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CHEBYSHEV TYPE ESTIMATES FOR BEURLING GENERALIZED PRIME NUMBERS. II

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ABSTRACT. Let $N(x)$ be the distribution function of the integers in a Beurling generalized prime system. The Chebyshev type estimates for Beurling generalized prime numbers in the general case

$$N(x) = x \sum_{\nu=1}^{n} A_\nu \log^{\rho_\nu - 1} x + O(x \log^{-\gamma} x)$$

is a long standing question. In this paper we shall give an affirmative answer to the question by proving that the Chebyshev type estimates

$$0 < \liminf_{x\to\infty} \frac{\psi(x)}{x}, \quad \limsup_{x\to\infty} \frac{\psi(x)}{x} < \infty$$

hold even under weaker condition

$$\int_1^\infty x^{-1} \left\{ \sup_{x<y} y^{-1} \left| N(y) - y \sum_{\nu=1}^{n} A_\nu \log^{\rho_\nu - 1} y \right| \right\} dx < \infty$$

with $\rho_n = \tau \geq 1$, $0 < \rho_1 < \rho_2 < \cdots < \rho_n$, and $A_n > 0$. This generalizes a result of Diamond and a result of the present author.

1. INTRODUCTION

Let $\psi(x)$ be the weighted counting function of the ordinary prime numbers. Chebyshev was the first to establish the correct order of magnitude of $\psi(x)$ by showing that there exist two numbers $\alpha > 0$ and $\beta < \infty$ such that

$$\liminf_{x\to\infty} \frac{\psi(x)}{x} \geq \alpha, \quad \limsup_{x\to\infty} \frac{\psi(x)}{x} \leq \beta. \tag{1.1}$$

The prime number theorem (P.N.T.) asserts that $\alpha = \beta = 1$. In [4], Diamond established Chebyshev type estimates for Beurling generalized primes. Here we shall generalize Diamond's result.

Let $\mathscr{P} = \{p_i\}_{i=1}^\infty$, where $1 < p_1 \leq p_2 \leq \dots$, $p_i \to \infty$, be a set of Beurling generalized (henceforth, g-) prime numbers and $\mathscr{N} = \{N_i\}_{i=0}^\infty$ be the associated set of g-integers (see [1, 2]). Define

$$N(x) = \sum_{\substack{i \\ n_i \leq x}} 1, \quad \psi(x) = \sum_{\substack{i,\alpha \\ p_i^\alpha \leq x}} \log p_i.$$

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Beurling [2] proved that if

$$N(x) = Ax + O(x \log^{-\gamma} x) \tag{1.2}$$

for some constants $A > 0$ and $\gamma > 3/2$, then the P.N.T. holds for $\mathscr{P}$. If $\gamma = 3/2$ in (1.2), the P.N.T. need not hold as Diamond [4] showed by an example based on a continuous example of Beurling. Diamond [5] also showed that if $\gamma > 1$ in (1.2) then (1.1) holds. On the other hand, (1.1) is not generally true if $\gamma < 1$, as an example of Hall [7] shows.

Beurling investigated also the more general case in which

$$N(x) = x \sum_{\nu=1}^{n} A_\nu \log^{\rho_\nu - 1} x + O(x \log^{-\gamma} x), \tag{1.3}$$

where $\rho_1 < \rho_2 < \cdots < \rho_n$ and $A_1, A_2, \ldots, A_n$ are arbitrary real numbers. He showed that if (1.3) with $1 \le \rho_n = \tau < 2$ holds for some $A_n > 0$ and $\gamma > 1 + \tau/2$ then $\lim_{x\to\infty} \psi(x)/x = \tau$, a generalization of the P.N.T. However, if $\tau \ge 2$, even an $O(1)$ error term in (1.3) does not guarantee $\psi(x) \sim \tau x$. Still, Beurling proved that if (1.3) holds with $\rho_n = \tau \ge 2$ for some $A_n > 0$ and $\gamma > 1 + \tau/2$ then there exist $0 < t_1 \le t_2 \le \cdots \le t_q < \infty$ with $q \le [\tau/2]$ such that

$$\psi(x) \sim x \left\{ \tau - 2 \sum_{\nu=1}^{q} \cos(t_\nu \log x - \operatorname{arctg} t_\nu) \right\}.$$

This gave rise to the long standing question of generalizing Chebyshev type estimates for the case (1.3). In the present paper, we shall prove the following theorem which gives an affirmative answer to the question.

Theorem. *Let $0 < \rho_1 < \rho_2 < \cdots < \rho_n$ and $A_1, A_2, \ldots, A_n$ be arbitrary real numbers. If*

$$\int_1^\infty x^{-1} \left\{ \sup_{x \le y} y^{-1} \left| N(y) - y \sum_{\nu=1}^{n} A_\nu \log^{\rho_\nu - 1} y \right| \right\} dx < \infty \tag{1.4}$$

holds with $\rho_n = \tau \ge 1$ and $A_n = A > 0$, then there exist numbers $\alpha > 0$ and $\beta < \infty$ for which (1.1) *holds.*

This theorem is a generalization of the results in [5, 8]. In particular, it has the following immediate consequence.

Corollary. *If* (1.3) *holds with $\rho_n = \tau \ge 1$ for some $A_n = A > 0$ and $\gamma > 1$, then* (1.1) *is true.*

The proof of the Theorem will utilize Beurling's asymptotic analysis [2], Diamond's approximate convolution inverse [5, 6], and the author's idea in [8]. Elementary convolution techniques [3] play the main role in the proof. It would also be interesting to find an analytic proof.

We remark that (1.1) is not generally true if $\gamma < 1$ in (1.3) as the example of Hall mentioned above shows.

2. Preliminaries

In the proof of the Theorem, we shall make frequent use of multiplicative convolution techniques which have been described in detail in [3]. In particular, we need the following preliminaries.

Let dt denote the Lebesgue measure on the Borel subsets of $[1, \infty)$. Let $\rho > 0$, $0 \le \varepsilon \le 1$. We define

$$(\delta - \varepsilon t^{-\varepsilon}\,dt)^\rho = \delta + \sum_{k=1}^{\infty} \frac{\rho(\rho-1)\cdots(\rho-k+1)}{k!}(-\varepsilon t^{-\varepsilon}\,dt)^k, \tag{2.1}$$

where δ represents point mass 1 at 1. We note that the left-hand side is a noninteger convolution power if ρ is not an integer. Hence,

$$\begin{aligned}(\delta - \varepsilon t^{-\varepsilon}\,dt)^\rho \\ = \delta + \left(\sum_{k=1}^{\infty} \frac{\rho(\rho-1)\cdots(\rho-k+1)}{k!}(-\varepsilon)^k \frac{\log^{k-1} t}{(k-1)!}\right) t^{-\varepsilon}\,dt.\end{aligned} \tag{2.2}$$

Similarly,

$$\begin{aligned}(\delta + (1-\varepsilon) t^{-\varepsilon}\,dt)^\rho \\ = \delta + \sum_{k=1}^{\infty} \frac{\rho(\rho-1)\cdots(\rho-k+1)}{k!}((1-\varepsilon)t^{-\varepsilon}\,dt)^k \\ = \delta + \left(\sum_{k=1}^{\infty} \frac{\rho(\rho-1)\cdots(\rho-k+1)}{k!}(1-\varepsilon)^k \frac{\log^{k-1} t}{(k-1)!}\right) t^{-\varepsilon}\,dt.\end{aligned} \tag{2.3}$$

We note that the sum on the right-hand side of (2.2) and (2.3) has only a finite number of terms and is easy to deal with if ρ is a positive integer. In the discussion below, special analysis will be required if ρ is not a positive integer.

By the exponential representation [3], it is easy to prove the following formulas

$$(\delta - \varepsilon t^{-\varepsilon}\,dt)^\rho = \exp\left\{-\rho \frac{1 - t^{-\varepsilon}}{\log t}\,dt\right\}$$

and

$$(\delta + (1-\varepsilon) t^{-\varepsilon}\,dt)^\rho = \exp\left\{\rho t^{-\varepsilon} \frac{1 - t^{-(1-\varepsilon)}}{\log t}\,dt\right\};$$

which are generalizations of (3.2d) in [3]. Therefore,

$$\begin{aligned}(\delta - \varepsilon t^{-\varepsilon}\,dt)^{\rho_1+\rho_2} = (\delta - \varepsilon t^{-\varepsilon}\,dt)^{\rho_1} * (\delta - \varepsilon t^{-\varepsilon}\,dt)^{\rho_2}, \\ (\delta + dt)^\rho * (\delta - \varepsilon t^{-\varepsilon}\,dt)^\rho = (\delta + (1-\varepsilon) t^{-\varepsilon}\,dt)^\rho,\end{aligned} \tag{2.4}$$

and

$$(\delta + dt)^\rho * (\delta - t^{-1}\,dt)^\rho = \delta. \tag{2.5}$$

Moreover, we have

$$(\delta + dt)^\rho * (\delta - \varepsilon t^{-\varepsilon}\,dt)^\tau = (\delta + (1-\varepsilon) t^{-\varepsilon}\,dt)^\rho * (\delta - \varepsilon t^{-\varepsilon}\,dt)^{\tau-\rho} \tag{2.6}$$

if $\tau > \rho$.

We then consider the Mellin transform

$$\widehat{F}(s) := \int_1^\infty x^{-s}(\delta - \varepsilon x^{-\varepsilon}\,dx)^\rho$$

(following [3], in this paper, an integral $\int_x^y$ means $\int_{x+}^{y+}$ except when $x = 1$, in which case we take the lower limit to be $1-$), which is convergent in the half-plane $\operatorname{Re} s = \sigma > 1$ if ρ is not a positive integer. (Actually, it is easy to show $|\int_1^x(\delta-\varepsilon t^{-\varepsilon}\,dt)^\rho| = O(x)$; see the proof of Lemma 4.5.) By (2.1), we have

$$\begin{aligned}\widehat{F}(s) &= 1+\sum_{k=1}^{\infty}\frac{\rho(\rho-1)\cdots(\rho-k+1)}{k!}\left(-\varepsilon\int_1^\infty x^{-s-\varepsilon}\,dx\right)^k\\ &= 1+\sum_{k=1}^{\infty}\frac{\rho(\rho-1)\cdots(\rho-k+1)}{k!}\left(\frac{-\varepsilon}{s-(1-\varepsilon)}\right)^k\\ &= \exp\left\{\rho\log\left(1-\frac{\varepsilon}{s-(1-\varepsilon)}\right)\right\} = \left(1-\frac{\varepsilon}{s-(1-\varepsilon)}\right)^\rho,\end{aligned}$$

where $\log z = \log|z| + i\arg z$ with $-\pi < \arg z \le \pi$. Therefore

$$\int_1^\infty x^{-s}(\delta-\varepsilon x^{-\varepsilon}\,dx)^\rho = \frac{(s-1)^\rho}{(s-(1-\varepsilon))^\rho} \quad \text{for } \sigma>1. \tag{2.7}$$

Similarly, by (2.3), we have

$$\int_1^\infty x^{-s}(\delta+(1-\varepsilon)x^{-\varepsilon}\,dx)^\rho = \frac{s^\rho}{(s-(1-\varepsilon))^\rho} \tag{2.8}$$

for $\sigma > 1-\varepsilon$. Furthermore,

$$\int_1^\infty x^{-s}\log x\;(\delta+dx)^\rho = \rho\left(\frac{s^\rho}{(s-1)^{\rho+1}}-\frac{s^{\rho-1}}{(s-1)^\rho}\right).$$

3. Asymptotic analysis

Instead of $x\sum_{\nu=1}^n A_\nu \log^{\rho_\nu-1}x$ on the right-hand side of (1.3) (or (1.4)), we shall use $\sum_{\mu=1}^m B_\mu\int_1^x(\delta+dt)^{\tau_\mu}$ which is easy to deal with by using convolution techniques. We first show that the latter is a good approximation of the former by using Beurling's idea [2].

