

RATIONAL ORBITS ON THREE-SYMMETRIC PRODUCTS OF ABELIAN VARIETIES

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ABSTRACT. Let A be an n -dimensional Abelian variety, $n \geq 2$; let $\text{CH}_0(A)$ be the group of zero-cycles of A , modulo rational equivalence; by regarding an effective, degree k , zero-cycle, as a point on $S^k(A)$ (the k -symmetric product of A), and by considering the associated rational equivalence class, we get a map $\gamma: S^k(A) \rightarrow \text{CH}_0(A)$, whose fibres are called γ -orbits.

For any $n \geq 2$, in this paper we determine the maximal dimension of the γ -orbits when $k = 2$ or 3 (it is, respectively, 1 and 2), and the maximal dimension of families of γ -orbits; moreover, for generic A , we get some refinements and in particular we show that if $\dim(A) \geq 4$, $S^3(A)$ does not contain any γ -orbit; note that it implies that a generic Abelian four-fold does not contain any trigonal curve. We also show that our bounds are sharp by some examples.

The used technique is the following: we have considered some special families of Abelian varieties: $A_t = E_t \times B$ (E_t is an elliptic curve with varying moduli) and we have constructed suitable projections between $S^k(A_t)$ and $S^k(B)$ which preserve the dimensions of the families of γ -orbits; then we have done induction on n . For $n = 2$ the proof is based upon the papers of Mumford and Roitman on this topic.

1. INTRODUCTION

Let X be a d -dimensional smooth algebraic variety; a *cycle* Z of codimension r in X is defined to be an element of the free Abelian group $\mathbf{C}^r(X)$ generated by the irreducible subvarieties of codimension r on X . We are interested in zero-cycles, i.e. when $r = d$. Two zero-cycles Z_1 and Z_2 of X are *rationaly equivalent* if there exists a cycle Z on $X \times \mathbf{A}^1$, which intersects each fibre $X \times \{t\}$ in some points such that Z_1 and Z_2 are obtained respectively by intersecting Z with the fibres $X \times \{0\}$ and $X \times \{1\}$. Note that this is in fact an equivalence relation and that the zero-cycles rationaly equivalent to 0 (the zero of $\mathbf{C}^d(X)$) form a subgroup of $\mathbf{C}^d(X)$, (see [H, R₁]).

We denote by $\text{CH}_0(X)$ the (Chow) group of zero-cycles on X , modulo rational equivalence. If $Z = \sum n_i P_i$ is a zero-cycle, where the P_i are points of X , we define the *degree* of Z to be $\sum n_i$. It is convenient to regard an *effective*

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zero-cycle $Z = \sum n_i P_i$ i.e. one where all the $n_i > 0$, as a point on the k th symmetric product $S^k(X)$ of X , where $k = \deg(Z)$. Then by taking the associated rational equivalence class, we obtain a map $\gamma: S^k(X) \rightarrow \text{CH}_0(X)$; the fibres of this map will be called γ -orbits; the irreducible, connected, components of a γ -orbit will be called γ -components, (γ -curves if they have dimension 1, γ -surfaces if they have dimension 2, etc.).

Now let A be an Abelian variety, if we consider the Albanese morphism $\alpha_k: S^k(A) \rightarrow \text{Alb}[S^k(A)] = A$ (i.e. $\alpha_k(x_1, x_2, \dots, x_k) = x_1 + x_2 + \dots + x_k$), we have that the fibres of α_k are all isomorphic and that every γ -orbit of $S^k(A)$ is contained in exactly one fibre of α_k . Then, if we want to study the γ -orbits of $S^k(A)$, we have only to consider the γ -orbits contained in $K_k(A) = \ker(\alpha_k)$.

In [P] the author showed that for a generic Abelian variety A , with $\dim(A) \geq 3$, its Kummer variety, $K(A)$, does not contain any rational curve. By remarking that $K(A)$ is $K_2(A)$ in the previous notations, you can think that in $S^2(A)$ there are no one-dimensional γ -orbits, (where “dimension” means: maximal dimension of the γ -components of the γ -orbit, see §3). In fact, as Clemens pointed out, the technique used in [P] is related to the famous Mumford’s paper [M] about the rational equivalence of zero cycles on a surface. So that, by those arguments, it is possible to show:

Theorem (1.1). *Let A be an Abelian variety, $\dim(A) \geq 2$, then*

- (a) *$S^2(A)$ does not contain any two-dimensional γ -orbit;*
- (b) *if A is generic and $\dim(A) \geq 3$, $S^2(A)$ does not contain any one-dimensional γ -orbit.*

The proof of (1.1) is essentially contained in [P]: you have only to change the words “rational curve” into “ γ -curve”, (see also (7.1)).

In this paper we study the γ -orbits of $S^3(A)$, $\dim(A) \geq 2$, and we obtain the following results:

Theorem (1.2). *Let A be an Abelian variety, $\dim(A) \geq 2$, then*

- (a) *in $S^3(A)$ there are no d -dimensional γ -orbits with $d \geq 3$;*
- (b) *in $K_3(A)$ there are no one-dimensional families of two-dimensional γ -orbits;*
- (c) *if $\dim(A) = 2$, in $K_3(A)$ there are no three-dimensional families of one-dimensional γ -orbits.*

Remark (1.3). If $\dim(A) = 2$, in $S^3(A)$ there are some two-dimensional γ -orbits and some two-dimensional families of one-dimensional γ -orbits, see Examples (5.2) and (5.3); so that (1.2) is sharp.

Theorem (1.4). *Let A be a generic Abelian variety, $\dim(A) \geq 3$, then*

- (a) *if $\dim(A) = 3$, in $S^3(A)$ there are no two-dimensional γ -orbits;*
- (b) *if $\dim(A) = 3$, in $K_3(A)$ there are no two-dimensional families of one-dimensional γ -orbits;*
- (c) *if $\dim(A) \geq 4$, in $S^3(A)$ there are no one-dimensional γ -orbits.*

The proof of (1.2), in §5, is based upon the results of Mumford and Roitman (see §3); but, to apply them, we have needed some linear algebra which we have condensed in §4.

To prove (1.4) we have considered some special families of Abelian varieties of this type: $A_t = E_t \times B$ (where E is usually an elliptic curve with varying moduli), and we have used the projections between $S^3(A_t)$ and $S^3(B)$ which preserve the dimension of the families of γ -orbits, then we have applied (1.2) to $S^3(B)$, (see §7).

Unfortunately we did not find an easy way to show that such projections do exist, not even when A is isogenous to a product of elliptic curves. So we were forced to prove the lemmas in §6; actually some proof could be shortened by using the De Franchis-Severi theorem (for curves and for surfaces, see [D-M]), but we have avoided this theorem, firstly since it is not strictly necessary, secondly since we hope to generalize our results to $S^k(A)$, $k \geq 4$.

Our theorems have the following corollary, which solves the problem put at the end of [P]:

Corollary (1.5). *Let A be a generic g -dimensional Abelian variety, $g \geq 4$. Then A is not a quotient of a Jacobian of a trigonal curve, in other words A does not contain trigonal curves.*

Proof. Let C be a trigonal curve such that there exists a surjective map

$$f: J(C) \rightarrow A.$$

By composing f with the Abel-Jacobi map, we get a nontrivial map $C \rightarrow A$, hence we have a finite map: $S^3(C) \rightarrow S^3(A)$; as C is trigonal we have another obvious map: $\mathbf{P}^1 \rightarrow S^3(C) \rightarrow S^3(A)$; this gives rise to a rational curve in $S^3(A)$, but it is not possible by (1.4)(c). \square

Remark (1.6). Obviously the Jacobian of a trigonal curve contains a trigonal curve: the curve itself; (1.5) shows that, among Abelian varieties, the Jacobians of genus 4 curves are special also under this point of view.

