TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 337, Number 1, May 1993

THE CANONICAL COMPACTIFICATION OF A FINITE GROUP OF LIE TYPE

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ABSTRACT. Let G be a finite group of Lie type. We construct a finite monoid $\mathscr{M}$ having G as the group of units. $\mathscr{M}$ has properties analogous to the canonical compactification of a reductive group. The complex representation theory of $\mathscr{M}$ yields Harish-Chandra's philosophy of cuspidal representations of G. The main purpose of this paper is to determine the irreducible modular representations of $\mathscr{M}$. We then show that all the irreducible modular representations of G come (via the 1942 work of Clifford) from the one-dimensional representations of the maximal subgroups of $\mathscr{M}$. This yields a semigroup approach to the modular representation theory of G, via the full rank factorizations of the 'sandwich matrices' of $\mathscr{M}$. We then determine the irreducible modular representations of any finite monoid of Lie type.

INTRODUCTION

The purpose of this paper is to construct, for each finite group of Lie type, a certain monoid modeled on the canonical compactification of DeConcini and Procesi [6] for reductive groups. The natural way of accomplishing this is to start with a linear algebraic monoid M with Frobenius endomorphism $\sigma : M \to M$ yielding a finite monoid $M_\sigma = \{x \in M \mid \sigma(x) = x\}$. This provides us with a highly structured procedure for constructing a large class of monoids of Lie type. The correct choice of M then yields our canonical compactification M_σ. For the purpose of relating the modular and complex representation theories of M_σ and its unit group G, we are naturally led to considering a universal central extension $\mathscr{M}$ of M_σ. We are able to construct this universal compactification $\mathscr{M} = \mathscr{M}(G)$ abstractly starting with any group G of Lie type. For clarity in exposition we do this first, saving a discussion of the monoids M_σ to the last section.

Using our universal monoid we count up the irreducible modular representations whose restriction to the unit group is irreducible. Using another counting method, we then find that any irreducible representation of $\mathscr{M}$ restricts to an irreducible representation of G. This essentially reduces the enumeration of irreducible modular representations of any finite monoid of Lie type to a combinatorial problem. Using the results of Clifford [2, 3], we are then able to show that the degree of a modular irreducible representation φ of G is equal to the rank of a certain matrix associated with φ and $\mathscr{M}$. In fact the full rank

Received by the editors May 21, 1990 and, in revised form, February 10, 1991.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 20G40, 20M30, 20G05.
First author partially supported by NSF, second author partially supported by NSERC.

factorization of the 'sandwich' matrix yields φ. In particular, we show that the modular Steinberg representation of G has an idempotent separating extension to $\mathcal{M}$.

We also include a brief discussion of complex representations of $\mathcal{M}$ mainly for the purpose of pointing out that $\mathcal{M}$ also has natural connections with the complex representation theory of G. In particular, every irreducible representation of G is a component of the restriction to G of an irreducible cuspidal representation of $\mathcal{M}$, yielding a new interpretation of Harish-Chandra's philosophy of cuspidal representations.

1. The universal canonical compactification

Let M be a finite monoid with group of units G. If $X \subseteq M$, we let $E(X) = \{e \in X \mid e^2 = e\}$. M is *regular* if $a \in aMa$ for all $a \in M$. For $e \in E = E(M)$, the following three subgroups of G are of critical importance in our work:

$$P = P(e) = C_G^r(e) = \{x \in G \mid xe = exe\},$$
$$P^- = P^-(e) = C_G^l(e) = \{x \in G \mid ex = exe\},$$
$$L = L(e) = C_G(e) = \{x \in G \mid xe = ex\}.$$

Let G be a finite group of Lie type with Coxeter graph Γ, cf. [1]. For a parabolic subgroup P of G, we let $R_u(P)$ denote its unipotent radical. By a (finite) *monoid of Lie type* on G, we mean a finite regular monoid M with zero 0 and having group of units G such that M is generated by $E = E(M)$ and G, and

(1) For all $e \in E$, $P(e)$, $P^-(e)$ are opposite parabolic subgroups of G and for all $u \in R_u(P(e))$, $v \in R_u(P^-(e))$, $ue = e = ev$.

(2) For all $e, f \in E$, $eM = fM$ or $Me = Mf$ implies $x^{-1}ex = f$ for some $x \in G$.

Monoids of Lie type were introduced in [11] with a different name (regular split monoids) as an abstraction of certain properties of linear algebraic monoids [9, 13, 16]. Section 4 and [17] provide examples of special monoids of Lie type.

Let M be a monoid of Lie type on G. Then G has a two-sided action on M. Let $\mathcal{U}$ denote the set of $G \times G$ orbits of M. These are the $\mathcal{J}$-classes of M in the usual sense [4]. Define $GaG \geq GbG$ if $b \in MaM$. In this way $\mathcal{U}$ becomes a lattice. There is a cross section of idempotents of e_J $(J \in \mathcal{U})$ such that $J = Ge_JG$ and for all $J_1, J_2 \in \mathcal{U}$, $e_{J_1}e_{J_2} = e_{J_2}e_{J_1} = e_{J_1 \wedge J_2}$. $\Lambda = \{e_J \mid J \in \mathcal{U}\}$ is called a *cross section lattice*. Moreover, $M = E(M)G$ and for all $J \in \mathcal{U}$,

$$E(J) = \{x^{-1}e_Jx \mid x \in G\}.$$

There is a *type map* $\lambda : \mathcal{U} \to 2^\Gamma$ such that for all $J \in \mathcal{U}$, $P(e_J) = P_{\lambda(J)}$, $P^-(e_J) = P^-_{\lambda(J)}$. We write P_J, P_J^-, L_J, U_J, U_J^- for $P_{\lambda(J)}$, $P^-_{\lambda(J)}$, $L_{\lambda(J)}$, $U_{\lambda(J)}$, $U^-_{\lambda(J)}$, respectively. We call $\lambda(J)$ the *type* of J (and e_J), λ the *type* of M. This type map completely determines the system of idempotents of M. For $J \in \mathcal{U}$, let

$$K_J = \{x \in G \mid xe_J = e_Jx = e_J\}.$$

Then $K_J \triangleleft L_J$. Let

$$\mathcal{K} = \mathcal{K}(M) = \{(J, K_J) \mid J \in \mathcal{U}\}.$$

The type map λ along with the data $\mathscr{K}(M)$, completely classifies a monoid of Lie type. We refer to [11, 12, 13] for details.

Let G be a group of Lie type with Coxeter graph Γ, $I \subseteq \Gamma$. Then $P_I^- P_I = U_I^- L_I U_I$. We therefore have a natural map $\theta_I : P_I^- P_I \to L_I$. θ_I restricted to P_I or P_I^- is a homomorphism. Moreover,

$$\theta_I(ax) = a\theta_I(x), \quad \theta_I(xa) = \theta_I(x)a \quad \text{for all } x \in P_I^- P_I,\ a \in L_I.$$

We are now in a position to construct the *universal compactification* $\mathscr{M} = \mathscr{M}(G)$. This is the universal monoid of Lie type with the properties of some very special monoids M_σ that we will construct in Theorems 4.7 and 4.10. The monoids M_σ are derived from linear algebraic monoids M which are cones on the compactification of a reductive group. The importance of $\mathscr{M}$ will be clear in Sections 2 and 3.

Theorem 1.1. *Let G be a group of Lie type with Coxeter graph Γ. Start with idempotents e_I $(I \subseteq \Gamma)$. Let $J_I = Ge_IG/\equiv$, $\mathscr{M} = \bigsqcup_{I\subseteq\Gamma} J_I \sqcup \{0\}$ where for $x, y, x_1, y_1 \in G$, define $xe_Iy \equiv x_1e_Iy_1$ if $x_1^{-1}x \in P_I$, $y_1y^{-1} \in P_I^-$ and $\theta_I(x_1^{-1}x) = \theta_I(y_1y^{-1})$. For $a = xe_Iy \in J_I$, $b = se_Kt \in J_K$, define*

$$ab = \begin{cases} xle_{I\cap K}mt & \text{if } ys \in U_I^- lmU_K,\ l \in L_I,\ m \in L_K, \\ 0 & \text{if } ys \notin P_I^- P_K. \end{cases}$$

Then $\mathscr{M} = \mathscr{M}(G)$ is a monoid of Lie type.

