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ATTRACTING AND REPELLING POINT PAIRS FOR VECTOR FIELDS ON MANIFOLDS. I

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ABSTRACT. Consider a compact, connected, n-dimensional, triangulable manifold M without boundary, embedded in $\mathbf{R}^{n+1}$ and a continuous vector field on M, given as a map f from M to S^n of degree not equal to 0 or $(-1)^{n+1}$. In this paper it is shown that there exists at least one pair of points $x, y \in M$ satisfying both $f(x) = -f(y)$ and $f(x) = \frac{x-y}{\|x-y\|}$. Geometrically, this means, that the points and the vectors lie on one straight line and the vector field is "repelling". Similarly, if the degree of f is not equal to 0 or 1, then there exists at least one "attracting" pair of points $x, y \in M$ satisfying both $f(x) = -f(y)$ and $f(x) = \frac{y-x}{\|y-x\|}$. The total multiplicities are $\frac{k\cdot(k+(-1)^n)}{2}$ for repelling pairs and $\frac{k\cdot(k-1)}{2}$ for attracting pairs.

In the proof, we work with close simplicial approximations of the map f, using Simplicial, Singular and Čech Homology Theory, Künneth's Theorem, Hopf's Classification Theorem and the algebraic intersection number between two n-dimensional homology cycles in a $2n$-dimensional space. In the case of repelling pairs, we intersect the graph of f in $M \times S^n$ with the set of points $(x, \frac{x-y}{\|x-y\|}) \in M \times S^n$, where x and y satisfy that $f(x) = -f(y)$. In order to show that this set carries the homology $(k, k) \in H_n(M \times S^n, \mathbf{Z})$, we study the set $A_f \equiv \{(x, y) \in M \times M | f(x) = -f(y)\}$ in a simplicial setting. Let f_j be a close simplicial approximation of f. It can be shown, that A_{f_j} is a homology cycle of dimension n with a natural triangulation and a natural orientation and that A_f and A_{f_j} carry the same homology.

0. INTRODUCTION

Consider a compact, connected, n-dimensional, triangulable manifold M without boundary, embedded in $\mathbf{R}^{n+1}$ and a continuous vector field on M, given as a map f from M to S^n of degree not equal to 0 or $(-1)^{n+1}$. In this paper it is shown that there exists at least one pair of points $x, y \in M$ satisfying both $f(x) = -f(y)$ and $f(x) = \frac{x-y}{\|x-y\|}$. Geometrically, this means, that the points and the vectors lie on one straight line and the vector field is "repelling". Similarly, if the degree of f is not equal to 0 or 1, then there exists at least one "attracting" pair of points $x, y \in M$ satisfying both $f(x) = -f(y)$ and $f(x) = \frac{y-x}{\|y-x\|}$. The total multiplicities are $\frac{k\cdot(k+(-1)^n)}{2}$ for repelling pairs and $\frac{k\cdot(k-1)}{2}$ for attracting pairs.

This problem setting came up naturally, when D. W. Henderson and the author considered the following problem (see [2]): Is it possible to find appropri-

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ate notions of gradient vector field, gradient flows and Morse theory for a nonsmooth function $f \equiv \max\{f_i | i = 1, \ldots, m\}$, where $f_1, \ldots, f_m$ are rational real-valued functions defined on an open subset O of $\mathbf{R}^{n+2}$? In [2], we proved, that for every point x in O the function f has one-sided directional derivatives; moreover, if at $x \in O$ f has a direction of descent, then, at $x \in O$, f has a unique direction of steepest descent: $-\nabla f(x)$. We then proved that this noncontinuous vector field $-\nabla f$ has continuous solution curves everywhere in O and that a version of Morse theory holds. In attempting to prove the uniqueness, of the solution curves, one naturally looks at the "regions" $A(a)$ in O near x, where a certain subset of functions $f_{i_1}, \ldots, f_{i_j}$, $a = \{i_1, \ldots, i_j\}$, assumes the maximum and one tries to determine, in which of the regions—if there is a unique one—the solution curve to $-\nabla f$ through x lies, immediately after passing through x. For that purpose, one takes $(n+1)$-dimensional, affine hyperplanes H_k perpendicular to $-\nabla f(x)$ and approaching x and considers the projection onto H_k of $-\nabla f(y)$, where y lies in one of the regions $A(i) \cap H_k$. If for infinitely many hyperplanes H_k approaching x, there is an $A(i)$ such that on the boundary of $A(i) \cap H_k$ one obtains an attracting point pair, it is to be hoped that then there is a solution curve of $-\nabla f$ through x in $A(i)$. If for infinitely many hyperplanes H_k approaching x, there is an $A(i)$ such that on the boundary of $A(i) \cap H_k$ one obtains a repelling point pair, it is hoped that then the solution curve to $-\nabla f$ through x bifurcates at x.