Lemma 3.1. *Let $\rho>0$, $0\le\varepsilon<1$. If $\rho=r$ is an integer, then, for $x>1$,*

$$\begin{aligned}\int_1^x(\delta+(1-\varepsilon)t^{-\varepsilon}\,dt)^r &= \frac{(1-\varepsilon)^{r-1}x^{1-\varepsilon}(\log x)^{r-1}}{\Gamma(r)}\\ &+\sum_{k=1}^{r-1}\frac{(r-1)(r-2)\cdots(r-k)}{k!\,\Gamma(r-k)}(1-\varepsilon)^{r-k-1}x^{1-\varepsilon}(\log x)^{r-k-1}.\end{aligned}\tag{3.1}$$

If ρ is not an integer, then, for any positive integer m,

$$\begin{aligned}\int_1^x(\delta+(1-\varepsilon)t^{-\varepsilon}\,dt)^\rho &= \frac{(1-\varepsilon)^{\rho-1}x^{1-\varepsilon}(\log x)^{\rho-1}}{\Gamma(\rho)}\\ &+\sum_{k=1}^{m-1}\frac{(\rho-1)(\rho-2)\cdots(\rho-k)}{k!\,\Gamma(\rho-k)}(1-\varepsilon)^{\rho-k-1}x^{1-\varepsilon}(\log x)^{\rho-k-1}\\ &+O_m(x^{1-\varepsilon}(\log x)^{\rho-m-1})\end{aligned}\tag{3.2}$$

holds as $x\to\infty$, where the O_m-constant is uniform for ε satisfying $0\le\varepsilon\le\varepsilon_0$ for each fixed $\varepsilon_0<1$.

Proof. It suffices to prove (3.1) and (3.2) with $m\ge\rho$. The method of our proof is standard. We consider the Mellin transform

$$\widehat{F}(s) = \int_1^\infty x^{-s}(\delta+(1-\varepsilon)x^{-\varepsilon}\,dx)^\rho = \frac{s^\rho}{(s-1+\varepsilon)^\rho}$$

for $\sigma > 1-\varepsilon$. Therefore, for $x > 1$,

$$\int_1^x (\delta + (1-\varepsilon)t^{-\varepsilon}\,dt)^\rho = \frac{1}{2\pi i}\int_{\sigma=\sigma_0} \frac{x^s s^{\rho-1}}{(s-1+\varepsilon)^\rho}\,ds, \tag{3.3}$$

where $\sigma_0 > 1-\varepsilon$, by Perron's inversion formula.

In the case $\rho = r$, the right-hand side of (3.3) equals

$$\begin{aligned}\sum_{k=0}^{r-1}\binom{r-1}{k}&(1-\varepsilon)^{r-k-1}\frac{1}{2\pi i}\int_{\sigma=\sigma_0}\frac{x^s}{(s-1+\varepsilon)^{r-k}}\,ds\\ &=\sum_{k=0}^{r-1}\binom{r-1}{k}(1-\varepsilon)^{r-k-1}\frac{1}{\Gamma(r-k)}x^{1-\varepsilon}(\log x)^{r-k-1}\end{aligned}$$

and (3.1) follows.

Hence, in the sequel, we assume that ρ is not an integer. As usual, we can shift the integration contour of the integral on the right-hand side of (3.3) to a loop, denoted by $l_\eta(1-\varepsilon)$, which consists of the half-line on the lower edge of the real axis from $-\infty$ to $1-\varepsilon-\eta$, the circle $C_\eta(1-\varepsilon)$, cut at the point $s = 1-\varepsilon-\eta$, with center $s = 1-\varepsilon$ and radius η sufficiently small, and the half-line on the upper edge of the real axis from $1-\varepsilon-\eta$ to $-\infty$. Thus we have

$$\int_1^x (\delta + (1-\varepsilon)t^{-\varepsilon}\,dt)^\rho = \frac{1}{2\pi i}\int_{l_\eta(1-\varepsilon)} \frac{x^s s^{\rho-1}}{(s-1+\varepsilon)^\rho}\,ds. \tag{3.4}$$

Let ε_0 be fixed and $0 \le \varepsilon_0 < 1$. Assume $0 \le \varepsilon \le \varepsilon_0$. Let $\alpha = \frac{1}{2}(1-\varepsilon_0)$. Let $0 < \eta < \alpha$ and $l'_\eta(1-\varepsilon) = l_\eta(1-\varepsilon) \cap \{|s-1+\varepsilon| \le 1-\varepsilon-\alpha\}$. Let $l''_\eta(1-\varepsilon)$ denote the remaining part of $l_\eta(1-\varepsilon)$. Then it is easy to see that

$$\int_{l''_\eta(1-\varepsilon)} \frac{x^s s^{\rho-1}}{(s-1+\varepsilon)^\rho}\,ds = O(x^\alpha/\log x), \tag{3.5}$$

where, and thereafter without repeat, the 0-constant depends only on ε_0 and is uniform for ε satisfying $0 \le \varepsilon \le \varepsilon_0$. In the disk $\{|s-1+\varepsilon| \le 1-\varepsilon-\alpha\}$, we have

$$s^{\rho-1} = (1-\varepsilon)^{\rho-1} + \sum_{k=1}^{m-1}\frac{(\rho-1)(\rho-2)\cdots(\rho-k)}{k!}(1-\varepsilon)^{\rho-k-1}(s-1+\varepsilon)^k + R(s)$$

where

$$R(s) = O(|s-1+\varepsilon|^m).$$

Therefore,

$$\begin{aligned}\frac{1}{2\pi i}\int_{l'_\eta(1-\varepsilon)}\frac{x^s s^{\rho-1}}{(s-1+\varepsilon)^\rho}\,ds &= \frac{(1-\varepsilon)^{\rho-1}}{2\pi i}\int_{l'_\eta(1-\varepsilon)}\frac{x^s}{(s-1+\varepsilon)^\rho}\,ds\\ &\quad+\sum_{k=1}^{m-1}\frac{(\rho-1)(\rho-2)\cdots(\rho-k)}{k!}(1-\varepsilon)^{\rho-k-1}\\ &\quad\cdot\frac{1}{2\pi i}\int_{l'_\eta(1-\varepsilon)} x^s(s-1+\varepsilon)^{k-\rho}\,ds\\ &\quad+\frac{1}{2\pi i}\int_{l'_\eta(1-\varepsilon)}\frac{x^s R(s)}{(s-1+\varepsilon)^\rho}\,ds.\end{aligned} \tag{3.6}$$

We denote by $l_\eta(0)$ and $l'_\eta(0)$ the respective translations of $l_\eta(1-\varepsilon)$ and $l'_\eta(1-\varepsilon)$ via $w = s-1+\varepsilon$. It is easy to see that

$$\begin{aligned}\frac{1}{2\pi i}\int_{l'_\eta(1-\varepsilon)} \frac{x^s}{(s-1+\varepsilon)^\rho}\,ds &= \frac{x^{1-\varepsilon}}{2\pi i}\int_{l'_\eta(0)} e^{w\log x} w^{-\rho}\,dw \\ &= x^{1-\varepsilon}\log^{\rho-1} x\left(\frac{1}{2\pi i}\int_{l'_\eta(0)} e^w w^{-\rho}\,dw + O\left(\frac{x^{-1+\varepsilon+\alpha}}{\log^\rho x}\right)\right) \\ &= \frac{x^{1-\varepsilon}\log^{\rho-1} x}{\Gamma(\rho)} + O(x^\alpha \log^{-1} x).\end{aligned} \tag{3.7}$$

In the same way we have

$$\frac{1}{2\pi i}\int_{l'_\eta(1-\varepsilon)} x^s(s-1+\varepsilon)^{k-\rho}\,ds = \frac{x^{1-\varepsilon}(\log x)^{\rho-k-1}}{\Gamma(\rho-k)} + O(x^\alpha\log^{-1} x). \tag{3.8}$$

Finally, we have

$$\begin{aligned}\int_{l'_\eta(1-\varepsilon)} \frac{x^s R(s)}{(s-1+\varepsilon)}\,ds &= O\left(\int_\alpha^{1-\varepsilon} x^\sigma|\sigma-1+\varepsilon|^{m-\rho}\,d\sigma\right) \\ &= O(x^{1-\varepsilon}(\log x)^{\rho-m-1}).\end{aligned} \tag{3.9}$$

Now, (3.2) follows from (3.4) to (3.9). □

Lemma 3.2. *Let* $\rho > 0$, $0 < \varepsilon \le 1$. *If* $\rho = r$ *is an integer, then*

$$\begin{aligned}&\int_1^x (\delta - \varepsilon t^{-\varepsilon}\,dt)^\rho \\ &= x^{1-\varepsilon}\sum_{n=0}^{r-1}\frac{(-1)^{n-1}}{n!}\left(\sum_{k=n+1}^{r}\binom{r}{k}\frac{\varepsilon^k}{(1-\varepsilon)^{k-n}}\right)\log^n x + \frac{1}{(1-\varepsilon)^r}.\end{aligned} \tag{3.10}$$

If ρ *is not an integer, then, for any positive integer* m,

$$\begin{aligned}&\int_1^x (\delta - \varepsilon t^{-\varepsilon}\,dt)^\rho \\ &= x\sum_{n=0}^{m-1}(-1)^{n-1}\pi^{-1}\sin\pi(\rho+n)\Gamma(\rho+n+1) \\ &\quad\cdot\left(\sum_{k=0}^{n}\frac{\rho(\rho+1)\cdots(\rho+k-1)}{k!}\varepsilon^{-k-\rho}\right)(\log x)^{-\rho-n-1} \\ &\quad + O_m(x(\log x)^{-\rho-m-1})\end{aligned} \tag{3.11}$$

holds as $x\to\infty$ (*where the* O_m*-constant is uniform for* ε *satisfying* $\varepsilon_0 \le \varepsilon \le 1$ *for each fixed* $\varepsilon_0 > 0$).