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2. NOTATIONS AND CONVENTIONS

\oplus	direct sum of vector spaces,
$\langle \mathbf{x}_1, \mathbf{x}_2, \dots \rangle$	\mathbf{C} -vector space generated by $\mathbf{x}_1, \mathbf{x}_2, \dots$,
variety	by this term we mean a projective complex variety,
n -fold	n -dimensional variety (not necessarily smooth),
surface	two-fold,
curve	one-fold,
generic	by this word we mean: outside a countable union of proper analytic subvarieties,
K_V	canonical divisor of the variety V when it is smooth,
$V \times V$	Cartesian product of the variety V with itself,
V^k	k -Cartesian product of the variety V ,
$S^k(V)$	k -symmetric product of the variety V ,
\mathcal{H}_n	Siegel space of n -dimensional Abelian varieties.

3. RATIONAL EQUIVALENCE OF ZERO-CYCLES

In this paragraph we recall the results of Roitman and Mumford we need in the sequel.

Proposition (3.1) (see [R₂]). *Let Z be a degree k effective zero-cycle on a smooth variety X , then the γ -orbit of X containing Z is a countable union of closed subsets of $S^k(X)$; such a set is usually called c -closed.*

We can define the *dimension* of a c -closed set as the maximal dimension of its irreducible components. In this way it is possible to define the dimension of the image: $\gamma(S^k(X)) \subseteq \text{CH}_0(X)$, even though it is not an algebraic variety, as

$$d_k = \dim(S^k(X)) - \min\{\text{dimension of a fibre of } \gamma\}.$$

We say that $\text{CH}_0(X)$ is *finite dimensional* if the set of integers d_k is bounded, otherwise we say that $\text{CH}_0(X)$ is *infinite dimensional*.

In [M] Mumford proved that if X is a surface with geometric genus $p_g > 0$, then $\text{CH}_0(X)$ is infinite dimensional. In [R₂] Roitman gave the following generalization:

Theorem (3.2). *Let X be a smooth variety; then there are integers $d(X)$ and $j(X) \geq 0$, and an integer k_0 , such that for all $k \geq k_0$ we have $d_k = kd(X) + j(X)$. Moreover $d(X) \leq \dim(X)$, and $d(X) = 0$ if and only if $\text{CH}_0(X)$ is finite dimensional.*

In [R₁ and R₂] Roitman proved the following:

Theorem (3.3). *Let X be a smooth variety, suppose that, for some positive integer q , there exists a nonzero global q -form ω on X . Then ω induces a q -form ω_k on $S^k(X)$ whose restriction to any γ -component of $S^k(X)$ is zero. Hence $d(X) \geq q$.*

We recall that the q -form ω_k quoted in (3.3) is defined as follows: we consider X^k and for any $i = 1, 2, \dots, k$ we consider the natural projection onto the i th factor $p_i: X^k \rightarrow X$, now the q -form $\sum p_i^* \omega$ is well defined at the generic point of $S^k(X)$ because it is invariant under the action of the symmetric group; so we set $\omega_k = \sum p_i^* \omega$. In the same papers Roitman also shows the following:

Theorem (3.4). *Let f_1, f_2 be two maps between a smooth variety V and $S^k(X)$ such that $\forall v \in V$ $f_1(v)$ is rationally equivalent to $f_2(v)$; let ω be a q -form defined on X ; then $f_1^*(\omega_k) = f_2^*(\omega_k)$.*

The previous theorem allows us to prove this corollary.

Corollary (3.5). *Let V be a smooth n -dimensional variety; let $f: V \rightarrow S^k(X)$ be a map; suppose that there exists a map $p: V \rightarrow B$, where B is an $n - t$ dimensional variety, such that $\forall b \in B$, $f[p^{-1}(b)]$ is a t -dimensional γ -component of $S^k(X)$; let ω be a q -form defined on X . Then $f^* \omega_k = 0$ if $q > n - t$.*

Proof. We can always choose a suitable subvariety W of V such that $p|_W$ is finite over B ; let $V^\#$ be $V \times_B W$ (fibre product). Let $p^\#: V^\# \rightarrow W$ and $\pi^\#: V^\# \rightarrow V$ the induced projections and $\sigma: W \rightarrow V^\#$ be the canonical section

of $p^\#$; now we consider the maps $h, g: V^\# \rightarrow S^k(X)$ such that $h(v) = f[\pi^\#(v)]$ and $g(v) = h\{\sigma[p^\#(v)]\}$. Obviously $h(v)$ is rationally equivalent to $g(v)$ $\forall v \in V^\#$, and therefore, by (3.4), $h^*\omega_k = g^*\omega_k$. But $g^*\omega_k = (p^\#)^*\sigma^*h^*\omega_k$ and $\sigma^*h^*\omega_k = 0$ if $q > n - t$, as $\pi^\#$ is finite on V , $f^*\omega_k = 0$. \square

4. SOME LINEAR ALGEBRA

Let V be \mathbb{C}^2 , and let $\{dz, dw\}$ be a basis for V^* . Let L_2 be the kernel of the map $\sigma: V \oplus V \oplus V \rightarrow V$ given by summation. Consider the following two-form on L_2 :

$$\begin{aligned} (\neg) \quad & [dz_1 \wedge dw_1 + dz_2 \wedge dw_2 + dz_3 \wedge dw_3]_{|L_2} \\ &= [2dz_1 \wedge dw_1 + 2dz_2 \wedge dw_2 + dz_1 \wedge dw_2 + dz_2 \wedge dw_1]_{|L_2} \\ &= [dz_1 \wedge d(2w_1 + w_2) + d(z_1 + 2z_2) \wedge dw_2]_{|L_2}. \end{aligned}$$

As (\neg) has maximal rank on L_2 , we have that any locally isotropic subspace of $V \oplus V \oplus V$ for (\neg) , has dimension 2 at most. In fact there are such two-dimensional maximal subspaces, for instance: $\{(\mathbf{v}, \rho\mathbf{v}, \rho^2\mathbf{v}), \mathbf{v} \in V, \rho \in \mathbb{C} \text{ with } 1 + \rho + \rho^2 = 0\}$.

Now let W be \mathbb{C}^n , $n \geq 2$, and let L_n be the kernel of the map $\sigma: W \oplus W \oplus W \rightarrow W$ as before. Let U be a linear subspace of L_n such that for all projections $W \rightarrow V$, the induced map $L_n \rightarrow L_2$ sends U into a totally isotropic subspace of L_2 for (\neg) . Then $\dim(U) \leq n$. In fact, for $n = 2$ this is true, for $n \geq 3$ we can proceed by induction on n : every projection $L_n \rightarrow L_{n-1}$ has kernel of dimension 2, so that $\dim(U) \leq n + 1$; moreover if $\dim(U) = n + 1$, the kernel of every projection $L_n \rightarrow L_{n-1}$ would lie in U , and this is not possible.

