecting.

At stage 3, each state in graph \mathcal{G}_1 of the form ${}_0[b]_{N-1}$, $b \in \mathcal{B}$ is split into the states

$${}_0[b\mathcal{S}(b_N; i, j)]_{N+J-1}, \quad 1 \leq i \leq \beta, \quad 1 \leq j \leq S_{b_N, i}.$$

To show that this splitting is $\tilde{\varphi}$ -respecting, we need to examine the daughter states having nonempty intersection with $\Sigma_{\tilde{A}}$. These are the states

$${}_0[b\mathcal{S}(b_N; \tilde{\varphi}(b_N), 1)]_{N+J-1},$$

where $b \in \mathcal{B}$ is an N path in $\mathcal{G}_{\tilde{A}}$. Abbreviate the displayed state by s . Now

$$\begin{aligned} w(s) &= e_{\tilde{\varphi}(b_N)} B^{J-1} = \Sigma[e_{t_{J-1}} : \tilde{\varphi}(b_N) t_1 t_2 \cdots t_{J-1} \text{ is a path in } \mathcal{G}_B] \\ &\geq \Sigma[e_{\tilde{\varphi}(s_{J-1})} : b_N s_1 s_2 \cdots s_{J-1} \text{ is a path in } \mathcal{G}_{\tilde{A}}] \\ &= \Sigma[e_{\tilde{\varphi}(s_{L+1})} : {}_0[s_1 s_2 \cdots s_{L+1}]_L \cap s \cap \Sigma_{\tilde{A}} \neq \emptyset], \end{aligned}$$

the inequality holds because $\tilde{\varphi} : \Sigma_{\tilde{A}} \rightarrow \Sigma_B$ is right resolving, and the last equality holds because $s \cap \Sigma_{\tilde{A}} = {}_0[b]_{N-1} \cap \Sigma_{\tilde{A}}$, which in turn follows from the fact that the set $\mathcal{S}(b_N; \tilde{\varphi}(b_N), 1)$ contains all the J paths in $\mathcal{G}_{\tilde{A}}$ following state b_N .

Thus, the round of splitting at stage 3 is $\tilde{\varphi}$ -respecting.

Similarly, the rounds of splitting at stage 4 are $\tilde{\varphi}$ -respecting because

$${}_0[db]_{N-1+k} \cap \Sigma_{\tilde{A}} = {}_0[db\mathcal{S}(b_N; \tilde{\varphi}(b_N), 1)]_{N+J-1+k} \cap \Sigma_{\tilde{A}},$$

and

$$w({}_0[db\mathcal{S}(b_N; \tilde{\varphi}(b_N), 1)]_{N+J-1+k}) = e_{\tilde{\varphi}(b_N)} B^{J-1-k},$$

where $b \in \mathcal{B}$, $0 < k \leq J-1$, and db is an $(N+k)$ -path in $\mathcal{G}_{\tilde{A}}$. This shows that the graph \mathcal{H} satisfies property (1).

One way to show (2), that \mathcal{H} has only a finite number of fat cycles, is to show that each fat cycle in \mathcal{H} is *terminal*: the only infinite fat path from any state on a fat cycle is the path winding forever around the cycle.

Let $x \in \Sigma_{\mathcal{H}}$ be a point such that ${}_{-\infty}(x)_0$ winds around a fat cycle $s_0 s_1 \cdots s_{p-1}$ and x_i is fat for $i > 0$. To show that the cycle $s_0 s_1 \cdots s_{p-1}$ is terminal, we must show that x is actually a periodic point winding around the cycle $s_0 s_1 \cdots s_{p-1}$.

Now x avoids the lean states ${}_0[db\mathcal{S}(b_N; i, j)]_{N+2J-2}$, where $b \in \mathcal{B}$ and db is an $(M+J-1)$ path in \mathcal{E}_A . Since the union over all such d, b, i, j of these states is

$$\bigcup_{b \in \mathcal{B}} {}_{J-1}[b]_{N+J-2}$$

the point $\alpha(x) \in \Sigma_A$ avoids the N paths in \mathcal{B} , and so it is a periodic point in Σ_A of period less than J . Since α is a conjugacy $x \in \Sigma_{\mathcal{H}}$ is a periodic point also, so x must be some shift of $(s_0 s_1 \cdots s_{p-1})^\infty$. This shows that any fat path leaving the cycle $s_0 s_1 \cdots s_{p-1}$ in \mathcal{H} has finite length, which means the cycle is terminal.

We show property (3). Any fat cycle \mathcal{C} in the graph \mathcal{H} avoids the marker states ${}_0[db\mathcal{S}(b_N; i, j)]_{N+J-1+|d|}$ so the states along \mathcal{C} must be of the form ${}_0[c]_{N+J-2} = {}_0[c]_L$, where c is j periodic for some $j < J$. Now

$$w({}_0[c_1 c_2 \cdots c_{L+1}]_L) = S_{c_{L+1}, *};$$

this is property (4).

We now describe rounds of legal, $\tilde{\varphi}$ -respecting splittings that are designed to isolate the fat cycles in \mathcal{H} . Because we will use this construction again, we set it off here as a claim.

(3.5) *Claim.* Suppose a graph \mathcal{H} is obtained from \mathcal{E}_A and \mathcal{H} satisfies property (1): If s is a state of \mathcal{H} with $s \cap \Sigma_{\tilde{A}} \neq \emptyset$, then $s \cap \Sigma_{\tilde{A}}$ is a union of $(L+1)$ blocks in $\Sigma_{\tilde{A}}$ and

$$w(s) \geq \Sigma[e_{\tilde{\varphi}(a_L)} : {}_0[a_0 a_1 \cdots a_L]_L \cap s \cap \Sigma_{\tilde{A}} \neq \emptyset].$$

Furthermore, assume that s is a state of \mathcal{H} with lean followers only. Then state s can be split by a legal, $\tilde{\varphi}$ -respecting splitting entirely into lean states. Therefore, property (1) is preserved by this splitting.

Proof of Claim (3.5). Now the total mass of all the followers of state s is $w(s)B$. A partition of the followers of state s defining a merely legal splitting of s entirely into lean states is easy to concoct: just partition the followers into

$$\begin{aligned} &w(s)_1 \text{ sets, each with total mass } B_{1,*}, \\ &w(s)_2 \text{ sets, each with total mass } B_{2,*}, \\ &\dots \\ &w(s)_\beta \text{ sets, each with total mass } B_{\beta,*}. \end{aligned}$$

As we now show, with a little more care we can ensure that such a partition defines a $\tilde{\varphi}$ -respecting splitting.

If $s \cap \Sigma_{\tilde{A}} = \emptyset$ there is nothing more to prove, so suppose $s \cap \Sigma_{\tilde{A}} \neq \emptyset$. By assumption, $s \cap \Sigma_{\tilde{A}}$ is a union of $(L+1)$ blocks in $\Sigma_{\tilde{A}}$. By property (1), any lean state of \mathcal{H} can intersect $\Sigma_{\tilde{A}}$ in at most one $(L+1)$ block of \mathcal{H} . In particular, this is true of each follower of state s . Let

$${}_0[a^k]_L = {}_0[a_0^k a_1^k \cdots a_L^k]_L, \quad 1 \leq k \leq K,$$

be the $(L+1)$ blocks such that ${}_0[a^k]_L \cap \Sigma_{\tilde{A}} \cap s \neq \emptyset$.

We now show that one can partition the followers of state s into sets with mass distributed as above, but with the added constraint that for each $1 \leq k \leq$

K , the set of states

$$\mathcal{Q}_k = \{t : t \text{ follows } s \text{ in } \mathcal{H} \text{ and } \emptyset \neq t \cap \Sigma_{\tilde{A}} \subseteq {}_0[a_1^k a_2^k \cdots a_L^k]_{L-1} \cap \Sigma_{\tilde{A}}\}$$

be contained entirely in a single partition set. Because \mathcal{H} satisfies property (1), $w(s) \geq \sum_{k=1}^K e_{\tilde{\varphi}(a_L^k)}^{\sim}$, so we have a lower bound for the total mass of the followers of s :

$$w(s)B \geq \sum_{k=1}^K e_{\tilde{\varphi}(a_L^k)}^{\sim} B.$$

Because $\tilde{\varphi} : \Sigma_{\tilde{A}} \rightarrow \Sigma_B$ is right resolving, we have

$$\begin{aligned} e_{\tilde{\varphi}(a_L^k)}^{\sim} B &\geq \Sigma[e_{\tilde{\varphi}(f)}^{\sim} : \text{state } f \text{ follows state } a_L^k \text{ in graph } \mathcal{G}_{\tilde{A}}] \\ &= \Sigma[w(t) : t \in \mathcal{Q}_k] = w(\mathcal{Q}_k), \end{aligned}$$

where the first equality holds because a follower t of state s in graph \mathcal{H} is in \mathcal{Q}_k iff

$$t \cap \Sigma_{\tilde{A}} = {}_0[a_1^k a_2^k \cdots a_L^k f]_L \cap \Sigma_{\tilde{A}}$$

for some follower f of state a_L^k in graph $\mathcal{G}_{\tilde{A}}$. Thus we can augment each set \mathcal{Q}_k by throwing in more followers of state s , to build a set $\mathcal{P}_k \supseteq \mathcal{Q}_k$ such that $w(\mathcal{P}_k) = e_{\tilde{\varphi}(a_L^k)}^{\sim} B$, while the \mathcal{P}_k 's remain, as the \mathcal{Q}_k 's, pairwise disjoint.

The total mass of the followers of state s that are not elements of $\bigcup_{k=1}^K \mathcal{P}_k$ is

$$w(s)B - \sum_{k=1}^K e_{\tilde{\varphi}(a_L^k)}^{\sim} B,$$

and these can be partitioned as before into $[w(s) - \sum_{k=1}^K e_{\tilde{\varphi}(a_L^k)}^{\sim}]_i$ sets, each with total mass $B_{i,*}$, for $1 \leq i \leq \beta$.

If state s is split according to this partition, and the daughter states are assigned elementary masses in the natural way, then the daughter state s^k of s corresponding to partition set \mathcal{P}_k has

$$s^k \cap \Sigma_{\tilde{A}} = {}_0[a_0^k a_1^k \cdots a_L^k]_L \cap \Sigma_{\tilde{A}},$$

and $w(s^k) = e_{\tilde{\varphi}(a_L^k)}^{\sim}$, so the splitting is $\tilde{\varphi}$ -respecting. This completes the proof of the claim.

Recall that each fat cycle \mathcal{C} in \mathcal{H} is terminal: all fat paths in \mathcal{H} leaving \mathcal{C} have finite length. Using Claim (3.5) we can split using a round of $\tilde{\varphi}$ -respecting splitting the terminal fat state of all maximal-length fat paths leaving the fat cycle \mathcal{C} , thereby shortening all maximal-length fat paths leaving \mathcal{C} by length 1. Not only is this splitting $\tilde{\varphi}$ -respecting (preserving property (1)), but it also preserves properties (2), (3), and (4) of the graph \mathcal{H} . By induction, we can continue shortening the fat paths leaving the fat cycle \mathcal{C} until \mathcal{C} is isolated. Doing this for all fat cycles, we reach our goal, graph \mathcal{G}' in which each fat cycle is isolated. This completes the proof of Lemma (3.4). \square

We now deal with the problem of splitting isolated fat cycles in \mathcal{G}' by rounds of $\tilde{\varphi}$ -respecting splitting.

We can treat the problem of splitting an isolated fat cycle \mathcal{C} in isolation from the rest of the graph \mathcal{G}' . This is because any state s of \mathcal{G}' immediately

forwardly accessible from \mathcal{E} is lean and hence cannot be further split by legal splitting; thus what follows state s in \mathcal{E}' can have no bearing on how states along \mathcal{E} can be legally split.

For this reason we regard the isolated fat cycle \mathcal{E} as sitting in an isolated fat cycle graph comprising a single (simple) cycle $s'_0 s'_1 \cdots s'_{p-1}$ together with the immediate followers of state s'_i (all lean except s'_{i+1}), $0 \leq i \leq p-1$.

