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VARIETIES OF TOPOLOGICAL GEOMETRIES

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ABSTRACT. A variety of topological geometries is either
A. a *projective* variety $\mathscr{L}(F)$ over some topological field F, or
B. a *matchstick* variety $\mathscr{M}(X)$ over some topological space X. As a main tool for showing this, we prove a structure theorem for arbitrary topological geometries.

1. INTRODUCTION.

a. The Euclidean plane, the real hyperbolic plane, the real affine 3-space, and the real projective n-spaces are all strongly interrelated by forming subgeometries and contractions. Therefore it seems natural to consider *classes* of geometries. In [KK 82] (reprinted in [Ku 86]), Kahn and Kung defined the notion "variety of (combinatorial) geometries". They succeeded in determining all varieties of *finite* geometries. On p. 498 they list some difficulties associated with the determination of all varieties of *arbitrary* geometries. Therefore it seems reasonable to add an additional structure: In this paper we determine all varieties of *topological* geometries:

5.4. **Main Theorem II.** *Let $\mathscr{T}$ be a variety of topological geometries containing a nondiscrete one. Then $\mathscr{T}$ is either*
A. *a projective variety $\mathscr{L}(F)$ over some topological field F, or*
B. *a matchstick variety $\mathscr{M}(X)$ over some Hausdorff topological space X.*

Comparing this with the finite case we observe that none of the three types: free variety, origami varieties, voltage-graphic varieties, has a topological analogue. If we view matchstick geometries as trivial, then 5.4 even says that only "classical" geometries (i.e. geometries coordinatizable over some topological field) can be members of a variety. Furthermore, of the multitude of nondesarguesian topological projective planes, none belongs to a variety.

There exist projective as well as matchstick varieties admitting more than one sequence of universal models:

A. Let $\mathbb{R}(x)$ be the field of rational functions over $\mathbb{R}$ (reals). There is a linear ordering on it with $r < x$ for all $r \in \mathbb{R}$, which makes it an ordered field. Endowed with the open interval topology induced by this ordering, $\mathbb{R}(x)$ becomes a topological field F. $\mathbb{R}(x^2)$ is a proper subfield isomorphic to F. Therefore (see [KK 82, p. 498]), defining $\mathbf{T}_n$ to be the affine n-space over

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F, we obtain another sequence of universal models for the projective variety $\mathscr{L}(F)$.

B. Let $X = \mathbb{R}$ and $Y := [0, 1]$, the closed unit interval. Since X is homeomorphic to the subspace $]0, 1[$ of Y, by 5.3 we have $\mathscr{M}(X) = \mathscr{M}(Y)$. Therefore the matchstick variety $\mathscr{M}(X)$ has at least two sequences of universal models: The powers of the line over $\mathbb{R}$, and the powers of the line over $[0, 1]$.

Our proof for the Main Theorem II is not analogous to the corresponding one in [KK 82]. We derive it from a structure theorem on arbitrary topological geometries, which is of interest in its own right:

4.6. **Main Theorem I.** *Each topological geometry* **G** *is isomorphic to the direct product of finitely many*

(1) *open subgeometries of projective spaces over topological fields,*
(2) *topological planes (not necessarily desarguesian or projective),*
(3) *lines over Hausdorff topological spaces, or*
(4) *discrete geometries.*

b. Notions. Notation. Facts.

b1. Let **G** be a *geometric lattice* (also called geometry, or matroid; see e.g. [Bi 67, CR70, Ku 86, We 76, Wh 86, Wh 87]). For $x \in \mathbf{G}$, we define the *dimension* of x—denoted by $\dim x$—to be the cardinality of a maximal chain from the smallest element 0 to x, minus 2. The dimension of **G** is defined by $\dim \mathbf{G} := \dim 1 =: n$, where 1 is the largest element. Elements of dimension $0, 1, 2, k, n-2, n-1$ are called *points, lines, planes, k-flats, colines, hyperplanes (or copoints)*. Furthermore, if $n = 0, 1, 2$, resp. k, then **G** itself is called a point, line, plane, resp. k-space.

For $x \in \mathbf{G}$, the intervals $\mathbf{G}_x := [0, x]$ and $\mathbf{G}^x := [x, 1]$ (*contraction*) are again geometric lattices, as is the following: For a set M of points, the *subgeometry* $\mathbf{G}(M)$ induced on M is the set of flats of **G** for which M contains a spanning set, with the ordering inherited from **G**. Furthermore we use the following symbols:

$x \lessdot y$	for: x is a lower neighbor of y,
$\underset{\bullet}{x}$	set of upper neighbors of x,
$\overset{\circ}{x}$	set of points below x,
${}_k\mathbf{G}$	set of k-flats of **G**,
${}_i\overset{j}{\bigvee}{}_k$	restriction of $\bigvee$ to $\{(x, y) \in {}_i\mathbf{G} \times {}_j\mathbf{G} \mid x \vee y \in {}_k\mathbf{G}\}$,
${}^i\underset{j}{\bigwedge}{}^k$	is defined dually.

b2. A *topological geometry* (or *topological n-space*) is a geometric lattice **G** where each "layer" ${}_j\mathbf{G}$ of j-flats carries a topology such that

(C$\vee$) Joining a point with a j-flat is continuous.