Note that $\dim(U) = n$ is possible, for instance if $U = \{(\mathbf{w}, \rho\mathbf{w}, \rho^2\mathbf{w}), \mathbf{w} \in W, \rho \in \mathbb{C} \text{ with } 1 + \rho + \rho^2 = 0\}$; we will see in (4.2) that it is the only possibility. Now we can prove the following:

Proposition (4.1). *In the same notation as before, let $n = 3$, let $\{dz, dw, du\}$ be a basis for W^* ; consider the following three-form:*

$$(\neg\neg) \quad dz_1 \wedge dw_1 \wedge du_1 + dz_2 \wedge dw_2 \wedge du_2 + dz_3 \wedge dw_3 \wedge du_3$$

and suppose that U is totally isotropic for $(\neg\neg)$. Then $\dim(U) \leq 2$.

Proof. By contradiction we suppose that $\dim(U) = 3$, then by projecting W to V three times along the respective axes we see that:

$$\begin{aligned} U = & \langle (a_1, 0, 0), (a_2, 0, 0), (a_3, 0, 0) \rangle + \langle (0, b_1, 0), (0, b_2, 0), (0, b_3, 0) \rangle \\ & + \langle (0, 0, c_1), (0, 0, c_2), (0, 0, c_3) \rangle \end{aligned}$$

with: $\sum a_i = \sum b_i = \sum c_i = 0$. So the vectors $\mathbf{a} = (a_i)$, $\mathbf{b} = (b_i)$, $\mathbf{c} = (c_i)$ in \mathbb{C}^3 lie in the plane P defined by the equation: $\sum x_i = 0$. Since for all projections $W \rightarrow V$, the induced map $L_n \rightarrow L_2$ sends U into a totally isotropic subspace of L_2 for (\neg) , we have: $\sum a_i b_i = \sum b_i c_i = \sum c_i a_i = 0$.

Since the symmetric bilinear form on \mathbb{C}^3 which has the identity associated matrix (with respect to the standard base) has rank 2 on P , we conclude from the above equations that either \mathbf{a} , \mathbf{b} or \mathbf{c} is $\mathbf{0}$, (this is impossible as we have supposed that $\dim(U) = 3$) or \mathbf{a} , \mathbf{b} and \mathbf{c} are all multiples of the same vector

\mathbf{w} with $\sum w_i = \sum (w_i)^2 = 0$. So that \mathbf{w} can be taken to be some permutation of $(1, \rho, \rho^2)$. Hence we can write: $\mathbf{a} = A\mathbf{w}$, $\mathbf{b} = B\mathbf{w}$ and $\mathbf{c} = C\mathbf{w}$ for some nonzero complex numbers A, B, C . But if we apply (\wedge) to these three vectors we have that the result is zero if and only if $ABC = 0$, contradiction! \square

By (4.1) it is very easy to prove the following:

Proposition (4.2). *In the previous notation: let $n \geq 4$. Then $U \subseteq \{(\mathbf{w}, \rho\mathbf{w}, \rho^2\mathbf{w}), \mathbf{w} \in W, \rho \in \mathbb{C} \text{ with } 1 + \rho + \rho^2 = 0\}$ and if all projections of W into \mathbb{C}^3 send U into a totally isotropic subspace for (\wedge) , we have that $\dim(U) \leq 2$.*

5. PROOF OF (1.2) AND SOME EXAMPLES

Let A be an n -dimensional Abelian variety. Firstly we want to recall some useful facts about $S^k(A)$.

There is an action of the additive group A on the variety $S^k(A)$: for every $a \in A$ we have $T_a: S^k(A) \rightarrow S^k(A)$ such that for every $(x_1, x_2, \dots, x_k) \in S^k(A)$ $(T_a(x_1, x_2, \dots, x_k) = (x_1 + a, x_2 + a, \dots, x_k + a)$. For every $a \in A$, T_a is an isomorphism of $S^k(A)$ which we will call translation, by abuse of language.

If we consider the nk -dimensional Abelian variety A^k , we have that there is a $(k!)$ -covering $p: A^k \rightarrow S^k(A)$ which is obviously ramified on the points (x_1, x_2, \dots, x_k) of $S^k(A)$ such that the x_i are not all distinct. Moreover there is another obvious $(k!)$ -covering $\pi: A^{k-1} \rightarrow K_k(A)$ ($K_k(A)$ is the kernel of the Albanese map, see §1) such that $\pi(x_1, x_2, \dots, x_{k-1}) = (x_1, x_2, \dots, x_{k-1}, -x_1 - x_2 - \dots - x_{k-1})$. Remark that any d -dimensional γ -component in $K_k(A)$ gives rise to a d -fold in A^{k-1} via π .

Now we are able to prove (1.2); recall that, by the argument of §1, we have to study the γ -orbits contained in $K_3(A)$.

Proof of (1.2)(a). Let V be the dual of the Lie algebra of A , $\dim(V) = \dim(A) = n$, and we recall that, for any Abelian variety A , $\forall q \geq 1$, $H^{q,0}(A) = \Lambda^q(V)$.

For any $\omega \in \Lambda^q(V)$, $q \geq 2$, we consider the q -form $\phi(\omega)$ induced by ω on $S^3(A)$ in the following way: $\phi(\omega) = p_*(p_1^*\omega + p_2^*\omega + p_3^*\omega)$, where

$$p: A^3 \rightarrow S^3(A)$$

and p_1, p_2, p_3 are the projections of $A \times A \times A$ on A .

The tangent space U at every smooth point of any γ -orbit of $K_3(A)$ lies in L_n (see §4); $\phi(\omega)$ has to vanish on U , by Theorem (3.3), for any $\omega \in \Lambda^q(V)$, $q = 2, 3, \dots, n$; this means that the assumptions of (4.2) about the projections of U are satisfied. Hence $\dim(U) \leq 2$; therefore every γ -orbit has dimension 2 at most. \square

Remark (5.1). The previous proof is based on the fact that all the forms belonging to $\phi(\Lambda^q(V))$, $q = 2, 3, \dots, n$, have to vanish on the tangent spaces at the smooth points of any γ -component of $K_3(A)$. So we can say that, if a d -fold, contained in $K_3(A)$, has the same properties, then $d \leq 2$.

Proof of (1.2)(b). If there would be such a family $\{S_t\}$, $t \in \mathbb{C}$, then in $K_3(A)$ we would get a three-fold T which would be filled by two-dimensional γ -components. By using the same notations as in the proof of (1.2)(a), we have that, by Corollary (3.5), the forms belonging to $\phi(\Lambda^q(V))$, $q = 2, 3, \dots, n$, have to vanish on the tangent spaces at the smooth points of T , but this implies that $\dim(T) \leq 2$ by Remark (5.1): contradiction! \square

Proof of (1.2)(c). If there would be a family $\{C_r\}$, $r \in \mathbb{C}^3$, of one-dimensional γ -orbits in $K_3(A)$ then $K_3(A)$ would be filled by one-dimensional γ -components and this is not possible by (3.2) and (3.3). \square

Now we prove, by some examples, that, when $\dim(A) = 2$, the one-dimensional γ -orbits can span a three-fold in $S^3(A)$, and that there are two-dimensional γ -orbits.

Example (5.2). Let A be an Abelian surface; let C be a nonhyperelliptic genus 3 (smooth, irreducible) curve on A . If we consider the divisor L supported by C , we get $L^2 = 4$ by the genus formula, and $h^0(L) = 2$ by the Riemann-Roch and Kodaira vanishing theorems.

