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KLOOSTERMAN SUMS FOR CHEVALLEY GROUPS

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ABSTRACT. A generalization of Kloosterman sums to a simply connected Chevalley group G is discussed. These sums are parameterized by pairs (w, t) where w is an element of the Weyl group of G and t is an element of a $\mathbf{Q}$-split torus in G. The $SL(2, \mathbf{Q})$-Kloosterman sums coincide with the classical Kloosterman sums and $SL(r, \mathbf{Q})$-Kloosterman sums, $r \geq 3$, coincide with the sums introduced in [B-F-G, F, S]. Algebraic properties of the sums are proved by root system methods. In particular an explicit decomposition of a general Kloosterman sum over $\mathbf{Q}$ into the product of local p-adic factors is obtained. Using this factorization one can show that the Kloosterman sums corresponding to a toral element, which acts trivially on the highest weight space of a fundamental irreducible representation, splits into a product of Kloosterman sums for Chevalley groups of lower rank.

1. INTRODUCTION

The classical Kloosterman sums [Kl] are defined for any triplet of integers m, n, and c, $c > 0$, by the formula

$$S(m, n, c) = \sum_{\substack{x, y \bmod c \\ xy=1 \pmod c}} e^{2\pi i} \frac{mx + ny}{c}.$$

These sums are connected with many problems in number theory (for an overview of number theoretical applications of Kloosterman sums see [D-I]. In [B-F-G, F, S] generalizations of Kloosterman sums to certain trigonometric sums related to the Bruhat decomposition of $GL(n)$ are introduced. These sums are defined as follows. Let T and U_+ denote the diagonal and the upper triangular unipotent subgroups of $GL(n, \mathbf{R})$, respectively, and let θ_1, θ_2 be characters of U_+ that are trivial on the subgroup $U_+(\mathbf{Z})$ consisting of elements of U_+ with integral matrix coefficients. Let $t \in T$ and let ω be a generalized permutation matrix in $GL(n, \mathbf{R})$ (with ± 1 nonzero entries) corresponding to a Weyl group element w. Then the Kloosterman sum corresponding to the data θ_1, θ_2, t and w is defined by the formula

$$(1.1) \qquad S(\theta_1, \theta_2; t, w) = \sum \theta_1(b_1)\theta_2(b_2)$$

where the summation ranges over representatives $b_1 t\omega b_2$ of distinct elements in finite set $U_+(\mathbf{Z})\backslash U_+(\mathbf{Q})t\omega U_+(\mathbf{Q}) \cap SL(n + 1, \mathbf{Z})/U_w(\mathbf{Z})$ where $U_w(\mathbf{Z}) =$

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$U(\mathbf{Q}) \cap \omega^{-1} U(\mathbf{Z}) \omega$. This sum is well defined if certain conditions are imposed on θ_1, θ_2 and t. In particular t has to be of the form

$$t = \operatorname{diag}(1/c_1, c_1/c_2, c_2/c_3, \dots, c_{n-2}/c_{n-1}),$$

where $c_1, c_2, \dots, c_{n-1}$ are nonzero integers. One easily shows that $GL(2)$ sums coincide with the classical Kloosterman sums. The related Kloosterman zeta function is defined by the formula

$$Z_w(\theta_1, \theta_2; s) = \sum_t S(\theta_1, \theta_2; t, w) t^{-s}.$$

The summation ranges over all $t = \operatorname{diag}(1/c_1, c_1/c_2, c_2/c_3, \dots, c_{n-2}/c_{n-1})$, where $c_1, c_2, \dots, c_{n-1}$ are positive integers, $s = (s_1, s_2, \dots, s_{n-1})$ is an $(n-1)$-tuple of complex numbers and $t^{-s} = c_1^{-ns_1} c_2^{-ns_2} \cdots c_{n-1}^{-ns_{n-1}}$. It is observed in [B-F-G] that the generalized Ramanujan conjecture about Fourier coefficients of nonholomorphic cusp forms follows if the zeta functions $Z_w(\theta_1, \theta_2; s)$ have a suitable meromorphic continuation.

In hope of obtaining better understanding of Kloosterman sums (1.1) we develop a theory of Kloosterman sums for a Chevalley group scheme G. These sums are associated with the following data: a field k and a subring R of k, a subset C of positive roots in the root system of G, an element t of the split torus in the group $G(k)$ of k-points of G and a pair of characters ϕ and ψ of certain unipotent subgroups $U_-(k)$ and $U_C(k)$ of $G(k)$ (ϕ and ψ are required to be trivial on subgroups $U_-(R)$ and $U_C(R)$, respectively). The Kloosterman sum corresponding to the data is

$$S_C(t, \phi, \psi) = \sum \phi(u') \psi(u) \tag{1.2}$$

where the summation ranges over a full set of representatives of

$$U_-(R) \backslash U_-(k) t U_C(k) \cap G(R) / U_C(R)$$

of the form $u'tu$, where $u' \in U_-(k)$ and $u \in U_C(k)$ (we show in what circumstances the sum is finite). It turns out that the sums $S_C(t, \phi, \psi)$ are generalizations the sums defined by (1.2). We prove various algebraic properties of sums (1.2) by root system methods. In particular, we obtain a decomposition of a generalized Kloosterman sum over $\mathbf{Q}$ into a product of "local" Kloosterman sums defined over $\mathbf{Q}_p$. Using this decomposition we give an explicit decomposition of Kloosterman sums into a product of classical Kloosterman sums in some special cases.

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2. Generalized Kloosterman sets and sums

Let L be a semisimple Lie algebra over C of rank r. Suppose that H is a fixed Cartan subalgebra of L, $\Phi \subset H^*$ is the root system corresponding to H with the Weyl group W, Δ is a fixed basis of Φ, $\Phi^+ = \{\alpha_1, \alpha_2, \dots, \alpha_m\}$ is the set of positive roots, h_i, $1 \leq i \leq r$, x_α, $\alpha \in \Phi$, is a fixed Chevalley basis for L. We also let Λ denote the lattice of integral weights of Φ (for definitions see [H]). Let G denote the ("simply connected") Chevalley group scheme [Ch, K, Bo] associated with the above data. Let k be a field. Then the group $G(k)$ of k-points of G can be described as follows. Let V be any complex, faithful representation of L, such that **Z**-linear span of the set of weights of V coincides with Λ. Recall that $\mu \in H^*$ is a weight of V if

$$V_\mu = \{v \in V,\ hv = \mu(h)v\} \neq \{0\}.$$

Let M be any admissible lattice in V [H, §27] and let $V(k) = k \otimes_Z M$. For any $\alpha \in \Phi$ and $\xi \in k$ let $U_\alpha(\xi)$ be an element of $\mathrm{End}_k(V(k))$ given by the formula