Proof. The proof is very close to the one of Lemma 3.1 and hence a sketch of it will be sufficient. The Mellin transform

$$\widehat{F}(s) := \int_1^\infty x^{-s}(\delta - \varepsilon x^{-\varepsilon}\,dx)^\rho$$

is convergent in the half-plane $\operatorname{Re} s = \sigma > 1-\varepsilon$ if $\rho = r$ is an integer. Hence,

$$\widehat{F}(s) = \left(1 - \frac{\varepsilon}{s-1+\varepsilon}\right)^r = \frac{(s-1)^r}{(s-1+\varepsilon)^r}. \tag{3.12}$$

Therefore, by Perron's inversion formula,

$$\begin{aligned}\int_1^x (\delta - \varepsilon t^{-\varepsilon}\, dt)^\rho &= \frac{1}{2\pi i}\int_{\sigma=\sigma_0} \frac{x^s}{s}\frac{(s-1)^r}{(s-1+\varepsilon)^r}\, ds \\ &= \sum_{k=0}^{r}\binom{r}{k}(-\varepsilon)^k \frac{1}{2\pi i}\int_{\sigma=\sigma_0}\frac{x^s}{s}\frac{1}{(s-1+\varepsilon)^k}\, ds \\ &= \sum_{k=0}^{r}\binom{r}{k}(-\varepsilon)^k\left(x^{1-\varepsilon}\sum_{\substack{m+n=k-1\\ m,n\geq 0}}\frac{(-1)^m\log^n x}{(1-\varepsilon)^{m+1}n!}+\frac{(-1)^k}{(1-\varepsilon)^k}\right),\end{aligned}$$

where $\sigma_0 > 1-\varepsilon$, and (3.10) follows.

If ρ is not an integer, then $\widehat{F}(s)$ is convergent in the half-plane $\sigma > 1$ and (3.12) holds for $\sigma > 1$. Therefore, we have

$$\int_1^x (\delta - \varepsilon t^{-\varepsilon}\, dt)^\rho = \frac{1}{2\pi i}\int_{\sigma=\sigma_0}\frac{x^s}{s}\frac{(s-1)^\rho}{(s-1+\varepsilon)^\rho}\, ds$$

where $\sigma_0 > 1$. Then we shift the integration contour to a new one, denoted by $l_\eta(1)$ with $0 < \eta < \varepsilon_0/4$, which consists of the half-lines on the lower edge of the real axis from $-\infty$ to $-\eta$ and on the upper edge from $-\eta$ to $-\infty$, the circle C, cut at points $s = -\eta$ and $s = 1-\varepsilon+\eta$, with center $s = \frac{1}{2}(1-\varepsilon)$ and radius $\frac{1}{2}(1-\varepsilon)+\eta$, the two line segments on the lower edge of the real axis from $1-\varepsilon+\eta$ to $1-\eta$ and on the upper edge from $1-\eta$ to $1-\varepsilon+\eta$, and the small circle $c_\eta(1)$, cut at the point $s = 1-\eta$, with center $s = 1$ and radius η. Let $l'_\eta(1) = l_\eta(1)\cap\{|s-1|\leq \varepsilon_0/2\}$. Then, it is easy to see that

$$\begin{aligned}\int_1^x (\delta - \varepsilon t^{-\varepsilon}\, dt)^\rho &= \frac{1}{2\pi i}\int_{l_\eta(1)}\frac{x^s}{s}\frac{(s-1)^\rho}{(s-1+\varepsilon)^\rho}\, ds \\ &= \frac{1}{2\pi i}\int_{l'_\eta(1)}\frac{x^s}{s}\frac{(s-1)^\rho}{(s-1+\varepsilon)^\rho}\, ds + O_{\varepsilon_0}(x^{1-\varepsilon_0/2}\log^{-1} x)\end{aligned} \tag{3.13}$$

and

$$\begin{aligned}&\frac{1}{2\pi i}\int_{l'_\eta(1)}\frac{x^s}{s}\frac{(s-1)^\rho}{(s-1+\varepsilon)^\rho}\, ds \\ &\quad = \sum_{n=0}^{m}(-1)^n\left(\sum_{k=0}^{n}\frac{\rho(\rho+1)\cdots(\rho+k-1)}{k!}\varepsilon^{-k-\rho}\right) \\ &\qquad \cdot\frac{1}{2\pi i}\int_{l'_\eta(1)} x^s(s-1)^{\rho+n}\, ds \\ &\qquad + \frac{1}{2\pi i}\int_{l'_\eta(1)} x^s(s-1)^\rho R(s)\, ds\end{aligned} \tag{3.14}$$

where

$$R(s) = O_{m,\varepsilon_0}(|s-1|^{m+1}).$$

Evaluating the integrals on the right-hand side of (3.14) as we did in the proof of Lemma 3.1, we arrive at (3.11). □

The following lemma is the main result of this section.

Lemma 3.3. *Let* $0 < \rho_1 < \rho_2 < \cdots < \rho_n$. *Then there exist* $0 < \tau_1 < \tau_2 < \cdots < \tau_m$ *with* $\tau_m = \rho_n$ *and* $\tau_\mu = \rho_\nu - k$ *for some nonnegative integer* k *such that, as* $x \to \infty$,

$$x\sum_{\nu=1}^{n} A_\nu (\log x)^{\rho_\nu - 1} = \sum_{\mu=1}^{m} B_\mu \int_1^x (\delta + dt)^{\tau_\mu} + R(x), \tag{3.15}$$

where $B_m = A_n\Gamma(\rho_n)$ *and* $R(x) = O(x(\log x)^{-2+\alpha})$ *with* $\alpha = \max\{\rho_\nu - [\rho_\nu],\ \nu = 1, 2, \ldots, n\}$ *if* ρ_ν, $\nu = 1, 2, \ldots, n$, *are not all integers and* $R(x) = 0$ *otherwise.*

Proof. Without loss of generality, we may assume $n = 1$. We shall prove (3.15) by induction. We first consider the case that ρ_1 is not an integer. For $0 < \rho_1 < 1$, we have, by (3.2) with $\varepsilon = 0$, $m = 1$, and $\rho = \rho_1$,

$$A_1 x(\log x)^{\rho_1 - 1} = A_1\Gamma(\rho_1)\int_1^x (\delta + dt)^{\rho_1} + O(x(\log x)^{-2+\rho_1})$$

and (3.15) is true. Then, for $k < \rho_1 < k+1$ with $k \geq 1$, by (3.2) with $\varepsilon = 0$, $m = [\rho_1] + 1$, and $\rho = \rho_1$, we have

$$\begin{aligned} A_1 x(\log x)^{\rho_1 - 1} = {} & A_1\Gamma(\rho_1)\int_1^x (\delta + dt)^{\rho_1} \\ & - \sum_{\mu=1}^{[\rho_1]} \overline{B}_\mu x(\log x)^{\rho_1 - \mu - 1} + O(x(\log x)^{-2+\rho_1-[\rho_1]}). \end{aligned}$$

Note that, for $\mu = 1, 2, \ldots, [\rho_1]$, $\rho_1 - \mu$ are all nonintegers, $\rho_1 - \mu < k$, and $\rho_1 - \mu - [\rho_1 - \mu] = \rho_1 - [\rho_1]$. Therefore, by hypothesis of induction, we have

$$\begin{aligned} A_1 x(\log x)^{\rho_1 - 1} = {} & A_1\Gamma(\rho_1)\int_1^x (\delta + dt)^{\rho_1} \\ & + \sum_{\mu=1}^{[\rho_1]} B_\mu \int_1^x (\delta + dt)^{\tau_\mu} + O(x(\log x)^{-2+\rho_1-[\rho_1]}) \end{aligned}$$

and (3.15) holds.

Similarly, by using (3.1), we can prove the truth of (3.15) when ρ_1 is an integer. □

4. Approximate convolution inverse

Let

$$P(x) = \sum_{\mu=1}^{m} B_\mu \int_1^x (\delta + dt)^{\tau_\mu} \quad \text{for } x \geq 1 \tag{4.1}$$

which is defined in (3.15) of Lemma 3.3, and $P(x) = 0$ for $x < 1$. Define a function $Q_1(x)$ as follows. If τ_m and $\tau_m - \tau_\mu$, $\mu = 1, 2, \ldots, m-1$, are all integers, let

$$Q_1(x) = \max\left\{\sup_{x \leq y} \frac{|N(y) - P(y)|}{y},\ x^{-1/2}\right\}. \tag{4.2$_1$}$$

If $\tau_m - \tau_\mu$, $\mu = 1, 2, \dots, m-1$, are all integers but τ_m is not, let

$$(4.2)_2 \qquad Q_1(x) = \max\left\{\sup_{x\le y}\frac{|N(y)-P(y)|}{y}, (\log ex)^{-1-\tau_m+[\tau_m]}\right\}.$$

If τ_m is an integer but $\tau_m - \tau_\mu$, $\mu = 1, 2, \dots, m-1$, are not all integers, let

$$(4.2)_3 \qquad Q_1(x) = \max\left\{\sup_{x\le y}\frac{|N(y)-P(y)|}{y}, (\log ex)^{-1-\beta}, (\log ex)^{-3/2+\gamma/2}\right\},$$

where

$$(4.3) \qquad \beta = \min\{\tau_m - \tau_\mu - k : \tau_m - k - 1 < \tau_\mu < \tau_m - k, 1 \le \mu \le m-1\}$$

and

$$(4.4) \qquad \gamma = \max\{\tau_\mu - \tau_m + k + 1 : \tau_m - k - 1 < \tau_\mu < \tau_m - k, 1 \le \mu \le m-1\}$$

so that $0 < \beta < 1$, $0 < \gamma < 1$. Finally, if τ_m is not an integer and $\tau_m - \tau_\mu$, $\mu = 1, 2, \dots, m-1$, are not all integers, let

$$(4.2)_4 \qquad Q_1(x) = \max\left\{\sup_{x\le y}\frac{|N(y)-P(y)|}{y}, (\log ex)^{-1-\tau_m+[\tau_m]}, (\log ex)^{-1-\beta}, (\log ex)^{-3/2+\gamma/2}\right\}.$$

Lemma 4.1. *Let* $Q_1(x)$ *be the function defined in* (4.2). *Assume* (1.4). *Then there exists a function* $Q(x)$ *defined for* $x \ge 0$ *such that*

$$(4.5) \qquad Q(x) \text{ is nonincreasing};$$

$$(4.6) \qquad \int_1^\infty Q(x)x^{-1}\,dx < \infty;$$

$$(4.7) \qquad Q(x) \le 4Q(x^2) \quad \text{for all } x \ge 1;$$

$$(4.8) \qquad Q(x) \ge Q_1(x).$$

Moreover,

$$(4.9) \qquad Q(x) = o(\log^{-1} ex);$$

for $x \ge 1$,

$$(4.10) \qquad \int_1^x xt^{-1}Q(x/t)Q(t)\,dt \le C_1 \min\left\{xQ(x), \int_1^x Q(t)\,dt\right\},$$

where C_1 *is a constant.*

Proof. By Lemma 3.3, we have

$$(4.11) \qquad \sup_{x\le y}\frac{|N(y)-P(y)|}{y} \le \sup_{x\le y} y^{-1}\left|N(y) - \sum_{\nu=1}^n A_\nu(\log y)^{\rho_\nu - 1}\right| + R(x)$$

where $R(x) = O((\log x)^{-2+\alpha})$ with $\alpha = \max\{\rho_\nu - [\rho_\nu]\}$ if ρ_ν, $\nu = 1, \dots, n$, are not all integers and $R(x) = 0$ otherwise. Therefore, by (1.4), (4.2), and (4.11),

$$\int_1^\infty Q_1(x)x^{-1}\,dx < \infty.$$

Moreover, $Q_1(x)$ is nonnegative and nonincreasing. Define $Q(x)$ recursively by setting

$$Q(x)=\begin{cases} Q_1(1) & \text{for } 0\le x<2\,,\\ \max\{Q_1(2^{2^{m-1}}),\,4^{-1}Q(2^{2^{m-1}})\} & \text{for } 2^{2^{m-1}}<x\le 2^{2^m}\,,\ m\in\mathbf{N}\,.\end{cases}$$

Then we can verify that this function satisfies condition (4.5) to (4.9) as we did in the proof of Lemmas 1 and 2 of [8].