Proof. Since θ_I is a homomorphism when restricted to P_I or P_I^-, we see that $\equiv$ is an equivalence relation. We next show that multiplication in $\mathscr{M}$ is well-defined. So let $a = xe_Iy$, $a_1 = x_1e_Iy_1 \in J_I$, $b = se_Kt$, $b_1 = s_1e_Kt_1 \in J_K$ such that $a \equiv a_1$ and $b \equiv b_1$. Then

$$x_1^{-1}x \in P_I, \quad y_1y^{-1} \in P_I^-, \quad \theta_I(x_1^{-1}x) = \theta_I(y_1y^{-1}),$$
$$s_1^{-1}s \in P_K, \quad t_1t^{-1} \in P_K^-, \quad \theta_K(s_1^{-1}s) = \theta_K(t_1t^{-1}).$$

It follows that $ys \in P_I^- P_K$ if and only if $y_1s_1 \in P_I^- P_K$. Let ys, $y_1s_1 \in P_I^- P_K = U_I^- L_I L_K U_K$. Then for some l, $l_1 \in L_I$, $m, m_1 \in L_K$, $ys \in U_I^- lmU_K$ and $y_1s_1 \in U_I^- l_1m_1U_K$. So $ab = xle_{I\cap K}mt$, $a_1b_1 = x_1l_1e_{I\cap K}m_1t_1$. Now since $x_1^{-1}x \in P_I$ and $U_I \lhd P_I$,

$$\begin{aligned} l_1^{-1}x_1^{-1}xl &\in U_I l_1^{-1}\theta_I(x_1^{-1}x)l = U_I l_1^{-1}\theta_I(y_1y^{-1})l \\ &= U_I\theta_I(l_1^{-1}y_1y^{-1}l), \quad \text{since } l, l_1 \in L_I. \end{aligned}$$

Then

$$\begin{aligned} z = l_1^{-1}y_1y^{-1}l &\in l_1^{-1}U_I^- l_1m_1U_Ks_1^{-1}sU_Km^{-1}l^{-1}U_I^- l \\ &= U_I^- m_1U_Ks_1^{-1}sU_Km^{-1}U_I^- \\ &= U_I^- m_1s_1^{-1}sm^{-1}U_KU_I^-, \quad \text{since } m, s_1^{-1}s \in P_K \end{aligned}$$

So for some $u_1, u_2 \in U_I^-$, $v \in U_K$,

$$z = u_1m_1s_1^{-1}sm^{-1}vu_2.$$

Let $z_1 = m_1s_1^{-1}sm^{-1}v$. Then since $z \in P_I^-$, we see that $z_1 \in P_I^-$ and $\theta_I(z) = \theta_I(z_1)$. Since $s_1^{-1}s \in P_K$ we see that $z_1 = m_1\theta_K(s_1^{-1}s)m^{-1}w$ for some $w \in U_K$. Let $z_2 = m_1\theta_K(s_1^{-1}s)m^{-1} \in L_K$. Then

$$z_1 = z_2w \in P_K \cap P_I^- = L_{I\cap K}(L_K \cap U_I^-)(U_K \cap P_I^-).$$

So $z_2 \in L_{I\cap K}(L_K \cap U_I^-)$, $w \in U_K \cap P_I^-$. Hence

$$\theta_I(z) = \theta_I(z_1) = \theta_I(z_2 w) = \theta_I(z_2)\theta_I(w), \quad \theta_I(z_2) \in L_{I\cap K}, \ \theta_I(w) \in U_K \cap L_I.$$

In particular, $\theta_I(z) \in P_I \cap P_K = P_{I\cap K}$. So

$$l_1^{-1}x_1^{-1}xl \in U_I\theta_I(z) \subseteq U_I P_{I\cap K} \subseteq P_{I\cap K}.$$

Moreover, since $U_I \cup U_K \subseteq U_{I\cap K}$,

$$\begin{aligned}\theta_{I\cap K}(l_1^{-1}x_1^{-1}xl) &= \theta_{I\cap K}(\theta_I(z)) = \theta_{I\cap K}(\theta_I(z_2)\theta_I(w)) = \theta_{I\cap K}(\theta_I(z_2)) \\ &= \theta_{I\cap K}(z_2), \quad \text{since } U_I^- \subseteq U_{I\cap K}^- \\ &= \theta_{I\cap K}(m_1\theta_K(s_1^{-1}s)m^{-1}).\end{aligned}$$

Similarly $m_1t_1t^{-1}m^{-1} \in P_{I\cap K}^-$. Now since $t_1t^{-1} \in P_K^-$, $m_1t_1t^{-1}m^{-1} \in m_1\theta_K(t_1t^{-1})m^{-1}U_K^-$. Since $U_K^- \subseteq U_{I\cap K}^-$,

$$\begin{aligned}\theta_{I\cap K}(m_1t_1t^{-1}m^{-1}) &= \theta_{I\cap K}(m_1\theta_K(t_1t^{-1})m^{-1}) \\ &= \theta_{I\cap K}(m_1\theta_K(s_1^{-1}s)m^{-1}) \\ &= \theta_{I\cap K}(l_1^{-1}x_1^{-1}xl).\end{aligned}$$

It follows that the multiplication on $\mathscr{M}$ is well defined. We next prove associativity. Let $a = xe_Iy \in J_I$, $b = se_Kt \in J_K$, $c = ue_Nv \in J_N$, $H = I \cap K \cap N$. Suppose $(ab)c \neq 0$. Then $ys \in U_I^- l_1 l_2 U_K$ for some $l_1 \in L_I$, $l_2 \in L_K$ and $ab = xl_1e_{I\cap K}l_2t$. Since $(ab)c \neq 0$,

$$l_2tu \in U_{I\cap K}^- l_3l_4U_N \quad \text{for some } l_3 \in L_{I\cap K}, \ l_4 \in L_N.$$

So $(ab)c = xl_1l_3e_Hl_4v$. Now $U_{I\cap K}^- = U_K^-(U_I^- \cap L_K)$. So

$$\begin{aligned}tu &\in l_2^{-1}U_{I\cap K}^- l_3l_4U_N = l_2^{-1}U_K^-(U_I^- \cap L_K)l_3l_4U_N \\ &= U_K^- l_2^{-1}(U_I^- \cap L_K)l_3l_4U_N.\end{aligned}$$

So for some $z \in U_I^- \cap L_K$,

$$tu \in U_K^- l_2^{-1}zl_3l_4U_N \quad \text{and} \quad bc = sl_2^{-1}zl_3e_{K\cap N}l_4v.$$

Now

$$\begin{aligned}ysl_2^{-1}zl_3 &\in U_I^- l_1l_2U_Kl_2^{-1}zl_3 = U_I^- l_1U_Kzl_3 \\ &= U_I^- l_1zl_3U_K, \quad \text{since } zl_3 \in L_K \\ &= U_I^- l_1zl_1^{-1}l_1l_3U_K \\ &= U_I^- l_1l_3U_K, \quad \text{since } z \in U_I^-, \ l_1 \in L_I.\end{aligned}$$

Since $U_K \subseteq U_{K\cap N}$ and $l_1l_3 \in L_I$, we see that

$$a(bc) = xl_1l_3e_Hl_4v = (ab)c.$$

Similarly $(ab)c \neq 0$ implies that $a(bc) = (ab)c$. This shows that $\mathscr{M}$ is a monoid with group of units $G = J_\Gamma$. We proceed to show that $\mathscr{M}$ satisfies the two conditions for being a monoid of Lie type.

Let $x \in P(e_I)$. Then $xe_I = e_Ixe_I$. So $e_Ixe_I \neq 0$, $x \in P_I^-P_I$. Thus $xe_I = e_Ixe_I = \theta_I(x)e_I$. Hence $x^{-1}\theta_I(x) \in P_I$. So $x \in P_I$. Clearly $L_I \subseteq P(e_I)$. If $u \in U_I$, then $\theta_I(u) = 1$. So $ue_I = e_I$ and $U_I \subseteq P(e_I)$. Hence $P_I = L_IU_I \subseteq P(e_I)$. So $P(e_I) = P_I$. Similarly $P^-(e_I) = P_I^-$ and $e_IU_I^- = \{e_I\}$.