$$U_\alpha(\xi)(1 \otimes v) = \sum_{m=1}^{\infty} \xi^m \otimes \frac{x_\alpha^m}{m!} v$$

where v is any element of M. Then $U_\alpha(\xi)$ is in fact an automorphism of $V(k)$. Moreover

$$\begin{gathered} U_\alpha(\xi)^{-1} = U_\alpha(-\xi) \quad \text{and} \quad U_\alpha(\xi+\eta) = U_\alpha(\xi)U_\alpha(\eta), \\ U_\alpha(\xi)U_\beta(\eta)U_\alpha(-\xi)U_\beta(-\eta) = \prod U_{i\alpha+j\beta}(c_{i,j}\xi^i\eta^j) \end{gathered} \tag{2.1}$$

for any $\alpha, \beta \in \Phi$, $\alpha+\beta \neq 0$, $\xi, \eta \in k$. The product on the right-hand side of (2.1) is taken over all roots of the form $i\alpha + j\beta$ (i, j are positive integers) arranged in some fixed ordering, $c_{i,j}$ are integers depending on α, β, and the chosen ordering, but not on η and ξ. Moreover, $[x_\alpha, x_\beta] = c_{11}x_{\alpha+\beta}$ if $\alpha+\beta$ is a root (for proofs see [St]). Then $G(k)$ coincides with the subgroup of $\mathrm{Aut}_k(V(k))$, generated by the set $\{U_\alpha(\xi);\, \alpha \in \Phi, \xi \in k\}$. We also let $T(k)$ denote the split torus in $G(k)$.

Let V' be any complex finite dimensional representation of L, and let M' be an admissible lattice in V'. Then $V'(k) = k \otimes_Z M'$ admits a k-linear action of $G(k)$ such that

$$U_\alpha(\xi)(1 \otimes v) = \sum_{m=1}^{\infty} \xi^m \otimes \frac{x_\alpha^m}{m!} v$$

for any $\alpha \in \Phi$ and $\xi \in k$ and $v \in M'$. For any weight μ, the corresponding $T(k)$-weight space is $V'_\mu(k) = k \otimes_Z (V'_\mu \cap M')$. We recall how $T(k)$ and $U_\alpha(\xi)$ act on spaces $V'_\mu(k)$, $\mu \in \Lambda$. Suppose that $v \in V'_\mu(k)$. Then

$$tv = t(\mu)v \quad \text{for any } t \in T(k),$$

where the map $(\mu, t) \to t(\mu)$ defines a group homomorphism form $\Lambda \times T(k)$ to k^* (the homomorphism is independent of V' and M'). If $\alpha \in \Phi$, $\xi \in k$, then

$$U_\alpha(\xi)v - v \in \bigoplus_{n \geq 1} V'_{\mu+n\alpha}(k).$$

Let R be any subring of k containing 1. Then

$$G(R) = \{g \in G(k);\ g(R \otimes_Z M) = R \otimes_Z M, \text{ for any admissible lattice } M \text{ in any complex representation of } L\}.$$

A result of Chevalley [Ch] states that

$$G(R) = \{g \in G(k);\ g(R \otimes_Z M) = R \otimes_Z M\}$$

where M is an admissible lattice in a faithful representation V of L such that the **Z**-linear span of the weights of V is Λ.

Let C be subset of Φ. C is said to be *closed* if $\alpha, \beta \in C$, $\alpha+\beta \in \Phi$, implies $\alpha+\beta \in C$. A subset I of a closed set C is called an *ideal* of C if $\alpha \in I$, $\beta \in C$, $\alpha+\beta$ implies $\alpha+\beta \in I$.

Example. Let w be an element in the Weyl group W. Then

$$P(w) = \{\alpha \in \Phi^+ ; w(\alpha) \in \Phi^-\}$$

is a closed subset of Φ^+.

Let C be a closed set and let $U_C(R)$ denote the subgroup of $G(k)$ generated by elements $U_\alpha(\xi)$, $\alpha \in C$, $\xi \in R$ (we write $U_+ = U_{\Phi_+}$ and $U_- = U_{\Phi_-}$). Clearly, $U_C(R) \subseteq G(R)$. If $C \cap (-C) = \varnothing$, (2.1) implies that any element u of $U_C(R)$ can be written uniquely as the product

$$u = U_{\alpha'_1}(\xi_1)U_{\alpha'_2}(\xi)\cdots U_{\alpha'_d}(\xi_d)$$

where $(\alpha'_1, \alpha'_2, \ldots, \alpha'_d)$, $d = \#(C)$, is any ordering of C and $\xi_1, \xi_2, \ldots, \xi_d \in R$ ($\xi_1, \xi_2, \ldots, \xi_d$ are called the *coordinates* of u).

Definition. For any $t \in T(k)$ we define the sets

$$Y_C(t) = \{g \in G(R);\ g = xty,\ x \in U_-(k),\ y \in U_C(k)\},$$

$$Y(t) = Y_{\Phi^+}(t) = U_-(k)tU_+(k) \quad \text{and} \quad K_C(t) = U_-(R)\backslash Y_C(t)/U_C(R).$$

If w is an element of the Weyl group, we write $Y_w = Y_{P(w)}$ and $K_w = K_{P(w)}$. $K_w(t)$ is called a *Kloosterman set.* The following proposition summarizes basic properties of sets Y_C and K_C.

Proposition 2.2. (i) *If $C \subset C'$ then $Y_C(t) \subset Y_{C'}(t)$ and $K_C(t) \subset K_{C'}(t)$ (under the obvious inclusion maps).*

(ii) *$t \neq t'$ then $Y_C(t) \cap Y_{C'}(t') = \varnothing$ for any C, C'. If $t^{-1}t' \in T(R) = T(k) \cap G(R)$ then the elements of $K_C(t)$ and $K_C(t')$ are in a one-to-one correspondence via the map $y \to yt^{-1}t'$.*

(iii) *$Y_C(t) \neq \varnothing$ then $t(\lambda) \in R$ for all dominant weights $\lambda \in \Lambda$.*

(iv) *Let $u'tu \in Y(t)$ for some $t \in k$, and assume $u' = \prod_{\alpha\in\Phi^+} U_{-\alpha}(\xi_{-\alpha})$ $u = \prod_{\alpha\in\Phi^+} U_\alpha(\xi_\alpha)$ (products taken in a fixed order), $\xi_{-\alpha}, \xi_\alpha$ in k. Let $\rho = (\sum_{\alpha\in\Phi^+}\alpha)/2$ and let $\{\lambda_\alpha ; \alpha \in \Delta\}$ be the set of fundamental weights relative to Δ. Then there exists a positive integer N such that $t(N\rho)\xi_{-\alpha}$ and $t(N\rho)\xi_\alpha$ are in R for all $\alpha \in \Phi^+$. Moreover $t(\lambda_\alpha)\xi_{-\alpha}$ and $t(\lambda_\alpha)\xi_\alpha$ are in R for all $\alpha \in \Delta$.*

(v) *Let v denote the quotient (abelian group) homomorphism $k \to k/R$. Assume that R satisfies*

(P_1) *For any $x \in R$ the set $v(\{y \in k ; xy \in R\})$ is finite.*

Then $K_C(t)$ is either finite or empty.