Moreover, from Lemma 3 of [8], we have, for $x\ge 1$,

$$\int_1^x xt^{-1}Q(x/t)Q(t)\,dt\le C_1'xQ(x)\,. \tag{4.12}$$

If $x\le 2$, then

$$\int_1^x xt^{-1}Q(x/t)Q(t)\,dt\le C_2'\int_1^x Q(t)\,dt\,.$$

If $x>2$, then, by (4.12),

$$\int_1^x xt^{-1}Q(x/t)Q(t)\,dt\le C_1'xQ(x)\le 2C_1'\int_1^x Q(t)\,dt$$

since $Q(x)$ is nonincreasing. This proves (4.10). □

Lemma 4.2. *Let $0<\rho<1$, $\varepsilon>0$, and $T>1$. Let*

$$\alpha(x)=\alpha_\rho(x)=\begin{cases}\dfrac{x(\log ex)^{\rho-1}}{\Gamma(\rho)}\,, & \text{if } x\ge T\,,\\ 0\,, & \text{if } x<T\,.\end{cases}$$

Then, for $x>T$,

$$\left|\int_1^x d\alpha(t)*(\delta-\varepsilon t^{-\varepsilon}\,dt)\right|\le\frac{T^\varepsilon}{(\log eT)^{(1-\rho)/2}\Gamma(\rho)}\cdot\left\{x^{1-\varepsilon}+\int_1^x\left(\frac{x}{t}\right)^{1-\varepsilon}(\log et)^{-3/2+\rho/2}\,dt\right\}. \tag{4.13}$$

Proof. We have

$$\begin{aligned}I(x)&:=\int_1^x d\alpha(t)*(\delta-\varepsilon t^{-\varepsilon}\,dt)\\ &=\alpha(x)-\varepsilon x^{1-\varepsilon}\int_1^x\alpha(t)t^{-2+\varepsilon}\,dt\end{aligned}$$

and

$$\begin{aligned}\int_1^x\alpha(t)t^{-2+\varepsilon}\,dt&=\frac{1}{\Gamma(\rho)}\int_T^x t^{-1+\varepsilon}(\log et)^{\rho-1}\,dt\\ &=\frac{1}{\varepsilon\Gamma(\rho)}\left\{x^\varepsilon(\log ex)^{\rho-1}-T^\varepsilon(\log eT)^{\rho-1}\right.\\ &\qquad\left.+(1-\rho)\int_T^x t^{-1+\varepsilon}(\log et)^{\rho-2}\,dt\right\}\end{aligned}$$

by integration by parts. Hence

$$\begin{aligned}|I(x)|&=\frac{1}{\Gamma(\rho)}\left|\frac{T^\varepsilon x^{1-\varepsilon}}{(\log eT)^{1-\rho}}-(1-\rho)\int_T^x\left(\frac{x}{t}\right)^{1-\varepsilon}(\log et)^{\rho-2}\,dt\right|\\ &\le\frac{T^\varepsilon}{\Gamma(\rho)(\log eT)^{(1-\rho)/2}}\left\{x^{1-\varepsilon}+\int_1^x\left(\frac{x}{t}\right)^{1-\varepsilon}(\log et)^{-3/2+\rho/2}\,dt\right\}.\ \square\end{aligned}$$

Lemma 4.3. *Let* $0 < \tau_1 < \tau_2 < \cdots < \tau_m$ *and* $0 < \varepsilon \le 1/2$. *Assume* $\tau_m = \tau \ge 1$. *If* $\tau - \tau_\mu$ *is a positive integer then*

$$\left|\int_1^x (\delta + dt)^{\tau_\mu} * (\delta - \varepsilon t^{-\varepsilon}\, dt)^\tau\right| \le \int_1^x (\delta + (1-\varepsilon)t^{-\varepsilon}\, dt)^{\tau-1} + 2\varepsilon \int_1^x (\delta + (1-\varepsilon)t^{-\varepsilon}\, dt)^\tau . \tag{4.14}$$

If $\tau - \tau_\mu$ *is not an integer and* $\tau_\mu > \tau - [\tau]$, *then for any given* $T > 1$, *there exists a constant* C_2 *dependent on* T *but independent of* ε *for which*

(4.15)

$$\begin{aligned}
&\left|\int_1^x (\delta + dt)^{\tau_\mu} * (\delta - \varepsilon t^{-\varepsilon}\, dt)^\tau\right| \\
&\quad\le C_2\left\{\int_1^x \frac{x}{t}Q(x/t)(\delta + (1-\varepsilon)t^{-\varepsilon}\, dt)^{\tau-1} + \int_1^x (\delta + (1-\varepsilon)t^{-\varepsilon}\, dt)^{\tau-1}\right\} \\
&\qquad + 2\varepsilon C_2\left\{\int_1^x xt^{-1}Q(x/t)(\delta + (1-\varepsilon)t^{-\varepsilon}\, dt)^\tau + \int_1^x (\delta + (1-\varepsilon)t^{-\varepsilon}\, dt)^\tau\right\} \\
&\qquad + \frac{T^\varepsilon}{C_3(\log eT)^{(1-\gamma)/2}}\left\{\int_1^x (\delta + (1-\varepsilon)t^{-\varepsilon}\, dt)^\tau \right. \\
&\qquad\qquad \left. + \int_1^x (\delta + (1-\varepsilon)t^{-\varepsilon}\, dt)^\tau * (\log et)^{-3/2+\gamma/2}\, dt\right\},
\end{aligned}$$

where $Q(x)$ *and* γ *are defined in Lemma* 4.1 *and in* (4.4) *respectively,* $C_3 = \min\{\Gamma(\tau_\mu - \tau + k + 1) : \tau - k - 1 < \tau_\mu < \tau - k,\ 1 \le \mu \le m-1\}$.

Proof. If $\tau - \tau_\mu$ is a positive integer, $\tau_\mu = \tau - k - 1$. Then we have, by (2.6),

$$\begin{aligned}
&\left|\int_1^x (\delta + dt)^{\tau_\mu} * (\delta - \varepsilon t^{-\varepsilon}\, dt)^\tau\right| \\
&\quad = \left|\int_1^x (\delta + (1-\varepsilon)t^{-\varepsilon}\, dt)^{\tau-k-1} * (\delta - \varepsilon t^{-\varepsilon}\, dt)^{k+1}\right| \\
&\quad \le \int_1^x (\delta + (1-\varepsilon)t^{-\varepsilon}\, dt)^{\tau-k-1} \\
&\qquad * \left(\delta + 2\varepsilon \sum_{l=1}^{k+1} \binom{k+1}{l} (1-\varepsilon)^l \frac{\log^{l-1} t}{(l-1)!} t^{-\varepsilon}\, dt\right) \\
&\quad \le \int_1^x (\delta + (1-\varepsilon)t^{-\varepsilon}\, dt)^{\tau-k-1} + 2\varepsilon \int_1^x (\delta + (1-\varepsilon)t^{-\varepsilon}\, dt)^\tau .
\end{aligned}$$

This proves (4.14).

If $\tau - \tau_\mu$ is not an integer and $\tau_\mu > \tau - [\tau]$, then there exists a nonnegative integer k such that $0 \le \tau - k - 1 < \tau_\mu < \tau - k$ since $\tau \ge 1$. Let

$$\alpha(x) = \begin{cases} \dfrac{x(\log ex)^{\tau_\mu - \tau + k}}{\Gamma(\tau_\mu - \tau + k + 1)} & \text{for } x \ge T, \\ 0 & \text{for } x < T. \end{cases}$$

Then we have

$$
\begin{aligned}
&\int_1^x (\delta + dt)^{\tau_\mu} * (\delta - \varepsilon t^{-\varepsilon}\, dt)^{\tau} \\
&= \int_1^x (\delta + (1-\varepsilon)t^{-\varepsilon}\, dt)^{\tau-k-1} * d\alpha(t) * (\delta - \varepsilon t^{-\varepsilon}\, dt)^{k+1} \\
&\quad + \int_1^x (\delta + (1-\varepsilon)t^{-\varepsilon}\, dt)^{\tau-k-1} \\
&\quad * \left(d\frac{t(\log et)^{\tau_\mu-\tau+k}}{\Gamma(\tau_\mu-\tau+k+1)} - d\alpha(t) \right) * (\delta - \varepsilon t^{-\varepsilon}\, dt)^{k+1} \qquad (4.16) \\
&\quad + \int_1^x (\delta + (1-\varepsilon)t^{-\varepsilon}\, dt)^{\tau-k-1} \\
&\quad * \left((\delta + dt)^{\tau_\mu-\tau+k+1} - d\frac{t(\log et)^{\tau_\mu-\tau+k}}{\Gamma(\tau_\mu-\tau+k+1)} \right) \\
&\quad * (\delta - \varepsilon t^{-\varepsilon}\, dt)^{k+1} \\
&= \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3 ,
\end{aligned}
$$

say. By Lemma 3.1 with $\varepsilon = 0$ and Lemma 4.1,

$$
\begin{aligned}
&\left| \int_1^x (\delta + dt)^{\tau_\mu-\tau+k+1} - \frac{x(\log ex)^{\tau_\mu-\tau+k}}{\Gamma(\tau_\mu-\tau+k+1)} \right| \\
&\qquad \le C_1' x(\log x)^{\tau_\mu-\tau+k-1} \le C_2' x Q(x) ,
\end{aligned}
$$

since $\tau - \tau_\mu - k \ge \beta$, where β is defined in (4.3). Therefore,

$$
\begin{aligned}
|\mathrm{I}_3| &\le C_2' \int_1^x xt^{-1}Q(x/t)(\delta + (1-\varepsilon)t^{-\varepsilon}\, dt)^{\tau-k-1} * (\delta + \varepsilon t^{-\varepsilon}\, dt)^{k+1} \\
&\le C_2' \int_1^x xt^{-1}Q(x/t)(\delta + (1-\varepsilon)t^{-\varepsilon}\, dt)^{\tau-k-1} \qquad (4.17) \\
&\quad + 2\varepsilon C_2' \int_1^x xt^{-1}Q(x/t)(\delta + (1-\varepsilon)t^{-\varepsilon}\, dt)^{\tau} .
\end{aligned}
$$