1. The main theorem and a proof outline

The main result is

Theorem 1.1. *Let M be a compact, connected, triangulable, n-dimensional manifold without boundary embedded in $\mathbf{R}^{n+1}$ and let $f: M \to S^n$ be a continuous map of degree k.*

If $k \neq 0$ or $(-1)^{n+1}$, then there is at least one pair of points $x, y \in M$, such that

1. $f(x) = -f(y)$,
2. $f(x) = \frac{x-y}{\|x-y\|}$,

and the total multiplicity of such point pairs is $\frac{k\cdot(k+(-1)^n)}{2}$.

Corollary 1.1. *If $k \neq 0$ or 1, then there is at least one pair of points $x, y \in M$, such that*

1. $f(x) = -f(y)$,
2. $f(x) = \frac{y-x}{\|y-x\|}$,

and the total multiplicity of such point pairs is $\frac{k\cdot(k-1)}{2}$.

Proof of Corollary 1.1. Consider the map $f' \equiv -f$, whose degree is $(-1)^{n+1} \cdot \deg(f)$. Finding a pair of points $x_i, y_i \in M$, satisfying

(a1) $f'(x_i) = -f'(y_i)$,
(a2) $f'(x_i) = \frac{y_i - x_i}{\|y_i - x_i\|}$,

is equivalent to finding a pair of points $x_i, y_i \in M$, satisfying

(b1) $-f(x_i) = f(y_i)$,
(b2) $f(x_i) = \frac{x_i - y_i}{\|x_i - y_i\|}$.

In case (b) we can apply Theorem 1.1 to obtain the multiplicity

$$\frac{(-1)^{n+1}k \cdot ((-1)^{n+1} \cdot k + (-1)^n)}{2} = \frac{k \cdot (k-1)}{2}.$$

This number is not 0, whenever $k \neq 0$ or $(-1)^{n+1}$. □

Remark. Theorem 1.1 and Corollary 1.1 do not have gaps for the degrees 0, $(-1)^{n+1}$ and 0, 1 respectively: there are easy examples of functions with those degrees defined on S^n, for which there are no repelling or attracting pairs.

If $f \equiv$ constant, then the degree of f is 0 and there are neither attracting nor repelling pairs.

If $f \equiv$ identity map, then the degree of f is 1 and there are no attracting pairs.

If $f \equiv$ antipodal map, then the degree of f is $(-1)^{n+1}$ and there are no repelling pairs.

Proof Outline for Theorem 1.1. We algebraically intersect the sets

$$K_f \equiv \left\{ \left(x, \frac{x-y}{\|x-y\|} \right) \in M \times S^n \,|\, f(x) = -f(y),\ x, y \in M \right\},$$

and

$$G_f \equiv \{(x, f(x)) \in M \times S^n | x \in M\},$$

because any element in the intersection of these two sets automatically satisfies the conditions needed for a repelling pair. The algebraic intersection between the homology classes carried by K_f and G_f counts these occurrences with multiplicity. If the number is not 0, then we can guarantee at least one repelling pair.

G_f, being the graph of f, can easily be shown to carry the homology

$$(1, k) \in H_n(M \times S^n, \mathbf{Z}),$$

and so the main effort is to show, that K_f carries the homology

$$(k, k) \in H_n(M \times S^n, \mathbf{Z}).$$

This will be done in several steps:

In §2, we will deal with the case of a particularly nice map $g_0 \colon S^n \to S^n$ of degree k, where K_{g_0} is the union of the graphs of k functions of degree 1; this implies that K_{g_0} carries the desired homology.

In §3, we introduce geodesically simplicial maps and state the key technical lemma:

Lemma 3.1. *Let $f_0, f_1 \colon TM \to GS^n$ be geodesically simplicial and noncollapsing maps of degree $k \neq 0$, then the following two sets*

$$A_{f_0} \equiv \{(x, y) \in M \times M | f_0(x) = -f_0(y)\} \quad \textit{and}$$
$$A_{f_1} \equiv \{(x, y) \in M \times M | f_1(x) = -f_1(y)\}$$

can be triangulated and oriented naturally as n-dimensional homology cycles and an $(n+1)$-dimensional homology cobordism A_H can be constructed which has A_{f_0} and A_{f_1} as its boundary.