Proof. (i) and (ii) are straightforward. For (iii) let λ be a dominant weight and let V be the irreducible representation of weight λ with a highest weight vector v. We can assume that v is contained in a $\mathbf{Z}$-basis of an admissible lattice M of V [H, §27]. Therefore $1 \otimes v$ belongs to a free R-basis of $R \otimes_Z M$. For any $g \in G(k)$, we let $f(g)$ denote the $1 \otimes v$-component of $g(1 \otimes v)$. If $g = u'tu \in Y(t)$ then $f(g) = t(\lambda) \in R$ by properties of the action of $T(k)$ and $U_\pm(k)$ on weight spaces of $V(k)$.

Part (iv). We need the following fact due to Chevalley [Ch, Bo].

Lemma 2.3. *For any $\alpha \in \Phi^+$, there exist functions $F_{\pm\alpha}$ defined on $G(k)$ with values in k and polynomials $P_{\pm\alpha}$ with integer coefficients in respective indeterminants $x_{\pm\beta}$, $0 < \beta < \alpha$, such that*

(i) $P_{\pm\alpha} = 0$ *if α is a simple root,*

(ii) *if $g \in G(R)$ then $F_{\pm\alpha}(g) \in R$,*

(iii) *if* $g = U_{-\alpha_1}(\eta_{-\alpha_1})U_{-\alpha_2}(\eta_{-\alpha_2})\cdots U_{-\alpha_m}(\eta_{-\alpha_m})tU_{\alpha_1}(\eta_{\alpha_1})U_{\alpha_2}(\eta_{-\alpha_2})\cdots U_{-\alpha_m}(\eta_{\alpha_m})$ *where $t \in T(k)$ and $\eta_\alpha \in k$, $\alpha \in \Phi$, then*

$$F_{\pm\alpha}(g) = t(2\rho)\eta_{\pm\alpha} + P_{\pm\alpha}(\eta_{\pm\beta}, 0 < \beta < \alpha).$$

The lemma implies the first part of (iv) by induction on the height of a given root (height of a root is the sum of its simple root components). For the second part we consider a simple root α and the irreducible representation V of the highest weight λ_α. Let v be the highest weight vector. Let v_α be a nonzero vector in the weight space $V_{\sigma_\alpha(\lambda_\alpha)} = V_{\lambda_\alpha - \alpha}$, where $\sigma_\alpha \in W$ is the reflection corresponding to α. We may assume that v and v_α, belong to a $\mathbf{Z}$-basis of an admissible lattice M in V (see [H, 27.1]). Therefore $1 \otimes v$ and $1 \otimes v_\alpha$ belong to a basis of $k \otimes_Z M$. For any $g \in G(k)$, we let $f_\alpha(g)$ denote the $1 \otimes v_\alpha$-component of $g(1 \otimes v)$. A calculation based on properties of weight spaces [H, 20.1] shows $f_\alpha(u'tu) = t(\lambda_\alpha)\xi_{-\alpha} \in R$. Finally, let ϑ denote the automorphism of $G(k)$ induced by the automorphism of Φ sending a root α to $-\alpha$. Then $f_\alpha(\vartheta(u'tu)^{-1}) = f_\alpha(\vartheta(u^{-1})t\vartheta((u')^{-1})) = -t(\lambda_\alpha)\eta_\alpha \in R$.

Part (v). In view of (iv) it is enough to prove the following lemma.

Lemma 2.4. *Let $(\alpha_1, \alpha_2, \ldots, \alpha_m)$ be an ordering of Φ^+ such that $\{\alpha_1, \alpha_2, \ldots, \alpha_q\}$, $1 \le q \le m$, is an ideal in Φ^+. For any finite subset S of k/R we set*

$$Y_S^q = \left\{ \prod_{i=1}^q U_{\alpha_i}(\xi_i);\ v(\xi_i) \in S \right\}.$$

If R satisfies property (P1) *then $\pi(Y_S^q)$ is finite* (*here $\pi\colon U_+(k) \to U_+(k)/U_+(R)$ is the quotient homomorphism*).

Proof of the lemma. The lemma is trivial in the case $q = 1$. Assume $q > 1$. For any $\xi \in k$ we define $Z_S^q(\xi) = Y_S^{q-1}U_{\alpha_q}(\xi)$. The assumptions of the lemma imply that $\pi(Y_S^q) \subset \bigcup_{j=1}^d \pi(Z_S^q(\xi_j))$ for some $\xi_j \in k$. Consequently it is enough to show that $\pi(Z_S^q(\xi))$ is finite for any $\xi \in k$. We observe that $U_{\alpha_q}(-\xi)Z_S^q(\xi) \subset Y_{S'}^{q-1}$ for some finite set S' of k/R since by identity (2.1) the coordinates of elements in $U_{\alpha_q}(-\xi)Z_S^q(\xi)$ are polynomial functions of the coordinates of elements in Y_S^q, and the polynomials involved depend only on the ordering of the roots. By induction on q we obtain $\#(\pi(Z_S^q(\xi))) = \#(\pi(U_{\alpha_q}(-\xi)Z_S^q(\xi))) \le \#(\pi(Y_{S'}^{q-1})) < \infty$.