Plainly, $(\delta + (1-\varepsilon)t^{-\varepsilon}\, dt)^{\tau-k-1}$ is a nonnegative measure since $\tau - k - 1 \ge 0$ and we have

$$
\begin{aligned}
|\mathrm{I}_2| &\le C_3' \int_1^x (\delta + (1-\varepsilon)t^{-\varepsilon}\, dt)^{\tau-k-1} * (\delta + \varepsilon t^{-\varepsilon}\, dt)^{k+1} \\
&\le C_3' \int_1^x (\delta + (1-\varepsilon)t^{-\varepsilon}\, dt)^{\tau-k-1} \qquad (4.18) \\
&\quad + 2\varepsilon C_3' \int_1^x (\delta + (1-\varepsilon)t^{-\varepsilon}\, dt)^{\tau} ,
\end{aligned}
$$

where the constant C_3' depends only on T. Finally, by Lemma 4.2 with $\rho =$

$\tau_\mu - \tau + k + 1$,

$$|\mathrm{I}_1| = \left| \int_1^x \left(\int_1^{x/t} d\alpha(u) * (\delta - \varepsilon u^{-\varepsilon}\, du) \right) ((\delta + (1-\varepsilon)t^{-\varepsilon}\, dt)^{\tau-k-1} * (\delta - \varepsilon t^{-\varepsilon}\, dt)^k) \right|$$

(4.19)
$$\begin{aligned} &\le \frac{T^\varepsilon}{C_3(\log eT)^{(1-\gamma)/2}} \int_1^x \left\{ \left(\frac{x}{t}\right)^{1-\varepsilon} + \int_1^{x/t} (\delta + (1-\varepsilon)u^{-\varepsilon}\, du) * (\log eu)^{-3/2+\gamma/2}\, du \right\} \\ &\quad \cdot (\delta + (1-\varepsilon)t^{-\varepsilon}\, dt)^{\tau-k-1} * (\delta + \varepsilon t^{-\varepsilon}\, dt)^k \\ &\le \frac{T^\varepsilon}{C_3(\log eT)^{(1-\gamma)/2}} \left\{ \int_1^x (\delta + (1-\varepsilon)t^{-\varepsilon}\, dt)^\tau \right. \\ &\quad \left. + \int_1^x (\delta + (1-\varepsilon)t^{-\varepsilon}\, dt)^\tau * (\log et)^{-3/2+\gamma/2}\, dt \right\}, \end{aligned}$$

since $\tau_\mu - \tau + k + 1 < \gamma$, where γ is defined in (4.4). Now, from (4.16) to (4.19), (4.15) follows. □

Lemma 4.4. *Let* $0 < \rho < 1$. *Then*

(4.20)
$$\int_1^x |(\delta - t^{-1}\, dt)^\rho| = O(x(\log x)^{-\rho-1}).$$

Proof. We have

$$\begin{aligned} (\delta - t^{-1}\, dt)^\rho &= \delta + \sum_{k=1}^\infty \frac{\rho(\rho-1)(\rho-2)\cdots(\rho-k+1)}{k!} (-1)^k t^{-1} \frac{(\log t)^{k-1}}{(k-1)!}\, dt \\ &= \delta - \left\{ \rho + \sum_{k=2}^\infty \frac{\rho(1-\rho)(2-\rho)\cdots(k-\rho-1)}{k!} \frac{\log^{k-1} t}{(k-1)!} \right\} t^{-1}\, dt \\ &= \delta - f(t)t^{-1}\, dt, \end{aligned}$$

say. Then $f(t) \ge 0$. Therefore,

$$\int_1^x (\delta - t^{-1}\, dt)^\rho = 1 - \int_1^x f(t)t^{-1}\, dt$$

and hence

$$\begin{aligned} \int_1^x |(\delta - t^{-1}\, dt)^\rho| = 1 + \int_1^x f(t)t^{-1}\, dt &= 2 - \int_1^x (\delta - t^{-1}\, dt)^\rho \\ &= O(x(\log x)^{-\rho-1}) \end{aligned}$$

by (3.11) with $\varepsilon = 1$. □

The establishment of the following lemma is a main step in the proof of the Theorem.

Lemma 4.5. *Assume* $\tau = \tau_m \ge 1$ *and* (4.5), (4.6), (4.7), (4.8). *Then, for fixed and sufficiently small* $\varepsilon > 0$, *we have*

$$U_\varepsilon(x) := \int_1^x dN * (\delta - \varepsilon t^{-\varepsilon}\, dt)^\tau * Q(t)\, dt \ge 0$$

for all $x \ge 1$ *and* $U_\varepsilon(x) \to \infty$ *as* $x \to \infty$.

Proof. We have

$$(\delta - \varepsilon t^{-\varepsilon}\, dt)^\tau = \delta + t^{-\varepsilon} f(t)\, dt$$

where

$$f(t) = \sum_{k=1}^{\infty} \frac{\tau(\tau-1)\cdots(\tau-k+1)}{k!}(-\varepsilon)^k \frac{(\log t)^{k-1}}{(k-1)!}.$$

Since $\tau > 0$, there exists a constant K such that

$$\left|\frac{\tau(\tau-1)\cdots(\tau-k+1)}{k!}\right| \le K$$

holds for $k = 1, 2, \ldots$. Therefore, we have

$$t^{-\varepsilon}|f(t)| \le K\varepsilon t^{-\varepsilon} \sum_{k=1}^{\infty} \frac{(\varepsilon \log t)^{k-1}}{(k-1)!} = K\varepsilon.$$

Hence,

$$\begin{aligned}\int_1^x dN * (\delta - \varepsilon t^{-\varepsilon}\, dt)^\tau &= \int_1^x dN * (\delta + t^{-\varepsilon} f(t)\, dt) \\ &\ge N(x) - \int_1^x dN * K\varepsilon\, dt \ge N(x) - K\varepsilon N(x)(x-1) \ge 0\end{aligned}$$

for $1 \le x \le (K\varepsilon)^{-1}$. The lemma is certainly true for $1 \le x \le (K\varepsilon)^{-1}$, since the third convolution factor is everywhere nonnegative.

Therefore, we assume $x \ge (K\varepsilon)^{-1}$ and $\varepsilon < 1/2$ and shall utilize all the convolution factors. We write, by (2.4),

$$\begin{aligned}U_\varepsilon(x) &= A\int_1^x (\delta + (1-\varepsilon)t^{-\varepsilon}\, dt)^\tau * Q(t)\, dt \\ &\quad + \sum_{\mu=1}^{m-1} B_\mu \int_1^x (\delta + dt)^{\tau_\mu} * (\delta - \varepsilon t^{-\varepsilon}\, dt)^\tau * Q(t)\, dt \\ &\quad + \int_1^x \left(dN - \sum_{\mu=1}^{m} B_\mu (\delta + dt)^{\tau_\mu}\right) * (\delta - \varepsilon t^{-\varepsilon}\, dt)^\tau * Q(t)\, dt \\ &= \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3,\end{aligned} \tag{4.21}$$

say, where $A = A_n \Gamma(\rho_n) > 0$ and $\sum_{\mu=1}^{m} B_\mu \int_1^x (\delta + dt)^{\tau_\mu}$ is defined in (3.15). We shall show that I_1 is positive, that $\mathrm{I}_1 \to \infty$ as $x \to \infty$, and that I_2 and I_3 are negligible. Actually, we have

$$\begin{aligned}\mathrm{I}_1 &\ge A\int_1^x (\delta + (1-\varepsilon)t^{-\varepsilon}\, dt) * Q(t)\, dt = A\int_1^x \left(\frac{x}{t}\right)^{1-\varepsilon} Q(t)\, dt \\ &\ge Ax^{1-\varepsilon} \int_1^x t^{-1} Q(t)\, dt \to \infty \quad \text{as } x \to \infty.\end{aligned}$$

We then estimate $|\mathrm{I}_2|$ and $|\mathrm{I}_3|$. By (4.8),

$$\left|\int_1^x \left(dN - \sum_{\mu=1}^{m} B_\mu (\delta + dt)^{\tau_\mu}\right) * Q(t)\, dt\right| \le \int_1^x xt^{-1} Q(x/t) Q(t)\, dt. \tag{4.22}$$

If τ is an integer, we have

$$(4.23)\quad \begin{aligned} |\mathrm{I}_3| &\le \int_1^x \left| \int_1^{x/t} \left(dN - \sum_{\mu=1}^m B_\mu(\delta + du)^{\tau_\mu} \right) * Q(u)\, du \right| (\delta + \varepsilon t^{-\varepsilon}\, dt)^\tau \\ &\le C_1 x Q(x) + 2C_1 \varepsilon \int_1^x (\delta + (1-\varepsilon)t^{-\varepsilon}\, dt)^\tau * Q(t)\, dt \end{aligned}$$

by (4.22) and (4.10). If τ is not an integer, then $\tau > 1$, and we have

$$\begin{aligned} &\left(dN - \sum_{\mu=1}^m B_\mu(\delta + dt)^{\tau_\mu} \right) * (\delta - \varepsilon t^{-\varepsilon}\, dt)^\tau * Q(t)\, dt \\ &\quad = \left(dN - \sum_{\mu=1}^m B_\mu(\delta + dt)^{\tau_\mu} \right) * (\delta + (1-\varepsilon)t^{-\varepsilon}\, dt)^{\tau-[\tau]} \\ &\qquad * (\delta - t^{-1}\, dt)^{\tau-[\tau]} * (\delta - \varepsilon t^{-\varepsilon}\, dt)^{[\tau]} * Q(t)\, dt, \end{aligned}$$

since

$$(\delta - \varepsilon t^{-\varepsilon}\, dt)^{\tau-[\tau]} = (\delta + (1-\varepsilon)t^{-\varepsilon}\, dt)^{\tau-[\tau]} * (\delta - t^{-1}\, dt)^{\tau-[\tau]}.$$