In §4, we show that for the continuous function f of Theorem 1.1, A_f carries the same homology in $H_n(M \times M - \Delta, \mathbf{Z})$ as each one of a sequence of

A_{f_i}'s, where the f_i's are geodesically simplicial maps uniformly approaching f. The idea is to show that A_f contains an inverse limit involving the A_{f_i}'s.

In §5, we define the map

$$\theta\colon M \times M - \Delta \to M \times S^n$$
$$(x, y) \mapsto \left(x, \frac{x-y}{\|x-y\|}\right),$$

which sends A_f to K_f. Using the result of §4, we know that A_f carries the same homology in $H_n(M \times M - \Delta, \mathbf{Z})$ as A_{f_0}, where f_0 is a close geodesically simplicial and noncollapsing approximation of the ideal map $g_0 \circ rp$, where rp is a very close geodesically simplicial and noncollapsing approximation of the radial projection from M onto a geometric sphere S_0^n, sitting in the bounded complement of M in $\mathbf{R}^{n+1}$ and g_0 is the ideal map defined on S^n with the same degree as f_0.

We prove that

(a) K_f carries the homology $\theta_*[A_{f_0}] \in H_n(M \times S^n, \mathbf{Z})$ and that

(b) $K_{f_0} = \theta(A_{f_0})$ carries the homology $(k, k) \in H_n(M \times S^n, \mathbf{Z})$.

The latter is accomplished by reducing the problem (M, f_0) to the case (S^n, g_0), and this case has been dealt with in §2.

Thus, one obtains that K_f carries the desired homology

$$(k, k) \in H_n(M \times S^n, \mathbf{Z}).$$

Finally, we give a proof of Theorem 1.1, by intersecting $K_f = \theta(A_f)$ with the graph of f, G_f. This ends the proof outline.

2. Motivation and example

We define a map g_0 on S^n, where the assertion of the theorem can be verified easily.

Lemma 2.1. *There is a map $g_0\colon S^n \to S^n$ of degree $k \neq 0$, where the set*

$$K_{g_0} \equiv \left\{\left(x, \frac{x-y}{\|x-y\|}\right) \in S^n \times S^n \,|\, g_0(x) = g_0(x),\ x \in S^n\right\}$$

carries the homology $(k, k) \in H_n(S^n \times S^n, \mathbf{Z})$.

Moreover, there are exactly $\frac{k\cdot(k+(-1)^n)}{2}$ many pairs of points $x_i, y_i \in S^n$, satisfying both

1. $g_0(x_i) = -g_0(y_i)$,
2. $g_0(x_i) = \frac{x_i - y_i}{\|x_i - y_i\|}$.

Proof. Define g_0 inductively:

$n = 1$: Let $\theta \in [0, 2\pi)$, $x \in S^1$ and $x = e^{i\theta}$, then define $g_0(x) = e^{i(-1)^{n-1}k\theta}$. Suppose, that this initial $S^1 = \{(x_1, x_2, 0, \dots, 0) \in \mathbf{R}^{n+1} | x_1^2 + x_2^2 = 1\}$. Extend g_0 to S^n by successively suspending with the antipodal maps.

Now define maps $y_j\colon S^n \to S^n$, $j = 1, \dots, k$, also inductively

$n = 1$: $y_j\colon S^1 \to S^1$, let $x = e^{i\theta}$. Define $y_j(x) = e^{i(\theta + \frac{(2j-1)\pi}{k})}$, then y_j is a homeomorphism on S^1 and

$$g_0(y_j(x)) = -g_0(x).$$

Inductively define y_j on S^n by suspending with the antipodal map.

Then $A_{g_0} \equiv \{(x, y_j(x)) | x \in S^n, j = 1, \ldots, k\}$ is the union of k graphs of y_j and is represented by $[A_{g_0}] = k \in H_n(S^n \times S^n - \Delta, \mathbf{Z}) \cong \mathbf{Z}$, where Δ is the diagonal in $S^n \times S^n$.

The vector $\frac{x - y_j(x)}{\|x - y_j(x)\|}$ can be homotoped to x, because it never equals $-x$. The set

$$K_{g_0, j} \equiv \left\{ \left(x, \frac{x - y_j(x)}{\|x - y_j(x)\|} \right) \middle| x \in S^n \right\}$$

is represented by $[K_{g_0, j}] = (1, 1) \in H_n(S^n \times S^n, \mathbf{Z})$. Thus $[K_{g_0}] \equiv [\bigcup_{j=1}^{k} K_{g_0, j}] = (k, k) \in H_n(S^n \times S^n, \mathbf{Z})$.