So C moves in a pencil $\{C_\mu\}$ which has four base points: A, B, C, D . The adjunction formula yields: $K_L = L|_C$; so that $A+B+C+D$ is a canonical divisor on every curve C_μ of the pencil.

The canonical model C'_μ of C_μ is a smooth plane quartic whose canonical series is cut by the lines, therefore the divisor of C'_μ corresponding to $A+B+C+D$ is cut on C'_μ by a line.

Now we consider a point P_μ on C_μ and the linear series g_3^1 corresponding to the linear series g_3^1 cut on C'_μ by the lines passing through the point corresponding to P_μ . So that for every $\lambda \in \mathbb{P}^1$ we have a divisor: $P_\mu + Q_{\mu\lambda} + R_{\mu\lambda} + S_{\mu\lambda}$ on C_μ . We choose an Abel map $\alpha_\mu: C_\mu \rightarrow J(C_\mu)$ such that $\alpha_\mu(P_\mu) = 0$, hence, by Abel theorem, $\alpha_\mu(Q_{\mu\lambda} + R_{\mu\lambda} + S_{\mu\lambda}) = \tau_{P,\mu}$ is constant with respect to λ . The 3-ple: $\alpha_\mu(Q_{\mu\lambda}), \alpha_\mu(R_{\mu\lambda}), \alpha_\mu(S_{\mu\lambda})$ in $J(C_\mu)$ gives rise to a rational curve in $S^3[J(C_\mu)]$ as λ moves in \mathbb{P}^1 .

We consider the following commutative diagram

$$\begin{array}{ccc} C_\mu & \xrightarrow{i_\mu} & A \\ \alpha_\mu \downarrow & \nearrow f_\mu & \\ J(C_\mu) & & \end{array}$$

in which i_μ is the embedding of C_μ in A and f_μ is the homomorphism between Abelian varieties induced by α_μ . By using f_μ we get a rational curve in $S^3(A)$; by translating this curve by $f_\mu(\tau_{P,\mu})$ we get a rational curve $\gamma_{P,\mu}$ in $K_3(A)$.

Now we let P vary on C_μ : for every point P we get a curve $\gamma_{P,\mu}$ in $K_3(A)$; these curves are all distinct because the used linear series g_3^1 on C'_μ are distinct. Now let P vary on C_μ and let μ vary in \mathbb{P}^1 : for every couple P, μ we get a curve $\gamma_{P,\mu}$ in $K_3(A)$; these curves are all distinct because they are made by points lying on different curves C_μ of A .

Obviously every curve $\gamma_{P,\mu}$ is contained in a γ -orbit of $K_3(A)$ and this example shows that in $K_3(A)$ there exist γ -orbits whose span is a three-fold.

Example (5.3). The previous example also shows that in $K_3(A)$ there exist some γ -orbits whose span is a surface. In fact for every curve C_μ of the previous example we can fix the point A , (one of the base points of the pencil $\{C_\mu\}$), and for every $\mu \in \mathbf{P}^1$ we get a rational curve $\gamma_{A,\mu} = \gamma_\mu$ in $K_3(A)$.

In this case, by recalling the construction of the linear series g_3^1 , we have that for every $\mu \in \mathbf{P}^1$ there exists a $\lambda \in \mathbf{P}^1$ such that $Q_{\mu\lambda} = B$, $R_{\mu\lambda} = C$, $S_{\mu\lambda} = D$. Therefore: $\alpha_\mu(B+C+D) = \tau_{A,\mu}$ and $f_\mu[\alpha_\mu(B+C+D)] = f_\mu(\tau_{A,\mu}) = i_\mu(B+C+D)$ is independent from μ , hence the obtained curves in $S^3(A)$ belong to

$$\{(x, y, z) \in S^3(A) | x + y + z = i_\mu(B + C + D)\}$$

and all pass through the point: $(i_\mu(B), i_\mu(C), i_\mu(D))$ in $S^3(A)$.

So that the translated curves γ_μ in $K_3(A)$ all intersect between them. Therefore the curves γ_μ span a rational surface in $K_3(A)$ which is contained in a γ -orbit.

6. THE LEMMAS

In this paragraph we prove some lemmas which will be useful in §7. We will need to study the projections of d -dimensional γ -components which are induced by natural projections between $K_3(V \times W)$ and $K_3(W)$, where V and W will be suitable Abelian varieties.

By the commutativity of the following diagram

$$(6.1) \quad \begin{array}{ccc} (V \times W) \times (V \times W) & \longrightarrow & W \times W \\ \downarrow \pi & & \downarrow \pi \\ K_3(V \times W) & \longrightarrow & K_3(W) \end{array}$$

we have to study the natural projections $(V \times W) \times (V \times W) \rightarrow W \times W$, this is the aim of the following two lemmas.

Let X be a smooth irreducible d -fold and let A be an n -dimensional Abelian variety; let $\sigma: X \rightarrow A \times A$ be a map, birational onto its image, such that $\sigma(X)$ generates $A \times A$. Assume that A is isogenous to $D \times D \times B$ where D and B are Abelian varieties of dimension q and $(n - 2q)$ respectively. We fix two “dual” isogenies $D \times D \times B \rightarrow A \rightarrow D \times D \times B$ such that their composition is the multiplication by an integer; in this way we get a map $f \circ \sigma: X \rightarrow B \times B$ by composing the natural projection f with σ ; let Y be $f[\sigma(X)]$; assume that

- (*) the natural projection $f: A \times A \rightarrow B \times B$ is such that $Y = f[\sigma(X)]$ is a d -dimensional subvariety of $B \times B$.

Now let $\nu_i: D \rightarrow D \times D \rightarrow A$ be the composition of an embedding of D in $D \times D$ with the previously chosen isogeny; we can suppose that i varies in a countable set, in fact among all embeddings $D \rightarrow D \times D$ there are the following morphisms of algebraic groups: $\mathbf{d} \rightarrow (a\mathbf{d}, b\mathbf{d})$ (for any $\mathbf{d} \in D$ and for a fixed couple of coprime integers a, b). We set $B_i = [(D \times D)/\nu_i(D)] \times B$ and let X_i be the image of X under the composition of the natural projection $A \times A \rightarrow B_i \times B_i$ with σ .

In this situation we have the maps: $q_i^*: H^1(X_i, \mathbb{Q}) \rightarrow H^1(X, \mathbb{Q})$ and $\sigma^*: H^1(A \times A, \mathbb{Q}) \rightarrow H^1(X, \mathbb{Q})$; let Λ_i be the image of q_i^* , then

Lemma (6.2). *With the previous notations, there exists an index i at least (hence an embedding of D in $D \times D$) such that Λ_i contains the image of σ^* .*

Proof. Note that this proof actually shows more, i.e. Λ_i contains the image of $H^1(A \times A, \mathbb{Q})$ in $H^1(X, \mathbb{Q})$ save for a finite number of i .