Therefore,

$$\begin{aligned} |\mathrm{I}_3| &\le \int_1^x \left| \int_1^{x/t} \left(dN - \sum_{\mu=1}^m B_\mu(\delta + du)^{\tau_\mu} \right) * Q(u)\, du \right| \\ &\qquad ((\delta + (1-\varepsilon)t^{-\varepsilon}\, dt)^{\tau-[\tau]} * |(\delta - t^{-1}\, dt)^{\tau-[\tau]}| * (\delta + \varepsilon t^{-\varepsilon}\, dt)^{[\tau]}) \\ &\le C_1 \int_1^x \left(\int_1^{x/t} Q(u)\, du \right) ((\delta + (1-\varepsilon)t^{-\varepsilon}\, dt)^{\tau-[\tau]} \\ &\qquad * |(\delta - t^{-1}\, dt)^{\tau-[\tau]}| * (\delta + \varepsilon t^{-\varepsilon}\, dt)^{[\tau]}) \\ &= C_1 \int_1^x Q(t)\, dt * |(\delta - t^{-1}\, dt)^{\tau-[\tau]}| \\ &\qquad * (\delta + (1-\varepsilon)t^{-\varepsilon}\, dt)^{\tau-[\tau]} * (\delta + \varepsilon t^{-\varepsilon}\, dt)^{[\tau]}, \end{aligned}$$

by (4.22) and (4.10). Then, by Lemma 4.4 with $\rho = \tau - [\tau]$ and Lemma 4.1,

$$(4.24)\quad \begin{aligned} \int_1^x |(\delta - t^{-1}\, dt)^{\tau-[\tau]}| * Q(t)\, dt &\le C_4 \int_1^x x t^{-1} Q(x/t) Q(t)\, dt \\ &\le C_1 C_4 \int_1^x Q(t)\, dt. \end{aligned}$$

It follows that

$$(4.25)\quad \begin{aligned} |\mathrm{I}_3| \le C_1^2 C_4 \Bigg\{ &\int_1^x (\delta + (1-\varepsilon)t^{-\varepsilon}\, dt)^{\tau-[\tau]} * Q(t)\, dt \\ &+ 2\varepsilon \int_1^x (\delta + (1-\varepsilon)t^{-\varepsilon}\, dt)^\tau * Q(t)\, dt \Bigg\}. \end{aligned}$$

To evaluate $|I_2|$, we note that, by (4.10),

$$\begin{aligned}\int_1^x \left(\int_1^{x/t} \frac{x}{ut} Q\left(\frac{x}{ut}\right)(\delta + (1-\varepsilon)u^{-\varepsilon}\, du)^{\tau-1}\right) Q(t)\, dt \\ = \int_1^x d(tQ(t)) * (\delta + (1-\varepsilon)t^{-\varepsilon}\, dt)^{\tau-1} * Q(t)\, dt \\ \le C_1 \int_1^x (\delta + (1-\varepsilon)t^{-\varepsilon}\, dt)^{\tau-1} * Q(t)\, dt\end{aligned} \tag{4.26}$$

and

$$\begin{aligned}\int_1^x \left(\int_1^{x/t} \frac{x}{ut} Q\left(\frac{x}{ut}\right)(\delta + (1-\varepsilon)u^{-\varepsilon}\, du)^{\tau}\right) Q(t)\, dt \\ \le C_1 \int_1^x (\delta + (1-\varepsilon)t^{-\varepsilon}\, dt)^{\tau} * Q(t)\, dt.\end{aligned} \tag{4.27}$$

Let C_5 be a constant such that

$$\int_1^x (\log et)^{-3/2+\gamma/2}\, dt \le C_5 x (\log ex)^{-3/2+\gamma/2} \quad \text{for all } x \ge 1,$$

where γ is defined in (4.4). Then

$$\int_1^x (\log et)^{-3/2+\gamma/2}\, dt \le C_5 x Q(x) \quad \text{for all } x \ge 1$$

by (4.8). Hence,

$$\begin{aligned}\int_1^x (\log et)^{-3/2+\gamma/2}\, dt * Q(t)\, dt \le C_5 \int_1^x xt^{-1} Q(x/t) Q(t)\, dt \\ \le C_1 C_5 \int_1^x Q(t)\, dt\end{aligned} \tag{4.28}$$

by (4.10). Also, we note that if $\tau_\mu < \tau - [\tau]$ then, by an analogue of (4.24),

$$\begin{aligned}&\left|\int_1^x (\delta + dt)^{\tau_\mu} * (\delta - \varepsilon t^{-\varepsilon}\, dt)^{\tau} * Q(t)\, dt\right| \\ &\le \int_1^x (\delta + (1-\varepsilon)t^{-\varepsilon}\, dt)^{\tau-[\tau]} * |(\delta - t^{-1}\, dt)^{\tau-[\tau]-\tau_\mu}| \\ &\qquad * (\delta + \varepsilon t^{-\varepsilon}\, dt)^{[\tau]} * Q(t)\, dt \\ &\le C_1 C_4 \int_1^x (\delta + (1-\varepsilon)t^{-\varepsilon}\, dt)^{\tau-[\tau]} * Q(t)\, dt * (\delta + \varepsilon t^{-\varepsilon}\, dt)^{[\tau]} \\ &\le C_1 C_4 \left\{\int_1^x (\delta + (1-\varepsilon)t^{-\varepsilon}\, dt)^{\tau-[\tau]} * Q(t)\, dt \right. \\ &\qquad \left. + 2\varepsilon \int_1^x (\delta + (1-\varepsilon)t^{-\varepsilon}\, dt)^{\tau} * Q(t)\, dt\right\}.\end{aligned} \tag{4.29}$$

Therefore, by (4.14), (4.15), (4.26), (4.27), (4.28), and (4.29), we have

$$
\begin{aligned}
|\mathrm{I}_2| \le \sum_{\mu=1}^{m-1}{}' |B_\mu| \Big\{ &\int_1^x (\delta + (1-\varepsilon)t^{-\varepsilon}\,dt)^{\tau-1} * Q(t)\,dt \\
&+ 2\varepsilon \int_1^x (\delta + (1-\varepsilon)t^{-\varepsilon}\,dt)^{\tau} * Q(t)\,dt \Big\} \\
+ \sum_{\mu=1}^{m-1}{}'' |B_\mu| \Big\{ &C_2(C_1+1) \int_1^x (\delta + (1-\varepsilon)t^{-\varepsilon}\,dt)^{\tau-1} * Q(t)\,dt \\
&+ 2\varepsilon C_2(C_1+1) \int_1^x (\delta + (1-\varepsilon)t^{-\varepsilon}\,dt)^{\tau} * Q(t)\,dt \\
&+ \frac{T^\varepsilon(1+C_1C_5)}{C_3(\log eT)^{(1-\gamma)/2}} \int_1^x (\delta + (1-\varepsilon)t^{-\varepsilon}\,dt)^{\tau} * Q(t)\,dt \Big\} \\
+ \sum_{\mu=1}^{m-1}{}''' |B_\mu| C_1 C_4 \Big\{ &\int_1^x (\delta + (1-\varepsilon)t^{-\varepsilon})^{\tau-[\tau]} * Q(t)\,dt \\
&+ 2\varepsilon \int_1^x (\delta + (1-\varepsilon)t^{-\varepsilon}\,dt)^{\tau} * Q(t)\,dt \Big\},
\end{aligned} \tag{4.30}
$$

where

$$\sum_{\mu=1}^{m-1}{}', \quad \sum_{\mu=1}^{m-1}{}'', \quad \text{and} \quad \sum_{\mu=1}^{m-1}{}'''$$

denote the respective sums over all τ_μ such that $\tau - \tau_\mu$ is an integer, such that $\tau - \tau_\mu$ is not an integer but $\tau_\mu > \tau - [\tau]$, and such that $\tau_\mu < \tau - [\tau]$.

We now choose a number T sufficiently large so that

$$\frac{\sum_{\mu=1}^{\prime\prime m-1} |B_\mu|(1+C_1C_5)}{C_3(\log eT)^{(1-\gamma)/2}} < \frac{A}{16} \tag{4.31}$$

and fix it. If τ is an integer, we can choose a positive number ε_0 sufficiently small so that $(K\varepsilon_0)^{-1} > T$, $T^{\varepsilon_0} < 2$ and that

$$2\varepsilon_0 \left(\sum_{\mu=1}^{m-1}{}' |B_\mu| + C_2(C_1+1) \sum_{\mu=1}^{m-1}{}'' |B_\mu| + C_1C_4 \sum_{\mu=1}^{m-1}{}''' |B_\mu| + C_1 \right) < \frac{A}{8}. \tag{4.32}$$

Then, for $\varepsilon \le \varepsilon_0$, from (4.21), (4.23), (4.30), (4.31), and (4.32), we have

$$
\begin{aligned}
U_\varepsilon(x) \ge \frac{3A}{4} &\int_1^x (\delta + (1-\varepsilon)t^{-\varepsilon}\,dt)^{\tau} * Q(t)\,dt \\
&- C_6 \int_1^x (\delta + (1-\varepsilon)t^{-\varepsilon}\,dt)^{\tau-1} * Q(t)\,dt - C_1 x Q(x)
\end{aligned} \tag{4.33}
$$

for $x \ge (K\varepsilon)^{-1}$, where

$$C_6 = \sum_{\mu=1}^{m-1}{}' |B_\mu| + C_2(C_1+1) \sum_{\mu=1}^{m-1}{}'' |B_\mu| + C_1C_4 \sum_{\mu=1}^{m-1}{}''' |B_\mu|.$$

By (3.1), there exists a number x_0 sufficiently large and independent of ε such that

$$\frac{A}{4}\int_1^x (\delta + (1-\varepsilon)t^{-\varepsilon}\,dt)^{\tau} \geq C_6 \int_1^x (\delta + (1-\varepsilon)t^{-\varepsilon}\,dt)^{\tau-1}$$

holds for $x \geq x_0$ and all ε satisfying $0 < \varepsilon \leq \varepsilon_0$. Hence, from (4.33), for $\varepsilon \leq (Kx_0^2)^{-1}$ $(\leq \varepsilon_0)$ and $x > (K\varepsilon)^{-1}$, we have

$$\begin{aligned} U_\varepsilon(x) &\geq \frac{A}{2}\int_1^x (\delta + (1-\varepsilon)t^{-\varepsilon}\,dt)^{\tau} * Q(t)\,dt \\ &\quad - C_6 \int_1^{x_0} (\delta + (1-\varepsilon)t^{-\varepsilon}\,dt)^{\tau-1} \int_{x/x_0}^x Q(t)\,dt - C_1 x Q(x) \\ &\geq \frac{A}{2}\int_1^x (\delta + (1-\varepsilon)t^{-\varepsilon}\,dt)^{\tau} * Q(t)\,dt \\ &\quad - \left(4C_6 \int_1^{x_0} (\delta + (1-\varepsilon)t^{-\varepsilon}\,dt)^{\tau-1} + C_1\right) xQ(x) \end{aligned}$$

since $x \geq x_0^2$ and

$$\int_{x/x_0}^x Q(t)\,dt \leq Q(x/x_0)x \leq Q(\sqrt{x})x \leq 4xQ(x)$$

by (4.5) and (4.7). Now, for sufficiently small $\varepsilon > 0$,

$$\begin{aligned} &\frac{A}{4}\int_1^x (\delta + (1-\varepsilon)t^{-\varepsilon}\,dt)^{\tau} * Q(t)\,dt \\ &\quad \geq \frac{A}{4}\int_1^x (\delta + (1-\varepsilon)t^{-\varepsilon}\,dt) * Q(t)\,dt \\ &\quad = \frac{A}{4}x \int_1^x t^{-1-\varepsilon} Q(x/t)\,dt \geq \frac{A}{4}xQ(x)\int_1^{1/K\varepsilon} t^{-1-\varepsilon}\,dt \\ &\quad \geq \frac{A}{4}xQ(x)(K\varepsilon)^{\varepsilon}\log\frac{1}{K\varepsilon} \\ &\quad \geq \left(4C_6 \int_1^{x_0} (\delta + (1-\varepsilon)t^{-\varepsilon}\,dt)^{\tau-1} + C_1\right) xQ(x) \end{aligned}$$

since $(K\varepsilon)^{\varepsilon} \geq \varepsilon^{\varepsilon} \geq \exp(-e^{-1})$. Therefore, $U_\varepsilon(x) \geq \frac{1}{4}\mathrm{I}_1$ for $x \geq (K\varepsilon)^{-1}$.