The intersection of the graph G of g_0 and K_{g_0} all lie in the plane determined by the initial S^1. The algebraic intersection number between $[G] = (1, \deg(g_0)) = (1, k)$ and $[K_{g_0}] = (k, k)$ gives us the number of such occurrences: $k \cdot (k + (-1)^n)$. But this counts a pair x, y twice, so we get exactly $\frac{k \cdot (k + (-1)^n)}{2}$ many repelling pairs x, y.

3. The key technical lemma and its reduction to the simplicial case

In this section, we state and partly prove the key technical lemma, which says that any two geodesically simplicial and noncollapsing maps f_0, f_1 have homologous "pull-backs" A_{f_0} and A_{f_1}. This is the key constructive idea of the whole proof, because the set K_f can be obtained as the image of A_f under a continuous function defined later on. This lemma will allow us to reduce to the case of a function which is much easier to deal with than f.

Definitions and set up for the technical lemma. Let TM be a simplicial triangulation of M. Let GS^n be a symmetric, geodesic triangulation of S^n. For reference about geodesic triangulations of the sphere, see [1].

Definition 3.1. A triangulation G of S^n is called *symmetric* if for every simplex $\sigma = x_0 x_1 \cdots x_l \in G$, the simplex $-\sigma = (-x_0)(-x_1) \cdots (-x_l) \in G$.

From GS^n we construct a triangulated, symmetric, polyhedral sphere Σ^n by taking as its vertices the vertices of GS^n and as its l-simplices the convex hull in $\mathbf{R}^{n+1}$ of the vertices $x_0 x_1 \cdots x_l$, where $\sigma = x_0 x_1 \cdots x_l$ is a geodesic simplex in GS^n. Denote this triangulation of Σ^n by $T\Sigma^n$.

Let $cp \colon \Sigma^n \to S^n$ be the centric projection.

Definition 3.2. The map $f \colon TM \to GS^n$ is *geodesically simplicial*, if there exists a simplicial map $f^{\text{simp}} \colon TM \to G\Sigma^n$, so that $f = cp \circ f^{\text{simp}}$.

Note that $A_f = A_{f^{\text{simp}}}$. We will take f_0^{simp} to be f_0 and f_1^{simp} to be f_1.

Definition 3.3. The map $f \colon K \to L$ is *noncollapsing*, if for every simplex $\sigma \in K$, f restricted to $|\sigma|$ is a homeomorphism.

The key technical result is

Lemma 3.1. *Let $f_0, f_1 \colon TM \to GS^n$ be geodesically simplicial and noncollapsing maps of degree $k \neq 0$, then the following two sets*

$$A_{f_0} \equiv \{(x, y) \in M \times M | f_0(x) = -f_0(y)\} \quad \textit{and}$$
$$A_{f_1} \equiv \{(x, y) \in M \times M | f_1(x) = -f_1(y)\}$$

can be triangulated and oriented naturally as n-dimensional homology cycles and an $(n+1)$-dimensional homology cobordism A_H can be constructed which has A_{f_0} and A_{f_1} as its boundary.

In the proof of Lemma 3.1 we need a suitable homotopy H, which is constructed in

Proposition 3.1. *Let $f_0, f_1: TM \to T\Sigma^n$ be simplicial and noncollapsing, then there exists a simplicial map*

$$H: T(M \times I) \to T(\Sigma^n \times I),$$

such that

1. *H is noncollapsing,*
2. *H is level preserving,*
3. *$T(\Sigma^n \times I)$ is equivariant with respect to the antipodal map on S^n,*
4. *$H(x, 0) = (f_0(x), 0)$ and $H(x, 1) = (f_1(x), 1)$.*

Proof. By Hopf's Theorem and the Simplicial Approximation Theorem, there is a simplicial homotopy $H: M \times I \to \Sigma^n$ with $H(x, 0) = f_0(x)$ and $H(x, 1) = f_1(x)$. With appropriate subdivisions we can assume that H is level preserving. Using general position we can take H to be noncollapsing. If necessary we have to subdivide again to make $T(\Sigma^n \times I)$ equivariant with respect to the antipodal map on S^n.

The construction of the pull-back A_H for H involves a lot of checking, which is left to the reader:

$$A_H \equiv \{(x, y, t) \in M \times M \times I | H(x, t) = (z, t) \& H(y, t) = (-z, t)\}.$$

A_H can be triangulated as a simplicial complex and has A_{f_0} and A_{f_1} as its boundary.