For every i we have a diagram of equidimensional d -folds

$$\begin{array}{ccc} X & \longrightarrow & X_i \\ & \searrow & \swarrow \\ & Y & \end{array}$$

(the map $X_i \rightarrow Y$ is obtained by using the natural projection

$$B_i \rightarrow B_i / [(D \times D) / \nu_i(D)]$$

and by remarking that $B_i / [(D \times D) / \nu_i(D)]$ is isogenous to B). It follows that: $K[Y] \subset K[X_i] \subset K[X]$ so that there are only a finite number of birational models for the X_i . The maps in the following diagram are defined in the obvious way:

$$\begin{array}{ccccc} H^1(B \times B, \mathbb{Q}) & \longrightarrow & H^1(B_i \times B_i, \mathbb{Q}) & \longrightarrow & H^1(A \times A, \mathbb{Q}) \\ \downarrow & & \downarrow & & \downarrow \sigma^* \\ H^1(Y, \mathbb{Q}) & \longrightarrow & H^1(X_i, \mathbb{Q}) & \longrightarrow & H^1(X, \mathbb{Q}) \end{array}$$

and we remark that, as $\sigma(X)$ generates $A \times A$ and the natural projection $A \times A \rightarrow B_i \times B_i$ is surjective, the map $H^1(B_i \times B_i, \mathbb{Q}) \rightarrow H^1(X, \mathbb{Q})$ is injective for any i . Now if we choose two distinct, transverse, embeddings of D in $D \times D$ for which the corresponding fields $K[X_{i1}]$ and $K[X_{i2}]$, contained in $K[X]$, coincide, then we have that $H^1(X_{i1}, \mathbb{Q}) \rightarrow H^1(X, \mathbb{Q})$ and $H^1(X_{i2}, \mathbb{Q}) \rightarrow H^1(X, \mathbb{Q})$ are the same map; by the injectivity of $H^1(B_{ij} \times B_{ij}, \mathbb{Q}) \rightarrow H^1(X, \mathbb{Q})$, $j = 1, 2$, we have that $\Lambda_{i1} = \Lambda_{i2}$ must contain the span of the images of $H^1(B_{i1} \times B_{i1}, \mathbb{Q})$ and $H^1(B_{i2} \times B_{i2}, \mathbb{Q})$ in $H^1(X, \mathbb{Q})$, hence $\Lambda_{i1} = \Lambda_{i2}$ must contain the image of $H^1(A \times A, \mathbb{Q})$ in $H^1(X, \mathbb{Q})$. \square

Lemma (6.3). *With the same assumptions as in (6.2), we get the same thesis if we consider $F^1 H^1(-, \mathbb{C})$, (in the sense of mixed Hodge structures, see [G]), instead of $H^1(-, \mathbb{Q})$.*

Remark (6.4). Note that, if $\dim(X) = 1$, $(*)$ is always satisfied, (save, obviously, when $A = E \times E \times B$, E elliptic curve, and $X = E$).

Now let Δ be an analytic scheme ($0 \in \Delta$), and $h: \mathbf{A} \rightarrow \Delta$ a proper fibration such that $h^{-1}(t)$, $t \in \Delta$, is an Abelian variety isogenous to $\mathbf{D}_t \times \mathbf{B}$, \mathbf{B} fixed, ($h^{-1}(0)$ isogenous to $\mathbf{D}_0 \times \mathbf{B}$).

The infinitesimal variation of the Hodge structures induces the following map $\phi: H^{1,0}(\mathbf{D}_0) \rightarrow \text{Hom}(T_\Delta(0), H^{0,1}(\mathbf{D}_0))$, such that for any $\mu \in H^{1,0}(\mathbf{D}_0)$ and for any $t \in T_\Delta(0)$, $\phi(\mu)(t)$ is the derivative of μ along t . We have the following:

Lemma (6.5). *With the previous assumptions, consider the commutative diagram*

$$\begin{array}{ccc} q_t: X_t & \longrightarrow & Z \\ \sigma_t \downarrow & & \downarrow \iota \\ f_t: \mathbf{D}_t \times \mathbf{B} & \longrightarrow & \mathbf{B} \end{array}$$

where X_t are varieties parametrized by t , σ_t are maps birational onto their images, $\sigma_t(X_t)$ generates $\mathbf{D}_t \times \mathbf{B}$ for any t , f_t is the natural projection, q_t is induced by f_t , ι is an inclusion and Z is fixed. Assume that ϕ is injective; then

$$\sigma_0^*[H^{1,0}(\mathbf{D}_0)] \cap q_0^*F^1H^1(Z) = 0 \in F^1H^1(X_0).$$

Proof. If μ belongs to that intersection, $\phi(\mu) = 0$ as $F^1H^1(Z)$ is independent from t ; as ϕ is injective we have $\mu = 0$. \square

Now let Δ be an open set of \mathcal{H}_n , ($0 \in \Delta$), we will call a “ (Δ, m, G) -situation” (for \mathcal{H}_n) the following data:

(i) a bundle of Abelian varieties over Δ : $\mathbf{A} \times_{\Delta} \mathbf{A} \times_{\Delta} \cdots \times_{\Delta} G$ (m times) where \mathbf{A} is the tautological Abelian bundle over Δ and G is a constant Abelian variety; (by abuse of notation we write $G = G \times \Delta$ and $\mathbf{A}^m \times G = \mathbf{A} \times_{\Delta} \mathbf{A} \times_{\Delta} \cdots \times_{\Delta} G$ (m times));

(ii) a family of d -dimensional varieties $k: \mathbf{X} \rightarrow \Delta$ over Δ ;

(iii) a morphism of Δ families $\sigma: \mathbf{X} \rightarrow \mathbf{A}^m \times G$, i.e. a commutative diagram as follows:

$$\begin{array}{ccc} \mathbf{X} & \xrightarrow{\sigma} & \mathbf{A}^m \times G \\ & \searrow k \quad \swarrow h & \\ & \Delta & \end{array}$$

(we set $X_t = k^{-1}(t)$ and $\mathbf{h}^{-1}(t) = (A_t)^m \times G$ for any $t \in \Delta$);

(iv) the assumption that the image $\sigma_t(X_t)$ generates $(A_t)^m \times G$ as a group, for any $t \in \Delta$.

We remark that, if conditions (i), (ii), (iii) are satisfied, the bundle of Abelian varieties generated by the images $\sigma_t(X_t)$ must be isomorphic to $\mathbf{A}^{m'} \times G'$ where $m' \leq m$ and G' is an Abelian subvariety of G ; so that, by changing the bundle, we always get a (Δ, m', G') -situation. With the above warning we can say that to have a (Δ, m, G) -situation is equivalent to have a d -dimensional variety in $\mathbf{A}^m \times G$ where A is generic in Δ ; (i.e. for any $t \in \Delta$ we have a d -fold X_t in $(A_t)^m \times G$). Actually we usually will consider only the case: $m = 2$, $G = 0$, (hence $\mathbf{h} = h \times_{\Delta} h$); for the sake of simplicity, from now on, this case will be simply called “ Δ -situation.”

Lemma (6.6). *We suppose to be in a Δ -situation; we choose A isogenous to $D \times D \times B$, (as in Lemma (6.2)), and for any linear embedding $\nu_i: D \rightarrow D \times D$ we fix an isogeny between A and $\nu_i(D) \times [(D \times D)/\nu_i(D)] \times B$.*

Let $\Delta_i = \{t \in \Delta \mid \text{the fibre of } h \times_{\Delta} h \text{ is } A_t \times A_t \text{ where } A_t \text{ is isogenous to } \nu_i(D) \times D_t \times B, D_t \in \mathcal{H}_q\}$; let A_0 be isogenous to A by the isogeny induced by the previously fixed one. This defines an embedding $\nu_i^: \mathcal{H}_q \rightarrow \mathcal{H}_n$, such that $\Delta_i = \Delta \cap [\nu_i^*(\mathcal{H}_q)]$; we set $B_i = \nu_i(D) \times B$.*

For any $t \in \Delta_i$, let $f_{i,t}: A_t \times A_t \rightarrow B_i \times B_i$ be the natural projection; if we assume () for the natural projection $f_{i,0}: A \times A \rightarrow B \times B$ and $\sigma_0(X_0)$, we have*

that, save a finite number of i at most, $f_{i,t}[\sigma_t(X_t)]$ is not a fixed subvariety of $B_i \times B_i$.