The next proposition will allow us to prove properties of Kloosterman sets by induction on the rank of the Lie algebra. We recall that Φ is a subset of a real vector space E equipped with a scalar product (,). As usual we write $\langle\lambda, \mu\rangle = \frac{2(\lambda, \mu)}{(\mu, \mu)}$ for any elements $\lambda, \mu \in E$, such that $\mu \neq 0$. With this notation

$$\Lambda = \{\lambda \in \Lambda;\ \langle\lambda, \alpha\rangle \in \mathbf{Z} \text{ for all } \alpha \in \Phi\}$$

and the fundamental weights λ_α, $\alpha \in \Delta$, are defined by conditions

$$\langle\lambda_\alpha, \gamma\rangle = \delta_{\alpha,\gamma} \quad \text{for all } \alpha, \gamma \in \Delta.$$

Proposition 2.5. *Suppose that $R \subset k$ satisfies the property*

(P_2) *Let $x \in k$. If $x^n \in R$ for some nonnegative integer n then $x \in R$. Let $u'tu \in Y_C(t)$ and let $t(\lambda_\alpha)$ be a unit in R for some $\alpha \in \Delta$. Let $U_{C'}(k)$ (resp. $U'_-(k)$) denote the subgroup of $U_+(k)$ (resp. $U'_-(k)$) generated by elements $U_\beta(\eta), \eta \in k$, $\beta \in C$ (resp. $U_\beta(\eta)$, $\eta \in k$, $\beta \in \Phi^-$) such that $\langle \lambda_\alpha, \beta \rangle = 0$. Then there exist elements $y \in U_-(R)$ and $x \in U_C(R)$ such that $yu' \in U'_-(k)$ and $ux \in U_{C'}(k)$.*

Proof. We need the following general fact about root systems.

Lemma 2.6 (Special filtration of sets of roots). *For a simple root α we let*

$$B = B(\alpha) = \{\beta \in \Phi^+ ; \langle \lambda_\alpha , \beta \rangle > 0\}$$

(B is just the set of all positive roots with nonzero α-component). We define $B_1 = \{\alpha\}$ and, inductively, we let B_n denote the set of elements β in B such that if $\langle \lambda_\alpha , \beta \rangle \beta = \sum_j \beta_j$, $\beta_j \in B$, then either $\beta_j = \beta$ for all j or $\beta_j \in B_{n-1}$ for some j (observe that $B_{n-1} \subseteq B_n$). Then $B = \bigcup \{B_n ; n \geq 1\}$.

Sketch of a proof. Clearly, it is enough to consider irreducible root systems. In the case of a root system of type G_2, the lemma can be verified by inspection. In remaining cases, the lemma follows, since $\{\beta \in B ; \mathrm{ht}(\beta) \leq n\} \subseteq B_n$ for all n, $n = 1, 2, \ldots$ (here, $\mathrm{ht}(\beta) = \sum_{\alpha \in \Delta} c_\alpha$, if $\beta = \sum_{\alpha\in\Delta} c_\alpha \alpha$). The above statement can be derived from the following two facts:

(1) In any root system positive roots of equal height are linearly independent over the reals.

(2) If $\Phi \subseteq E$ is an irreducible root system of type different than G_2, than for any positive root β there exists a real valued function f defined in E, such that:

(i) $f(\gamma + \gamma') \leq f(\gamma) + f(\gamma')$ and $f(q\gamma) = qf(\gamma)$ for any $\gamma, \gamma' \in E$ and $q = 0, 1, 2, \ldots$,

(ii) $f(\beta) \neq 0$ and $f(\beta) \geq f(\beta')$ for all roots β' with $\mathrm{ht}(\beta') \geq \mathrm{ht}(\beta)$.

We now come back to the proof of Proposition 2.5. For a fixed $\alpha \in \Delta$, we let

$$B(\alpha) = \{\beta \in \Phi^+ ; \langle \lambda_\alpha , \beta \rangle > 0\} = \{\beta_1 , \beta_2 , \ldots , \beta_N\}$$

where the ordering $(\beta_1, \beta_2, \ldots, \beta_N)$ of $B(\alpha)$ is consistent with the filtration $\{B_n\}$ of the lemma (i.e. if $\beta_i \in B_n$ and $\beta_j \notin B_n$ for some n then $i < j$). Let $g = u'tu \in Y(t)$ where $u' = U_{-\beta_1}(\xi_1) U_{-\beta_2}(\xi_2) \cdots U_{-\beta_N}(\xi_N) u_1$ for some $u_1 \in U'_-(k)$ and $u = u_2 U_{\beta_{i_M}}(\eta_M) U_{\beta_{i_{M-1}}}(\eta_{M-1}) \cdots U_{\beta_{i_1}}(\eta_1) u_2$ where, $i_1 \leq i_2 \leq \cdots \leq i_M$, $\{\beta_{i_1}, \beta_{i_2}, \ldots, \beta_{i_M}\} = B(\alpha) \cap C$ and $u_2 \in U_{C'}(k)$. It is enough to show that ξ_i, $1 \leq i \leq N$ and η_i, $1 \leq i \leq M$, are in R. We will prove this using the following general fact about weight spaces of a fundamental representation of L.

Lemma 2.7. *Let $\alpha \in \Delta$, $\beta \in \Phi^+$, $c = \langle \lambda_\alpha , \beta \rangle$ and let V be the irreducible representation of weight λ_α. Then $\frac{x_\beta^c}{c!} \frac{x_{-\beta}^c}{c!} v = v$ on V_{λ_α}. Consequently $\frac{x_{-\beta}^c}{c!}$ maps V_{λ_α} isomorphically onto $V_{\lambda_\alpha - c\beta}$ since $\sigma_\beta(\lambda_\alpha) = \lambda_\alpha - c\beta$ and* $\dim(V_{\lambda_\alpha}) = \dim(V_{\lambda_\alpha - c\beta}) = 1$. *Moreover, if $\mu = \lambda_\alpha - \sum_{\beta \in \Delta - \{\alpha\}} n_\beta \beta$ where n_β are nonnegative integers, not all equal to zero, then $V_\mu = \{0\}$.*

Proof of the lemma. The first assertion follows directly from the identity

$$\frac{x_\beta^c}{c!}\frac{x_{-\beta}^a}{a!} = \sum_{k=0}^{\min(a,c)} \frac{x_{-\beta}^{(a-k)}}{(a-k)!}\binom{h_\beta - a - c + 2k}{k}\frac{x_\beta^{(c-k)}}{(c-k)!}$$

where β is any positive root and $h_\beta = [x_\beta, x_{-\beta}]$ [H, §26.2]. The second assertion follows from the Freudenthal's multiplicity formula [H, §26.2].