4. The reduction from the continuous f to the geodesically simplicial and noncollapsing f_i via an inverse limit

Consider the following subset of $(M \times M - \Delta) \times [m, \infty)$:

$$B'_m \equiv \{(x, y, j) \in (M \times M - \Delta) \times [m, \infty) | j \in \mathbf{Z},\ j \geq m \text{ and } (x, y) \in A_{f_j}\}.$$

Let B_m be the closure of B'_m in $(M \times M - \Delta) \times [m, \infty]$, and let $i_{m+k, m}: B_{m+k} \hookrightarrow B_m$, $k \geq 0$, be the inclusion map, then $(B_m, i_{m, m-1})$ forms an inverse system with limit $B_\infty = \bigcap_{m=1}^{\infty} B_m$.

Lemma 4.1. *B_∞ in $(M \times M - \Delta) \times \{\infty\}$ carries the homology $z_0 - [A_{f_m}]$, for all $m \in \mathbf{N}$, in $H_n((M \times M - \Delta), \mathbf{Z})$.*

Proof. It can be checked that $A_{H_{m, m+1}}$ with boundary $A_{f_m} \cup A_{f_{m+1}}$ does not intersect the diagonal $\Delta \times [0, 1]$ in $M \times M \times [0, 1]$.

The following diagram commutes for all $m \in \mathbf{N}$:

$$\begin{tikzcd}
B_m \arrow[d, "i"'] & B_{m+1} \arrow[l, "i_{m+1, m}"'] \arrow[d, "i"] \\
(M \times M - \Delta) \times [m, \infty] & (M \times M - \Delta) \times [m+1, \infty] \arrow[l, "i"']
\end{tikzcd}$$

The bottom inclusion map induces the identity in homology, because $(M \times M - \Delta) \times [m, \infty]$ can be deformation retracted to $(M \times M - \Delta) \times [m+1, \infty]$.

From the above we obtain an inverse system in homology:

$$\begin{tikzcd} \check{H}_n(B_m, \mathbf{Z}) \arrow[d, "i_*"'] & & \check{H}_n(B_{m+1}, \mathbf{Z}) \arrow[ll, "i_{m+1,m *}"'] \arrow[d, "i_*"] \\ \check{H}_n((M \times M - \Delta) \times [m, \infty], \mathbf{Z}) & & \check{H}_n((M \times M - \Delta) \times [m+1, \infty], \mathbf{Z}) \arrow[ll, "i_*"'] \end{tikzcd}$$

where $\check{H}_n$ means Čech homology. By passing to the limit, we get by Theorem 5.8 (p. 195) in [6], that

$$\begin{aligned} \check{H}_*((M \times M - \Delta) \times [m, \infty], \mathbf{Z}) &= \check{H}_*((M \times M - \Delta) \times \{\infty\}, \mathbf{Z}) \\ &= \check{H}_*((M \times M - \Delta), \mathbf{Z}) \\ &= H_*((M \times M - \Delta), \mathbf{Z}) \end{aligned}$$

here, the Čech homology groups are the same as the singular homology groups, because all the spaces are $2n$-dimensional manifolds, which are paracompact and Hausdorff (see [3, p. 220]). By the same theorem in [6], we also get that

$$\check{H}_*(B_\infty, \mathbf{Z}) = \varprojlim \check{H}_*(B_m, \mathbf{Z}).$$

For each m, consider $\{i_* g \in \check{H}_n((M \times M - \Delta) \times [m, \infty], \mathbf{Z}) | g \in \check{H}_n(B_m, \mathbf{Z})\}$.

The following diagram commutes, since all the maps are induced by inclusions:

$$\begin{tikzcd} H_n(A_{f_m}, \mathbf{Z}) \arrow[r, "i''_*"] \arrow[dr, "i'_*"'] & \check{H}_n(B_m, \mathbf{Z}) \arrow[d, "i_*"] \\ & H_n((M \times M - \Delta) \times [m, \infty], \mathbf{Z}) \end{tikzcd}$$

Thus $i''_*[A_{f_m}]$ is an element in $i_*\check{H}_n(B_m, \mathbf{Z})$ and from Lemma 3.1, we know, that $[A_{f_m}]$ and $[A_{f_{m+1}}]$ are homologous in $H_n((M \times M - \Delta) \times [m, m+1], \mathbf{Z})$.