Proof. We proceed by contradiction: if (6.6) is false, then for any i , $f_{i,t}[\sigma_t(X_t)]$ is a fixed d -fold X_i in B_i for any t , and X_i generates B_i . Then we have the following commutative diagram:

$$\begin{array}{ccc} q_{i,0}: X_0 & \longrightarrow & X_i \\ \downarrow \sigma_0 & & \downarrow \iota_i \\ f_{i,0}: (D \times D \times B)^2 & \longrightarrow & B_i \times B_i. \end{array}$$

Note that we can apply Lemma (6.5) because we are in a Δ -situation, so we have that $\sigma_0^*[H^{1,0}(D^4)] \cap (q_{i,0})^*F^1H^1(X_i) = 0 \in F^1H^1(X_0)$ but, by Lemma (6.3), $(q_{i,0})^*F^1H^1(X_i)$ contains $\sigma_0^*[H^{1,0}(D^4)]$ except for a finite number of i , contradiction! \square

Lemma (6.7). *We are supposed to be in a Δ -situation; but now we choose A isogenous to $D^m \times B$, and we consider the countable set of the linear embeddings $\nu_i: D^p \rightarrow D^m$ ($p \leq m$, positive integers, $D \in \mathcal{H}_q$, $B \in \mathcal{H}_{n-mq}$). For any embedding ν_i we fix an isogeny between A and $\nu_i(D^p) \times [D^m/\nu_i(D^p)] \times B$; let $\Delta_i = \{t \in \Delta \mid \text{the fibre of } h \times_{\Delta} h \text{ is } A_t \times A_t \text{ where } A_t \text{ is isogenous to } F_t \times [D^m/\nu_i(D^p)] \times B, F_t \in \mathcal{H}_{pq}\}$, A_0 is isogenous to A as in the previous cases. This defines an embedding $\nu_i^*: \mathcal{H}_{pq} \rightarrow \mathcal{H}_n$ such that: $\Delta_i = \Delta \cap [\nu_i^*(\mathcal{H}_{pq})]$; we set: $B_i = [D^m/\nu_i(D^p)] \times B$.*

For any $t \in \Delta_i$, let $f_{i,t}: A_t \times A_t \rightarrow B_i \times B_i$ be the natural projection; if we assume (*) for the natural projection $f_{i,0}: A \times A \rightarrow B \times B$ and $\sigma_0(X_0)$, we have that, save a finite number of i at most, $f_{i,t}[\sigma_t(X_t)]$ is not a fixed subvariety of B_i .

Proof. See the proof of (6.6). \square

To apply the above lemmas we need condition (*); this is a crucial point: it allows us to avoid the use of the De Franchis-Severi theorem. When X is of general type and $d = 1$ or 2 , this theorem would assure the existence of a finite number of subfields $K[X_i]$ of $K[X]$ (see the proof of Lemma (6.2)), without the assumption that f is generically finite, i.e., roughly speaking, without fixing a shield $Y = f[\sigma(X)]$.

We use the following remark: consider diagram (6.1): our natural projections between $(V \times W) \times (V \times W)$ and $W \times W$ are induced by natural projections between $K_3(V \times W)$ and $K_3(W)$, so that to verify (*) it suffices to verify the corresponding statement for projections between $K_3(V \times W)$ and $K_3(W)$, and vice versa. This explains the statements of the following other lemmas.

Lemma (6.8). *Let S be a γ -surface in $K_3(E \times E)$ where E is a generic elliptic curve (in the sense of moduli); let S' be the pullback of S in $E^2 \times E^2$; let E_{pq} be a fixed embedding of $E \times E$ in $E^2 \times E^2$ such that $E_{pq} = \{px, qx, py, qy\}$ where $(x, y) \in E \times E$ and p, q are coprime integers. Then there exist infinitely many couples (p, q) such that E_{pq} intersects S' properly. In these cases the natural projection $E^2 \times E^2 \rightarrow (E^2 \times E^2)/E_{pq}$ is generically finite on S' (and the induced map $K_3(E \times E) \rightarrow K_3[(E \times E)/\{px, qy\}]$ is generically finite on S).*

Proof. We will prove that there exists a couple (p, q) at least, such that E_{pq} intersects S' properly, but, in fact, our proof will also show that the intersection is proper save for a finite number of couples.

We proceed by contradiction; we recall that if two surfaces in E^4 does not intersect properly then, for every generic point of the first surface, there passes a translate of the second one which intersects the former one along a curve. In fact the intersection cycle of two surfaces in E^4 depends only on their homology class, and the homology class is invariant under translations.

We fix a generic point P of S' , if every E_{pq} does not intersect S' properly then, $\forall p, q$, there exists a translate of E_{pq} passing through P and cutting S' along a curve; hence, by looking at the tangent spaces, we have that in the Lie algebra of E^4 there are: a vector space generated by $(p, q, 0, 0)$ and $(0, 0, p, q)$, $\forall p, q$, and the vector space $\langle (a_1, a_2, a_3, a_4), (b_1, b_2, b_3, b_4) \rangle$ (corresponding to the tangent space to S' at P), such that the matrix:

$$\begin{pmatrix} p & 0 & a_1 & b_1 \\ q & 0 & a_2 & b_2 \\ 0 & p & a_3 & b_3 \\ 0 & q & a_4 & b_4 \end{pmatrix}$$

is always singular. Now we show that, for generic E , this situation is not possible.

As (a_1, a_2, a_3, a_4) and (b_1, b_2, b_3, b_4) are independent, it is possible to choose a base for the Lie algebra such that: $a_1 = b_2 = 1$, $b_1 = a_2 = 0$; otherwise is not possible that the previous matrix is singular $\forall p, q$. Now it is easy to see that it is possible only if $b_3 = a_4 = 0$ and $b_4 = a_3 = \rho$, with $\rho \in \mathbb{C}$. As S' is the pullback in $E \times E \times E \times E$ of a γ -component S in $K_3(E \times E)$ which is not contained in the branching locus of π , the skew symmetric two-form (\wedge) considered in §4 has to vanish on the tangent space at the generic point P of S' by (3.3), hence: $1 + \rho + \rho^2 = 0$ and ρ is a constant, independent from P .