By property (3) of graph \mathcal{E}' given in the Marker Splitting Lemma (3.4) the sequence of state masses along a fat cycle $s'_1 s'_1 \cdots s'_{p-1}$ in \mathcal{E}' is given by $w(s'_i) = S_{s_i, *}$, where $s'_i = 0[s_{i-L} s_{i-L+1} \cdots s_i]_L$ and $s_0 s_1 \cdots s_{p-1}$ is a cycle in the original graph \mathcal{E}_A . We can replace S in the statement of the Marker Splitting Lemma (3.4) by SB^n for any $n \geq 0$, to obtain a graph \mathcal{E}' in which any fat cycle $s'_0 s'_1 \cdots s'_{p-1}$ has state masses given by $w(s'_i) = S_{s_i, *} B^n$, where again $s_0 s_1 \cdots s_{p-1}$ is a cycle in the original graph \mathcal{E}_A .

The following lemma will serve to eliminate all the fat cycles from \mathcal{E}' .

(3.6) **Isolated Cycle Lemma.** *Assume that $A, B, R, S, \Sigma_{\tilde{A}} \subseteq \Sigma_A, \Sigma_{\tilde{B}} \subseteq \Sigma_B$, and $\tilde{\varphi}: \Sigma_{\tilde{A}} \rightarrow \Sigma_{\tilde{B}}$ satisfy all the hypotheses of the Marker Splitting Lemma (3.4). In addition, assume that $\Sigma_A \xrightarrow{\text{per}} \Sigma_B$. Then there is an $N > 0$ such that for all $n \geq N$, for all cycles $s_0 s_1 \cdots s_{p-1}$ in \mathcal{E}_A , an isolated fat cycle $s'_0 s'_1 \cdots s'_{p-1}$ in \mathcal{E}' with $s'_i = 0[s_{i-L} \cdots s_i]_L$ and $w(s'_i) = S_{s_i, *} B^n$ can be split completely into lean states by rounds of legal, $\tilde{\varphi}$ -respecting splitting.*

To prove this lemma we study the class of graphs obtained by splitting isolated p -cycles: *p -cycles of trees*.

A *p -cycle of trees* is a union of p disjoint directed trees (with edges directed away from the root and toward the leaves), where the roots r_0, r_1, \dots, r_{p-1} of the trees are connected in a directed cycle $r_0 r_1 \cdots r_{p-1}$.

Notice that an isolated fat cycle together with the states immediately accessible from it form a cycle of trees.

We define the *phase* of a node s in a p -cycle of trees \mathcal{T} to be $|ws| \bmod p$ where $r_0 ws$ is any path in \mathcal{T} from a fixed root r_0 to the state s . If a round of state splitting is applied to a cycle of trees \mathcal{T} we obtain a new cycle of trees and we can consistently assign to each state in the new cycle of trees the phase of its ancestor.

We define a *resolving covering graph* of a graph \mathcal{E}_B to be a directed graph \mathcal{G} such that there is a map π from the states of \mathcal{G} to the states of \mathcal{E}_B that is right resolving at each *internal node* of \mathcal{G} , where an *internal node* is a state of \mathcal{G} having followers in \mathcal{G} . Further, for each internal node s of \mathcal{G} , and each follower t' of $\pi(s)$ in \mathcal{E}_B , there is a follower s' of s in \mathcal{G} with $\pi(s') = t'$.

If an isolated cycle $s_0 s_1 \cdots s_{p-1}$ is split by legal rounds of state splitting into a graph \mathcal{T} with all lean states, then \mathcal{T} is a cycle of trees that forms a resolving covering graph of \mathcal{E}_B , covering \mathcal{E}_B with the mass map.

Since state splitting preserves phase in a cycle of trees, we have

$$\Sigma[w(s) : s \text{ if an internal node of } \mathcal{T} \text{ having phase } i] = w(s_i).$$

We have proved the only if part of the following observation.

(3.7) *Observation.* An isolated fat cycle $s_0 s_1 \cdots s_{p-1}$ can be completely split into lean states by rounds of legal splitting if and only if there is a p cycle of

trees \mathcal{T} that is a resolving covering graph of \mathcal{G}_B by state map π such that

$$\Sigma[e_{\pi(s)} : s \text{ is an internal node of } \mathcal{T} \text{ having phase } i] = w(s_i).$$

The proof of the if part is similar to the proof of Observation (3.9) below, or see [A2].

Observation (3.7) characterizes the isolated fat cycles that can be split into lean states by *legal* splittings. We will need more than this in order to split certain isolated cycles in \mathcal{G}' by $\tilde{\varphi}$ -respecting splittings. Observation (3.9) below meets this need. The isolated fat cycles of \mathcal{G}' fall into two types.

(3.8) **Lemma.** *Let \mathcal{G}' be the graph produced by the Marker Splitting Lemma (3.4), and let $\mathcal{C} = s'_1 s'_1 \cdots s'_{p-1}$ be an isolated fat cycle in \mathcal{G}' . Then either each state along \mathcal{C} , regarded as a subset of Σ_A , is disjoint from $\Sigma_{\tilde{A}}$, or each state along \mathcal{C} intersects $\Sigma_{\tilde{A}}$ in an $(L+1)$ -block in $\Sigma_{\tilde{A}}$.*

Proof. Each state s'_i along an isolated fat cycle \mathcal{C} in \mathcal{G}' is an $(L+1)$ -block $0[s_i s_{i+1} \cdots s_{i+L}]_L$ in Σ_A , where $s_0 s_1 \cdots s_{p-1}$ is a p cycle in \mathcal{G}_A . If $s_0 s_1 \cdots s_{p-1}$ is a p -cycle in $\mathcal{G}_{\tilde{A}}$, then each s'_i intersects $\Sigma_{\tilde{A}}$ in an $(L+1)$ -block in $\Sigma_{\tilde{A}}$. If $s_0 s_1 \cdots s_{p-1}$ is not a p -cycle in $\mathcal{G}_{\tilde{A}}$ then at least one state s_{i_0} along the p -cycle is not in $\mathcal{G}_{\tilde{A}}$. Now since $L+1 = N+J-1 \geq J-1 \geq p$, we have

$$0[s_i s_{i+1} \cdots s_{i+L}]_L \subseteq \sigma^{-k} 0[s_{i_0}]_0,$$

where $0 \leq k \leq L$, and $k \equiv i_0 - i \pmod{p}$. But $0[s_{i_0}]_0 \cap \Sigma_{\tilde{A}} = \emptyset$. \square

If $\mathcal{C} = s'_0 s'_1 \cdots s'_{p-1}$ is an isolated cycle in \mathcal{G}' with $s'_i \cap \Sigma_{\tilde{A}} = \emptyset$, $0 \leq i \leq p-1$, then any legal splitting of the states along \mathcal{C} is $\tilde{\varphi}$ -respecting. We now consider the other possibility. Each state s'_i along \mathcal{C} intersects $\Sigma_{\tilde{A}}$ in an $(L+1)$ -block $0[s_{i-L} s_{i-L+1} \cdots s_i]_L$ in $\Sigma_{\tilde{A}}$.

(3.9) **Observation.** Let $\mathcal{C} = s'_0 s'_1 \cdots s'_{p-1}$ be an isolated cycle in \mathcal{G}' with $s'_i \cap \Sigma_{\tilde{A}} \neq \emptyset$, $0 \leq i \leq p-1$, (so $s'_i \cap \Sigma_{\tilde{A}} = 0[s_{i-L} s_{i-L+1} \cdots s_i]_L \cap \Sigma_{\tilde{A}}$ where $s_0 s_1 \cdots s_{p-1}$ is a p -cycle in $\mathcal{G}_{\tilde{A}}$ with $p < J$). For $0 \leq i \leq p-1$, let $\tilde{f}_{i,1}, \tilde{f}_{i,2}, \dots, \tilde{f}_{i,n_i}$ be the followers of s_i in the graph $\mathcal{G}_{\tilde{A}}$ distinct from the follower s_{i+1} , if such exist. The isolated fat cycle \mathcal{C} can be completely split into lean states by rounds of $\tilde{\varphi}$ -respecting splittings if there is a p -cycle of trees $\tilde{\mathcal{T}}$ that is a resolving graph covering \mathcal{G}_B by state map π satisfying

- (i) $\pi(r_i) = \tilde{\varphi}(s_i)$, where state r_i in $\tilde{\mathcal{T}}$ is the root at phase i , $0 \leq i \leq p-1$,
- (ii) the tree in $\tilde{\mathcal{T}}$ with root r_i has a leaf at depth 1 with π image $\tilde{\varphi}(\tilde{f}_{i,j})$, for $1 \leq j \leq n_i$,
- (iii) $w(s'_i) = \Sigma[e_{\pi(s)} : s \text{ is an internal node of } \tilde{\mathcal{T}} \text{ having phase } i]$.

Proof of (3.9). The proof is an induction on the maximum depth of an internal node of $\tilde{\mathcal{T}}$. If this maximum depth is 0, then the only internal nodes of $\tilde{\mathcal{T}}$ are the root nodes r_0, r_1, \dots, r_{p-1} , so by (i) and (iii), $w(s'_i) = e_{\pi(r_i)} = e_{\tilde{\varphi}(s_i)}$. Thus no splitting needs to be done to completely split \mathcal{C} into lean states.

Now suppose that the maximum depth of an internal node of $\tilde{\mathcal{T}}$ is $n > 0$.

Condition (iii) gives that for $0 \leq i \leq p-1$,

$$\begin{aligned}
& \Sigma[w(t) : t \text{ is a follower of } s'_i \text{ other than } s'_{i+1}] = w(s'_i)B - w(s'_{i+1}) \\
& = \Sigma[e_{\pi(s)}B : s \text{ is an internal node of } \widetilde{\mathcal{T}} \text{ having phase } i] \\
& \quad - \Sigma[e_{\pi(s)} : s \text{ is an internal node of } \widetilde{\mathcal{T}} \text{ having phase } i+1] \\
& = \Sigma[e_{\pi(s)} : s \text{ is any node of } \widetilde{\mathcal{T}} \text{ having phase } i+1] \\
& \quad - \Sigma[e_{\pi(s)} : s \text{ is an internal node of } \widetilde{\mathcal{T}} \text{ having phase } i+1] \\
& = \Sigma[e_{\pi(s)} : s \text{ is a leaf of } \widetilde{\mathcal{T}} \text{ having phase } i+1].
\end{aligned}$$

Thus the leaves of $\widetilde{\mathcal{T}}$ at phase $i+1$ can be put into 1-to-1 correspondence with the followers of state s'_i in \mathcal{G}' other than s'_{i+1} in such a way that $e_{\pi(s)} = w(t)$, where leaf s in $\widetilde{\mathcal{T}}$ corresponds to follower t of state s'_i in \mathcal{G}' . By condition (ii), we can arrange this correspondence so that for $1 \leq j \leq n_i$, there is a leaf $l_{i,j}$ at depth 1, phase $i+1$, in $\widetilde{\mathcal{T}}$ with $\pi(l_{i,j}) = \tilde{\varphi}(\tilde{f}_{i,j})$ corresponding to the follower $f_{i,j}$ of s'_i in \mathcal{G}' with

$$f_{i,j} \cap \Sigma_{\tilde{A}} = 0[s_{i-L+1}s_{i-L+2} \cdots s_i \tilde{f}_{i,j}]_L \cap \Sigma_{\tilde{A}}$$

and therefore with $w(f_{i,j}) = e_{\tilde{\varphi}(\tilde{f}_{i,j})}$.

Having fixed this correspondence, we define a $\tilde{\varphi}$ -respecting round of splitting of \mathcal{G}' as follows. For $0 \leq i \leq p-1$, let $s_{i,1}, \dots, s_{i,m_i}$ be the *internal* nodes of $\widetilde{\mathcal{T}}$ at phase i and depth n , if there are any. Partition the followers of state s'_i in \mathcal{G}' into m_i+1 subsets, $\mathcal{P}_{i,0}, \mathcal{P}_{i,1}, \dots, \mathcal{P}_{i,m_i}$, where for $1 \leq j \leq m_i$, the elements of $\mathcal{P}_{i,j}$ are those followers of state s'_i in \mathcal{G}' that correspond to the followers of state $s_{i,j}$ in $\widetilde{\mathcal{T}}$. This makes sense because all the followers of state $s_{i,j}$ in $\widetilde{\mathcal{T}}$ are leaves. The elements of $\mathcal{P}_{i,0}$ are the remaining followers of state s'_i in \mathcal{G}' (corresponding to leaves of $\widetilde{\mathcal{T}}$ at depth at most n together with state s'_{i+1}).