We continue the proof of Proposition 2.5. Let v be a highest weight vector contained in a basis of an admissible lattice M of V. Since $M = \bigoplus_{\mu\in\Lambda} M \cap V_\mu$ (see [H, §27]) we can assume that the basis consists of weight vectors. For any $\beta_i \in B$, $i = 1, 2, \dots, N$, let v_i denote a basis vector of weight $\sigma_{\beta_i}(\lambda_\alpha) = \lambda_\alpha - \langle\lambda_\alpha, \beta_i\rangle\beta_i$. For any $g \in G(k)$, let $f_i(g)$ denote the $1 \otimes v_i$ component of $g(1 \otimes v)$. By the definition of sets B_i, $i = 1, 2, \dots, N$, one obtains $f_1(g) = t(\lambda_\alpha)\xi_1$ and inductively

$$\begin{aligned} &f_j(U_{-\beta_{j-1}}(-\xi_{j-1})U_{-\beta_{j-2}}(-\xi_{j-2})\cdots U_{-\beta_1}(-\xi_1)g) \\ &\quad = t(\omega_\alpha)f_j(U_{-\beta_j}(\xi_j)U_{-\beta_{j+1}}(\xi_{j+1})\cdots U_{-\beta_N}(\xi_N)) = t(\lambda_\alpha)\xi_j^{\langle\lambda_\alpha, \beta_j\rangle}\end{aligned}$$

for $j = 1, 2, \dots, N$. Therefore property (P2) implies $\xi_j \in R$ for all j. Similarly, we obtain $\eta_j \in R$ for all j, replacing g with $\vartheta(g^{-1})$, where ϑ is the automorphism of $G(k)$ induced by the automorphism of the root system Φ sending a root β to $-\beta$.

Let J be a subset of the set of simple reflections and let $\Delta_J \subset \Delta$ be the corresponding set of simple roots. Let Φ_J denote the root system obtained by taking the set of roots in Φ which are linear combinations of roots in Δ_J. Let W_J denote the Weyl group of Φ_J (for a discussion of subgroups of the Weyl group, or more generally a Coxeter group, see [D1, D2]. Then the groups $G^J(k)$, $U^J_\pm(k)$, and $T^J(k)$, associated with Φ_J, can be naturally identified with subgroups of $G(k)$, $U_\pm(k)$, and $T(k)$, respectively. Note that with this identification $t \in T^J(R)$ if and only if $t(\lambda_\alpha) = 1$ for all $\alpha \in \Delta\backslash\Delta_J$. If $t \in T^J(R)$ and $C \subset \Phi_J^+$ is a closed subset then $Y_C^J(t) = \{g \in G^J(R);\ g = xty,\ x \in U_-^J(k),\ y \in U_C(k)\}$ and $K_C^J(t) = U_-^J(R)\backslash Y_C^J(t)/U_C(R)$ are subsets of $Y_C(t)$ and $K_C(t)$, respectively.

From now on we assume that $R \subset k$ satisfies properties (P1) and (P2) (for example $\mathbf{Z} \subset \mathbf{Q}$, and $\mathbf{Z}_p \subset \mathbf{Q}_p$ satisfy (P1) and (P2)).

Proposition 2.8. *Let C be a closed subset of Φ^+.* (i) *Let $C_J = C \cap \Phi_J$. If $t \in T^J(k)$ then $K^J_{C_J}(t) = K_C(t)$.*

(ii) *If $C \subset \Phi_J$ and $K_C(t)$ is not empty then $t(\lambda_\alpha)$ is a unit in R for every $\alpha \in \Delta\backslash\Delta_J$.*

Proof. Part (i) is an immediate consequence of Proposition 2.5. For part (ii) let $g = u'tu \in Y_C(t)$ where $t \in T(k)$, $u' \in U_-(k)$, and $u \in U_C(k)$. Then $t(\lambda_\alpha) \in R$, $\alpha \in \Delta$, by Proposition 2.2, part (iii). Fix $\alpha \in \Delta\backslash\Delta_J$. We can write $u' = xy$ where x is a product of elements $U_{-\beta}$, $\beta \in \Phi^+$, $\langle\lambda_\alpha, \beta\rangle = 0$ and y is a product of elements $U_{-\beta}$, $\beta \in \Phi^+$, $\langle\lambda_\alpha, \beta\rangle > 0$. Let V be the irreducible representation of weight λ_α with a highest weight vector v. We can assume that v belongs to a **Z**-basis of an admissible lattice in V. Therefore $1 \otimes v$ belongs to a basis of $V(k) = k \otimes_Z M$. For any $g \in G(k)$, we let $f(g)$ denote the $1 \otimes v$-component of $g(1 \otimes v)$. Then

$$f(g^{-1}) = f(u^{-1}t^{-1}y^{-1}x^{-1}) = f(u^{-1}t^{-1}y^{-1}) = t(-\lambda_\alpha) = (t(\lambda_\alpha))^{-1} \in R$$

since $x^{-1}(1\otimes v) = 1\otimes v$ by Lemma 2.7 and since no sum of roots in Φ_J equals a sum of roots in $B(\alpha)$.

It can be shown that $P(w) \subseteq \Phi_J$ for any $w \in W_J$. Therefore we have

Corollary. *If* $w \in W_J$ *and* $t \in T^J(k)$ *then* $K_w^J(t) = K_w(t)$.

Definition. For any closed subset C of Φ^+, $t \in T(k)$, such that $K_C(t) \neq \varnothing$, and a pair of characters $\phi: U_-(k) \to \{z \in \mathbf{C},\ |z| = 1\}$ and $\psi: U_C(k) \to \{z \in \mathbf{C},\ |z| = 1\}$ which are trivial on $U_-(R)$ and $U_C(R)$ respectively, we define

$$S_C(t, \phi, \psi) = \sum \phi(u')\psi(u)$$

where the summation ranges over a full set of representatives $u'tu$ of $K_C(t)$, where $u' \in U_-(k)$ and $u \in U_C(k)$ (note that $K_C(t)$ is finite, since we have assumed that $R \subset k$ satisfies property (P1)). We will call $S_C(t, \phi, \psi)$ a *generalized Kloosterman sum* associated with C, t, ϕ and ψ. We write $S_w(t, \phi, \psi) = S_{P(w)}(t, \phi, \psi)$.

Example 2.9. Let $\Phi = \{\alpha, -\alpha\}$ be the root system of type A_1, $R = \mathbf{Z}$ and $k = \mathbf{Q}$. Then $G(\mathbf{Q}) = SL(2, \mathbf{Q})$ and $T(\mathbf{Q})$, $U_-(\mathbf{Q})$, $U_+(\mathbf{Q})$ are, respectively, the diagonal, the lower triangular unipotent and the upper triangular unipotent subgroups of $SL(2, \mathbf{Q})$. A simple computation shows that if $w = \sigma_\alpha$,

$$t = \begin{bmatrix} c & 0 \\ 0 & \frac{1}{c} \end{bmatrix}, \qquad \phi\left(\begin{bmatrix} 1 & 0 \\ y & 1 \end{bmatrix}\right) = e^{2\pi i m y} \quad \text{and} \quad \psi\left(\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}\right) = e^{2\pi i n x}$$

then $S_w(t, \phi, \psi) = S(-m, n, c)$ where $S(-m, n, c)$ is the classical Kloosterman sum.