This means that the only surfaces in E^4 which does not intersect properly $E_{pq} \forall p, q$, are, up to translations, those Abelian surfaces S' which are the embeddings of $E \times E$ in E^4 such that $S' = \{x, y, \rho x, \rho y\}$ where $(x, y) \in E \times E$ and $\rho \in \mathbb{C}$ with $1 + \rho + \rho^2 = 0$; but this implies that E has an endomorphism: $x \rightarrow \rho x \forall x \in E$, with $1 + \rho + \rho^2 = 0$, and this is not possible for generic E . \square

Lemma (6.9). *Let S be a γ -surface in $K_3(E \times E \times E)$ where E is a generic elliptic curve; let S' be the pullback of S in $E^3 \times E^3$; let $E(p, q, r, p', q', r')$ be a fixed embedding of $E^2 \times E^2$ in $E^3 \times E^3$ such that $E(p, q, r, p', q', r') = \{px + p'y, qx + q'y, rx + r'y, pz + p'w, qz + q'w, rz + r'w\}$ where $(x, y, z, w) \in E^2 \times E^2$, and $(p, q, r), (p', q', r')$ are triple of coprime integers, and such that the following matrix has rank 2:*

$$\begin{pmatrix} p & q & r \\ p' & q' & r' \end{pmatrix}.$$

Then there exist infinitely many choices (p, q, r, p', q', r') such that $E(p, q, r, p', q', r')$ intersects S' properly. In these cases the natural projection

$$E^3 \times E^3 \rightarrow (E^3 \times E^3)/E(p, q, r, p', q', r')$$

is generically finite on S' ,

$$(K_3(E \times E \times E) \rightarrow K_3[(E \times E \times E)/\{px + p'y, qx + q'y, rx + r'y\}])$$

is generically finite on S).

Proof. We can proceed as in the proof of Lemma (6.8). \square

Lemma (6.10). *Let E be a generic elliptic curve and let T be a three-fold in $K_3(E^3)$ which is filled by a two-dimensional family of γ -curves; let T' be the pullback of T in $E^3 \times E^3$; let E_{pqr} be a fixed embedding of $E \times E$ in $E^3 \times E^3$ such that $E_{pqr} = \{px, qx, rx, py, qy, ry\}$ where $(x, y) \in E \times E$ and p, q, r are coprime integers. Then there exist infinitely many triples (p, q, r) such that E_{pqr} does not intersect T' or intersects T' in a finite number of points. In these cases the natural projection $E^3 \times E^3 \rightarrow (E^3 \times E^3)/E_{pqr}$ is generically finite on T' (and the induced map $K_3(E^3) \rightarrow K_3[E^3/\{px, qx, rx\}]$ is generically finite on T).*

Proof. By arguing as in Lemma (6.8) we get that the only three-folds in $E^3 \times E^3$ which does not intersect properly E_{pqr} $\forall p, q, r$ are, up to translations, those Abelian three-folds T' which are the embeddings of $E \times E \times E$ in $E^3 \times E^3$ such that $T' = \{x, y, z, sx, sy, sz\}$ where $(x, y, z) \in E \times E \times E$ and $s \in \mathbb{C}$ with $s(s+1) = 0$.

This would imply that, in $K_3(E^3)$, T would be given by the unordered triples: $\{P, sP, -(s+1)P\}$, where $s = 0$ or $s = -1$ and $P \in E^3$; in any case we could define an embedding $\lambda: T \rightarrow K_2(E^3)$ such that

$$\lambda(\{P, sP, -(s+1)P\}) = \{P, -P\};$$

$\lambda(T)$ would be a three-fold filled out by γ -curves; but this is not possible by (1.1)(b): recall that E is generic and the locus of nonsimple Abelian three-folds is dense in \mathcal{H}_3 .

7. PROOF OF (1.4)

For the sake of simplicity, in every Δ -situation considered in §7 we will identify X_t with $\sigma_t(X_t)$.

Proof of (1.4)(a). We proceed by contradiction: we assume that for any three-dimensional Abelian variety A , $S^3(A)$, and therefore $K_3(A)$, contains a γ -surface; by their pullback via π we have a surface in any A^2 , so we are in a Δ -situation. Then we can construct a fibration $h \times_{\Delta} h: A \times_{\Delta} A \rightarrow \Delta \subset \mathcal{H}_3$ as in §6. We want to apply Lemma (6.6) with $D = B = E$, E generic elliptic curve. To have (*) we use Lemma (6.9): we can fix an Abelian variety A isogenous to $E \times E \times E$, such that, when we project the γ -surface X contained in $K_3(E \times E \times E)$ into $K_3(E)$ (the last factor), by the natural projection, we obtain another γ -surface Y . This means that the natural projection $f: A \times A \rightarrow B \times B$ satisfies (*).

Now let E_{pq} be the image in $E \times E$ of the embedding ν_{pq} of E such that $\nu_{pq}(x) = (px, qx) \forall x \in E$, (p, q) is a couple of coprime integers. We fix an isogeny between A and $E_{pq} \times B_{pq}$ where B_{pq} is $[(E \times E)/E_{pq}] \times E$. Let $\Delta_{pq} = \Delta \cap [\nu_{pq}^*(\mathcal{H}_1)]$ the open subset of Δ such that the fibre over $t \in \Delta_{pq}$ is

$A_t \times A_t$ where A_t is isogenous to $E_t \times B_{pq}$ (A_0 isogenous to A by the previously fixed isogeny) and E_t is an elliptic curve whose moduli depend on t .

Let φ_t be the natural projection between $K_3(A_t)$ and $K_3(B_{pq})$, by our assumption there is a γ -surface X_t in every A_t and $X_0 = X$. For small t , we can assume that $Y_t = \varphi_t(X_t)$ is a γ -surface of $K_3(B_{pq})$; in fact $Y_0 = \varphi_0(X_0) = \varphi_0(X)$ is a surface in $K_3(B_{pq})$ because X projects into a surface in $K_3(E)$.

By Lemma (6.6), we can choose (p, q) such that $\{Y_t\}$ is a one-dimensional family of γ -surfaces of $K_3(B_{pq})$ (i.e. the union of the Y_t span a three-fold in $K_3(B_{pq})$); but $\dim(B_{pq}) = 2$ and this is a contradiction with (1.2)(b). \square

Remark (7.1). Here we want to give a short outline of the proof of (1.1)(b) when $\dim(A) = 3$. Firstly we need (1.1)(a) for dimension 2: this is just an application of (3.2) and (3.3): if (1.1)(a) were false, for the generic point of $S^2(A)$ would pass a positive dimensional γ -orbit, but then d_2 would be strictly less than 4.

Now we proceed by contradiction: we assume that for the generic Abelian three-fold A , $S^2(A)$, and therefore $K_2(A)$ (which is the Kummer variety $K(A)$ of A), contains a γ -curve. By their pullback via π we get a curve in any A ; by using these we can build a family of curves that gives rise to a Δ -situation. By arguing as in the proof of (1.4)(a) we can choose a suitable projection from $K_2(A_t) = K(A_t)$ onto $K(E \times E)$, where A_t is isogenous to $E_t \times E \times E$, E generic elliptic curve, in such a way that the images of our curves cover $K(E \times E)$. Since the image of a γ -orbit is a γ -orbit, we get a contradiction with (1.1)(a).

Proof of (1.4)(b). We proceed by contradiction: we assume that for any three-dimensional Abelian variety $S^3(A)$, and therefore $K_3(A)$ contains a three-fold filled by γ -curves: by their pull-back via π we have a three-fold in any A^2 . So we are in a Δ -situation and we can construct a fibration $h \times_{\Delta} h: \mathbf{A} \times_{\Delta} \mathbf{A} \rightarrow \Delta \subset \mathcal{H}_3$ as in §6. Pay attention: now we proceed in a very similar way to the proof of (1.4)(a), but we cannot use Lemma (6.6) in that manner.

We fix an Abelian variety A isogenous to $E \times E \times E$, E generic elliptic curve. Let E_{pqr} be the image in $E \times E \times E$ of the embedding ν_{pqr} of E such that $\nu_{pqr}(x) = (px, qx, rx) \quad \forall x \in E$, (p, q, r) is a triple of coprime integers; let F_{pqr} be $(E \times E \times E)/E_{pqr}$, we fix an isogeny between A and $E_{pqr} \times F_{pqr}$.