If we split s'_i according to this partition into states $s'_{i,j}$ corresponding to these sets $\mathcal{P}_{i,j}$, $0 \leq j \leq m_i$, and assign masses

$$w(s'_{i,j}) = e_{\pi(s_{i,j})}, \quad 1 \leq j \leq m_i,$$

and

$$w(s'_{i,0}) = \Sigma[e_{\pi(s)} : s \text{ is an internal node of } \widetilde{\mathcal{T}} \text{ having phase } i \text{ and depth } < n],$$

then the splitting is legal. We show that the splitting is $\tilde{\varphi}$ -respecting. The followers of s'_i in \mathcal{G}' having nonempty intersection with $\Sigma_{\tilde{A}}$, namely $f_{i,j}$, $1 \leq j \leq n_i$, and s'_{i+1} , are all elements of the partition set $\mathcal{P}_{i,0}$, giving

$$s'_{i,0} \cap \Sigma_{\tilde{A}} = s'_i \cap \Sigma_{\tilde{A}} = 0[s_{i-L}s_{i-L+1} \cdots s_i]_L \cap \Sigma_{\tilde{A}},$$

and $s'_{i,j} \cap \Sigma_{\tilde{A}} = \emptyset$, $1 \leq j \leq m_i$. Now $w(s'_{i,0}) \geq e_{\pi(r_i)} = e_{\varphi(s_i)}$, so the splitting is $\tilde{\varphi}$ -respecting.

After this round of splitting the cycle $s'_{0,0}s'_{1,0}s'_{2,0} \cdots s'_{p-1,0}$ is an isolated cycle in the new graph. The cycle of trees $\mathcal{T}' \subset \widetilde{\mathcal{T}}$ defined by omitting from $\widetilde{\mathcal{T}}$ all leaves at depth $n+1$ is a resolving covering graph of \mathcal{G}_B by state map

π (restricted to \mathcal{T}') satisfying properties (i), (ii), and (iii) for the new isolated cycle $s'_{0,0}s'_{1,0}\cdots s'_{p-1,0}$. The maximum depth of an internal node of \mathcal{T}' is $n-1$, so by the inductive hypothesis, the new isolated cycle can be completely split into lean states by rounds of $\tilde{\varphi}$ -respecting splitting. This completes the proof of Observation (3.9). \square

Observations (3.7) and (3.9) demonstrate that an isolated fat cycle in \mathcal{G}' can be split by $\tilde{\varphi}$ -respecting splittings when a cycle of trees satisfying certain properties exists. The remainder of the proof of the Isolated Cycle Lemma (3.6) focuses on the task of constructing this cycle of trees. We organize the construction as a series of technical lemmas.

For $n > 0$ and any cycle $\mathcal{C} = s_0s_1\cdots s_{p-1}$ in \mathcal{G}_A , and any graph homomorphism $\psi: \mathcal{C} \rightarrow \mathcal{G}_B$ (onto a q cycle, where $q|p$), we define a resolving cycle of trees $\mathcal{T}(n, \mathcal{C}, \psi)$ covering \mathcal{G}_B by state map π as follows:

- (i) $\mathcal{T}(n, \mathcal{C}, \psi)$ has a cycle of roots $r_0r_1\cdots r_{p-1}$ with $\pi(r_i) = \psi(s_i)$, $0 \leq i \leq p-1$,
- (ii) for each follower g of $\psi(s_i)$ in \mathcal{G}_B except $\psi(s_{i+1})$, the tree in $\mathcal{T}(n, \mathcal{C}, \psi)$ with root r_i has a complete resolving subtree of depth n covering \mathcal{G}_B , whose root covers g by state map π . Thus all the leaves of $\mathcal{T}(n, \mathcal{C}, \psi)$ are at depth $n+1$.

Similarly, for $n > 0$ and any cycle $\mathcal{C} = s_0s_1\cdots s_{p-1}$ in $\mathcal{G}_{\tilde{A}}$, define a resolving cycle of trees $\tilde{\mathcal{T}}(n, \mathcal{C})$ covering \mathcal{G}_B by state map π as follows:

- (i) $\tilde{\mathcal{T}}(n, \mathcal{C})$ has a cycle of roots $r_0r_1\cdots r_{p-1}$ with $\pi(r_i) = \tilde{\varphi}(s_i)$, $0 \leq i \leq p-1$,
- (ii) the tree in $\tilde{\mathcal{T}}(n, \mathcal{C})$ with root r_i has a leaf at depth 1 with π -image $\tilde{g}_{i,j}$, $1 \leq j \leq \tilde{n}_i$, where $\tilde{g}_{i,1}, \tilde{g}_{i,2}, \dots, \tilde{g}_{i,\tilde{n}_i}$ are the followers of state $\tilde{\varphi}(s_i)$ distinct from $\tilde{\varphi}(s_{i+1})$ in the graph $\mathcal{G}_{\tilde{B}}$ (if there are any),
- (iii) for each follower g of $\tilde{\varphi}(s_i)$ in \mathcal{G}_B that is *not* a state in $\mathcal{G}_{\tilde{B}}$, the tree in $\tilde{\mathcal{T}}(n, \mathcal{C})$ with root r_i has a complete resolving subtree of depth n whose root covers g by state map π .

Thus all the leaves of $\tilde{\mathcal{T}}(n, \mathcal{C})$ are at depth 1 or $n+1$.

For any path $w = w_1w_2\cdots w_n$ in the graph \mathcal{G}_B define the vector $I(w) \in \mathbb{N}^\beta$ by $I(w) = e_{w_1}B^{n-1}$. For any state t in $\mathcal{G}_{\tilde{B}}$ denote

$$u(t) = \Sigma[e_f : f \text{ follows } t \text{ in } \mathcal{G}_B \text{ and } f \text{ is not a state in } \mathcal{G}_{\tilde{B}}].$$

Note that $u(t) \neq 0$ for each state t of $\mathcal{G}_{\tilde{B}}$ by the escape hypothesis on $\mathcal{G}_{\tilde{B}} \subseteq \mathcal{G}_B$.

For any path $w = w_1w_2\cdots w_n$ in $\mathcal{G}_{\tilde{B}}$ define the vector $\tilde{I}(w) \in \mathbb{N}^\beta$ by

$$\tilde{I}(w) = e_{w_n} + u(w_{n-1}) + u(w_{n-2})B + \cdots + u(w_1)B^{n-2}.$$

For any p -cycle of trees that covers \mathcal{G}_B by state map π , define the p -by- β matrix $\mathcal{J}(\mathcal{T})$ by

$$\mathcal{J}(\mathcal{T})_{i,*} = \Sigma[e_{\pi(s)} : s \text{ is an internal state of } \mathcal{T} \text{ at phase } i].$$

(3.10) *Claim.* If $s_0s_1\cdots s_{p-1}$ is a p cycle in \mathcal{G}_A and $\psi(s_0\cdots s_{p-1})$ is a q -cycle in \mathcal{G}_B where $q|p$, then the cycle of trees $\mathcal{T}(n, s_0s_1\cdots s_{p-1}, \psi)$ covering \mathcal{G}_B by state map π satisfies

$$\mathcal{J}(\mathcal{T}(n, s_0s_1\cdots s_{p-1}, \psi))_{i,*} = I(\pi(s_{i-n}s_{i-n+1}\cdots s_i)) = e_{\pi(s_{i-n})}B^n.$$

Similarly, if $s_0 s_1 \cdots s_{p-1}$ is a p cycle in \mathcal{G}_A , then $\widetilde{\mathcal{T}}(n, s_0 s_1 \cdots s_{p-1})$ is the cycle of trees satisfying

$$\mathcal{F}(\widetilde{\mathcal{T}}(n, s_0 s_1 \cdots s_{p-1}))_{i,*} = \widetilde{I}(\widetilde{\varphi}(s_{i-n} s_{i-n+1} \cdots s_i)),$$

where the indices are regarded modulo p .

The proof is a simple calculation.

For two vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$ let $\angle(\vec{u}, \vec{v})$ be the angle θ formed by \vec{u} and \vec{v} , where $0 \leq \theta \leq \pi$.

(3.11) *Claim.* Let $\vec{l} > 0$ be a left Perron eigenvector for B . For all $\varepsilon > 0$ there exists $K > 0$ such that for all paths w in \mathcal{G}_B with $|w| \geq K$, $\angle(\vec{l}, \widetilde{I}(w)) < \varepsilon$ and for all paths w in \mathcal{G}_B with $|w| \geq K$, $\angle(\vec{l}, I(w)) < \varepsilon$.

Proof. Given $\varepsilon > 0$. By Perron-Frobenius theory [S, Theorem 1.2], there is $k_0 > 0$ such that for $k \geq k_0$, the angle formed by \vec{l} and each row of B^k is less than $\varepsilon/2$. Let M be the maximum of the Euclidean norms

$$\|\widetilde{I}(w)\|_2 = \|e_{w_{k_0+1}} + u(w_{k_0}) + u(w_{k_0-1})B + \cdots + u(w_1)B^{k_0-1}\|_2$$

over all words w of length $k_0 + 1$ in \mathcal{G}_B . Choose $k_1 > 0$ such that for all $k \geq k_1$, and for $1 \leq i \leq \beta$,

$$\|B_{i,*}^{k_0+k-1}\|_2 > \frac{M}{\sin(\varepsilon/2)}.$$

Let x be a path in \mathcal{G}_B with $|x| \geq k_0 + k_1 + 1$. Express $x = vw$ where $|v| = k \geq k_1$ and $|w| = k_0 + 1$. Now

$$\widetilde{I}(x) = \widetilde{I}(w) + [u(v_1)B^{k-1} + u(v_2)B^{k-2} + \cdots + u(v_k)]B^{k_0}.$$

Abbreviating the second term as \vec{v} , we have $\|\widetilde{I}(w)\|_2 \leq M$ and $\|\vec{v}\|_2 > M/\sin(\varepsilon/2)$. Also $\angle(\vec{l}, \widetilde{I}(w)) < \pi/2$ and $\angle(\vec{l}, \vec{v}) < \varepsilon/2$.

If $\theta = \angle(\vec{l}, \widetilde{I}(x))$, then

$$\begin{aligned} \tan \theta &< \frac{M + \|\vec{v}\|_2 \sin(\varepsilon/2)}{\|\vec{v}\|_2 \cos(\varepsilon/2)} = \frac{M}{\|\vec{v}\|_2 \cos(\varepsilon/2)} + \tan(\varepsilon/2) \\ &< \frac{\sin(\varepsilon/2)}{\cos(\varepsilon/2)} + \tan(\varepsilon/2) = 2 \tan(\varepsilon/2) < \tan(\varepsilon), \end{aligned}$$

so $\theta < \varepsilon$. The proof that $\angle(\vec{l}, I(w)) < \varepsilon$ for long paths w is easier and we omit it. \square

(3.12) *Claim.* For all $\varepsilon > 0$, there exists $L > 0$ such that for all $l \geq L$, for all states s of \mathcal{G}_A , $\angle(\vec{l}, w(s)B^l) < \varepsilon$. The proof follows from Perron-Frobenius theory [S].

Claims (3.13) and (3.14) give the base case of an inductive construction of cycles of trees whose existence by Observations (3.7) and (3.9), ensures that we can split isolate fat cycles in \mathcal{G}' . Claims (3.15) and (3.16) will give the inductive step in the construction. We bring the strands of the construction together in Claim (3.17).

(3.13) *Claim.* For all $\delta > 0$, there is $L > 0$ such that for all $l \geq L$, for all states s in \mathcal{G}_A , for all paths w in $\mathcal{G}_{\tilde{B}}$, if $\tilde{I}(w) \not\leq w(s)B^l$, then

$$(1 - \delta)w(s)B^l < \tilde{I}(w).$$

Similarly, for all $\delta > 0$, there is $L > 0$ such that for all $l \geq L$, for all states s in \mathcal{G}_A , for all paths w in \mathcal{G}_B , if $I(w) \not\leq w(s)B^l$, then

$$(1 - \delta)w(s)B^l < I(w).$$

Proof. The idea is that for large l , $\tilde{I}(w)$ and $w(s)B^l$ are nearly parallel.