It suffices now to show that all the idempotents of J_I are conjugate. So let $f = xe_Iy \in E(J)$. Then $e_Iyxe_I = e_I$. So $yx \in P_I^- P_I$, $\theta_I(yx) = 1$. Hence $yx \in U_I^- U_I$ and $yx = uv$ for some $u \in U_I^-$, $v \in U_I$. Thus $u^{-1}y = vx^{-1}$ and

$$f = xe_Iy = xv^{-1}e_Iu^{-1}y = (xv^{-1})e_I(vx^{-1}).$$

It follows that e_I and f are conjugate, proving the theorem. □

Let $I \subseteq \Gamma$. Then $J_I^0 = J_I \cup \{0\}$ is a subsemigroup of $\mathscr{M}$, which is *completely 0-simple*. This means (since $\mathscr{M}$ is finite and regular) that J_I^0 has no proper ideals, cf. [4]. The maximal subgroup of e_I is just $e_IL_I \cong L_I$ and $e_I\mathscr{M}e_I \cong \mathscr{M}(L_I)$. The Rees theorem [4, Theorem 3.5] gives the structure of J_I^0 as a matrix semigroup over L_I. The multiplication is twisted by a *sandwich matrix* which we can describe easily. Let

$$G/P_I^- = \{P_I^- a_1, \dots, P_I^- a_t\}, \quad G/P_I = \{b_1P_I, \dots, b_tP_I\}.$$

The sandwich matrix is the $t \times t$ matrix $S = (\theta_I(a_ib_j))$ where θ_I is taken to be zero on $G\backslash P_I^- P_I$. J_I^0 can be thought of as the set of all $t \times t$ matrices over $L_I \cup \{0\}$ with at most one nonzero entry and with the multiplication given by:

$$A \circ B = ASB$$

We will let $\mathscr{M}_I = G \cup J_I \cup \{0\}$. Then $\mathscr{M}_I$ is a submonoid of $\mathscr{M}$ and is itself a monoid of Lie type on G. We also note that $\mathscr{U}(\mathscr{M}) = \{J_I | I \subseteq \Gamma\} \cup \{0\}$ and for all $I, K \subseteq \Gamma$, $J_I^0 J_K^0 = J_{I\cap K}^0$.

Let M be any monoid of Lie type on G, $J \in \mathscr{U}(M)$, $J \neq 0$. Let $J^0 = J \cup \{0\}$ with

$$a \circ b = \begin{cases} ab & \text{if } ab \in J, \\ 0 & \text{if } ab \notin J. \end{cases}$$

Then J^0 becomes a completely 0-simple semigroup. $M(J) = G \cup J^0$ (with obvious multiplication) is a monoid of Lie type. We note that in general $M(J)$ is not a submonoid of M, but is a submonoid of the Rees quotient monoid M/Ω, where $\Omega = \{a \in M \mid J \not\subseteq MaM\}$. For instance, if $M = \mathscr{M}_n(F)$, then J consists of all matrices of a particular rank. If $x \in G$, then we see by [11] that $e_Jxe_J \in J$ if and only if $x \in P_J^- P_J$. We therefore have the following.

Corollary 1.2. *Let M be a monoid of Lie type, $J \in \mathscr{U}(M)$, $J \neq 0$, $\lambda(J) = I$. Then there is a unique surjective homomorphism $\varphi\colon \mathscr{M}_I \to M(J)$ such that $\varphi(e_I) = e_J$ and φ is the identity map on G.*

2. Modular representations

Let p denote the natural characteristic of G, $\kappa = \overline{\mathbb{F}}_p$. By a (modular) representations of G, we will mean in this section, a representation $\varphi : G \to GL(n, \kappa)$. We will let $\kappa[G]$ denote the group algebra of G over κ. By a G-module we will mean a $\kappa[G]$-module.

Let M be a monoid of Lie type on G. We will mean by a (modular) representation of M, a homomorphism $\varphi : M \to \mathscr{M}_n(\kappa)$ such that $\varphi(1) = 1$ and $\varphi(0) = 0$. We will let $\kappa_0[M]$ denote the contracted semigroup algebra of M over κ (i.e., the zero of M is the zero of $\kappa_0[M]$). Then the representations of M are in a natural 1-1 correspondence with the nonzero $\kappa_0[M]$-modules. By an M-module we will mean a $\kappa_0[M]$-module. We will use the same notation for any semigroup with zero.

Proposition 2.1. *Let G be a reductive group defined over κ and G_σ an associated finite group of Lie type. Let $\psi : G \to GL(n, \kappa)$ be a representation, φ the restriction of ψ to G_σ. Let B, B^- be opposite σ-stable Borel subgroups of G, $T = B \cap B^-$. Then there is a minimal nonzero idempotent $e_\varphi \in E(\overline{\kappa\psi(T)})$ such that $\psi(B) \subseteq P(e_\varphi)$, $\psi(B^-) \subseteq P^-(e_\varphi)$. For $g \in G$, $a \in \mathscr{M}_n(\kappa)$, let $ga = \psi(g)a$, $ag = a\psi(g)$. Then*

$$M_\varphi = G_\sigma \cup G_\sigma e_\varphi G_\sigma \cup \{0\}$$

is a monoid of Lie type on G_σ and there is a representation $\hat{\varphi} : M_\varphi \to \mathscr{M}_n(\kappa)$ with $\hat{\varphi}(g) = \varphi(g)$ for $g \in G_\sigma$, $\hat{\varphi}(e_\varphi) = e_\varphi$.

Proof. Let $G_1 = \kappa^*\psi(G)$. Then G_1 is a reductive group. Let $M_1 = \overline{G}_1$, the closure in $\mathscr{M}_n(\kappa)$. Then M_1 is a regular linear algebraic monoid and the existence of $e = e_\varphi$ follows from [10, Chapter 9]. Moreover, if $T_1 = \kappa^*\psi(T)$, then $e\psi(G)e = eT_1 \cup \{0\}$. In particular, $eU^- = \{e\} = Ue$ where U, U^- denote the unipotent radicals of B, B^-, respectively. Let $N = N_G(T)$. Then for all $n \in N$, either $en = ne$ or else $ene = (e \cdot nen^{-1})n = 0$. Let $P_1 = \{x \in G_\sigma | xe = exe\}$, $P_1^- = \{x \in G_\sigma | ex = exe\}$. Then P_1, P_1^- are parabolic subgroups of G_σ containing B_σ and B_σ^- respectively. Moreover, for $n \in N_\sigma$, $n \in P_1$ if and only if $n \in P_1^-$. It follows that P_1 and P_1^- are opposite parabolic subgroups of G_σ. Let $L_1 = P_1 \cap P_1^-$. Then $P_1 = L_1U_1$, $P_1^- = L_1U_1^-$, $U_1 \subseteq U$, $U_1^- \subseteq U^-$. So $U_1e = \{e\} = eU_1^-$. Let $x \in G_\sigma$. Suppose $x \notin P_1^- P_1$. Then by the Bruhat decomposition, $x = anb$ for some $a \in P_1^-$, $n \in N_\sigma$, $b \in P_1$, $n \notin L_1$. So $ene = 0$ and

$$exe = (eae)(ene)(ebe) = 0.$$

If $x \in P_1^- P_1 = U_1^- L_1 U_1$ then for some $l \in L_1$, $u \in U_1^-$, $v \in U_1$, $x = ulv$ and

$$exe = eulve = ele = el = le.$$

It follows that M_φ is a monoid. It suffices to show that all the idempotents of $G_\sigma e G_\sigma$ are conjugate in M_φ. So let $x, y \in G_\sigma$ such that $f = xey \in E(M_\varphi)$. Then $eyxe = e$. So $yx \in P_1^- P_1$. Hence $yx = ulv$, $u \in U_1^-$, $v \in U_1$, $l \in L_1$. So $ele = eyxe = e$, $u^{-1}y = lvx^{-1}$ and

$$f = xey = xv^{-1}l^{-1}eu^{-1}y = (xv^{-1}l^{-1})e(lvx^{-1})$$

Hence e and f are conjugate in M_φ. Thus M_φ is a monoid of Lie type and the proof is complete. □

Let G be a (finite) group of Lie type with Coxeter graph Γ. By [5, Theorems 5.7, 6.15] the irreducible modular representations of G are in 1-1 correspondence with ordered pairs (I, χ) where $I \subseteq \Gamma$ and $\chi : L_I \to \kappa^*$ any homomorphism. If V is an irreducible $\kappa[G]$-module, then there is a unique one-dimensional subspace Y of V which is stabilized by B. $I \subseteq \Gamma$ is such that $P_I = \{x \in G | xY = Y\}$. Then Y is a one-dimensional $\kappa[L_I]$ module, yielding $\chi : L_I \to \kappa^*$.