By Lemma (6.10) we can assume that, when we project the three-fold T , filled by γ -curves, contained in $K_3(E \times E \times E)$, into $K_3(F_{pqr})$, by the natural projection, we obtain another three-fold $T^{\#}$ with the same property.

Let $\Delta_{pqr} = \Delta \cap [\nu_{pqr}^*(\mathcal{H}_1)]$ the open subset of Δ such that the fibre over $t \in \Delta_{pqr}$ is $A_t \times A_t$ where A_t is isogenous to $E_t \times F_{pqr}$ (A_0 isogenous to A) and E_t is an elliptic curve whose moduli depend on t .

Let φ_t be the natural projection between $K_3(A_t)$ and $K_3(F_{pqr})$, by our assumption there is a three-fold T_t , filled by γ -curves, in every A_t and $T_0 = T$. Moreover $\varphi_0(T_0) = \varphi_0(T) = T^{\#}$ is a three-fold in $K_3(F_{pqr})$ by the previous remarks. Therefore, by choosing a smaller disk, we can assume that $T_t^{\#} = \varphi_t(T_t)$ is three-fold in $K_3(F_{pqr})$.

We can use Lemma (6.6) (and Remark (6.4)), to assure that there exist triples (p, q, r) (for instance with $r = 0$) such that every one-dimensional family $\{C_t\}$ of γ -curves of $K_3(A_t)$ projects into another similar family of $K_3(F_{pqr})$. We choose one of these triples.

Now we consider two cases: if $T_t^\#$ is a variable three-fold in $K_3(F_{pqr})$, by the previous condition, we would get a three-dimensional family of γ -curves in $K_3(F_{pqr})$, but $\dim(F_{pqr}) = 2$ and this is forbidden by (1.2)(c).

If $T_t^\# = T_0^\#$ is a fixed three-fold in $K_3(F_{pqr})$ then, by the previous condition, infinitely many γ -components pass through any point of $T_0^\#$, hence we would have a one-dimensional family of γ -surfaces in $K_3(F_{pqr})$ at least, and this is not possible by (1.2)(b). \square

Proof of (1.4)(c). Firstly we assume that $\dim(A) = 4$ and we proceed by contradiction: we assume that for any four-dimensional Abelian variety A , $S^3(A)$, and therefore $K_3(A)$ contains a γ -curve; by their pullback via π we have a curve in any A^2 . So we have a Δ -situation and then we can construct a fibration $h \times_\Delta h: A \times_\Delta A \rightarrow \Delta \subset \mathcal{H}_4$ as in §6.

We want to use Lemma (6.7) with $D = B = E$, E generic elliptic curve, $p = 2$, $m = 3$; note that (*) is satisfied by Remark (6.4).

We fix an Abelian variety A isogenous to $E \times E \times E \times E$. For any embedding $\nu_i: E^2 \rightarrow E^3$ let $\Delta_i = \Delta \cap [\nu_i^*(\mathcal{H}_2)]$ the open subset of Δ such that the fibre over $r \in \Delta_i$ is $A_r \times A_r$ where A_r is isogenous to $D_r \times [E^3/\nu_i(E^2)] \times E$ (A_0 isogenous to A as usual), D_r is an Abelian surface and D_0 is isogenous to $E \times E$; in this case we set: $B_i = [E^3/\nu_i(E^2)] \times E$. By our assumptions there exists a one-dimensional γ -component C_r in any $K_3(A_r)$.

Now we use Lemma (6.7) and we have that, for all i except a finite number, when we project $K_3(A_r)$ into $K_3(B_i)$, by the natural projection, we get that every one-dimensional family of γ -curves projects into a one-dimensional family of γ -curves.

So we get a three-dimensional family of γ -curves in $K_3(B_i)$; they cannot cover all $K_3(B_i)$ by Theorem (3.2) (recall that B_i is isogenous to E^2); they cannot cover a three-fold, otherwise this three-fold would be filled by γ -surfaces and this is not possible by (1.2)(b); so the only possibility is the following: they all project in a fixed surface S in $K_3(B_i)$, which is a γ -component.

Note that, by Lemma (6.8), we can suppose that S project into a fixed surface S^\wedge when we project $K_3(B_i) = K_3([E^3/\nu_i(E^2)] \times E)$ into $K_3(E)$ by the natural projection on the last factor, hence S^\wedge is $K_3(E)$.

Now we choose $D_r = E_\sigma \times E_s$ (σ, s belonging to the moduli space of elliptic curves) and generic embeddings ν_i in *infinitely many different ways*; for any choice, by using all the previous arguments, we get

- a γ -curve $C_{\sigma,s}$ in any $K_3(E_\sigma \times E_s \times E \times E)$,
 - a surface S_s in $K_3(E_s \times E \times E)$, (S_s covered by γ -curves),
 - a fixed surface S in $K_3(E \times E)$ into which *all* S_s project,
 - a fixed surface S^\wedge in $K_3(E)$ into which S projects,
- (we always use natural projections).

We want to prove that this is a contradiction to Lemma (6.6).

These facts create a situation which is very similar to a Δ -situation, (see §6): actually, in this case, we have only a one-dimensional family of Abelian varieties: $A_s \times A_s = (E_s \times E \times E) \times (E_s \times E \times E)$ and a surface in every $A_s \times A_s$ which is the pullback, via π , of the surface S_s contained in $K_3(E_s \times E \times E)$. So we have a fibration defined only over an open set $\Delta'_i \subset \mathcal{H}_1$ and the surfaces $\{S_s\}$ are the fibres over Δ'_i . However, by looking at the proof of (6.6), it is obvious that it is true even in this case.

But, in our case, we also have that, if we choose A isogenous to $E \times E \times E$, there are *infinitely many embeddings* $\mu_{pq}: E \rightarrow E^2$ ($\forall x \in E \quad \mu_{pq}(x) = (px, qx)$, p, q coprime integers, *the μ_{pq} are induced by the ν_i*) such that, when we choose a family $\{A_s \times A_s\}$, s belonging to a suitable open set $\Delta_{pq} \subset \mathcal{H}_1$, (depending on Δ'_i), such that $\forall s \in \Delta_{pq}$, A_s is isogenous to $E_s \times \mu_{pq}(E) \times E$ (as usual A_0 is isogenous to $[E^2/\mu_{pq}(E)] \times \mu_{pq}(E) \times E$, isogenous to A), then

— $K_3(A_s)$ contains a surface S_s for any s ,

— S_0 projects into a surface S' in $K_3(E)$ by the natural projection on the last factor, (hence condition $(*)$ is satisfied),

—all surfaces S_s project into a fixed surface S in $K_3[\mu_{pq}(E) \times E]$.

This is a contradiction to Lemma (6.6)!

Now we assume that $\dim(A) = n \geq 5$ and we proceed by induction on n . Suppose that for any n -dimensional Abelian variety A , $S^3(A)$, and therefore $K_3(A)$ contains a γ -curve; then it is true for those Abelian n -folds which are isogenous to $E \times B$ where E is a generic elliptic curve and B is a generic Abelian $(n-1)$ -fold. It is easy to see that, by choosing a suitable isogeny, we also get a γ -curve in $K_3(B)$, and this is a contradiction to our induction hypothesis. \square

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