Thus

$$i'_*[A_{f_1}] = i'_*[A_{f_2}] = \cdots = i'_*[A_{f_m}] = \cdots.$$

Denote this cycle in $H_*((M \times M - \Delta), \mathbf{Z})$ by z_0. So z_0 lies in

$$\{i_* g \in H_n((M \times M - \Delta) \times [m, \infty], \mathbf{Z}) | g \in \check{H}_n(B_m, \mathbf{Z}),$$

for all m. By passing to the limit, we can conclude by Theorem 5.8 (p. 195) in [6], that

$$z_0 \in \{i_* g | g \in \check{H}_n(B_\infty, \mathbf{Z})\} \subset H_n((M \times M - \Delta) \times \{\infty\}, \mathbf{Z}).$$

Thus B_∞ carries the homology z_0 in $H_n((M \times M - \Delta), \mathbf{Z})$. □

Lemma 4.2. $B_\infty \subset A_f$.

Proof. Let $(x, y, t) \in B_\infty$, then $(x, y, t) = \lim(x_j, y_j, t_j)$ of points in A_{f_j}. We have: $x = \lim x_j$, $y = \lim y_j$ in M and $\infty = \lim t_j$, $t_j \in \mathbf{Z} \cup \{\infty\}$. Since the functions f_m converge uniformly to f, for any convergent sequence of points $z_l \to z$, $z \in M$:

$$\lim_l \lim_m f_m(z_l) = \lim_m \lim_l f_m(z_l) = f(z).$$

In particular, this implies that

$$f(x) = \lim f_j(x_j) = \lim(-f_j(y_j)) = -f(y),$$

and thus $(x, y) \in A_f$. $\square$

Corollary 4.1. *A_f carries the homology*

$$z_0 = [A_{f_m}] \in H_n((M \times M - \Delta), \mathbf{Z})$$

and in fact, so does any other A_h, where $\deg(h) = \deg(f)$.

5. The homology carried by K_f and the proof of Theorem 1.1

Let M be as in Theorem 1.1 and without loss of generality assume that the unit n-sphere S_0^n sits inside the bounded complement of M. Let $rp\colon M \to S_0^n$ be a very close geodesically simplicial and noncollapsing approximation of the radial projection onto S_0^n. Finally, let $g_0\colon S^n \to S^n$ be the map defined in §2, so that $\deg(g_0) = \deg(f)$. Define $f_0 = g_0 \circ rp$. We want rp to satisfy the following condition:

There is a homotopy $G_1\colon M \times I \to \mathbf{R}^{n+1}$, $G_1(x, 0) = x$, and $G_1(x, 1) = rp(x) \in S_0^n$, such that if $x, y \in M$, $f_0(x) = -f_0(y)$, then for all $t \in [0, 1]$, $G_1(x, t) \neq G_1(y, t)$. Clearly, such a map rp and homotopy G_1 can be found. Then

(a) The map $f_0 \equiv g_0 \circ rp$ from M to S^n has the same degree as f.

(b) There exists a geodesically simplicial and noncollapsing map $f_0'\colon TM \to GS^n$ such that A_{f_0} is an n-cycle homologous to $A_{f_0'}$ in $H_n((M \times M - \Delta), \mathbf{Z})$.

Part (b) may require a subdivision of the geodesical triangulation GS^n of S^n as given in §2 and uses that g_0 is finite to one and open. (See Figure.)

Lemma 5.1. *Let f be a continuous map from M to S^n, let f_0' be the map from M to S^n constructed above, and let*

$$\theta\colon M \times M - \Delta \to M \times S^n \text{ be defined by}$$

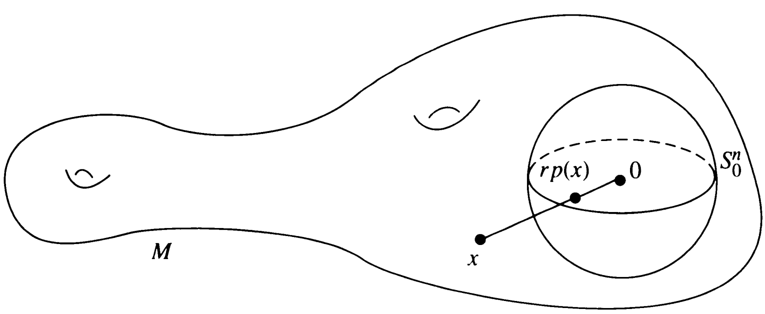
$$(x, y) \mapsto \left(x, \frac{x - y}{\|x - y\|}\right),$$

then $K_f = \theta(A_f)$ carries the homology $\theta_*([A_{f_0'}]) \in H_n(M \times S^n, \mathbf{Z})$.