By Theorem 1.1, the non-zero $\mathscr{J}$-classes of $\mathscr{M} = \mathscr{M}(G)$ are J_I $(I \subseteq \Gamma)$ and the maximal subgroup of e_I is $e_I L_I \cong L_I$. Hence by [4, Theorems 5.33, 5.51] the irreducible modular representations of $\mathscr{M}$ are in 1-1 correspondence with ordered pairs (I, φ) where $I \subseteq \Gamma$ and φ is a modular irreducible representation of L_I. Let V be an irreducible $\kappa_0[\mathscr{M}]$-module. Then there is a minimum $I \subseteq \Gamma$ such that $J_I V \neq 0$, and $e_I V$ is an irreducible L_I-module yielding φ.

Theorem 2.2. *Every irreducible modular representation of $\mathscr{M} = \mathscr{M}(G)$ restricts to an irreducible representation of G. An irreducible representation of G of type (I, χ) extends to $2^{|\Gamma\setminus I|}$ inequivalent irreducible representations of $\mathscr{M}$. If α_I denotes the number of homomorphisms from L_I to κ^*, then the number of inequivalent irreducible representations of $\mathscr{M}$ is $\sum_{I\subseteq\Gamma} 2^{|\Gamma\setminus I|}\alpha_I$.*

Proof. Let $I \subseteq \Gamma$, $\chi : L_I \to \kappa^*$ a homomorphism. We then have an associated irreducible G-module V. Let φ be the corresponding representation of G. By [20], φ is the restriction of a representation of the associated reductive group. By Proposition 2.1, we can construct the monoid of Lie type M_φ. Then V is an M_φ-module. By Corollary 1.2, V then becomes an irreducible $\mathscr{M}_I$-module such that B stabilizes the one-dimensional subspace $e_I V$ and P_I is the stabilizer of $e_I V$. Since J_I^0 is an ideal of $\mathscr{M}_I$, V is also an irreducible J_I^0-module. Let $\Omega = \bigcup\{J_K | I \not\subseteq K\} \cup \{0\}$. Then Ω is an ideal of $\mathscr{M}$ and J_I^0 is an ideal of the Rees quotient semigroup $\mathscr{M}/\Omega$. Thus V becomes an $\mathscr{M}/\Omega$-module and hence also an $\mathscr{M}$-module. Now $J_I V \neq 0$, $J_K V = 0$ for $I \not\subseteq K$. Now let $I \subseteq K$. Then since $e_K \geq e_I$, $J_K V \neq 0$. Since $\mathscr{M}_K$ is a submonoid of $\mathscr{M}$, containing G, V is also an irreducible $\mathscr{M}_K$-module. Hence as above, V becomes an irreducible $\mathscr{M}$-module in a new way such that $J_K V \neq 0$, $J_{K'} V = 0$ for $K \not\subseteq K'$. We have thus produced $2^{|\Gamma\setminus I|}$ inequivalent irreducible $\mathscr{M}$-modules which are all the same as G-modules. Hence we have produced

$$\sum_{I\subseteq\Gamma} 2^{|\Gamma\setminus I|}\alpha_I \tag{1}$$

inequivalent irreducible representations of $\mathscr{M}$, each of which restricts to an irreducible representation of G. The number of inequivalent irreducible representations of L_I is $\sum_{K\subseteq I}\alpha_K$. The number of irreducible representations of $\mathscr{M}$ is therefore

$$\sum_{I\subseteq\Gamma}\sum_{K\subseteq I}\alpha_K. \tag{2}$$

Since the numbers (1), (2) are equal, the proof is complete. □

Example 2.3. Let $G = SL(n, \mathbb{F}_q)$. The number of irreducible modular representations of G is equal to q^{n-1}, cf. [20]. The number of irreducible modular representations of $\mathscr{M}$ is equal to $\sum_{i=0}^{n-1} 2^i q^{n-1-i} = \frac{q^n - 2^n}{q-2}$.

We now use a result of Clifford [2, 3] to obtain some information about the degree of an irreducible modular representation of G.

Corollary 2.4. *Let φ be an irreducible modular representation of G of type (I, χ) where $I \subseteq \Gamma$, $\chi : L_I \to \kappa^*$ a homomorphism. Let*

$$G/P_I^- = \{P^- a_1, \ldots, P^- a_t\}, \quad G/P_I = \{b_1 P, \ldots, b_t P\}.$$

If $a_i b_j \in P_I^- P_I$, let $\alpha_{ij} = \chi(\theta_I(a_i b_j))$. If $a_i b_j \notin P_I^- P_I$, let $\alpha_{ij} = 0$. Then the degree of φ is equal to the rank of (α_{ij}).

Proof. Let V be the corresponding G-module. Then by Theorem 2.2, V is also a J_I^0-module. By Clifford [2, 3] or [4, Theorem 5.46], $\dim V$ is equal to the rank of $\chi(S)$, where S is the sandwich matrix of J_I^0. We have already seen that $S = (\theta_I(a_i b_j))$ where θ_I is taken to be zero on $G\setminus P_I^- P_I$. The result follows. □

Remark 2.5. Actually by [4, Chapter 5] we see that φ can be obtained from the full rank factorization of the $t \times t$ matrix (α_{ij}).

Corollary 2.6. *The modular Steinberg representation of G extends to an idempotent separating representation φ of $\mathscr{M}$. The representation φ can also be obtained* (*via Clifford* [2]) *from the trivial representation of $T = L_\varnothing$.*

Proof. Since the Steinberg representation is of type $(\varnothing, 1)$, we see by Theorem 2.2 that it has an extension to a representation φ of $\mathscr{M}$ such that $\varphi(e_\varnothing) \neq 0$ and the corresponding $L_\varnothing$-module is the trivial module. Hence it suffices to show that φ is idempotent separating. Suppose $\varphi(e) = \varphi(f)$ for some idempotents $e \neq f$. Then either $e\mathscr{M} \neq f\mathscr{M}$ or $\mathscr{M}e \neq \mathscr{M}f$. By symmetry assume $e\mathscr{M} \neq f\mathscr{M}$. By [11, Corollary 2.17] there exist $e_1, f_1 \in E(\mathscr{M})$ such that $e\mathscr{M} = e_1\mathscr{M}$, $f\mathscr{M} = f_1\mathscr{M}$ and $e_1 f_1 = f_1 e_1$. Let $h = e_1 f_1$. Then either $e_1 \neq h$ or $f_1 \neq h$. By symmetry let $e_1 \neq h$. Then $e_1 > h$ and $\varphi(e_1) = \varphi(h)$. So $h \neq 0$. By [11, Corollary 2.14] we can assume that $e_1 = e_I$, $h = e_K$, $K \subset I$. There exists $\sigma = nT \in I \backslash K$. So

$$0 = \varphi(e_K n e_K) = \varphi(e_I n e_I) = \varphi(e_I n) \neq 0.$$

This contradiction completes the proof. □

Corollary 2.7. *Let M be any monoid of Lie type on G. Then any irreducible modular representation of M restricts to an irreducible representation of G.*

Proof. Let V be an irreducible M-module. Then there is a minimum $J \in \mathscr{U}(M)$ such that $JV \neq 0$. Let $\Omega = \{a \in M \mid J \not\subseteq MaM\}$. Then Ω is an ideal of M and V is an irreducible M/Ω-module. Since J^0 is an ideal of M/Ω, V is an irreducible J^0-module. Since J^0 is an ideal of $M(J)$, V is also an irreducible $M(J)$-module. By Corollary 1.2, V is an irreducible $\mathscr{M}_I$-module where $I = \lambda(J)$. Hence V is an irreducible $\mathscr{M}$-module. By Theorem 2.2, V is an irreducible G-module. This completes the proof. □