(C$\wedge$) Meeting a hyperplane with a j-flat resulting in a $(j-1)$-flat, is continuous.

(S) Each hyperplane h and j-flat x with $j \geq 1$ meeting in a $(j-1)$-flat, have neighborhoods V, resp. W, with the same properties, i.e. each $h' \in V$ meets each $x' \in W$ in a $(j-1)$-flat.

This notion has been justified in [Gr 86a]. Furthermore, in [CG 89] it is shown that it would be too strong to postulate—as the analogy to topological groups etc. might suggest it—a topology on $\mathbf{G}$ such that the "full" functions of $\vee$ and $\wedge$ are continuous. Instead of postulating this, our three axioms only require all "slices" ${}_0\bigvee_j^{j+1}$ and ${}^{n-1}\bigwedge_{j-1}^{j}$ to be continuous, and the domain of ${}^{n-1}\bigwedge_{j-1}^{j}$ to be open.

We define the topology τ of $\mathbf{G}$ to be that of the free union of the ${}_j\mathbf{G}$ (cf. [Gr 86a, p. 115]). Thus, a topological geometry could equivalently be defined as a pair $(\mathbf{G}, \tau)$ of a geometric lattice and a topology, satisfying $(C\vee)$, $(C\wedge)$ and (S), where each ${}_j\mathbf{G}$ is open.

For a topological field F, the topological projective n-space over F, denoted by $\mathbf{P}_n(F) =: \mathbf{P}$, is the lattice of subspaces of the F-vector space F^{n+1}, given the following topology τ: For $M \subseteq F^{n+1}$, denote by $M_{\mathbf{P}}$ the set of subspaces meeting M. Define τ to be the topology having as a subbasis

$$\{M_{\mathbf{P}} | M \text{ open in } F^{n+1}\} \cup \{{}_i\mathbf{P} | -1 \leq i \leq n\}.$$

From [Sz 86, Proposition 3] it follows easily that $(\mathbf{P}, \tau)$ is a topological projective space in the sense of [Mi 68, p. 246], which by [Gr 86a, Theorem 3.3] is a topological geometry.

b3. The (direct) product $\mathbf{G}_1 \times \mathbf{G}_2$ of two topological geometries $\mathbf{G}_i$ is defined as the (direct) product (see e.g. [Bi 67, p. 8]) of the corresponding lattices, endowed with the product topology. Since a variety is closed under the formation of products, it is important to observe the following fact:

1.1. *The product of two topological geometries is again a topological geometry.*

The proof is a verification using generalized sequences (or nets, see e.g. [Du 66, p. 2.10]). □

b4. We write $\cong$ for "isomorphic" (see [Gr 86a, p. 115]), and $\approx$ for "homeomorphic".

2. Planes

The following proposition is a generalization of [Gr 86a, Corollary 5.3]. It is needed here for planes only. However, the proof for this special case would not be simpler.

2.1. *Let* $\mathbf{G}$ *be an arbitrary topological geometry. Let* h_i *be two hyperplanes,* l *a line such that* $h_i \wedge l =: p_i$ *are points. If either*

a. l *contains at least three points, or*

b. $p_1 = p_2$,

then there exist neighborhoods U_i *of* p_i *in* $\overset{\circ}{h}_i$ *with* $U_1 \approx U_2$.

Proof. By (S) there exists a neighborhood V of l all of whose lines meet each h_i in a point. In both cases **a** and **b**, l contains a point q different from both p_i. Since $V \cap \underset{\bullet}{q}$ is open in the set $\underset{\bullet}{q}$ of all lines through q, by [Gr 86a, Lemma 4.3] the set

$$U := \overset{\circ}{\overbrace{V \cap \underset{\bullet}{q}}} \setminus \{q\}$$

of all points $\neq q$ on these lines is an open point set. Now the sets $U_i := U \cap \overset{\circ}{h}_i$ are the required neighborhoods, and $p_1' \mapsto (p_1' \vee q) \wedge h_2$ is a homeomorphism from U_1 onto U_2. □

2.2. **Corollary.** *Let* $\mathbf{G}$ *be an arbitrary topological plane. Let* l *be a line. Let* p_i *be two points on* l *such that there exists a point* q *outside* l *satisfying:*

(i) *Each line* $p_i \vee q$ *contains at least three points.*

(ii) *At least three lines pass through* q.

Then there exist neighborhoods U_i *of* p_i *in* $\overset{\circ}{l}$ *with* $U_1 \approx U_2$.

Proof. Apply 2.1a, 2.1b, and then again 2.1a. □

For the remainder of §2, let $\mathbf{G}$ be an arbitrary topological plane.

2.3. *Let* p *be a cluster point. Then:*

a. *For at least one line* l *through* p, p *is a cluster point in the space* $\overset{\circ}{l}$ *of all points on* l.

b. *If at least three lines pass through* p, *then* p *is a cluster point on each line through* p.