Example 2.10. Let $R = \mathbf{Z}$, $k = \mathbf{Q}$ and let Φ be the root system of type A_r in its standard representation as a subset of $\mathbf{R}^{r+1}$ (see [B or H, p. 64]). We identify the Weyl group W of Φ with the group of permutation matrices in $GL(r+1, \mathbf{Q})$. Let $S(\theta_1, \theta_2; t, w)$ be the $GL(r+1, \mathbf{Q})$ Kloosterman sum defined by formula (1.1). We define $w_1 = w_0 w^{-1} w_0^{-1}$, $t_1 = w_0 t w_0^{-1}$, $\phi(y) = \theta_1(w_0^{-1} y w_0)$, $y \in U_-(\mathbf{Q})$ and $\psi(x) = \theta_2((w_0\omega)^{-1} y w_0 \omega)$, $x \in U_w(\mathbf{Q})$ where w_0 is the *long element* of the Weyl group (this is the unique element of W sending each positive root to a negative root). Then $S_{w_1}(t_1, \phi, \psi) = S(\theta_1, \theta_2; t, w)$ if the right-hand side of the equation is well defined. To verify this identity we may assume that the element b_2 is the definition of $S(\theta_1, \theta_2; t, \omega)$ belongs to $U_w(\mathbf{Q})$. Consequently

$$w_0 b_1 t \omega b_2 (w_0\omega)^{-1} = w_0 b_1 (w_0)^{-1} w_0 t (w_0)^{-1} w_0 \omega b_2 (w_0 \omega)^{-1} \in Y_{w_1}(t_1).$$

In fact the map $x \to w_0 x (w_0\omega)^{-1}$ gives the desired bijection between $K_{w_1}(t_1)$ and $U_+(\mathbf{Z})\backslash U_+(\mathbf{Q}) t\omega U_+(\mathbf{Q}) \cap SL(n+1, \mathbf{Z})/U_w(\mathbf{Z})$.

Next proposition summarizes elementary properties of generalized Kloosterman sums.

Proposition 2.11. *Let* C, t, ϕ *and* ψ *be as in the definition of generalized Kloosterman sums.*

(i) *If* $t \in T(R)$ *then* $S_C(t, \phi, \psi) = \#(K_C(t)) = 1$.

(ii) *If* $t_1 \in T(R)$ *then*

$$S_C(tt_1, \phi, \psi) = S_C(t_1, \phi^t, \psi) = S_C(t_1, \phi, \psi^{t^{-1}})$$

where $\phi^t(x) = \phi(txt^{-1})$, $x \in U_+(k)$, $\psi^{t^{-1}}(y) = \psi(t^{-1}yt)$ *and* $y \in U_C(k)$.

(iii) *Let* ϑ *be any automorphism or anti-automorphism of* $G(k)$, *defined over* R, *such that it preserves* $T(k)$ *and for any* $\alpha \in \Phi^+$ (*resp.* $\alpha \in \Phi^-$) *and* $\xi \in k$ *one has* $\vartheta(U_\alpha(\xi)) = U_{\vartheta(\alpha)}(r_\vartheta \xi)$ *for some* $r_\vartheta \in \mathbf{R}^\times$ *and* $\vartheta(\alpha) \in \Phi^+$ (*resp.* $\vartheta(\alpha) \in \Phi^-$). *Then*

$$S_C(t, \phi, \psi) = S_{\vartheta(C)}(\vartheta(t), \phi_\vartheta, \Psi_\vartheta)$$

where $\phi_\vartheta(x) = \phi(\vartheta^{-1}(x))$, $x \in U_-(k)$, *and* $\psi_\vartheta(y) = \psi(\vartheta^{-1}(y))$, $y \in U_{\vartheta(C)}(k)$.

The following proposition reveals when a Kloosterman sum related to a given root system Φ reduces to a Kloosterman sum related to a root subsystem Φ_J. Notation is the same as in Proposition 2.8.

Proposition 2.12. *Let* C, t, ϕ, ψ *be as in the definition of Kloosterman sums. We define* $t_1 \in T(k)$ *by the formula* $t_1(\lambda_\alpha) = t(\lambda_\alpha)$ *if* $\alpha \in \Delta_J$ *and* $t_1(\lambda_\alpha) = 1$ *for* $\alpha \in \Delta \backslash \Delta_J$.

(i) *If* $C \subset \Phi_J$ *then* $t_1 \in T(R)$ *and* $S_C(t, \phi, \psi) = S_C^J(tt_1^{-1}, \phi, \psi^{t_1^{-1}})$ *where the right-hand side is the Kloosterman sum in* $G^J(k)$.

(ii) *Let* $t \in T^J(k)$ *and let* $C_J = C \cap \Phi_J$. *Then* $S_C(t, \phi, \psi) S_{C_J}^J(t, \phi, \psi)$.

The proposition follows directly from Propositions 2.8 and 2.11.

Corollary 2.13. *Let* $t, t_1 \in T(R)$ *be as in Proposition* 2.12 *and let* w *be an element in* W. *Then* $S_w(t, \phi, \psi) = S_{w_J}^J(tt_1^{-1}, \phi, \psi^{t_1^{-1}})$ *where* W_J *is the unique element of* W_J *such that* $P(w) \cap \Phi_J$ (*it can be shown that such element* w_J *always exists*).

This is a generalization of a result about $GL(r+1)$-Kloosterman sums obtained in [F, Proposition 3.6 and S, Corollary (3.11)].

3. Global and local Kloosterman sums

In this section we assume that either $R = \mathbf{Z} \subset k = \mathbf{Q}$ or $R = \mathbf{Z}_p \subset k = \mathbf{Q}_p$ where p is a prime. We let Ψ and Ψ_p denote the fundamental characters of the additive group k, trivial on R, in the respective cases. Explicitly, if $k = \mathbf{Q}$ then $\Psi(x) = e^{2\pi i x}$ and if $k = \mathbf{Q}_p$ then

$$\Psi_p\left(\sum_{n=-N}^{\infty} a_n p^n\right) = e^{2\pi i}\left(\sum_{n=-N}^{-1} a_n p^n\right).$$

We observe that if $x \in \mathbf{Q}$ then

$$\Psi(x) = \prod_p \Psi_p(x).$$

Let C be a closed subset in either Φ^+ or Φ^- an let $U_C(k)$ be the corresponding subgroup of $G(k)$. We say that a group homomorphism $\varphi: U_C(k) \to \{z \in \mathbf{C};\ |z| = 1\}$ is a *rational character*, defined over R, if

$$\varphi\left(\prod_{\alpha \in C} U_\alpha(\xi_\alpha)\right) = \Psi\left(\sum_{\alpha \in C} d_\alpha \xi_\alpha\right) \quad \text{where } d_\alpha \in R,\ \xi_\alpha \in k$$

(the product is taken in a fixed order, and Ψ is the fundamental character of k/R).