For

$$0 \leq \varepsilon < \min_{1 \leq i \leq \beta} \sin^{-1}(l_i / \|\vec{l}\|_2)$$

and $1 \leq i, j \leq \beta$, define

$$\rho_{ij}(\varepsilon) = \frac{\min\{x_i : x_j = 1 \text{ and } \angle(\vec{l}, \vec{x}) \leq \varepsilon\}}{\max\{x_i : x_j = 1 \text{ and } \angle(\vec{l}, \vec{x}) \leq \varepsilon\}}.$$

Note that for ε in the range of definition, the set $\{\vec{x} : x_j = 1 \text{ and } \angle(\vec{l}, \vec{x}) \leq \varepsilon\}$ is a solid ellipsoid. As $\varepsilon \rightarrow 0$, the major diameter of this ellipsoid goes to zero while the point \vec{l}/l_j remains in its interior. Thus $\rho_{ij}(\varepsilon) \rightarrow 1$ as $\varepsilon \rightarrow 0$. For

$$0 \leq \varepsilon < \min_{1 \leq j \leq \beta} \sin^{-1}(l_j / \|\vec{l}\|_2),$$

the function $\rho(\varepsilon) = \min_{1 \leq i, j \leq \beta} \rho_{ij}(\varepsilon)$ is continuous and $\rho(0) = 1$.

Given $\delta > 0$, choose $\varepsilon > 0$ such that $1 - \delta < \rho(\varepsilon)$. For this $\varepsilon > 0$, fix $K > 0$ as in Claim (3.11). Now fix $L > 0$ such that (i) for all states s of \mathcal{G}_A , and for all $l \geq L$, $\angle(\vec{l}, w(s)B^l) < \varepsilon$ (use Claim (3.12)), (ii) for all paths w in $\mathcal{G}_{\tilde{B}}$ with $|w| < K$, and for all states s in \mathcal{G}_A , $\tilde{I}(w) \leq w(s)B^L$.

Let $l \geq L$, let s be a state in \mathcal{G}_A , and let w be a path in $\mathcal{G}_{\tilde{B}}$ such that $\tilde{I}(w) \not\leq w(s)B^l$. For any state t in \mathcal{G}_A following s along a path of length $l - L$, we have $\tilde{I}(w) \not\leq w(t)B^L$. This and property (ii) give $K \leq |w|$. By Claim (3.11) and the choice of K , we have $\angle(\vec{l}, \tilde{I}(w)) < \varepsilon$. Since $\tilde{I}(w) \not\leq w(s)B^l$, there is an index $1 \leq j \leq \beta$ such that $[w(s)B^l]_j < \tilde{I}(w)_j$. We need to show that $(1 - \delta)[w(s)B^l]_i < \tilde{I}(w)_i$ for $i \neq j$. Now $\tilde{I}(w) \in \{\vec{x} : x_j \geq [w(s)B^l]_j \text{ and } \angle(\vec{l}, \vec{x}) \leq \varepsilon\}$.

Since the coordinate function x_i increases in the direction of any vector \vec{u} with $\angle(\vec{l}, \vec{u}) \leq \varepsilon$, the minimum of x_i over the above set (an infinite truncated cone) is achieved on its flat boundary face $\{\vec{x} : x_j = [w(s)B^l]_j \text{ and } \angle(\vec{l}, \vec{x}) \leq \varepsilon\}$. Hence

$$\begin{aligned} \tilde{I}(w)_i &\geq \min\{x_i : x_j = [w(s)B^l]_j \text{ and } \angle(\vec{l}, \vec{x}) \leq \varepsilon\} \\ &= [w(s)B^l]_j \min\{x_i : x_j = 1 \text{ and } \angle(\vec{l}, \vec{x}) \leq \varepsilon\}. \end{aligned}$$

Since $\angle(\vec{l}, w(s)B^l) < \varepsilon$ (property (i)), we have

$$\begin{aligned} [w(s)B^l]_i &\leq \max\{x_i : x_j = [w(s)B^l]_j \text{ and } \angle(\vec{l}, \vec{x}) \leq \varepsilon\} \\ &= [w(s)B^l]_j \max\{x_i : x_j = 1 \text{ and } \angle(\vec{l}, \vec{x}) \leq \varepsilon\}. \end{aligned}$$

Thus

$$\frac{\tilde{I}(w)_i}{[w(s)B^l]_i} \geq \frac{[w(s)B^l]_j}{[w(s)B^l]_j} \rho_{ij}(\varepsilon) = \rho_{ij}(\varepsilon) \geq \rho(\varepsilon) > 1 - \delta.$$

Thus $\tilde{I}(w) > (1 - \delta)w(s)B^l$. The statement regarding $I(w)$ is proved similarly. \square

(3.14) *Claim.* For all $\delta > 0$, there exists $L > 0$ such that for all $l \geq L$, for all cycles $\mathcal{C} = s_0 s_1 \cdots s_{p-1}$ in \mathcal{G}_A^\sim , there is a resolving cycle of trees $\tilde{\mathcal{T}}$ covering \mathcal{G}_B by state map π satisfying

(i) $\pi(r_i) = \tilde{\varphi}(s_i)$, $0 \leq i \leq p-1$, where $r_0 r_1 \cdots r_{p-1}$ is the cycle in $\tilde{\mathcal{T}}$,
(ii) the tree with root r_i in $\tilde{\mathcal{T}}$ has a leaf at depth 1 with π image $\tilde{g}_{i,j}$, for each $1 \leq j \leq \tilde{n}_i$, where $\tilde{g}_{i,1}, \dots, \tilde{g}_{i,\tilde{n}_i}$ are the followers of state $\tilde{\varphi}(s_i)$ in the graph \mathcal{G}_B^\sim distinct from $\tilde{\varphi}(s_{i+1})$,

(iii) $\mathcal{I}(\tilde{\mathcal{T}})_{i,*} \leq w(s_i)B^l$, $0 \leq i \leq p-1$,

(iv) there exists i_0 , $0 \leq i_0 \leq p-1$, such that $(1 - \delta)w(s_{i_0})B^l \leq \mathcal{I}(\tilde{\mathcal{T}})_{i_0,*}$.

Furthermore, for all $l \geq L$, for all graph homomorphisms $\psi: \mathcal{C} \rightarrow \mathcal{G}_B$ (mapping \mathcal{C} onto a q cycle in \mathcal{G}_B with $q|p$) there is a resolving cycle of trees \mathcal{T} covering \mathcal{G}_B by state map π satisfying

(i') $\pi(r_i) = \psi(s_i)$, $0 \leq i \leq p-1$, where $r_0 r_1 \cdots r_{p-1}$ is the cycle in \mathcal{T} ,

(ii') $\mathcal{I}(\mathcal{T})_{i,*} \leq w(s_i)B^l$, $0 \leq i \leq p-1$,

(iii') there exists i_0 , $0 \leq i_0 \leq p-1$ such that $(1 - \delta)w(s_{i_0})B^l \leq \mathcal{I}(\mathcal{T})_{i_0,*}$.

Proof. Given $\delta > 0$. For this δ , choose $L > 0$ as in Claim (3.13), and fix $l \geq L$. Also fix a cycle $\mathcal{C} = s_0 s_1 \cdots s_{p-1}$ in \mathcal{G}_A^\sim . Let n be the largest integer such that

$$\tilde{I}(\tilde{\varphi}(s_{i-n} s_{i-n+1} \cdots s_i)) \leq w(s_i)B^l, \quad 0 \leq i \leq p-1.$$

We can assume $n \geq 1$ by increasing L if necessary. By this choice of n , there is i_0 , $0 \leq i_0 \leq p-1$, such that

$$\tilde{I}(\tilde{\varphi}(s_{i_0-n-1} s_{i_0-n} \cdots s_{i_0})) \not\leq w(s_{i_0})B^l.$$

Because $l \geq L$, by Claim (3.13) we have

$$(1 - \delta)w(s_{i_0})B^l < \tilde{I}(\tilde{\varphi}(s_{i_0-n-1} s_{i_0-n} \cdots s_{i_0})),$$

so

$$(1 - \delta)w(s_{i_0})B^l < \min\{\tilde{I}(\tilde{\varphi}(s_{i_0-n-1} s_{i_0-n} \cdots s_{i_0})), w(s_{i_0})B^l\}.$$

Abbreviate the componentwise minimum occurring on the right-hand side of the above inequality by \vec{m} .

We define the cycle of trees $\tilde{\mathcal{T}}$ as follows: $\tilde{\mathcal{T}}$ is a cycle of trees such that

$$\tilde{\mathcal{T}}(n, s_0 s_1 \cdots s_{p-1}) \subseteq \tilde{\mathcal{T}} \subseteq \tilde{\mathcal{T}}(n+1, s_0 s_1 \cdots s_{p-1}),$$

where for $1 \leq j \leq \beta$, $[\vec{m} - \tilde{I}(\tilde{\varphi}(s_{i_0-n} \cdots s_{i_0}))]_j$ of the leaves of $\tilde{\mathcal{T}}(n, s_0 s_1 \cdots s_{p-1})$ having depth $n+1$, phase i_0 , and π -image j , are defined to be internal

nodes of $\widetilde{\mathcal{T}}$, and each is given B_{jk} followers (leaves in $\widetilde{\mathcal{T}}$) with π -image k . That this is possible follows from the fact that the number of leaves of $\widetilde{\mathcal{T}}(n, s_0s_1 \cdots s_{p-1})$ having depth $n+1$, phase i_0 , and π -image j is

$$\begin{aligned} [u(s_{i_0-n-1})B^n]_j &= [\widetilde{I}(\widetilde{\varphi}(s_{i_0-n-1} \cdots s_{i_0})) - \widetilde{I}(\widetilde{\varphi}(s_{i_0-n} \cdots s_{i_0}))]_j \\ &\geq [\vec{m} - \widetilde{I}(\widetilde{\varphi}(s_{i_0-n} \cdots s_{i_0}))]_j. \end{aligned}$$

Thus

$$\mathcal{S}(\widetilde{\mathcal{T}})_{i,*} = \begin{cases} \widetilde{I}(\widetilde{\varphi}(s_{i-n} \cdots s_i)) & \text{if } i \neq i_0, \\ \vec{m} & \text{if } i = i_0, \end{cases}$$

so $\mathcal{S}(\widetilde{\mathcal{T}})_{i,*} \leq w(s_i)B^l$, $0 \leq i \leq p-1$, and $(1-\delta)w(s_{i_0})B^l < \mathcal{S}(\widetilde{\mathcal{T}})_{i_0,*}$, which are (iii) and (iv). Properties (i) and (ii) follow because $\widetilde{\mathcal{T}}$ is just $\widetilde{\mathcal{T}}(n, s_0s_1 \cdots s_{p-1})$ with some leaves grown from nodes at depth $n+1 \geq 2$.

A similar construction using $\mathcal{S}(n, s_0s_1 \cdots s_{p-1}, \psi)$ for $\widetilde{\mathcal{T}}(n, s_0s_1 \cdots s_{p-1})$ and $I(w)$ in place of $\widetilde{I}(w)$ gives the tree \mathcal{T} of the claim. \square

The trees $\widetilde{\mathcal{T}}$ and \mathcal{T} of Claim (3.14) will serve as the bases for an inductive construction of cycles of trees satisfying the hypothesis of Observations (3.9) and (3.7) respectively. The induction will proceed by growing at successive steps, new leaves at successive phases.

Define $\vec{f} = (1, 1, \dots, 1)$ to be the row vector of β 1's.

(3.15) *Claim.* There exists $\delta > 0$ and $L > 0$ such that for all paths st of length 2 in \mathcal{G}_A where s has at least two followers (as is the case when s is a state of \mathcal{G}_A^-), if $l \geq L$ and $\vec{v} \in \mathbb{R}^\beta$ satisfies $(1-\delta)w(s)B^l \leq \vec{v}$, then $w(t)B^l \leq \vec{v}B - \vec{f}$.