Irreducible modular representations of $\mathscr{M}_n(\mathbb{F}_p)$ were determined by Harris and Kuhn [7]. By Corollary 1.2 and Theorem 2.2, we have

Corollary 2.8. *Let M be a monoid of Lie type on G and $\varphi : G \to GL(n, \kappa)$ an irreducible modular representation of type (I, χ). Then the inequivalent irreducible representations of M, extending φ, are in* 1-1 *correspondence with $0 \neq J \in \mathscr{U}$ such that $\lambda(J) = K \supseteq I$ and so that the following diagram can be completed*:

$$\begin{array}{ccc} \mathscr{M}_K & \longrightarrow & M(J) \\ \downarrow & \swarrow & \\ \mathscr{M}_n(\kappa) & & \end{array}$$

Remark 2.9. By Corollary 2.8, it follows that the irreducible modular representations of M can be enumerated by analyzing the combinatorial system $(\mathscr{U}, \lambda)$ and the irreducible modular representations of G. In particular the results of Harris and Kuhn [7, §6] can be recovered.

Remark 2.10. The referee points out that Corollary 2.7 is not valid for indecomposable representations. A counterexample is provided by $M = \mathscr{M}_2(\mathbb{F}_2)$, $G =$

$GL(2, \mathbb{F}_2)$, $V = \langle x^2, y^2, xy\rangle \subseteq \kappa[x, y]$. The referee further informs us that Leonid Krop has studied such examples in detail.

3. Complex representations

Let G be a finite group of Lie type. By a representation of G, we will mean in this section, a homomorphism $\varphi : G \to GL(n, \mathbb{C})$. If M is a finite monoid of Lie type on G, we will mean by a representation of M, a homomorphism $\varphi : M \to \mathscr{M}_n(\mathbb{C})$ such that $\varphi(1) = 1$ and $\varphi(0) = 0$.

An irreducible representation $\varphi : G \to GL(n, \mathbb{C})$ is *cuspidal* if for any $I \subsetneq \Gamma$,

$$\sum_{u \in U_I} \varphi(u) = 0.$$

Then Harish-Chandra observed, cf. [1, Chapter 9] that for any irreducible representation φ of G, there is a subset $I \subseteq \Gamma$ and an irreducible cuspidal representation θ of L_I such that if $\overline{\theta}$ is the lift of θ from L_I to P_I (i.e., $\overline{\theta}$ is trivial on U_I), then φ is a component of the induced representation $\overline{\theta}_{P_I}^G$. In this way the irreducible representations of G are classified according to which Levi subgroup L_I they come from. This philosophy of Harish-Chandra along with a submonoid of $\mathscr{M}(G)$ was used by Okniński and one of the authors [8] to show that any complex representation of a monoid of Lie type is completely reducible. This result along with the decomposition of [16] are used by Solomon [18] to begin a program of constructing Hecke algebras for monoids of Lie type.

Let $\mathscr{M} = \mathscr{M}(G)$. Then by Theorem 1.1 the nonzero $\mathscr{J}$-classes of $\mathscr{M}$ are J_I $(I \subseteq \Gamma)$. Moreover, the maximal subgroup of e_I is $e_I L_I \cong L_I$. Hence by [4, Theorems 5.33, 5.51], the irreducible representations φ of $\mathscr{M}$ are in a natural 1-1 correspondence with (I, θ) where $I \subseteq \Gamma$ and θ is an irreducible representation of L_I. We will say that φ is *cuspidal* if θ is cuspidal. By [8, Corollary 2.7] φ restricted to G is just the induced representation $\overline{\theta}_{P_I}^G$ where $\overline{\theta}$ is the lift of θ to P_I. We therefore have the following interpretation of Harish-Chandra's philosophy:

Theorem 3.1. *Any complex irreducible representation of G is a component of an irreducible cuspidal representation of $\mathscr{M}$, restricted to G.*

Remark 3.2. Unlike in the situation of modular representations, the above theorem only gives us induced representations and not irreducible representations of G. For example, unlike Corollary 2.6, it does not give us the Steinberg representation. For this reason it may be worthwhile to look for natural monoids having G as the group of units, whose complex semigroup algebras are <u>not</u> semisimple!

Conjecture 3.3. Theorem 2.1 is valid for representation over fields of characteristic not equal to the natural characteristic of G.

4. Reductive monoids and finite monoids

Let M be a reductive, algebraic monoid with unit group $G = G(M)$. By this we mean that $M = \overline{G}$ is a linear algebraic monoid, cf. [10], with G being a reductive group. If the center ZG of G is one dimensional, we say that M is *semisimple*. We assume throughout that the characteristic of the underlying ground field is $p > 0$. Let σ be an endomorphism of M. Let $G_\sigma = \{g \in G \mid \sigma(g) = g\}$ and $M_\sigma = \{x \in M \mid \sigma(x) = x\}$. Define $1 - \sigma : G \to G$ by $(1-\sigma)(g) = g\sigma(g)^{-1}$. For $X \subseteq M$, let $X_\sigma = \{x \in X \mid \sigma(x) = x\}$. If $G = T$ is a torus we say that $M = \overline{T}$ is a D-monoid. We denote by $X(T)$ and $X(\overline{T})$ the character group and monoid of T and $\overline{T}$, respectively.

Proposition 4.1. *The following are equivalent:*

(a) *σ is a finite morphism and G_σ is a finite group.*
(b) *$(1-\sigma)(G) = G$ and $\sigma(Z) = Z$ for some $Z = \overline{T} \subseteq M$ a maximal D-submonoid.*

Proof. Assume (a). If G_σ is finite then by [20] there exists $T \subseteq G$ a maximal torus such that $\sigma(T) = T$. If σ is finite then $\sigma(\overline{T}) = \overline{T}$ since finite morphisms are closed. Furthermore, by Lang's theorem [20], $(1-\sigma)(G) = G$. Assume (b). If $(1-\sigma)(G) = G$ then G_σ is finite, and by [19] there exists $T \subseteq G$ a maximal torus such that $\sigma(T) = T$. By assumption, $\sigma(\overline{T}) = \overline{T}$ since all maximal tori are conjugate. Thus, $\sigma(E(\overline{T})) = E(\overline{T})$. Hence, σ is injective on $E(\overline{T})$ and so $\sigma \mid \overline{T}$ is finite. Thus, by [15, 2.1 and 6.3] σ is finite. □

Corollary 4.2. *Assuming G_σ finite, the following are equivalent:*

(a) *$\sigma : M \to M$ is finite.*
(b) *If $\sigma(T) = T$ then in $\sigma^* : X(\overline{T}) \to X(\overline{T})$, $\sigma^*(X(\overline{T})) \subseteq X(\overline{T})$ satisfies $nX(\overline{T}) \subseteq \sigma^*(X(\overline{T}))$ for some $n > 0$.*

Proof. By [10, Theorem 8.14], $\sigma \mid \overline{T}$ is finite if and only if $nX(\overline{T}) \subseteq \sigma^*(X(\overline{T}))$ for some $n > 0$. □

Assume that σ is a finite morphism with G_σ finite. We now investigate the important structural properties of M_σ. We let $\Lambda = \Lambda(M)$ denote a cross section lattice of M, [10, Chapter 9]. Thus Λ consists of order-preserving idempotent representatives of $\mathcal{J}$-classes ($= G \times G$ orbits) of M. The Green's relation $\mathcal{R}$ on M is defined as: $a\mathcal{R}b$ if and only if $aM = bM$.