Remark. We say that a root in C is *indecomposable* in C if it is not a sum of roots in C. Let D be the set of indecomposable roots in C. Then one can show that the commutator group $[U_C(k), U_C(k)]$ coincides with the group $U_{C\setminus D}(k)$. Consequently any rational character of $U_C(k)$ defined over R is of the form

$$(*) \qquad \varphi\left(\prod_{\alpha\in D} U_\alpha(\xi_\alpha)\right) = \Psi\left(\sum_{\alpha\in D} d_\alpha \xi_\alpha\right) \quad \text{where } d_\alpha \in R,\ \xi_\alpha \in k,$$

and vice versa, given numbers $d_\alpha \in R$, $\alpha \in D$, then formula $(*)$ gives a rational character of $U_C(k)$. It is clear that if $C = \Phi^+$ then $D = \Delta$. It has been pointed out to the author by V. Deodhar that in root systems whose elements are all of equal length, set D for $C = P(w)$ is given by the formula $D = \{\alpha \in P(w), l(w\sigma_\alpha) = l(w) - 1\}$, where $l(\cdot)$ is the length function on W. We observe that if $k = \mathbf{Q}$, then $\varphi(x) = \prod_p \varphi_p(x)$ where

$$\varphi_p\left(\prod_{\alpha\in C} U_\alpha(\xi_\alpha)\right) = \Psi_p\left(\sum_{\alpha\in C} a_\alpha \xi_\alpha\right).$$

We will assume that the characters ϕ, ψ appearing in definition of generalized Kloosterman sums are rational, defined over R.

Proposition 3.1. *Let C be a closed subset of Φ^+ and let $t \in T(\mathbf{Q})$ and let ϕ, ψ be rational characters defined over $\mathbf{Z}$ of $U_-(\mathbf{Q})$ and $U_C(\mathbf{Q})$, respectively. Then*

$$S_C(t, \phi, \psi) = \prod_p S_C^p(t, \phi_p, \psi_p)$$

(S_C^p denotes the $G(\mathbf{Q}_p)$ Kloosterman sum).

Proof. We first notice that if p does not divide $t(\rho)$ then $t \in T(\mathbf{Z}_p)$ and $S_C^p(t, \phi_p, \psi_p) = 1$ by Proposition 2.11. Therefore the proposition will follow if we show that the natural embedding

$$K_C(t) \to \prod_{p|t(\rho)} K_C^p(t)$$

is in fact a bijection. This fact is an immediate consequence of the following lemma.

Lemma 3.2. *Let C be a closed subset in either Φ^+ or Φ^- and let $U_C(\mathbf{Q}_p)$ be the corresponding subgroup of $G(\mathbf{Q}_p)$. Let S be a finite set of primes and let $x_p \in U_C(\mathbf{Q}_p)$ for $p \in S$. There exist elements y, $z \in U_C(\mathbf{Q})$ such that $x_p z$ and $y x_p$ belong to $U_C(\mathbf{Z}_p)$.*

Proof of the lemma. Let $(\alpha_1, \alpha_2, \ldots, \alpha_N)$ be an ordering of C such that $\{\alpha_1, \alpha_2, \ldots, \alpha_q\}$ is an ideal in C for any q, $1 \le q \le N$. We then have $x_p = U_{\alpha_1}(\xi_{1,p})U_{\alpha_2}(\xi'_{2,p}) \cdots U_{\alpha_q}(\xi_{q,p})$ for some $\xi_{1,p}, \xi_{2,p}, \ldots, \xi_{q,p} \in \mathbf{Q}_p$. We proceed by induction on q. Clearly, the lemma holds if $q = 1$. Assume $q > 1$. We may suppose that $\xi_{q,p} = \xi \in \mathbf{Q}$, $p \in S$. Consequently, $U_{\alpha_q}(-\xi)x_p = U_{\alpha_1}(\xi'_{1,p})U_{\alpha_2}(\xi'_{2,p}) \cdots U_{\alpha_{q-1}}(\xi'_{q-1,p})$ for some $\xi'_{1,p}, \xi'_{2,p}, \ldots, \xi'_{q-1,p} \in \mathbf{Q}_p$ by formula (2.1). By inductive hypothesis there exists $y' \in U_C(\mathbf{Q})$ such that $y' U_{\alpha_q}(-\xi)x_p \in U_C(\mathbf{Q})$ for all $p \in S$. Therefore $y = y' U_{\alpha_q}(-\xi)$ satisfies the desired property.

Corollary 3.3 (Multiplicativity of generalized Kloosterman sums). *Let t and t' be elements of $T(\mathbf{Q})$ such that $t(\rho)$ and $t'(\rho)$ are relatively prime. Then*

$$S_C(tt', \phi, \psi) = S_C(t, \phi', \psi)S_C(t', \phi'', \psi)$$

for some characters ϕ' and ϕ''.

Proof. By Propositions 3.1 and 2.11,

$$S_C(tt', \phi, \psi) = \prod_{p|t(\rho)} S_C^p(t, \phi_p^{t'}, \psi_p) \prod_{p|t'(\rho)} S_C^p(t', \phi_p^t, \psi_p).$$

Since the character ϕ is rational, there exist $d_\alpha \in \mathbf{Z}$, $\alpha \in \Delta$, such that

$$\phi(x) = \Psi\left(\sum_{\alpha\in\Delta} d_\alpha \eta_\alpha\right)$$

for any $x = \prod_{\alpha\in\Phi^+} U_{-\alpha}(\eta_\alpha) \in U_-(\mathbf{Q})$. Consequently

$$\phi^t(x) = \Psi\left(\sum_{\alpha\in\Delta} t(-\alpha)d_\alpha\eta_\alpha\right) \quad \text{and} \quad \phi^{t'}(x) = \Psi\left(\sum_{\alpha\in\Delta} t'(-\alpha)d_\alpha\eta_\alpha\right).$$

For each $\alpha \in \Delta$, we chose $b_\alpha, c_\alpha \in \mathbf{Z}$, such that

$$b_\alpha - d_\alpha t(-\alpha) \in t'(\lambda_\alpha)\mathbf{Z}_p \quad \text{for all } p|t'(\rho) \tag{3.4}$$

and

$$c_\alpha - d_\alpha t'(-\alpha) \in t(\lambda_\alpha)\mathbf{Z}_p \quad \text{for all } p|t(\rho). \tag{3.5}$$

Since $t(-\alpha)$ is in $\mathbf{Z}_p$) (resp. $t'(\alpha)$ is in $\mathbf{Z}_p$) whenever $p|t'(\rho)$ (resp. $p|t(\rho)$) such choice of b_α and c_α is possible by the Chinese Remainder Theorem. We define the characters ϕ' and ϕ'' by formulae $\phi'(x) = \Psi(\sum_{\alpha\in\Delta} c_\alpha\eta_\alpha)$ and $\phi''(x) = \Psi(\sum_{\alpha\in\Delta} b_\alpha\eta_\alpha)$. Assume now that $xty \in Y_C^p$ where $p|t(\rho)$. By Proposition 2.2 $t(\lambda_\alpha)\eta_\alpha \in \mathbf{Z}_p$. Hence $\phi_p^{t'}(x) = \phi'(x)$ and

$$S_C(t, \phi', \psi) = \prod_{p|t(\rho)} S_C^p(t, \phi_p^{t'}, \psi_p).$$

By a similar reasoning we obtain

$$S_C(t', \phi'', \psi) = \prod_{p|t'(\rho)} S_C^p(t', \phi_p^t, \psi_p)$$

and the proof is complete.