Proof. θ is well defined on $M \times M - \Delta$ and continuous. By Corollary 4.1, A_f carries the homology $[A_{f_0'}]$ in $H_n((M \times M - \Delta), \mathbf{Z})$. Thus, the image $K_f = \theta(A_f)$ carries the homology $\theta_*[A_{f_0'}]$, where

$$\theta_*\colon H_n((M \times M - \Delta), \mathbf{Z}) \to H_n(M \times S^n, \mathbf{Z})$$

is the homomorphism induced by θ. $\square$



FIGURE

Lemma 5.2. *$K_{f_0} = \theta(A_{f_0})$ carries the homology (k, k) in $H_n(M \times S^n, \mathbf{Z})$.*

Proof. Using Künneth's formula we obtain $H_n(M \times S^n, \mathbf{Z}) \cong \mathbf{Z} \times \mathbf{Z}$. Take the homotopy G_1 retracting M to S_0^n as stated above, namely $G_1(x, 0) = x$ and $G_1(x, 1) = rp(x) \in S_0^n$, such that if $x, y \in M$, $f_0(x) = -f_0(y)$, then for all $t \in [0, 1]$, $G_1(x, t) \neq G_1(y, t)$. Next define a homotopy of the map θ:

$$\begin{aligned} G_2 \colon A_{f_0} \times I &\to M \times S^n \\ (x, y, t) &\mapsto \left(x, \frac{G_1(x, t) - G_1(y, t)}{\|G_1(x, t) - G_1(y, t)\|}\right). \end{aligned}$$

G_2 is well defined. Hence,

$$G_2(x, y, 0) = \left(x, \frac{x - y}{\|x - y\|}\right) = \theta(x, y, 0),$$

$$G_2(x, y, 1) = \left(x, \frac{rp(x) - rp(y)}{\|rp(x) - rp(y)\|}\right),$$

and therefore $G_2(A_{f_0} \times \{0\}) = \theta(A_{f_0})$ is homologous to $G_2(A_{f_0} \times \{1\})$.

Consider the map

$$\begin{aligned} \xi \colon M \times M - \{(x, y) \in M \times M | y = \lambda \cdot x, \lambda > 0\} &\to S^n, \\ (x, y) &\mapsto \frac{rp(x) - rp(y)}{\|rp(x) - rp(y)\|}. \end{aligned}$$

ξ is well defined, because $rp(x) \neq rp(y)$, ξ is continuous and $\xi = G_2(\cdot, 1)$.

We know that $A_{f_0} \subset M \times M - \{(x, y) \in M \times M | y = \lambda \cdot x, \lambda > 0\}$, because if $f_0(x) = -f_0(y)$, then $y \neq \lambda \cdot x$, $\lambda > 0$. In terms of x, ξ is homotopic to $rp(x)$ and in terms of y, ξ is homotopic to $-rp(y)$. Thus ξ has the degree $(-1)^{n+1}$ with respect to y. Therefore the map $\theta' \equiv (\mathrm{id}_M, \xi)$ has degree $(1_x, (-1)_y^{n+1})$. The image of θ' is contained in the set

$$d \equiv \{(x, z) \in M \times S^n | z \neq -rp(x)\}.$$

In particular, if $M = S^n$, denote the diagonal in $S^n \times S^n$ by Δ and the antidiagonal by Δ^-; then $H_n(S^n \times S^n - \Delta, \mathbf{Z}) \cong \mathbf{Z}$ is generated by $[\Delta^-]$, $H_n(S^n \times S^n - \Delta^-, \mathbf{Z}) \cong \mathbf{Z}$ is generated by $[\Delta]$ and the map $\theta \equiv (\mathrm{id}_{S^n}, \xi)$ induces

$$\theta_* \colon H_n(S^n \times S^n - \Delta, \mathbf{Z}) \to H_n(S^n \times S^n - \Delta^-, \mathbf{Z}),$$

where $\theta_*[\Delta^-] = [\Delta]$.