Theorem 4.3. (a) *M_σ is finite.*

(b) *M_σ is unit regular, i.e. $M_\sigma = E(M_\sigma)G_\sigma$.*
(c) *If $T \subseteq B$ satisfies $\sigma(T) = T$ and $\sigma(B) = B$ then $M_\sigma = \bigcup_{e \in \Lambda_\sigma} G_\sigma e G_\sigma$ and $E_\sigma = E(M_\sigma) = \bigcup_{g \in G_\sigma} g\Lambda_\sigma g^{-1}$ where $\Lambda_\sigma = \{e \in E(\overline{T}) \mid \sigma(e) = e,\ Be = eBe\}$.*
(d) *If $e \in E_\sigma$ then $C^r_{G_\sigma}(e)$ and $C^l_{G_\sigma}(e)$ are opposite parabolic subgroups of G_σ.*
(e) *If $e, f \in E_\sigma$ and $eM_\sigma = fM_\sigma$ or $M_\sigma e = M_\sigma f$ then there exists $g \in G_\sigma$ such that $geg^{-1} = f$.*
(f) *If $e \in \Lambda_\sigma$ then $Ue = eU^- = \{e\}$, where $U \lhd C^r_{G_\sigma}(e)$ and $U^- \lhd C^l_{G_\sigma}(e)$ are the unipotent radicals.*

Proof. We first prove (b). Let $x \in M_\sigma$, so $\sigma(x) = x$. Thus, $\sigma(R) = R$ where $R = xG$ is the $\mathcal{R}$-class of x. If $x = ex$, $e \in E(R)$, then $\sigma(C^r_G(e)) = C^r_G(e)$. But then $C^r_G(e) \times E(R) \to E(R)$ is transitive, while $C_G(f)$ is connected for all $f \in E(R)$ [10, Corollary 6.18]. Thus, by [19, I, 2.7(b)], $C^r_{G_\sigma}(e) \times E(R)_\sigma \to E(R)_\sigma$ is transitive, and $E(R)_\sigma \neq \varnothing$. Hence, $x = ey$ for some $e \in E(R)_\sigma$, $y \in G$. Now let $X = \{g \in G \mid eg = x\}$, $H = \{h \in G \mid eh = e\}$. Then $H \times H \to X$

is transitive and σ acts accordingly. Thus, by [19, I, 2.7(a)], $X_\sigma \neq \varnothing$. This shows that we can choose $g \in G_\sigma$ and so M_σ is unit regular.

Proof of (a). Let $f \in E_\sigma$, and let $Z = Cl_G(f) = \{gfg^{-1} \mid g \in G\}$. Then $\sigma(Z) = Z$. But then $G \times Z \to Z$ is transitive and σ acts as in [19]. Thus by [19, I, 2.7(b)] $G_\sigma \times Z_\sigma \to Z_\sigma$ is transitive, and so Z_σ is finite. But the number of such Z's is finite. Hence, E_σ is finite. Finally, $M_\sigma = G_\sigma E_\sigma G_\sigma$ is finite.

Proof of (c). We have $M_\sigma = G_\sigma E_\sigma G_\sigma$. If $\sigma(e) = e$ then $\sigma(L) = L$ where $L = C_G(e)$. Thus, there exists $T \subseteq B_0 \subseteq L$, a maximal torus and Borel subgroup respectively, such that $\sigma(T) = T$ and $\sigma(B_0) = B_0$. But also $\sigma(R_u(C_G^r(e))) = R_u(C_G^r(e))$. Now $B = B_0 R_u(C_G^r(e))$ is a Borel subgroup of G, and $\sigma(B) = B$. Hence we have found a pair (T, B) such that $\sigma(T) \subseteq T$, $\sigma(B) \subseteq B$, $T \subseteq B$ and $T \subseteq C_G(e)$. Hence $\Lambda = \{f \in E(\overline{T}) \mid B \subseteq C_G^r(f)\}$ is a cross section lattice of M such that $\sigma(\Lambda) \subseteq \Lambda$ and $e \in \Lambda_\sigma$. From the proof of (a) we know that if $e, f \in M_\sigma$ are conjugate in M then they are conjugate in M_σ. So $E_\sigma = \bigcup_{g \in G_\sigma} g\Lambda_\sigma g^{-1}$. Hence, $M_\sigma = \bigcup_{e \in \Lambda_\sigma} G_\sigma e G_\sigma$. The union is disjoint since $\Lambda_\sigma \subseteq \Lambda$ and $G_\sigma \subseteq G$.

Proof of (d). This follows from [10, Theorem 7.1] and the observations (i) if $P \subseteq G$ is parabolic and $\sigma(P) = P$, then $P_\sigma \subseteq G_\sigma$ is parabolic, (ii) if P and P^- are opposite in G and $\sigma(P) = P$, $\sigma(P^-) = P^-$ then P_σ and P_σ^- are opposite in G_σ.

Proof of (e). This is another application of [19]. Let $e, f \in E_\sigma$, $e\mathscr{R}f$ in M_σ. Then $e\mathscr{R}f$ in M. Thus $G \times Z \to Z$ is transitive, where $Z = Cl_G(e) = Cl_G(f)$. Since $C_G(e)$ is connected, $G_\sigma \times Z_\sigma \to Z_\sigma$ is transitive.

Proof of (f). If $e \in \Lambda_\sigma$ and $P = C_G^r(e)$ then $U = R_u(P)_\sigma$. But $R_u(P)e = \{e\}$. □

Remark 4.4. Because of (b), (d), (e) and (f) M_σ is, in the terminology of §1, a monoid of Lie type. In particular, there is a Bruhat type decomposition for M_σ which follows formally from Theorem 4.3 above and the results of [11]. In detail, if $\sigma(T) = T$, $\sigma(B) = B$, $T \subseteq B$ and $\Lambda = \{e \in E(\overline{T}) \mid B \subseteq C_G^r(e)\}$, then by [11, §4], $M_\sigma = \bigcup_{r \in \mathfrak{R}'} B_\sigma r B_\sigma$, where $\mathfrak{R}' = \langle W_\sigma, E(\overline{T})_\sigma \rangle$. But the union is disjoint since $\mathfrak{R}' \subseteq \mathfrak{R} = \langle W, E(\overline{T}) \rangle$ and $B_\sigma \subseteq B$. Also $\mathfrak{R}' \subseteq \mathfrak{R}_\sigma$. Conversely, $\mathfrak{R}_\sigma \subseteq \mathfrak{R}'$ since $x \in \mathfrak{R}_\sigma$ implies $\sigma : T \cdot x \to T \cdot x$ has a fixed point [19, I, 2.7(a)] say x, and so $x \in B_\sigma r B_\sigma$ for some $r \in \mathfrak{R}'$. Thus, $x = r$ by the above disjointedness argument.

Hence we obtain

Corollary 4.5. *$M_\sigma = \bigcup_{r \in \mathfrak{R}_\sigma} B_\sigma r B_\sigma$. Furthermore, $\mathfrak{R}_\sigma = \langle W_\sigma, E(\overline{T})_\sigma \rangle$, and the union is disjoint.*

Remark 4.6. One can use Corollary 4.5 to obtain a formula for the order of M_σ [17]. One could probably refine that formula significantly.

Our task now is to construct, for each finite group G_σ of Lie type a monoid M_σ (for some reductive monoid M with $\sigma: M \to M$) with the following properties:

(1) $G(M_\sigma)$ is a central extension of G_σ.
(2) The map $e \mapsto C_{G_\sigma}(e) = P_{I(e)}$ induces an order preserving bijection from $\Lambda_\sigma \backslash \{0\}$ to 2^Γ. Here Γ is the set of simple involutions relative

to T_σ and B_σ, and $I(e) \subseteq \Gamma$ is such that $C^r_{G_\sigma}(e) = B_\sigma W_{I(e)} B_\sigma$. By the results of [13] one can obtain reductive algebraic monoids with these properties by choosing a high weight in general position. The resulting monoid is a cone on the canonical compactification. This compactification was introduced in [6] because of its superior intersection theoretic properties.

For most pairs (G, σ) the construction of M as above is straightforward, but for (G, σ) of Ree or Suzuki type, one cannot use a semisimple monoid, because the requirement that $\sigma : M \to M$ be a finite morphism would imply that $\sigma^2(t) = t^{p^{2a+1}}$ for all $t \in Z(G(M))^0$ and some $a > 0$. This is not possible.