Proof. Let $\vec{r}^{(B)}$ be a right Perron eigenvector of B (for eigenvalue λ). Assume $\vec{r}^{(B)}$ is scaled so that

$$\|B^n - \lambda^n \vec{r}^{(B)} \vec{l}\|_\infty < \kappa \rho^n, \quad n > 0,$$

where $\kappa > 0$ and $0 < \rho < \lambda$ [S, Theorem 1.2]. It is easy to check that the column vector $\vec{r}^{(A)}$ defined by $\vec{r}^{(A)} = S \vec{r}^{(B)}$ is a right Perron eigenvector for A . We have $w(s) \vec{r}^{(B)} = r_s^{(A)}$ for each state s of \mathcal{G}_A . Fix $\delta > 0$ such that $\nu = \min\{(1-\delta)\lambda r_s^{(A)} - r_t^{(A)} : st \text{ is a path of } \mathcal{G}_A \text{ and } s \text{ has followers distinct from } t\} > 0$. For $l > 0$ we have

$$\begin{aligned} &(1-\delta)w(s)B^{l+1} - w(t)B^l \\ &= (1-\delta)\lambda^{l+1}w(s) \vec{r}^{(B)} \vec{l} - \lambda^l w(t) \vec{r}^{(B)} \vec{l} \\ &\quad + (1-\delta)w(s)[B^{l+1} - \lambda^{l+1} \vec{r}^{(B)} \vec{l}] - w(t)[B^l - \lambda^l \vec{r}^{(B)} \vec{l}] \\ &\geq (1-\delta)\lambda^{l+1}w(s) \vec{r}^{(B)} \vec{l} - \lambda^l w(t) \vec{r}^{(B)} \vec{l} - \mu \rho^l \vec{f}, \end{aligned}$$

where $\mu > 0$ is fixed and independent of l, s , and t ; so

$$\begin{aligned} (1-\delta)w(s)B^{l+1} - w(t)B^l &\geq \lambda^l [(1-\delta)\lambda r_s^{(A)} - r_t^{(A)}] \vec{l} - \mu \rho^l \vec{f} \\ &\geq \lambda^l \nu \vec{l} - \mu \rho^l \vec{f}. \end{aligned}$$

Fix $L > 0$ such that for $l \geq L$, $\lambda^l \nu(\min_{1 \leq i \leq \beta} l_i) - \rho^l \mu > 1$. For $l \geq L$, if $\vec{v} \in \mathbb{R}^\beta$ is such that $(1 - \delta)w(s)B^l \leq \vec{v}$, then

$$\begin{aligned} \vec{v}B - w(t)B^l &\geq (1 - \delta)w(s)B^{l+1} - w(t)B^l \\ &\geq \lambda^l v \vec{l} - \mu \rho^l \vec{f} \geq \vec{f}, \end{aligned}$$

so $w(t)B^l \leq \vec{v}B - \vec{f}$ as claimed. \square

(3.16) *Claim.* There are $\delta > 0$ and $L > 0$ such that for all $l \geq L$, for all cycles $\mathcal{C} = s_0 s_1 \cdots s_{p-1}$ in \mathcal{G}_A , if there is a resolving cycle of trees $\widetilde{\mathcal{T}}$ covering \mathcal{G}_B by state map π satisfying (i), (ii), (iii), and (iv) of Claim (3.14) (for this δ and $l \geq L$), then there is a resolving cycle of trees $\widetilde{\mathcal{T}}' \supseteq \widetilde{\mathcal{T}}$ covering \mathcal{G}_B , still satisfying (i), (ii), (iii), (iv) of Claim (3.14), while also satisfying

$$(v) \quad w(s_{i_0+1})B^l = \mathcal{J}(\widetilde{\mathcal{T}})_{i_0+1,*}.$$

For the same $\delta > 0$ and $L > 0$, for all $l \geq L$, for all cycles $\mathcal{C} = s_0 s_1 \cdots s_{p-1}$ in \mathcal{G}_A , and all graph homomorphisms $\psi: \mathcal{C} \rightarrow \mathcal{G}_B$, if there is a resolving cycle of trees \mathcal{T} covering \mathcal{G}_B by state map π satisfying (i'), (ii'), and (iii') of Claim (3.14), then there is a resolving cycle of trees $\mathcal{T}' \supseteq \mathcal{T}$ covering \mathcal{G}_B still satisfying (i'), (ii'), and (iii') of Claim (3.14) while also satisfying

$$(iv') \quad \rho w(s_{i_0+1})B^l \leq \mathcal{J}(\mathcal{T}')_{i_0+1,*},$$

where $\rho = \min_{1 \leq j \leq \beta} \mathcal{J}(\mathcal{T})_{i_0,j} / [w(s_{i_0})B^l]_j \geq 1 - \delta$, and even satisfying

$$(v') \quad w(s_{i_0+1})B^l = \mathcal{J}(\mathcal{T}')_{i_0+1,*}$$

if state s_{i_0} in \mathcal{G}_A has followers distinct from s_{i_0+1} .

Proof. By Claim (3.15) we can fix $\delta > 0$ and $L > 0$ such that for all $l \geq L$, for all paths st of length 2 in \mathcal{G}_A where state s has followers distinct from t , if $\vec{v} \in \mathbb{R}^\beta$ is such that $(1 - \delta)w(s)B^l \leq \vec{v}$, then

$$w(t)B^l \leq \vec{v}B - \vec{f}.$$

Assume $l \geq L$, $\mathcal{C} = s_0 s_1 \cdots s_{p-1}$ is a cycle in \mathcal{G}_A , and $\widetilde{\mathcal{T}}$ is a resolving cycle of trees covering \mathcal{G}_B by state map π and satisfying (i), (ii), (iii), and (iv) for this δ and l . For the remainder of this proof assign mass $e_{\pi(s)}$ to each state s of $\widetilde{\mathcal{T}}$.

By (iv), the total mass $\mathcal{J}(\mathcal{T})_{i_0,*}$ of the internal nodes of $\widetilde{\mathcal{T}}$ at phase i_0 satisfies $(1 - \delta)w(s_{i_0})B^l \leq \mathcal{J}(\mathcal{T})_{i_0,*}$. Now $\mathcal{J}(\mathcal{T})_{i_0,*}B$ is the total mass of all nodes of $\widetilde{\mathcal{T}}$ at phase $i_0 + 1$. Thus the total mass of all nodes of $\widetilde{\mathcal{T}}$ at phase $i_0 + 1$ excluding the \tilde{n}_i leaves at depth 1 specified in (ii) satisfies

$$\mathcal{J}(\widetilde{\mathcal{T}})_{i_0,*}B - \sum_{j=1}^{\tilde{n}_i} e_{\tilde{g}_{i,j}} \geq \mathcal{J}(\widetilde{\mathcal{T}})_{i_0,*}B - \vec{f} \geq w(s_{i_0+1})B^l,$$

where the last inequality follows from $(1 - \delta)w(s_{i_0})B^l \leq \mathcal{J}(\mathcal{T})_{i_0,*}$ and the choice of $\delta > 0$ given by Claim (3.15). Thus the total mass of the leaves of $\widetilde{\mathcal{T}}$

at phase $i_0 + 1$ excluding the \tilde{n}_i leaves specified in (ii) is

$$\begin{aligned} \mathcal{J}(\widetilde{\mathcal{T}})_{i_0, *}\mathbf{B} - \sum_{j=1}^{\tilde{n}_i} e_{\tilde{g}_{i,j}} - \mathcal{J}(\widetilde{\mathcal{T}})_{i_0+1, *} \\ \geq w(s_{i_0+1})\mathbf{B}^l - \mathcal{J}(\widetilde{\mathcal{T}})_{i_0+1, *} \geq 0. \end{aligned}$$

Among these leaves, for each $1 \leq j \leq \beta$, choose $[w(s_{i_0+1})\mathbf{B}^l]_j - \mathcal{J}(\widetilde{\mathcal{T}})_{i_0+1, j}$ leaves, each having π image j and declare them to be internal nodes of $\widetilde{\mathcal{T}}'$ giving each B_{jk} children in $\widetilde{\mathcal{T}}'$ with π -image k , $1 \leq k \leq \beta$.

Now $\widetilde{\mathcal{T}}'$ satisfies (i), (ii), (iii), (iv), and $w(s_{i_0+1})\mathbf{B}^l = \mathcal{J}(\widetilde{\mathcal{T}}')_{i_0+1, *}$, which is (v).

Now assume $\mathcal{C} = s_0 s_1 \cdots s_{p-1}$ is a cycle in \mathcal{G}_A , and assume \mathcal{T} is a resolving cycle of trees covering \mathcal{G}_B by state map π satisfying (i'), (ii'), and (iii') of Claim (3.14) for the δ and $l \geq L$ specified above. As before, assign mass $e_{\pi(s)}$ to node s of \mathcal{T} . The total mass of all nodes of \mathcal{T} at phase $i_0 + 1$ is $\mathcal{J}(\mathcal{T})_{i_0, *}\mathbf{B}$. Setting $\vec{m} = \min\{w(s_{i_0+1})\mathbf{B}^l, \mathcal{J}(\mathcal{T})_{i_0, *}\mathbf{B}\}$ and arguing as in the case of $\widetilde{\mathcal{T}}$, we can declare enough leaves of \mathcal{T} at phase $i_0 + 1$ to be internal nodes of \mathcal{T}' to bring $\mathcal{J}(\mathcal{T}')_{i_0+1, *}$ up to equal \vec{m} . Setting

$$\rho = \min_{1 \leq j \leq \beta} \mathcal{J}(\mathcal{T})_{i_0 j} / [w(s_{i_0})\mathbf{B}^l]_j,$$

we have $\rho w(s_{i_0+1})\mathbf{B}^l \leq \rho w(s_{i_0})\mathbf{B}^{l+1} \leq \mathcal{J}(\mathcal{T})_{i_0, *}\mathbf{B}$, where by (iii'), $1 - \delta \leq \rho$. So $\rho w(s_{i_0+1})\mathbf{B}^l \leq \vec{m} = \mathcal{J}(\mathcal{T}')_{i_0+1, *}$. This is (iv').

If state s_{i_0} has followers in \mathcal{G}_A other than s_{i_0+1} , it follows from Claim (3.15), and $(1 - \delta)w(s_{i_0})\mathbf{B}^l \leq \mathcal{J}(\mathcal{T})_{i_0, *}$, and the choice of δ and $l \geq L$ that $w(s_{i_0+1})\mathbf{B}^l \leq \mathcal{J}(\mathcal{T})_{i_0, *}\mathbf{B}$, so $\mathcal{J}(\mathcal{T}')_{i_0+1, *} = \vec{m} = w(s_{i_0+1})\mathbf{B}^l$ in this case. This is (v'). \square

(3.17) *Claim.* There is $L > 0$ such that for all $l \geq L$, for all cycles $\mathcal{C} = s_0 s_1 \cdots s_{p-1}$ in \mathcal{G}_A , there is a resolving cycle of tress $\widetilde{\mathcal{T}}$ covering \mathcal{G}_B by state map π satisfying

- (i) $\pi(r_i) = \tilde{\varphi}(s_i)$, $0 \leq i \leq p - 1$, where $r_0 r_1 \cdots r_{p-1}$ is the cycle in \mathcal{T} ,
- (ii) the tree in $\widetilde{\mathcal{T}}$ with root r_i has a leaf at depth 1 with π image $\tilde{g}_{i,j}$ for each $1 \leq j \leq \tilde{n}_i$, where $\tilde{g}_{i,1}, \dots, \tilde{g}_{i,\tilde{n}_i}$ are the followers of state $\tilde{\varphi}(s_i)$ in the graph $\mathcal{G}_{\widetilde{B}}$ that are distinct from $\tilde{\varphi}(s_{i+1})$,
- (iii) $w(s_i)\mathbf{B}^l = \mathcal{J}(\widetilde{\mathcal{T}})_{i, *}$, $0 \leq i \leq p - 1$.

Furthermore, if $\Sigma_A \xrightarrow{\text{per}} \Sigma_B$, then for all $l \geq L$, for all cycles $\mathcal{C} = s_0 s_1 \cdots s_{p-1}$ in \mathcal{G}_A , there is a resolving p -cycle of trees \mathcal{T} covering \mathcal{G}_B by state map π satisfying

- (iii') $w(s_i)\mathbf{B}^l = \mathcal{J}(\mathcal{T})_{i, *}$, $0 \leq i \leq p - 1$.

Proof. Let $\delta > 0$ and $L_1 > 0$ satisfy the conditions on δ and L specified in the statement of Claim (3.16). For this δ , choose $L_2 > 0$ satisfying the conditions on L specified in Claim (3.14). Set $L_3 = \max\{L_1, L_2\}$.