If τ is not an integer, we choose a positive number ε_0 sufficiently small so that

$$(4.34)\quad 2\varepsilon_0 \left({\sum_{\mu=1}^{m-1}}' |B_\mu| + C_2(C_1+1) {\sum_{\mu=1}^{m-1}}'' |B_\mu| + C_1 C_4 {\sum_{\mu=1}^{m-1}}''' |B_\mu| + C_1^2 C_4 \right) < \frac{A}{8}.$$

Then, from (4.25), (4.24), (4.30), (4.31), and (4.34), we have

$$(4.35)\quad \begin{aligned} U_\varepsilon(x) &\geq \frac{3A}{4}\int_1^x (\delta + (1-\varepsilon)t^{-\varepsilon}\,dt)^{\tau} * Q(t)\,dt \\ &\quad - C_6 \int_1^x (\delta + (1-\varepsilon)t^{-\varepsilon}\,dt)^{\tau-1} * Q(t)\,dt \\ &\quad - C_1^2 C_4 \int_1^x (\delta + (1-\varepsilon)t^{-\varepsilon}\,dt)^{\tau-[\tau]} * Q(t)\,dt. \end{aligned}$$

In the same way as above, by (3.2),

$$\frac{A}{4}\int_1^x (\delta + (1-\varepsilon)t^{-\varepsilon}\,dt)^\tau \geq C_6 \int_1^x (\delta + (1-\varepsilon)t^{-\varepsilon}\,dt)^{\tau-1} + C_1^2 C_4 \int_1^x (\delta + (1-\varepsilon)t^{-\varepsilon}\,dt)^{\tau-[\tau]}$$

holds for $x \geq x_0$ and all ε satisfying $0 < \varepsilon \leq \varepsilon_0$ and hence, from (4.35),

$$\begin{aligned} U_\varepsilon(x) &\geq \frac{A}{2}\int_1^x (\delta + (1-\varepsilon)t^{-\varepsilon}\,dt)^\tau * Q(t)\,dt \\ &\quad - C_6 \int_1^{x_0} (\delta + (1-\varepsilon)t^{-\varepsilon}\,dt)^{\tau-1} \int_{x/x_0}^x Q(t)\,dt \\ &\quad - C_1^2 C_4 \int_1^{x_0} (\delta + (1-\varepsilon)t^{-\varepsilon}\,dt)^{\tau-[\tau]} \int_{x/x_0}^x Q(t)\,dt \\ &\geq \frac{A}{2}\int_1^x (\delta + (1-\varepsilon)t^{-\varepsilon}\,dt)^\tau * Q(t)\,dt \\ &\quad -4\left(C_6 \int_1^{x_0} (\delta + (1-\varepsilon)t^{-\varepsilon}\,dt)^{\tau-1} \right. \\ &\qquad \left. + C_1^2 C_4 \int_1^{x_0} (\delta + (1-\varepsilon)t^{-\varepsilon}\,dt)^{\tau-[\tau]}\right) xQ(x). \end{aligned}$$

Therefore, for sufficiently small $\varepsilon > 0$, $U_\varepsilon(x) > \frac{1}{4}\mathrm{I}_1$ for $x \geq (K\varepsilon)^{-1}$. □

5. The proof of the Theorem

The proof of the Theorem follows the general idea in [5, 8].

Lemma 5.1. *Suppose that*

$$N(x) = x\sum_{\nu=1}^{n} A_\nu (\log x)^{\rho_\nu - 1} + O(x\log^{-1} ex). \tag{5.1}$$

Then we have

$$\int_1^x L\,dN * (\delta - \varepsilon t^{-\varepsilon}\,dt)^\tau = O_\varepsilon(x),$$

where $\tau = \rho_n \geq 1$.

Proof. We have

$$\begin{aligned} &\int_1^x L\,dN * (\delta - \varepsilon t^{-\varepsilon}\,dt)^\tau \\ &= \int_1^x L\,dP * (\delta - \varepsilon t^{-\varepsilon}\,dt)^\tau + \int_1^x (L\,dN - L\,dP) * (\delta - \varepsilon t^{-\varepsilon}\,dt)^\tau \\ &= \mathrm{I}_1 + \mathrm{I}_2, \end{aligned} \tag{5.2}$$

say, where $P(x)$ is defined in (4.1). If suffices to show that both I_1 and I_2 are $O(x)$.

Actually, from (5.1) and (3.15), $N(x) - P(x) = O(x \log^{-1} ex)$. Therefore, we have

$$\int_1^x (L\,dN - L\,dP) = \log x(N(x) - P(x)) - \int_1^x (N(t) - P(t))t^{-1}\,dt = O(x).$$

If $\tau = r$ is an integer, then plainly,

$$\mathrm{I}_2 = O\left(\int_1^x xt^{-1}(\delta + \varepsilon t^{-\varepsilon}\,dt)^r\right) = O\left(x\left(1 + \sum_{k=1}^{r} \varepsilon^k \frac{1}{(k-1)!}\int_1^x t^{-1-\varepsilon}(\log t)^{k-1}\,dt\right)\right) = O(x). \tag{5.3}$$

If τ is not an integer, then $0 < \tau - [\tau] < 1$ and we have

$$\mathrm{I}_2 = \int_1^x (L\,dN - L\,dP) * (\delta - \varepsilon t^{-\varepsilon}\,dt)^{\tau - [\tau]} * (\delta - \varepsilon t^{-\varepsilon}\,dt)^{[\tau]}$$

and, by Lemma 4.4,

$$\left|\int_1^x (L\,dN - L\,dP) * (\delta - \varepsilon t^{-\varepsilon}\,dt)^{\tau-[\tau]}\right| \ll \int_1^x xt^{-1}|(\delta - \varepsilon t^{-\varepsilon}\,dt)^{\tau-[\tau]}|$$
$$= x\left\{x^{-1}\int_1^x |(\delta - \varepsilon t^{-\varepsilon}\,dt)^{\tau-[\tau]}| + \int_1^x \left(\int_1^t |(\delta - \varepsilon u^{-\varepsilon}\,du)^{\tau-[\tau]}|\right) t^{-2}\,dt\right\} \ll x,$$

by integration by parts. Therefore, (5.3) is still true.

To estimate I_1, we consider the Mellin transform

$$\int_1^\infty x^{-s} L\,dP * (\delta - \varepsilon x^{-\varepsilon}\,dx)^\tau = \tau B_m \left(\frac{s^\tau}{(s-1)(s-1+\varepsilon)^\tau} - \frac{s^{\tau-1}}{(s-1+\varepsilon)^\tau}\right) + \sum_{\mu=1}^{m-1} \tau_\mu B_\mu \left(\frac{s^{\tau_\mu}(s-1)^{\tau-\tau_\mu-1}}{(s-1+\varepsilon)^\tau} - \frac{s^{\tau_\mu-1}(s-1)^{\tau-\tau_\mu}}{(s-1+\varepsilon)^\tau}\right)$$

for $\sigma > 1$. Then, by Perron's inversion formula,

$$\mathrm{I}_1 = \int_1^x L\,dP * (\delta - \varepsilon t^{-\varepsilon}\,dt)^\tau$$
$$= \frac{\tau B_m}{2\pi i}\left(\int_{\sigma=\sigma_0} \frac{x^s}{s}\frac{s^\tau}{(s-1)(s-1+\varepsilon)^\tau}\,ds - \int_{\sigma=\sigma_0} \frac{x^s}{s}\frac{s^{\tau-1}}{(s-1+\varepsilon)^\tau}\,ds\right)$$
$$+ \sum_{\mu=1}^{m-1} \tau_\mu \frac{B_\mu}{2\pi i}\left(\int_{\sigma=\sigma_0} \frac{x^s}{s}\frac{s^{\tau_\mu}(s-1)^{\tau-\tau_\mu-1}}{(s-1+\varepsilon)^\tau}\,ds - \int_{\sigma=\sigma_0} \frac{x^s}{s}\frac{s^{\tau_\mu-1}(s-1)^{\tau-\tau_\mu}}{(s-1+\varepsilon)^\tau}\,ds\right),$$

where $\sigma_0 > 1$. As we did in the proof of Lemma 3.2, we shift the integration contour to $l_\eta(1)$ and obtain

$$\frac{1}{2\pi i}\int_{\sigma=\sigma_0} \frac{x^s}{s}\frac{s^\tau}{(s-1)(s-1+\varepsilon)^\tau}\,ds = \frac{x}{\varepsilon^\tau}(1+o(1)),$$

and

$$\frac{1}{2\pi i}\int_{\sigma=\sigma_0} \frac{x^s}{s}\frac{s^{\tau_\mu-1}(s-1)^{\tau-\tau_\mu}}{(s-1+\varepsilon)^\tau}\,ds = O(x(\log x)^{-\tau+\tau_\mu-1}),$$

$$\frac{1}{2\pi i}\int_{\sigma=\sigma_0} \frac{x^s}{s}\frac{s^{\tau_\mu}(s-1)^{\tau-\tau_\mu-1}}{(s-1+\varepsilon)^\tau}\,ds = O(x(\log x)^{-\tau+\tau_\mu})$$

if $\tau-\tau_\mu$ is not an integer. Also, as in the proof of Lemma 3.1, we obtain

$$\frac{1}{2\pi i}\int_{\sigma=\sigma_0} \frac{x^s}{s}\frac{s^{\tau_\mu-1}(s-1)^{\tau-\tau_\mu}}{(s-1+\varepsilon)^\tau}\,ds = O(x^{1-\varepsilon}(\log x)^{\tau-1}),$$

$$\frac{1}{2\pi i}\int_{\sigma=\sigma_0} \frac{x^s}{s}\frac{s^{\tau_\mu}(s-1)^{\tau-\tau_\mu-1}}{(s-1+\varepsilon)^\tau}\,ds = O(x^{1-\varepsilon}(\log x)^{\tau-1}),$$

if $\tau-\tau_\mu$ is an integer, and

$$\frac{1}{2\pi i}\int_{\sigma=\sigma_0} \frac{x^s}{s}\frac{s^\tau-1}{(s-1+\varepsilon)^\tau}\,ds = O(x^{1-\varepsilon}(\log x)^{\tau-1})$$

Therefore,

$$\mathrm{I}_1 = \frac{x}{\varepsilon^\tau}(1+o(1)) = O(x). \quad \square$$

We are now in the position to set up the upper estimate of the Theorem. The starting point is the identity $d\psi * dN = L\,dN$, an analogue of Chebyshev's one. We convolve each side of it by $(\delta-\varepsilon t^{-\varepsilon}\,dt)^\tau * Q(t)\,dt$, where $\tau=\tau_m=\rho_n$, and obtain

$$\int_1^x dN * (\delta-\varepsilon t^{-\varepsilon}\,dt)^\tau * Q(t)\,dt * d\psi$$
$$= \int_1^x L\,dN * (\delta-\varepsilon t^{-\varepsilon}\,dt)^\tau * Q(t)\,dt.$$

Proof of the upper estimate. Assume (1.4), then (4.5), (4.6), (4.7), and (4.8) hold by Lemma 4.1. By Lemmas 4.5 and 5.1,

$$\psi(x/B) \le \int_1^x U_\varepsilon(x/t)\,d\psi(t) = O(x)$$

since $U_\varepsilon(x)\ge 1$ for $x \ge B$. $\square$

To establish the lower estimate of the Theorem, we need one more lemma.