Suppose that G is a simple algebraic group and $\sigma : G \to G$ is of Chevalley or Steinberg type. So if $\sigma(T) = T$ and $\sigma(B) = B$ then $\sigma^*(\alpha) = q\rho^*(\alpha)$ for all $\alpha \in \Delta$ where $\rho : G \to G$ is some outer automorphism ($\rho = \text{id}$ is allowed) such that $\rho(T) = T$ and $\rho(B) = B$. Let (X, Φ) be the root system of $G \times k^*$ where $X = X(T) \oplus \mathbb{Z}$, and $\Phi \subseteq X(T)$ is the set of roots of G. Choose a dominant weight $\lambda = \sum_{i=1}^r a_i\lambda_i$ such that $\sigma^*(x) = q\lambda$ and $a_i > 0$ for all i. Let $(\lambda, 1) \in X$ and extend σ^* to X via $\sigma_1^*(\mu, \gamma) = (\sigma^*(\mu), q\gamma)$. Then σ_1^* is induced from $\sigma_1 : G \times k^* \to G \times k^*$, $\sigma_1(g, t) = (\sigma(g), t^q)$. Let $C \subseteq X$ be the smallest polyhedral cone containing $\{(w(\lambda), 1 \mid w \in W\}$. Then C is W-invariant and $\sigma_1^*(C) = qC$. By [15, Theorem 6.5] there exists a semisimple monoid M with unit group $G \times k^*$ such that $X(\overline{T}_1) = C \subseteq X$, where $\overline{T}_1 \subseteq M$ is the closure in M of T_1. By [15, Corollary 4.5] and Corollary 1.2 above, $\sigma_1 : G \times k^* \to G \times k^*$ extends to a finite morphism $\sigma_1 : M \to M$ such that $(G \times k^*)_\sigma$ is a finite group. Changing notation, let $\sigma = \sigma_1$ and $G = G_1$.

Theorem 4.7. *Let Γ be the set of simple involutions of W_σ relative to B_σ. Then M_σ satisfies the following properties:*

(a) $\Lambda_\sigma \backslash \{0\} \cong 2^\Gamma$, *via the map* $e \mapsto I(e)$, $C^r_{G_\sigma}(e) = P_{I(e)}$, $I(e) \subseteq \Gamma$.

(b) *If* $I(e) = I$, $I(f) = K \subseteq \Gamma$ *let* $J_I = G_\sigma e G_\sigma$ *and* $J_K = G_\sigma f G_\sigma$. *Then* $J_I J_K \subseteq J_{I \cap K} \cup \{0\}$.

Proof. By [1, 1.18], the standard parabolics of G_σ are $\{P_\sigma \mid P \supseteq B$ and $\sigma(P) = P\}$. Conversely, $\Lambda_\sigma = \{e \in \Lambda \mid \sigma(e) = e\} = \{e \in \Lambda \mid \sigma(C^r_G(e)) = C^r_G(e)\}$. Hence $\Lambda_\sigma \backslash \{0\} \to 2^\Gamma$ is bijective, and order preserving since by [13, Lemma 4.12], $\Lambda \backslash \{0\} \to \{P \mid P \supseteq B\}$ is order preserving. This proves (a).

For (b) it suffices to prove the corresponding statement for M, since for $e \in \Lambda_\sigma \backslash \{0\}$, $GeG \cap M_\sigma = G_\sigma e G_\sigma$. Let $e_0 \in \Lambda \backslash \{0\}$ be the minimal element. Since $C_W(e_0) = \{1\}$, it follows that $\Lambda \backslash \{0\} = \{f \in E(\overline{T}) \mid fe_0 = e_0 f = e_0\}$. So let $x \in J_I$, $y \in J_K$. Then we can write $x = geg'$, $y = h'fh$ where $e, f \in \Lambda \backslash \{0\}$ and $g, h, g', h' \in G$. Thus, $xy = geg'h'fh = geh_1wh_2fh$, where $b_1 \in B^-$, $b_2 \in B$ and $w \in W$. But $e, f \in \Lambda$, so $xy = gb_1'ewfb_2'h$ for some $b_1' \in B^-$, $b_2' \in B$. If $ewfw^{-1} = 0$ then $xy = 0$. If $ewfw^{-1} \neq 0$ then $e, wfw^{-1} \in \Lambda'$ for some cross section lattice $\Lambda' \subseteq E(\overline{T})$. But $\Lambda'^v = \Lambda$ for some $v \in W$. Thus, $e^v, e, f^{vw}, f \in \Lambda$, and so $e^v = e$ and $f^{vw} = f$. Hence, $xy \in J_{I \cap K}$ because $(ef^w)^v = ef$. $\square$

We now turn to the task of constructing monoids as in Theorem 4.7 for the groups of Ree and Suzuki type. As we have already pointed out, it is not possible to find a reductive monoid M with $0 \in M$ and $\dim ZG = 1$ such that $\sigma : G \to G$ extends to a finite morphism $\sigma : M \to M$.

So assume G is an algebraic group of type C_2, F_4 or G_2 and $\sigma : G \to G$ is an endomorphism of Ree or Suzuki type. Assuming $\sigma(T) = T$ and $\sigma(B) = B$, with $T \subseteq B$, we can arrange the simple roots $\Delta = \{\alpha_i, \beta_i\}_{i=1}^{s}$ ($s = 1$ or 2) so that $\sigma^*(\alpha_i) = p^a\beta_i$ and $\sigma^*(\beta_i) = p^{a+1}\alpha_i$. (See [19, 11.6] for more details.) Choose $\mu = \sum_{i=1}^{2s} a_i\lambda_i$, a dominant weight so that $a_i > 0$ for $i > 0$ and $a\lambda_i \in X(T)$ for all $i > 0$. Then $\sigma^*(\mu)$ is also a dominant weight, and $\sigma^*(\mu)$ is not a multiple of μ since σ^* has no rational eigenvalues on $X(T) \otimes \mathbb{R}$. Let $\overline{\mu} = \sigma^*(\mu)/p^a \in X(T)$. Let $\theta_1 : G \to Gl(V)$ and $\theta_2 : G \to Gl(W)$ be the irreducible representation of high weight μ and $\overline{\mu}$ respectively, and let $\rho = \theta_1 \oplus \theta_2$. Define

$$M_1 = \overline{\rho(G)(k^* \times k^*)} \subseteq \mathrm{End}(V \oplus W)$$

and let M be the normalization of M_1. Using 6.3 and 6.4 of [15] and 4.1 of [14] it follows that if $\overline{T} \subseteq M$ is the closure of a maximal torus then $X(\overline{T})$ is the smallest polyhedral cone in $X(T) = X(T_0) \oplus \mathbb{Z}^2$ containing $W \cdot \xi \cup W \cdot \overline{\chi}$ where $\chi = (\mu, 1, 0)$ and $\overline{\chi} = (\overline{\mu}, 0, 1)$. The morphism $\sigma : G_0 \to G_0$ extends to $\sigma : G \to G$ via $\sigma(g, s, t) = (\sigma(g), t^{p^{a+1}}, s^{p^a})$ and this yields (by duality) $\sigma : T \to T$ via $\sigma^*(\gamma, \alpha, \beta) = (\sigma^*\gamma, p^{a+1}\beta, p^a\alpha)$. Since $\sigma^*(\chi) = p^a\overline{\chi}$ and $\sigma^*(\overline{\chi}) = p^{a+1}\chi$, $\sigma^*(X(\overline{T})) \subseteq X(\overline{T})$ with $p^{2a+1}X(\overline{T}) \subseteq \sigma^*(X(\overline{T}))$. Thus, $\sigma : T \to T$ extends to $\sigma : \overline{T} \to \overline{T}$. Hence, by 4.5 of [15], there exists a unique $\sigma : M \to M$ extending $\sigma : \overline{T} \to \overline{T}$ and $\sigma : G \to G$. By Corollary 4.2 above σ is a finite morphism.

By Corollary 3.2 of [15] it follows that Λ has the following properties:

(1) $\Lambda_1 = \{e_1, e_2\}$ and $\sigma(e_1) = e_2$, $\sigma(e_2) = e_1$.
(2) If $e \in \Lambda_1$, then $C_G(e) = T$ a maximal torus.
(3) $\Lambda\backslash\{0\} = \{f \in E(\overline{T}) \mid fe_1 = e_1 \text{ or } fe_2 = e_2\}$.