Corollary 3.6. *Let C be any closed subset of Φ^+ and let $t \in T(\mathbf{Q})$ satisfy $(t(\lambda_\alpha), t(\lambda_\beta)) = 1$ for all $\alpha, \beta \in \Delta$, $\alpha \neq \beta$. Then*

$$S_C(t, \phi, \psi) = \prod_{\alpha\in\Delta} S(m_\alpha, n_\alpha, t(\lambda_\alpha))$$

where $n_\alpha = 0$ if $\alpha \notin C$, $\psi(U_\alpha(\xi)) = \Psi(n_\alpha\xi)$, $\xi \in \mathbf{Q}$, if $\alpha \in C$ and $m_\alpha = -d_\alpha t(-\alpha+2\lambda_\alpha)$, $\alpha \in \Delta$. In particular (under the above assumptions about t) the Kloosterman sums $S_w(t, \phi, \psi)$, $w \in W$ are products of classical Kloosterman sums.

Proof. Let $\alpha \in \Delta$ be fixed. We observe that $t(-\alpha + 2\lambda_\alpha)$ is an integer since $-\alpha + 2\lambda_\alpha = \sum_{\beta\in\Delta-\{\alpha\}} -\langle\alpha, \beta\rangle\lambda_\beta$ is a dominant weight. We define the element

$t_\alpha \in T(\mathbf{Q})$ by the formula $t_\alpha(\lambda_\beta) = t(\lambda_\beta)^{\delta_{\alpha\beta}}$, $\alpha, \beta \in \Delta$. Proposition 3.3 implies that

$$S_C(t, \phi, \psi) = S_C(t_\alpha, \phi', \psi) S_C(t(t_\alpha)^{-1}, \phi'', \psi)$$

where $\phi'(x) = \Psi(\sum_{\alpha\in\Delta} c_\alpha \eta_\alpha)$ and $\phi''(x) = \Psi(\sum_{\alpha\in\Delta} b_\alpha \eta_\alpha)$ with b_α and c_α determined by the formulae (3.4) and (3.5). In fact we can take $c_\alpha = d_\alpha(tt_\alpha^{-1}(-\alpha)) = t(-\alpha + 2\lambda_\alpha)$, $b_\alpha = 0$ and $c_\beta = 0$, $b_\beta = d_\beta t_\alpha(-\beta)$ for $\beta \in \Delta$, $\beta \neq \alpha$. Consequently, Example 2.9 and Proposition 2.8 imply

$$S_C(t, \phi, \psi) = S(m_\alpha, n_\alpha, t(\lambda_\alpha)) S'_{C'}(tt_\alpha^{-1}, \phi'', \psi)$$

where $S'_{C'}(tt_\alpha^{-1}, \phi'', \psi)$ is the Kloosterman sum for the group G' corresponding to the weight lattice of the root subsystem Φ' of Φ whose base is $\Delta' = \Delta - \{\alpha\}$ and the closed set $C' = C \cap \Phi'$. We observe that if $\beta \neq \alpha$ then $t_\alpha(-\beta)tt_\alpha^{-1}(-\beta + 2\lambda_\beta) = t(-\beta + 2\lambda_\beta)$ and therefore the induction on the rank of Φ implies the statement of the corollary.

Example 3.7 (Kloosterman sums for the long element). We assume that Φ is an irreducible root system and we let $\alpha_1, \alpha_2, \dots, \alpha_r$ be the basis of Φ as described in root systems tables in [B, Chapter VII]. We let $m_{ij} = \langle \alpha_i, \alpha_j \rangle$ be the Cartan integers. Suppose that t, ϕ, ψ are as in the definition of a Kloosterman sum, and we let $C = P(w_0) = \Phi^+$. We also put $d_i = d_{\alpha_i}$, $n_i = n_{\alpha_i}$, and $c_i = t(\lambda_{\alpha_i})$, $1 \le i \le r$. If c_i are *pairwise relatively prime*, Corollary 3.4 implies

$$S_{w_0}(t, \phi, \psi) = \prod_{i=1}^{r} S(m_i, n_i, c_i) \quad \text{where } m_i = d_i \prod_{j\neq i} c_j^{-m_{ij}}, \quad i = 1, 2, \dots, r.$$

For rank 2 root systems we have

$$\begin{aligned}
S_{w_0}(t, \phi, \psi) &= S(-d_1c_2, n_1, c_1)S(-d_2c_1, n_2, c_2) && \text{if } \Phi \text{ is of type } \mathbf{A}_2,\\
S_{w_0}(t, \phi, \psi) &= S(-d_1c_2, n_1, c_1)S(-d_2(c_1)^2, n_2, c_2) && \text{if } \Phi \text{ is of type } \mathbf{B}_2,\\
S_{w_0}(t, \phi, \psi) &= S(-d_1(c_2)^2, n_1, c_1)S(-d_2c_1, n_2, c_2) && \text{if } \Phi \text{ is of type } \mathbf{C}_2,\\
S_{w_0}(t, \phi, \psi) &= S(-d_1(c_2)^3, n_1, c_1)S(-d_2c_1, n_2, c_2) && \text{if } \Phi \text{ is of type } \mathbf{G}_2,
\end{aligned}$$

if $(c_1, c_2) = 1$.

The problem of evaluation of Kloosterman sums in the general case (that is, if numbers $t(\lambda_\alpha)$, $\alpha \in \Delta$, are not necessarily pairwise relatively prime) is much more difficult. The strongest known results concerning evaluation of $GL(n, \mathbf{Q}_p)$-Kloosterman sums can be found in [S, §4]. Also in [S] a necessary and sufficient condition for existence of $GL(n, \mathbf{Q}_p)$-Kloosterman sums is given. Such a condition for Chevalley group Kloosterman sums is discussed in [D].

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