Let $l \geq L_3$ and let $\mathcal{C} = s_0 s_1 \cdots s_{p-1}$ be any cycle in \mathcal{G}_A . By Claim (3.14), there is a resolving cycle of trees $\widetilde{\mathcal{T}}^{(0)}$ covering \mathcal{G}_B by state map π satisfying

conditions (i), (ii), (iii), and (iv) of Claim (3.14). By p successive applications of Claim (3.16), there are cycles of trees $\widetilde{\mathcal{T}}^{(0)} \subseteq \widetilde{\mathcal{T}}^{(1)} \subseteq \dots \subseteq \widetilde{\mathcal{T}}^{(p)}$ all satisfying (i), (ii), and (iii) of Claim (3.14); furthermore, for $1 \leq k \leq p$, $\widetilde{\mathcal{T}}^{(k)}$ satisfies

$$w(s_{i_0+j})B^l = \mathcal{T}(\widetilde{\mathcal{T}}^{(k)})_{i_0+j,*}, \quad 1 \leq j \leq k.$$

This follows directly from Claim (3.16) for $k = 1$. For $k > 1$, at the k th application of Claim (3.16),

$$(1 - \delta)w(s_{i_0+k-1})B^l \leq w(s_{i_0+k-1})B^l = \mathcal{T}(\widetilde{\mathcal{T}}^{(k-1)})_{i_0+k-1,*},$$

which is condition (iv) of (3.14) (with $i_0 + k - 1$ playing the role of i_0), so Claim (3.16) gives $\widetilde{\mathcal{T}}^{(k)}$ as described. Now $\widetilde{\mathcal{T}} = \widetilde{\mathcal{T}}^{(p)}$ satisfies properties (i), (ii), and (iii) of the present claim.

Now let $\mathcal{C} = s_0s_1 \dots s_{p-1}$ be any cycle in \mathcal{G}_A . Because $\Sigma_A \xrightarrow{\text{per}} \Sigma_B$, there is a graph homomorphism $\psi: \mathcal{C} \rightarrow \mathcal{G}_B$. By Claim (3.14) there is a resolving cycle of trees $\mathcal{T}^{(0)}$ covering \mathcal{G}_B by state map π satisfying conditions (i'), (ii'), and (iii') of Claim (3.14).

By the irreducibility of \mathcal{G}_A , there is a state on \mathcal{C} with at least two followers; we let $s_{i_0+k_0}$ be the first such state following s_{i_0-1} on \mathcal{C} . Note $0 \leq k_0 \leq p - 1$. By k_0 successive applications of Claim (3.16), there are cycles of trees $\mathcal{T}^{(0)} \subseteq \mathcal{T}^{(1)} \subseteq \dots \subseteq \mathcal{T}^{(k_0)}$ satisfying conditions (i') and (ii') of Claim (3.14); furthermore, for $1 \leq k \leq k_0$, $\mathcal{T}^{(k)}$ satisfies

$$\rho w(s_{i_0+j})B^l \leq \mathcal{T}(\mathcal{T}^{(k)})_{i_0+j,*}, \quad 1 \leq j \leq k,$$

where $1 - \delta \leq \rho$. In particular, $(1 - \delta)w(s_{i_0+k_0})B^l \leq \mathcal{T}(\mathcal{T}^{(k_0)})_{i_0+k_0,*}$.

Since state $s_{i_0+k_0}$ has at least two followers, we can apply Claim (3.16) once again to produce $\mathcal{T}^{(k_0+1)}$ containing $\mathcal{T}^{(k_0)}$ satisfying $w(s_{i_0+k_0+1})B^l = \mathcal{T}(\mathcal{T}^{(k_0+1)})_{i_0+k_0+1,*}$. Now we apply Claim (3.16) $p - 1$ more times to produce cycles of trees $\mathcal{T}^{(k_0+1)} \subseteq \mathcal{T}^{(k_0+2)} \subseteq \dots \subseteq \mathcal{T}^{(k_0+p)}$ all satisfying (i') and (ii') of Claim (3.14); furthermore, for $1 \leq k \leq p$, $\mathcal{T}^{(k_0+k)}$ satisfies $w(s_{i_0+k_0+j})B^l = \mathcal{T}(\mathcal{T}^{(k_0+k)})_{i_0+k_0+j,*}$, $1 \leq j \leq k$. We show this as follows. Just before the k th of these $p - 1$ applications of Claim (3.16), we have $\rho = 1$ (by inductive hypothesis), so conclusion (iv') of Claim (3.16) becomes

$$w(s_{i_0+k_0+1+k})B^l \leq \mathcal{T}(\mathcal{T}^{(k_0+1+k)})_{i_0+k_0+1+k,*},$$

while property (ii') of Claim (3.14) is preserved, giving

$$w(s_{i_0+k_0+1+k})B^l = \mathcal{T}(\mathcal{T}^{(k_0+1+k)})_{i_0+k_0+1+k,*}$$

as asserted. Now $\mathcal{T} = \mathcal{T}^{(k_0+p)}$ satisfies property (iii') of the present claim. \square

Using Observations (3.7), (3.9), and Claim (3.17), we can prove the Isolated Cycle Lemma (3.6).

Proof of (3.6). Let $N > 0$ satisfy the conditions on L specified in the statement of Claim (3.17). Let $\mathcal{C} = s'_0s'_1 \dots s'_{p-1}$ be an isolated fat cycle in \mathcal{G}' . Now $s'_i = {}_0[s_{i-L} \dots s_i]_L$ for some p cycle $s_0s_1 \dots s_{p-1}$ in \mathcal{G}_A . By Lemma (3.8), either $s'_i \cap \Sigma_{\widetilde{A}} = \emptyset$, $0 \leq i \leq p - 1$, or $s'_i \cap \Sigma_{\widetilde{A}}$ is an $(L + 1)$ -block ${}_0[s_{i-L} \dots s_i]_L \cap \Sigma_{\widetilde{A}}$ in $\Sigma_{\widetilde{A}}$, $0 \leq i \leq p - 1$. We treat the case of nonempty intersection first.

As in the statement of Observation (3.9), let $\tilde{f}_{i,1}, \dots, \tilde{f}_{i,n_i}$ be the followers of state s_i in the graph $\mathcal{G}_{\tilde{A}}$ distinct from the follower s_{i+1} , if there are any. Let $l \geq N$. For this l , let $\tilde{\mathcal{T}}$ be a p -cycle of trees covering \mathcal{G}_B by state map π whose existence is asserted by Claim (3.17). We show that $\tilde{\mathcal{T}}$ satisfies the hypothesis of Observation (3.9). $\tilde{\mathcal{T}}$ satisfies property (i) of Observation (3.9): $\pi(r_i) = \tilde{\varphi}(s_i)$, $0 \leq i \leq p-1$, where $r_0 r_1 \cdots r_{p-1}$ is the cycle in $\tilde{\mathcal{T}}$, because this is just property (i) of Claim (3.17). Also $\tilde{\mathcal{T}}$ satisfies property (ii) of Observation (3.9):

The tree in $\tilde{\mathcal{T}}$ with root r_i has a leaf at depth 1 with π -image $\tilde{\varphi}(\tilde{f}_{i,j})$, $1 \leq j \leq n_i$,

because $\tilde{\mathcal{T}}$ satisfies property (ii) of Claim (3.17):

The tree in $\tilde{\mathcal{T}}$ with root r_i has a leaf at depth 1 with π -image $\tilde{g}_{i,j}$ for each $1 \leq j \leq \tilde{n}_i$, where $\tilde{g}_{i,1}, \dots, \tilde{g}_{i,\tilde{n}_i}$ are the followers of state $\tilde{\varphi}(s_i)$ in the graph $\mathcal{G}_{\tilde{B}}$ that are distinct from $\tilde{\varphi}(s_{i+1})$,

and each of the states $\tilde{\varphi}(\tilde{f}_{i,j})$ is among $\tilde{g}_{i,1}, \dots, \tilde{g}_{i,\tilde{n}_i}$. Finally $\tilde{\mathcal{T}}$ satisfies property (iii) of Observation (3.9): $w(s'_i) = \mathcal{J}(\tilde{\mathcal{T}})_{i,*}$, $0 \leq i \leq p-1$, when $w(s'_i) = S_{s_i,*} B^l = w(s_i) B^l$, $0 \leq i \leq p-1$, because this is property (iii) of Claim (3.17). So by Observation (3.9), the isolated fat cycle $s'_0 s'_1 \cdots s'_{p-1}$ in \mathcal{G}' with masses given by $w(s'_i) = S_{s_i,*} B^l$ can be split into lean states by rounds of legal, $\tilde{\varphi}$ -respecting splittings.

Now we treat the case where $s'_i \cap \Sigma_{\tilde{A}} = \emptyset$, $0 \leq i \leq p-1$. Let $l \geq N$ and let \mathcal{T} be the cycle of trees covering \mathcal{G}_B by state map π satisfying $w(s_i) B^l = \mathcal{J}(\mathcal{T})_{i,*}$, $0 \leq i \leq p-1$, whose existence is asserted by Claim (3.17). Now \mathcal{T} satisfies the hypothesis of Observation (3.7) (when $w(s'_i) = S_{s_i,*} B^l$), so the isolated fat cycle $s'_0 s'_1 \cdots s'_{p-1}$ in \mathcal{G}' can be split by legal (and vacuously $\tilde{\varphi}$ -respecting) rounds of splitting into lean states. This completes the proof of the Isolated Cycle Lemma (3.6). \square

We now complete the proof of Theorem (1.1).

Recall that by Lemma (3.1) we can assume without loss of generality that $A, B, R, S, \Sigma_{\tilde{A}} \subseteq \Sigma_A$, $\Sigma_{\tilde{B}} \subseteq \Sigma_B$, and $\tilde{\varphi}: \Sigma_{\tilde{A}} \rightarrow \Sigma_{\tilde{B}}$ satisfy all the hypotheses of the Marker Splitting Lemma (3.4). In addition we are assuming that $\Sigma_A \xrightarrow{\text{per}} \Sigma_B$.

Let $N > 0$ be as specified in the Isolated Cycle Lemma (3.6). Fix any $n \geq N$. Replace the matrix S by the matrix $S B^n$ in the statement of the Marker Splitting Lemma (3.4) to obtain a graph \mathcal{G}' in which any fat cycle $s'_0 s'_1 \cdots s'_{p-1}$ has state masses given by $w(s'_i) = S_{s_i,*} B^n$, where $s'_i = {}_0[s_{i-L} \cdots s_i]_L$, and $s_0 s_1 \cdots s_{p-1}$ is a p cycle in \mathcal{G}_A .

By the Isolated Cycle Lemma (3.6) and the choice of $n \geq N$, the isolated fat cycle $s'_0 s'_1 \cdots s'_{p-1}$ in \mathcal{G}' can be split by rounds of $\tilde{\varphi}$ -respecting splittings entirely into lean states.

Let \mathcal{G}'' be a graph obtained from \mathcal{G}' by splitting all the isolated fat cycles in \mathcal{G}' entirely into lean states by $\tilde{\varphi}$ -respecting splittings. Because \mathcal{G}'' has no fat cycles, all remaining fat paths in \mathcal{G}'' have finite length. By Claim (3.5), the terminal fat state of any maximal length fat path in \mathcal{G}'' can be split by a legal,

$\tilde{\varphi}$ -respecting splitting entirely into lean states, thereby shortening the maximal length fat path by length 1. Iteratively eating up fat paths in this way, we can split all fat states in \mathcal{G}'' into lean states in at most M rounds of $\tilde{\varphi}$ -respecting splittings, where M is the length of the longest fat path in \mathcal{G}'' .