Here Λ_1 consists of the minimal nonzero elements of Λ. Similarly Λ_2 consists of those elements of Λ covering some element of Λ_1, etc. The sets $E_i = E_i(M)$, $i = 1, 2, \ldots$, are similarly defined.

Proposition 4.8. *Let $\Lambda \subseteq M$ be a cross section lattice such that $\sigma(\Lambda) \subseteq \Lambda$. Then $\Lambda_\sigma\backslash\{0\} = \{e \in \Lambda \mid \sigma(e) = e,\ e \neq 0\}$ has a unique nonzero minimal element e_0. Further, $e_0 \in \Lambda_2$.*

Proof. Let $\Lambda_1 = \{e_1, e_2\}$. It suffices to show that $e_1 \vee e_2 \in \Lambda_2$ since $\sigma(e_1 \vee e_2) = \sigma(e_1) \vee \sigma(e_2) = e_2 \vee e_1 = e_1 \vee e_2$, and if $e \in \Lambda_\sigma\backslash\{0\}$ then $e > e_1$ or $e > e_2$. In the former case $e = \sigma(e) > \sigma(e_1) = e_2$ and so $e > e_1 \vee e_2$.

Consider $\Lambda_2^{e_1} = \{f \in E(\overline{T}) \mid f \text{ covers } e_1\} \subseteq \Lambda$. Then $|\Lambda_2^{e_1}| \geq rk_{ss}G + 1$ since $\dim T_{e_1} = rk_{ss}G + 1$. Further, each $f \in \Lambda_2^{e_1}$, $f \neq e_1 \vee e_2$ is of the form $f = e_1 \vee e_f$ for some unique $e_f \in E_1(\overline{T})$. Notice that if $e_f \neq e_2$ then e_f is W-conjugate to e_1 since otherwise there exists $\tau \in W$ such that $(e_1 \vee e_f)^\tau = e_2 \vee e_1^\tau$. But then $e_2 \vee e_1^\tau \in \Lambda_2$, a contradiction. Thus, either $f = e_1 \vee e_2$ or $f = e_1 \vee e_1^\tau$ for some $\tau \in W$.

Since $\dim ZG = 2$, $E(\overline{Z(G)^0}) = \{1, e, f, 0\}$ by [10, Remark 8.8]. So consider

$$\varphi : M \to eM \times fM, \quad \varphi(x) = (ex, fx).$$

Then $\sigma(e) = f$, so $e > e_1$ and $f > e_2$ (or vice versa) and so φ is a finite morphism. Also $ee_2 = 0 = fe_1$ since $ef = 0$. Notice that eMe and fMf

are $\mathscr{J}$-irreducible with $J_0 = \varnothing$ [13]. So let $h = e_1 \vee e_1^\tau \in \Lambda_2^{e_1}$. Then $eh = h$ and $fh = 0$ since $eE_1(M) = Cl_G(e_1)$ and $fE_1(M) = Cl_G(e_2)$ and $ef = 0$. Since eMe is $\mathscr{J}$-irreducible with $J_0 = \varnothing$ there exists a unique $\alpha \in \Delta$ such that $\sigma_\alpha h \sigma_\alpha = h$ (4.13 of [13]). Thus $e_1^\tau = e^{\sigma_2}$ and so $\tau = \sigma_\alpha$. Therefore

$$\Lambda_2^{e_1} \subseteq \{e_1 \vee e_2,\ e_1 \vee e_1^{\sigma_{\alpha_1}},\ \dots,\ e_1 \vee e_1^{\sigma_{\alpha_r}}\}$$

where $r = rk_{ss}G$. But from before, $|\Lambda_2^{e_1}| \geq rk_{ss}G + 1$. Thus $e_1 \vee e_2 \in \Lambda_2^{e_1}$. □

Proposition 4.9. *Let $e \in E(\overline{T})_\sigma$, $e \neq 0$. Then $e = g_1 \vee \cdots \vee g_t$ where each g_i is W-conjugate to $e_0 = e_1 \vee e_2$.*

Proof. First note that $e \in \Lambda_{2i}$ for some $i \geq 0$, since $X(T)$ has no σ^*-invariant subgroup of odd rank. If G is of type C_2 or G_2 then by Proposition 4.8, $\Lambda_\sigma = \{1, e_1 \vee e_2, 0\}$ and $W_\sigma = \{1, s\}$ with $(e_1 \vee e_2)^s \neq e_1 \vee e_2$. If G is of type F_4 then $\Lambda_\sigma = \{0\} \cup \{e_1 \vee e_2\} \cup (\Lambda_4)_\sigma \cup \{1\}$. So it suffices to show that any $e \in (\Lambda_4)_\sigma$ satisfies the claim, since the other cases are obvious. For this it suffices to show that there are at least two elements in the set $\{f \in E_2(\overline{T}) \mid f \leq e,\ \sigma(f) = f\}$, since these idempotents are all conjugate. Consider eMe. Then $\mathscr{U}(eMe) = e\Lambda$ and σ acts on $e\Lambda$. eMe is not a D-monoid since its rank is four while $|(e\Lambda)_1| = 2$. Thus, there exists $w \in \Lambda_\sigma$ such that $we = ew \neq e$. But if $e \geq e_1 \vee e_2$ then $e = e^w \geq (e_1 \vee e_2)^w$. But $C_W(e_1 \vee e_2) = \{1\}$ and so $e_1 \vee e_2 \neq (e_1 \vee e_2)^w$. □

Theorem 4.10. *Let $M_\sigma = \{x \in M \mid \sigma(x) = x\}$ where M_σ is as above. Let Γ be the set of simple reflections of W_σ relative to B_σ. Then M_σ satisfies the following properties:*

(a) $\Lambda_\sigma \backslash \{0\} \cong 2^\Gamma$ *via* $e \mapsto I(e)$, $C^r_{G_\sigma}(e) = P_{I(e)}$.

(b) *If* $I(e) = I$, $I(f) = K \subseteq \Gamma$ *let* $J_I = G_\sigma e G_\sigma$ *and* $J_K = G_\sigma f G_\sigma$. *Then* $J_I J_K \subseteq J_{I\cap K} \cup \{0\}$.

Proof. The proof follows formally from what has been written down in [13], once a few observations are made. 4.3 of [13] is true of M_σ by 1.3, 1.10 above and [11, Theorem 3.3]. 4.4 of [13] with Λ_σ in place of Λ follows from the fact that if $e \in \Lambda_\sigma$ then (eMe, σ) is a monoid of the same type as (M, σ), and hence has no central σ-stable idempotents $\neq 0, 1$. 4.5 is true vacuously since $J(e) = \varnothing$ for all $e \in \Lambda \backslash \{0\}$. 4.6 and 4.12 follow from 4.3 and $J(e) = \varnothing$. Finally 4.13 follows routinely. In our case it states the following: Let $e \in \Lambda_\sigma \backslash \{0\}$. Then there is a one-to-one correspondence between $\{\sigma_\alpha \in S \mid \sigma_\alpha e \neq e \sigma_\alpha\}$ and $\{f \in \Lambda_\sigma \mid f \text{ covers } e\}$. σ_α corresponds to the unique f such that $C^r_{G_\sigma}(f) = P_{I(e) \cup \{\sigma_\alpha\}}$.

Starting with $e_0 = e_1 \vee e_2$ and applying this to Λ_σ yields part (a). For part (b), notice that the claim is valid for M, while if $x \in M_\sigma$ then, as before, $GxG \cap M_\sigma = G_\sigma x G_\sigma$. □

There are many finite monoids having the properties of Theorems 4.3, 4.7, 4.10. The abstract universal monoids $\mathscr{M}$ constructed in §1 are such that the unit groups H_e of each $e\mathscr{M}e$, $e \in E(\mathscr{M})$ are as large as possible. In the monoids M_σ above, a local group H_e, instead of being isomorphic to some Levi subgroup L of $G(M_\sigma)$ is instead isomorphic to L/N for some central subgroup N of L. This creates problems in relating the representation theories of G_σ and M_σ. We are therefore naturally led to considering the universal central extension $\mathscr{M}$ of M_σ.

Finally the authors would like to thank the referee for useful suggestions.

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