Lemma 5.2. *Suppose that*

$$N(x) = x\sum_{\nu=1}^{n} A_\nu(\log x)^{\rho_\nu-1} + o(x\log^{-1}ex). \tag{5.4}$$

Then we have

$$\int_1^x L\,dN * (\delta - t^{-1}\,dt)^\tau = Ax + o(x) \tag{5.5}$$

where $\tau=\rho_n\ge 1$ *and* $A=\tau A_n\Gamma(\tau)>0$.

Proof. As in the proof of Lemma 5.1, the integral on the right-hand side of (5.5) equals

$$\int_1^x L\,dP*(\delta-t^{-1}\,dt)^{\tau}+\int_1^x (L\,dN-L\,dP)*(\delta-t^{-1}\,dt)^{\tau}=\mathrm{I}_1+\mathrm{I}_2,$$

say. From (5.4) and (3.15), $N(x)-P(x)=o(x\log^{-1}ex)$ and hence

$$\int_1^x (L\,dN-L\,dP)=o(x).$$

It follows that

$$\begin{aligned}\left|\int_1^x (L\,dN-L\,dP)*(\delta-t^{-1}\,dt)^{\tau-[\tau]}\right| &\le \varepsilon\int_1^{x/M} xt^{-1}|(\delta-t^{-1}\,dt)^{\tau-[\tau]}| \\ &\quad +K\int_{x/M}^x xt^{-1}|(\delta-t^{-1}\,dt)^{\tau-[\tau]}| \\ &\le \varepsilon(C+1)x\end{aligned}$$

for x sufficiently large, since $C=\int_1^\infty t^{-1}|(\delta-t^{-1}\,dt)^{\tau-[\tau]}|$ is convergent by (4.20) with $\rho=\tau-[\tau]$. Hence

$$\mathrm{I}_2=\int_1^x (L\,dN-L\,dP)*(\delta-t^{-1}\,dt)^{\tau-[\tau]}*(\delta-t^{-1}\,dt)^{[\tau]}=o(x). \tag{5.6}$$

To estimate I_1, we consider the Mellin transform

$$\begin{aligned}&\int_1^x x^{-s}L\,dP*(\delta-x^{-1}\,dx)^{\tau}\\ &\quad=\tau B_m\left(\frac{1}{s-1}-\frac{1}{s}\right)+\sum_{\mu=1}^{m-1}\tau_\mu B_\mu\left(\frac{(s-1)^{\tau-\tau_\mu-1}}{s^{\tau-\tau_\mu}}-\frac{(s-1)^{\tau-\tau_\mu}}{s^{\tau-\tau_\mu+1}}\right)\end{aligned}$$

for $\sigma>1$. As we did in the proof of Lemmas 3.2 and 3.1, we have

$$\frac{1}{2\pi i}\int_{\sigma=\sigma_0}\frac{x^s}{s}\frac{(s-1)^{\tau-\tau_\mu-1}}{s^{\tau-\tau_\mu}}\,ds=O((\log x)^{\tau-\tau_\mu})\quad\text{or}\quad O(x(\log x)^{-\tau+\tau_\mu})$$

according as $\tau-\tau_\mu$ is an integer or not and a similar estimate for

$$\frac{1}{2\pi i}\int_{\sigma=\sigma_0}\frac{x^s}{s}\frac{(s-1)^{\tau-\tau_\mu}}{s^{\tau-\tau_\mu+1}}\,ds.$$

Therefore, by Perron's inversion formula,

$$\begin{aligned}\mathrm{I}_1&=\int_1^x L\,dP*(\delta-t^{-1}\,dt)^{\tau}\\ &=\tau B_m\left(\int_{\sigma=\sigma_0}\frac{x^s}{s}\frac{1}{s-1}\,ds-\int_{\sigma=\sigma_0}\frac{x^s}{s^2}\,ds\right)\\ &\quad+\sum_{\mu=1}^{m-1}\tau_\mu B_\mu\left(\int_{\sigma=\sigma_0}\frac{x^s}{s}\frac{(s-1)^{\tau-\tau_\mu-1}}{s^{\tau-\tau_\mu}}\,ds-\int_{\sigma=\sigma_0}\frac{x^s}{s}\frac{(s-1)^{\tau-\tau_\mu}}{s^{\tau-\tau_\mu+1}}\,ds\right)\\ &=\tau B_m x+o(x).\end{aligned}\tag{5.7}$$

Now, (5.5) follows from (5.6) and (5.7). □

Proof of the lower estimate. Assume (1.4). Then from (3.15), (4.8), (4.9), and (4.2), (5.4) is true. Hence, (5.5) holds by Lemma 5.2. Let

$$\begin{aligned} U_1(x) &:= \int_1^x dN * (\delta - t^{-1}\,dt)^\tau \\ &= \int_1^x dP * (\delta - t^{-1}\,dt)^\tau + \int_1^x (dN - dP) * (\delta - t^{-1}\,dt)^\tau \\ &= \mathrm{I}_1 + \mathrm{I}_2, \end{aligned} \tag{5.8}$$

say. We shall first show that both I_1 and I_2 are $O(1 + \int_1^x Q(t)\,dt)$ and hence so is $U_1(x)$.

Actually, if τ is not an integer, we have

$$\left| \int_1^x (dN - dP) * (\delta - t^{-1}\,dt)^{\tau - [\tau]} \right| \ll \int_1^x x t^{-1} Q(x/t) |(\delta - t^{-1}\,dt)^{\tau-[\tau]}|$$

by (4.2) and (4.8). By integration by parts, the right-hand side equals

$$\begin{aligned} &Q(1) \int_1^x |(\delta - t^{-1}\,dt)^{\tau-[\tau]}| - x \int_1^x \left(\int_1^t |(\delta - u^{-1}\,du)^{\tau-[\tau]}| \right) d(t^{-1}Q(x/t)) \\ &\quad \le Q(1) \int_1^x |(\delta - t^{-1}\,dt)^{\tau-[\tau]}| \\ &\qquad + x \int_1^x \left(\int_1^t |(\delta - u^{-1}\,du)^{\tau-[\tau]}| \right) t^{-2} Q(x/t)\,dt \end{aligned}$$

since $Q(x/t)$ is nondecreasing in t. Therefore we have

$$\begin{aligned} &\left| \int_1^x (dN - dP) * (\delta - t^{-1}\,dt)^{\tau-[\tau]} \right| \\ &\quad \ll x(\log ex)^{-\tau+[\tau]-1} + x \int_1^x t^{-1} (\log et)^{-\tau+[\tau]-1} Q(x/t)\,dt \\ &\quad \ll xQ(x) + x \int_1^x t^{-1} Q(t) Q(x/t)\,dt \\ &\quad \ll xQ(x), \end{aligned} \tag{5.9}$$

by Lemmas 4.4 and 4.1. If τ is an integer, (5.9) is plainly true. It follows that

$$\begin{aligned} \mathrm{I}_2 &= \int_1^x (dN - dP) * (\delta - t^{-1}\,dt)^{\tau-[\tau]} * (\delta - t^{-1}\,dt)^{[\tau]} \\ &\ll \int_1^x x t^{-1} Q(x/t) (\delta + t^{-1}\,dt)^{[\tau]} \\ &= xQ(x) + x \sum_{k=1}^{[\tau]} \frac{1}{(k-1)!} \int_1^x t^{-2} Q(x/t) \log^{k-1} t\,dt \\ &\ll xQ(x) + \int_1^x x t^{-1} Q(x/t) Q(t)\,dt \end{aligned}$$

since $t^{-1}(\log t)^{k-1} \ll Q(t)$ by (4.2). Hence,

$$\mathrm{I}_2 \ll 1 + \int_1^x Q(t)\,dt \tag{5.10}$$

by (4.10).

To evaluate I_1, we have, by Lemma 3.2 with $\rho = \tau - \tau_\mu$ and $\varepsilon = 1$,

$$\int_1^x (\delta - t^{-1}\,dt)^{\tau-\tau_\mu} = O((\log x)^{\tau-\tau_\mu-1}) \quad \text{or} \quad O(x(\log x)^{-\tau+\tau_\mu-1})$$

according as $\tau - \tau_\mu$ is an integer or not. It follows that

$$\begin{aligned} \mathrm{I}_1 &= 1 + \sum_{\mu=1}^{m-1} B_\mu \int_1^x (\delta - t^{-1}\,dt)^{\tau-\tau_\mu} \\ &\ll xQ(x) \ll 1 + \int_1^x Q(t)\,dt \end{aligned} \tag{5.11}$$

by (4.8). Now, from (5.8), (5.10), and (5.11),

$$U_1(x) \ll 1 + \int_1^x Q(t)\,dt.$$

From the analogue of Chebyshev's identity, we have

$$\int_1^x dN * (\delta - t^{-1}\,dt)^\tau * d\psi = \int_1^x L\,dN * (\delta - t^{-1}\,dt)^\tau.$$

By (5.5), it turns out that

$$\int_1^x U_1(x/t)\,d\psi(t) = Ax + o(x).$$

We note that, by the upper estimate of $\psi(x)$,

$$\begin{aligned} \int_1^x U_1(x/t)\,d\psi(t) &\le K \int_1^x \left(1 + \int_1^{x/t} Q(u)\,du\right) d\psi(t) \\ &\le K\left\{\psi(x) + \int_1^x \psi(x/t)Q(t)\,dt\right\} \\ &\le K\left\{\psi(x)\left(1 + \int_1^B Q(t)\,dt\right) + C_7 x \int_B^x t^{-1}Q(t)\,dt\right\} \\ &\le C_8\psi(x) + C_7 K x \int_B^x t^{-1}Q(t)\,dt. \end{aligned}$$

For B sufficiently large, $C_7 K \int_B^x t^{-1}Q(t)\,dt \le \frac{1}{3}A$. Fixing B, for x sufficiently large, we have $C_8\psi(x) \ge \frac{1}{3}Ax$. This completes the proof of the Theorem. □

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