Denote the resulting graph by $\mathcal{G}_{A'}$, where A' is its transition matrix. Because the graph $\mathcal{G}_{A'}$ is obtained from \mathcal{G}' by rounds of $\tilde{\varphi}$ -respecting splittings, $\mathcal{G}_{A'}$, like \mathcal{G}' , satisfies property (1) of the Marker Splitting Lemma (3.4):

If s is a state of $\mathcal{G}_{A'}$ and $s \cap \Sigma_{\tilde{A}} \neq \emptyset$ then $s \cap \Sigma_{\tilde{A}}$ is a union of $(L+1)$ -blocks in $\Sigma_{\tilde{A}}$ and

$$w(s) \geq \Sigma[e_{\tilde{\varphi}(s_L)} : {}_0[s_0 s_1 \cdots s_L]_L \cap s \cap \Sigma_{\tilde{A}} \neq \emptyset].$$

Together with the fact that all states of $\mathcal{G}_{A'}$ are lean, this gives that each state s of $\mathcal{G}_{A'}$ intersects $\Sigma_{\tilde{A}}$ in at most one $(L+1)$ -block in $\Sigma_{\tilde{A}}$. If $s \cap \Sigma_{\tilde{A}} \neq \emptyset$, then

$$s \cap \Sigma_{\tilde{A}} = {}_0[s_0 s_1 \cdots s_L]_L \cap \Sigma_{\tilde{A}},$$

where $s_0 s_1 \cdots s_L$ is an $(L+1)$ -path in $\mathcal{G}_{\tilde{A}}$ and $w(s) = e_{\tilde{\varphi}(s_L)}$. Let $\mu: \Sigma_{A'} \rightarrow \Sigma_B$ be the right resolving 1-block mass map defined by the state map $\mu(s) = t$ where $w(s) = e_t$. If $\alpha: \Sigma_{A'} \rightarrow \Sigma_A$ is the ancestor map, we have for each state s of $\mathcal{G}_{A'}$ with $s \cap \Sigma_{\tilde{A}} \neq \emptyset$, $\mu(s) = \tilde{\varphi}(s_L)$ and $\alpha(s) = s_0$, where

$$s \cap \Sigma_{\tilde{A}} = {}_0[s_0 s_1 \cdots s_L]_L \cap \Sigma_{\tilde{A}}.$$

Thus

$$(\mu \circ \alpha^{-1})|_{\Sigma_{\tilde{A}}} = \tilde{\varphi} \circ \sigma^L.$$

So $\mu \circ \alpha^{-1}: \Sigma_A \rightarrow \Sigma_B$ is an extension of $\tilde{\varphi} \circ \sigma^L$. If we define $\varphi: \Sigma_A \rightarrow \Sigma_B$ by $\varphi = \mu \circ \alpha^{-1}$, then $\varphi \circ \sigma^{-L}$ is an extension of $\tilde{\varphi}: \Sigma_{\tilde{A}} \rightarrow \Sigma_B$.

Now we can apply [A1] to conclude that there is a 1-to-1 almost everywhere right-closing map $\hat{\varphi}: \Sigma_A \rightarrow \Sigma_B$. The construction in [A1] lowers the degree of a given map $\varphi: \Sigma_A \rightarrow \Sigma_B$ by altering the image of $x \in \Sigma_A$ between certain marker blocks in x (these are preimages of *resolving blocks*). With some care, we can choose these marker blocks to be outside of $\Sigma_{\tilde{A}}$, thus modifying the construction in [A1] to ensure that $\hat{\varphi}$, like φ , is an extension of $\tilde{\varphi} \circ \sigma^L$.

This completes the construction of $\varphi: \Sigma_A \rightarrow \Sigma_B$ and the proof of Theorem (1.1).

4. THE SOFIC RESULT

The proof of Theorem (1.2) from Theorem (1.1) closely follows the proof of the special case given in [KM], where the range shift of finite type Σ_B is the full n -shift.

We first give the part of the argument borrowed directly from [KM]. Lemmas (4.1) and (4.2) are a new complication.

Let m be a *resolving block* for the canonical right cover $\pi_R: \Sigma_R \rightarrow S$ (all paths w in Σ_R with $\pi_R(w) = m$ end in the same state). Since π_R is right resolving and left closing, any point in S that contains an occurrence of m has a single preimage under π_R . If we define the sofic system $T \subseteq S$ by

$$T = \{y \in S : \text{the word } m \text{ does not occur in } y\},$$

then the points of S with more than one preimage under π_R are all in T .

Now the preimage of T under π_R , call it $\Sigma = \pi_R^{-1}(T) \subseteq \Sigma_A$ is a shift of finite type because it is obtained from Σ_A by deleting any point of Σ_A containing a word with π_R image m . The shift of finite type Σ contains all points x in Σ_A with at least two points in $\pi_R^{-1}\pi_R(x)$.

Using Lemmas (4.1) and (4.2) below, we construct a right-closing map $\psi: T \rightarrow \Sigma_B$ and then define $\tilde{\varphi}: \Sigma \rightarrow \Sigma_B$ by

$$\tilde{\varphi} = \psi \circ (\pi_R|_{\Sigma}).$$

We then use the extension Theorem (1.1) to extend $\tilde{\varphi}$ to a right-closing map $\varphi: \Sigma_R \rightarrow \Sigma_B$. Finally we define $\pi: S \rightarrow \Sigma_B$ by $\pi = \varphi \circ \pi_R^{-1}$. The map π is well defined, for if $x, x' \in \Sigma_R$ are distinct points with $\pi_R(x) = \pi_R(x')$, then $x, x' \in \Sigma$ and

$$\varphi(x) = \tilde{\varphi}(x) = \psi(\pi_R(x)) = \psi(\pi_R(x')) = \tilde{\varphi}(x') = \varphi(x').$$

By a compactness argument, π is a block map. Also, π never identifies two left asymptotic points in S whose common left tail contains an occurrence of the resolving block m because their (unique) preimages under π_R are themselves left asymptotic in Σ_R , and $\varphi: \Sigma_R \rightarrow \Sigma_B$ is right closing. Thus π is right closing almost everywhere.

The only remaining job is to construct a right-closing map $\psi: T \rightarrow \Sigma_B$. To do this we need two technical lemmas.

(4.1) **Lemma.** *Let T be a sofic system and let Γ be an aperiodic shift of finite type with $h(T) < h(\Gamma)$ and $T \xrightarrow{\text{per}} \Gamma$. Then there is a shift of finite type Λ with $T \subseteq \Lambda$, $h(\Lambda) < h(\Gamma)$, and $\Lambda \xrightarrow{\text{per}} \Gamma$.*

Proof. Define Λ_n to be the shift of finite type whose allowed n blocks are all the n blocks occurring in points of T . Note that $T \subseteq \Lambda_n$.

At this point we prove a claim to be used only in this proof.

(4.2) **Claim.** For all $L > 0$, there exists $N > 0$ such that for all $n \geq N$ and for all $l \leq L$, Λ_n and T have the same points of period l .

Proof of Claim. Let $\pi_R: \Sigma_R \rightarrow T$ be the canonical right cover of T and let d be the number of states of \mathcal{G}_R . We show that $\Lambda_{(d+1)L}$ and T have the same points of period at most L as follows. Let $y \in \Lambda_{(d+1)L}$ be a point of period $l \leq L$. Denote

$${}_0(y)_{l-1} = y_0 y_1 \cdots y_{l-1} = c.$$

There is a $(d+1)L$ path $w = w_0 w_1 \cdots w_{(d+1)L-1}$ in \mathcal{G}_R that is π_R -labelled by ${}_0(y)_{(d+1)L-1} = c^m r$, where $m = \lfloor (d+1)L/l \rfloor \geq d+1$ and $|r| < l$. Now the states $w_0, w_l, \dots, w_{(m-1)l}$ in \mathcal{G}_R cannot all be distinct since $m > d$, so $w_{il} = w_{jl}$ for some i, j with $0 \leq i < j \leq m-1$, and there is a cycle through the state w_{il} of \mathcal{G}_R that is π_R -labelled by c^{j-i} . Thus the periodic point y is an element of $\pi_R(\Sigma_R) = T$. This establishes the claim.

We finish the proof of Lemma (4.1). Because the shift of finite type Γ is aperiodic, there is $L > 0$ such that for all $l \geq L$, Γ has a point of period l . For this L , choose N as in the above claim, and fix $n \geq N$; the periodic points of period at most L in Λ_n and T coincide. Now $T \xrightarrow{\text{per}} \Gamma$, so for each point y of Λ_n of period $p \leq L$, there is a point in Γ whose period divides p . Also, by the choice of L , for each periodic point in Λ_n of period exceeding

L , there is a point in Γ of equal period. Thus $\Lambda_n \xrightarrow{\text{per}} \Gamma$ for all $n \geq N$. Since $h(T) < h(\Gamma)$ and $h(\Lambda_n) \rightarrow h(T)$ as $n \rightarrow \infty$, we can choose $n \geq N$ such that $h(\Lambda_n) < h(\Gamma)$ and set $\Lambda = \Lambda_n$. \square

(4.3) **Lemma.** *Let Λ and Γ be shifts of finite type with Γ aperiodic, $h(\Lambda) < h(\Gamma)$, and $\Lambda \xrightarrow{\text{per}} \Gamma$. Then there is a right-closing map $\tilde{\psi}: \Lambda \rightarrow \Gamma$.*

Proof. Following [B], denote by $\pi_j(\Sigma)$ the number of periodic points having least period j in a shift of finite type Σ . Since $h(\Lambda) < h(\Gamma)$ there is some $L > 0$ such that $\pi_l(\Lambda) \leq \pi_l(\Gamma)$, $l \geq L$. Because $\Lambda \xrightarrow{\text{per}} \Gamma$, for each $l > 0$, if $\pi_l(\Lambda) > 0$, then $\pi_{q(l)}(\Gamma) > 0$ for some $q(l)$ with $q(l)|l$.

At this point, we use [B, Lemma (2.1)]. We state the lemma here.

(4.4) **Lemma [B, Lemma (2.1)].** *Suppose Γ is an irreducible shift of finite type of positive entropy and $\pi_q(\Gamma) > 0$, and M_0, M_1, \dots, M_k are positive integers. Then there is an irreducible shift of finite type $\hat{\Gamma}$ such that*

- (i) *there is a right-closing factor map $\hat{\phi}: \hat{\Gamma} \rightarrow \Gamma$,*
- (ii) *if Γ is aperiodic, then $\hat{\Gamma}$ is aperiodic,*
- (iii) *$\pi_j(\hat{\Gamma}) = \pi_j(\Gamma)$ if j is not among q, qM_0, \dots, qM_k ,*
- (iv) *$\pi_q(\hat{\Gamma}) = \pi_q(\Gamma) - q + q|\{i : M_i = 1\}|$,*
- (v) *$\pi_{qM_i}(\hat{\Gamma}) = \pi_{qM_i}(\Gamma) + qM_i|\{j : M_j = M_i\}|$, for those i with $M_i > 1$.*

As remarked in [B], $\hat{\phi}: \hat{\Gamma} \rightarrow \Gamma$ as constructed there is right closing.

For each $1 \leq l < L$, apply the lemma iteratively, at the l th stage ‘blowing up’ a periodic orbit of least period $q(l)$ into $\pi_l(\Lambda)$ periodic points of least period l . Specifically, set $M_0 = 1$, $M_1 = M_2 = \dots = M_{\pi_l(\Lambda)/l} = l/q(l)$. The net effect of this is to construct a shift of finite type $\hat{\Gamma}$ satisfying

- (i) *there is a right-closing factor map $\hat{\phi}: \hat{\Gamma} \rightarrow \Gamma$,*
- (ii) *$\hat{\Gamma}$ is aperiodic,*
- (iii) *$\pi_l(\hat{\Gamma}) = \pi_l(\Gamma) \geq \pi_l(\Lambda)$, $l \geq L$,*
- (iv) *$\pi_l(\hat{\Gamma}) = \pi_l(\Gamma) + \pi_l(\Lambda)$, $1 \leq l < L$.*

In particular, $\pi_l(\hat{\Gamma}) \geq \pi_l(\Lambda)$ for $l \geq 1$. This, together with $h(\hat{\Gamma}) = h(\Gamma) > h(\Lambda)$, allows us to use the Embedding Theorem [Kr2, Theorem 3] to conclude that there is an embedding $\theta: \Lambda \rightarrow \hat{\Gamma}$. We now define $\tilde{\psi}: \Lambda \rightarrow \Gamma$ by $\tilde{\psi} = \hat{\phi} \circ \theta$. Since θ is 1-to-1 and $\hat{\phi}$ is right closing, $\tilde{\psi}$ is right closing also. \square

Recall that to prove Theorem (1.2), it remains to construct a right-closing map $\psi: T \rightarrow \Sigma_B$ using Lemmas (4.1) and (4.3).

Now

$$h(T) = h(\Sigma) < h(\Sigma_R) = h(S) = h(\Sigma_B).$$

The inequality holds because the shift of finite type $\Sigma \subseteq \Sigma_R$ is obtained from the irreducible shift of finite type Σ_R by disallowing the blocks whose π_R image is the resolving block m . Also $T \xrightarrow{\text{per}} \Sigma_B$ because $T \subseteq S$ and $S \xrightarrow{\text{per}} \Sigma_B$. Thus Lemma (4.1) gives a shift of finite type Λ with $T \subseteq \Lambda$, $h(\Lambda) < h(\Sigma_B)$, and $\Lambda \xrightarrow{\text{per}} \Sigma_B$. Now Lemma (4.3) gives a right closing map $\tilde{\psi}: \Lambda \rightarrow \Sigma_B$. Finally, we define $\psi: T \rightarrow \Sigma_B$ as $\psi = \tilde{\psi}|_T$. This completes the proof of Theorem (1.2).