Now consider the following diagram:

$$\begin{array}{ccc} M \times M - \{(x, y) \in M \times M | y = \lambda \cdot x, \lambda > 0\} & \xrightarrow{\theta'} & D = \{(x, z) \in M \times S^n | z \neq -rp(x)\} \\ \Big\downarrow{\scriptstyle (rp, rp)} & & \Big\downarrow{\scriptstyle (rp, \mathrm{id}_{S^n})} \\ S^n \times S^n - \Delta & \xrightarrow{\theta} & S^n \times S^n - \Delta^- \end{array}$$

This diagram commutes on the set level:

$$\begin{aligned} (rp, \mathrm{id}_{S^n}) \circ \theta'(x, y) &= (rp, \mathrm{id}_{S^n}) \left(x, \frac{rp(x) - rp(y)}{\|rp(x) - rp(y)\|}\right) \\ &= \theta(rp(x), rp(y)) = \theta \circ (rp, rp)(x, y). \end{aligned}$$

Therefore it commutes in homology:

$$\begin{array}{ccc} H_n(M \times M - \{\cdots\}, \mathbf{Z}) & \xrightarrow{\theta'_*} & H_n(D, \mathbf{Z}) \\ {\scriptstyle (rp, rp)_*} \downarrow & & \downarrow {\scriptstyle (rp, \mathrm{id}_{S^n})_*} \\ H_n(S^n \times S^n - \Delta, \mathbf{Z}) & \xrightarrow{\theta_*} & H_n(S^n \times S^n - \Delta^-, \mathbf{Z}) \end{array}$$

D is homotopy equivalent to $\Delta' \equiv \{(x, rp(x)) | x \in M\} \subset D$:

$$R\colon D \times I \to D$$

$$(x, z, t) \mapsto \left(x, \frac{t \cdot rp(x) + (1-t) \cdot z}{\|t \cdot rp(x) + (1-t) \cdot z\|}\right).$$

R is well defined, because $-rp(x) \neq z$, if $(x, z) \in D$. R is a deformation retraction, because it is the identity on Δ' for all $t \in I$. Therefore $H_n(D, \mathbf{Z}) = H_n(\Delta', \mathbf{Z}) \cong \mathbf{Z}$, generated by Δ'. Moreover $\theta_*[\Delta^-] = [\Delta]$, $(rp, \mathrm{id}_{S^n})_*[\Delta'] = [\Delta]$, thus they both induce the identity isomorphism on $\mathbf{Z}$. By construction $(rp, rp)(A_{f_0}) = A_{g_0}$ and $[A_{g_0}] = k \in H_n(S^n \times S^n - \Delta, \mathbf{Z}) \cong \mathbf{Z}$, by §2. Therefore we get

$$\theta'_*[A_{f_0}] = (rp, \mathrm{id}_{S^n})_* \circ \theta'_*[A_{f_0}] = \theta_* \circ (rp, rp)_*[A_{f_0}] = \theta_*[A_{g_0}] = k.$$

Consider the homomorphism i_*, induced by the inclusion map $i\colon D \to M \times S^n$. i_* sends the generator $[\Delta']$ of $H_n(D, \mathbf{Z})$ to the homology class generated by $1_n \otimes 1_0 + 1_0 \otimes 1_n$, namely $(1, 1)$. Therefore

$$i_* \circ \theta'_*[A_{f_0}] = (k, k) \in H_n(M \times S^n, \mathbf{Z}).$$

Thus $\theta'(A_{f_0})$ carries the homology (k, k). □

Corollary 5.1. *$K_f = \theta(A_f)$ carries the homology (k, k) in $H_n(M \times S^n, \mathbf{Z})$.*

Lemma 5.3. *The algebraic intersection number of the homology classes $i_1 \cdot (1_n \otimes 1_0) + j_1 \cdot (1_0 \otimes 1_n)$ and $i_2 \cdot (1_n \otimes 1_0) + j_2 \cdot (1_0 \otimes 1_n)$ in $H_n(M^n \times S^n, \mathbf{Z})$ is $i_1 \cdot j_2 + (-1)^n j_1 \cdot i_2$.*

Proof. See [3, p. 174].

Proof (*of Theorem* 1.1). Denote the graph of f by

$$G_f \equiv \{(x, f(x)) \in M \times S^n | x \in M\}.$$

By hypothesis f has degree $k \neq 0$. Therefore $[G_f] = (1, k) \in H_n(M \times S^n, \mathbf{Z})$. The intersection number of (k, k) with $[G_f] = (1, k)$ is $k \cdot (k + (-1)^n)$. Since $k \cdot (1 + (-1)^n \cdot k) \neq 0$, we get at least one instance, where $f(x) = \frac{x-y}{\|x-y\|}$ and $f(x) = -f(y)$ hold simultaneously. However, the total multiplicity of such point pairs is $\frac{k \cdot (k + (-1)^n)}{2}$, since the algebraic intersection of (k, k) and $(1, k)$ counts each pair (x, y) at least twice. □

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