5. AN EXAMPLE

Let $S \subseteq \{0, 1\}^{\mathbb{Z}}$ be sofic system defined as follows. Between any two 1's in a point of S , there occurs an even number of 0's. The sofic system S is known as the *even system*.

The canonical right cover of S is the *golden mean system* Σ_A defined by the graph \mathcal{G}_A depicted in Figure 1.

The 1-block canonical covering map $\pi_R: \Sigma_A \rightarrow S$ is given by $\pi_R(a) = 1$, $\pi_R(b) = \pi_R(c) = 0$. Since π_R is left resolving, S is almost finite type. Since Σ_A has periodic points of all orders, we have $S \xrightarrow{\text{per}} \Sigma_A$. By Theorem (1.2) there is a 1-to-1 a.e. right-closing a.e. factor map $\pi: S \rightarrow \Sigma_A$. We will construct a specific π following the construction outlined in the proofs of Theorems (1.1) and (1.2).

The only points x in Σ_A with $|\pi_R^{-1}\pi_R(x)| > 1$ are the two points in the orbit of $(bc)^\infty$; π_R maps both of these points to $0^\infty \in S$. This is because the symbol 1 is a resolving block for π_R . Define $T \subseteq S$ as $T = \{0^\infty\}$ and $\Sigma \subseteq \Sigma_A$ as $\Sigma = \pi_R^{-1}(T) = \{(bc)^\infty, (cb)^\infty\}$. In this example, the construction of the right-closing map $\psi: T \rightarrow \Sigma_A$ is easy (in fact there is only one such map). We define $\psi(0^\infty) = a^\infty$. We define $\tilde{\varphi}: \Sigma \rightarrow \Sigma_A$ by $\tilde{\varphi} = \psi \circ (\pi_R|_\Sigma)$, so $\tilde{\varphi}((bc)^\infty) = \tilde{\varphi}((cb)^\infty) = a^\infty$.

We follow the construction of Theorem (1.1) in extending $\tilde{\varphi}: \Sigma \rightarrow \Sigma_A$ to $\varphi: \Sigma_A \rightarrow \Sigma_A$. Now $AI = IA$ and $AI = A$, so we can take $S = I$ and $R = A$. Since $S = I$, we have $w(b) = (0, 1, 0)$ and $w(c) = (0, 0, 1)$.

Now $w(b)$ does not dominate $e_{\tilde{\varphi}(b)} = e_{\tilde{\varphi}(c)} = (1, 0, 0)$ (neither does $w(c)$), so it is impossible to split \mathcal{G}_A with masses given by I by $\tilde{\varphi}$ -respecting splittings into lean states. Notice A^2 is the lowest possible power of A giving each state along the 2 cycle in \mathcal{G}_A enough mass to split them both into lean states by rounds of $\tilde{\varphi}$ -respecting splittings. We have

$$w(b) = (1, 1, 0) \geq (1, 0, 0), \quad w(c) = (1, 1, 1) \geq (1, 0, 0).$$

Figure 2 illustrates 3 rounds of $\tilde{\varphi}$ -respecting splittings leading from \mathcal{G}_A (with masses given by A^2) to a graph $\mathcal{G}_{A'}$ with all lean states. States of $\mathcal{G}_{A'}$ with a common ancestor in \mathcal{G}_A are in the same column; states of $\mathcal{G}_{A'}$ with a common elementary mass are in the same row.

The map $\varphi: \Sigma_A \rightarrow \Sigma_A$ is defined by $\mu \circ \alpha^{-1}$, where $\alpha: \Sigma_{A'} \rightarrow \Sigma_A$ is the ancestor map (vertical projection in Figure 2) and $\mu: \Sigma_{A'} \rightarrow \Sigma_A$ is the mass map (horizontal projection in Figure 2). Notice that the 2-cycle of Σ_A maps by φ onto the 1-cycle of Σ_A .

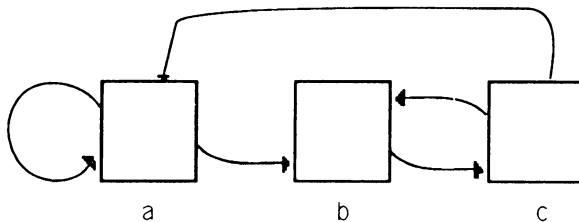


FIGURE 1

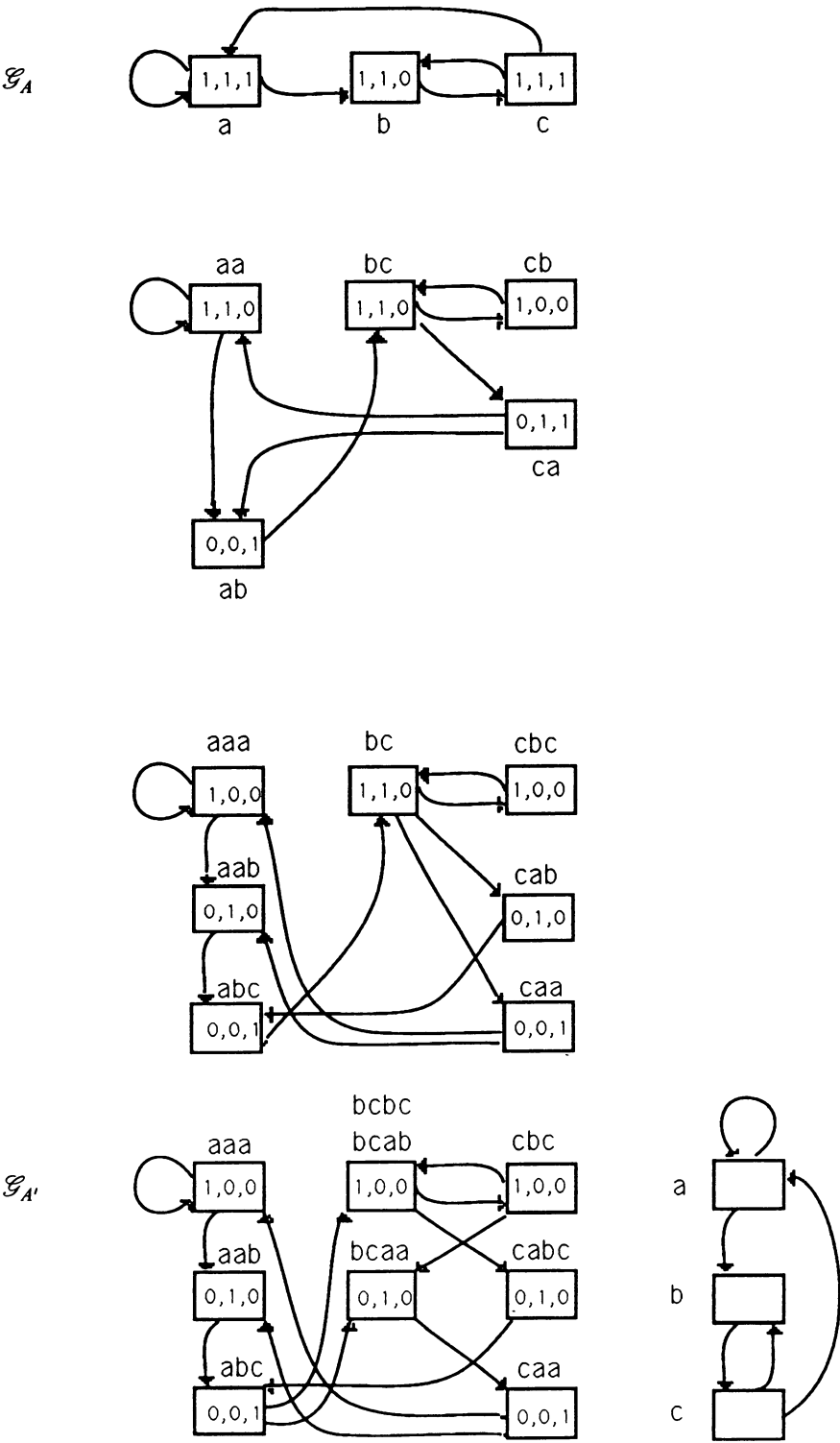


FIGURE 2

TABLE 1

	π_R		φ
111	\longleftrightarrow	aaa	\mapsto a
110	\longleftrightarrow	aab	\mapsto b
100	\longleftrightarrow	abc	\mapsto c
0011	\longleftrightarrow	bcaa	\mapsto b
0010	\longleftrightarrow	bcab	\mapsto a
0000	\longleftrightarrow	bcbc	\mapsto a
011	\longleftrightarrow	caa	\mapsto c
0100	\longleftrightarrow	cabc	\mapsto b
000	\longleftrightarrow	cbc	\mapsto a

The map $\varphi : \Sigma_A \rightarrow \Sigma_A$ is the 4-block map defined by the last two columns of Table 1. Each block w listed in the center column of Table 1 is contained in a state of $\mathcal{G}_{A'}$.

The table should be interpreted as follows. If $x \in \Sigma_A$ has ${}_0(x)_{|w|-1} = w$ for a word w listed in the second column, then ${}_0(\varphi(x))_0$ is the corresponding symbol listed in the last column.

The first column of Table 1 gives the π_R images of the corresponding words in the second column. Each word in the first column except words 0000 and 000 has a unique π_R preimage (moreover, it appears in the second column). If $x \in S$ has ${}_0(x)_2 = 000$, then ${}_0(\pi_R^{-1}x)_2$ is either bcb or cbc ; and if $y \in \Sigma_A$ has ${}_0(y)_2 = bcb$ or cbc , then ${}_0(\varphi y)_0 = a$. Hence, if ${}_0(x)_2 = 000$, then ${}_0(\pi x)_0 = a$. Thus $\pi : S \rightarrow \Sigma_A$ is a 4-block map and can be computed by reading from the first column of Table 1 all the way over to the third column.

By [BMT, Proposition 4.12], if $\pi : S \rightarrow \Sigma$ is a right-closing factor map from a sofic system S onto a shift of finite type Σ , then S is actually a shift of finite type. Thus, the map $\pi : S \rightarrow \Sigma_A$ constructed above cannot be right closing everywhere. We can see this directly by observing that $\pi(0^\infty \underline{1} 0^\infty) = \pi(0^\infty \underline{0} 1^\infty) = a^\infty b \underline{c} a^\infty$, where the underscored symbols occur at time 0.

REFERENCES

- [AM] R. Adler and B. Marcus, *Topological entropy and equivalence of dynamical systems*, Mem. Amer. Math. Soc., no. 219, 1979.
- [A1] J. Ashley, *Bounded-to-1 factors of an aperiodic shift of finite type are 1-to-1 almost everywhere factors also*, Ergodic Theory Dynamical Systems **10** (1990), 615–625.
- [A2] ———, *Resolving factor maps for shifts of finite type with equal entropy*, Ergodic Theory and Dynamical Systems **11** (1991), 219–240.
- [B] M. Boyle, *Lower entropy factors of sofic systems*, Ergodic Theory and Dynamical Systems **4** (1984), 541–557.
- [BMT] M. Boyle, B. Marcus, and P. Trow, *Resolving maps and the dimension group for shifts of finite type*, Mem. Amer. Math. Soc., no. 377, 1987.
- [KM] R. Karabed and B. Marcus, *Sliding-block coding for input-restricted channels*, IEEE Trans. Inform. Theory **34** (1988), 2–26.
- [Kr1] W. Krieger, *On dimension functions and topological Markov chains*, Invent. Math. **56** (1980), 239–250.
- [Kr2] ———, *On the subsystems of topological Markov chains*, Ergodic Theory Dynamical Systems **2** (1982), 195–202.

- [M] B. Marcus, *Factors and extensions of full shifts*, Monatsh. Math. **88** (1979), 239–247.
- [S] E. Seneta, *Non-negative matrices and Markov chains*, 2nd ed., Springer-Verlag, New York, 1981.
- [We] B. Weiss, *Subshifts of finite type and sofic systems*, Monatsh. Math. **77** (1973), 462–474.
- [Wi] R. Williams, *Classification of shifts of finite type*, Ann. of Math. **98** (1973), 120–153; *Errata*, Ann. of Math. **99** (1974), 380–381.

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