Proof. **a.** Our assumption implies the existence of a generalized sequence (or net; see e.g. [Du 66, p. 210]) $p_\nu \to p$, $p_\nu \neq p$. Because of $\dim \mathbf{G} \geq 2$ there exists a point q outside l. Since lines are hyperplanes, by (S) $p_\nu \vee q =: m_\nu$ finally meets l in a point p_ν. Now (C$\vee$) and (C$\wedge$) imply $p_\nu \to p$.

Case 1. There exist arbitrarily large ν with $p_\nu' \neq p$. Then p is a cluster point of $\overset{\circ}{l}$.

Case 2. $p_\nu' = p$ finally. Then $p_\nu' < m := p \vee q$ finally. Hence p is a cluster point of $\overset{\circ}{m}$.

b. Because of **a**, there exists a line l through p containing a generalized sequence $p_\nu \to p$ with $p_\nu \neq p$. Let $l' \neq l$ be any line through p. By assumption there exists a point q outside l and l'. Analogously to the proof for a, define $p_\nu' := (p_\nu \vee q) \wedge l'$. Again by (S), p_ν' finally is a point, and by (C$\vee$) and (C$\wedge$), $p_\nu' \to p$. By construction, $p_\nu' \neq p$. □

2.4. *Let* $l \neq m$ *be two lines meeting in a point* p *such that*

(1) p *is a cluster point on* l,

(2) m *contains at least three points.*

Then each point of l *is a cluster point on* l.

Proof. Let p' be a second point on l. Our statement will follow from Corollary 2.2, applied to $p_1 = p$, $p_2 = p'$, and $q =$ second point on m. We have to show conditions (i) and (ii):

We conclude (ii) from (1) using: (a) The point space of a topological geometry is Hausdorff [Gr 86a, 4.2]. (b) A T_1-space having a cluster point is infinite.

To show (i) we use (2) and choose a third point q' on m. By (1), and (a), (b) above there exists a generalized sequence $p_\nu \neq p$, p' on l with $p_\nu \to p$. By (C$\vee$) and (S) the line $q' \vee p_\nu$ finally meets $p' \vee q$ in a point, which must be different from q and p'. □

2.5. **Theorem.** *Let* $\mathbf{G}$ *be a nondiscrete topological plane containing a two-point line. Then* $\mathbf{G}$ *contains a line* l *and a point* q *such that* $\mathbf{G} \cong \mathbf{G}_l \times \mathbf{G}_q$.

Proof. Since **G** is not discrete, by [Gr 86a, 4.19] the point space $_\circ\mathbf{G}$ is not discrete, and therefore contains a cluster point p. By 2.3a, there exists a line l through p such that p is a cluster point on l. We prove 2.5 by showing: If the plane **G** contains at least two points outside l then it contains no two-point line. Actually we show that then each line has a cluster point on it, and hence (see 2.4, proof for (ii)) is infinite.

Assume **G** contains at least two points outside l. Either these are collinear with p, or there exist at least two lines $\neq l$ through p, and then by 2.3b l is a cluster point on each line through p. Hence in either case, there exists a line $m \neq l$ through p containing at least three points.

Therefore by 2.4 each point of l is a cluster point on l. Let now l' be any line meeting l in a point $p' \neq p$. Joining p' with the points of m we obtain at least three lines through p'. Hence by 2.3b, p' is a cluster point also on l'. Since l contains at least three points, by 2.4 each point of l' is a cluster point on it. Finally, let l' be any line $\neq l$ either meeting l in p, or disjoint to l (i.e. $l' \wedge l = 0$). Choose a point $q \neq p$ on l' and two points $p_i \neq p$ on l. By the preceding paragraph, q is a cluster point on $p_1 \vee q$, and hence by 2.3b on l'. □

3. Regular flats. Separators

In each *following section* of this paper, **G** denotes an *arbitrary* topological geometry, of (lattice theoretical) dimension n.

For 3.3 we need the following fact:

3.1. *Assume that* $(*)$ *each line contains at least three points. If there exist a* k*-flat* z *with* $0 \leq k \leq n-2$, *and an open set* $W \neq \varnothing$ *of upper neighbors, then* **G** *is discrete.*

Proof. Choose $y \in W$. Because of $k \neq n-1$ there exists a hyperplane h with $y \wedge h = z$. Because of $k \geq 0$ and (S) we may assume that each $y' \in W$ meets h in a k-flat, too. Since $z \leq y'$, this k-flat must be z. By [Gr 86a, 4.6] the set V of k-flats contained in some $(k+1)$-flat of W is open. Since $V \cap \mathbf{G}_h = \{z\}$ is open in the topological geometry [Gr 86a, 4.20a] $\mathbf{G}_h$, by $(*)$ and [Gr 86a, Theorem 5.4 (3) $\Rightarrow$ (6)], $\mathbf{G}_h$ is discrete. Hence by [Gr 86, Theorem 5.4 (2) $\Rightarrow$ (6)], **G** is discrete. □

Definition. A flat x is *regular* if and only if

$(*)$ Each line $l \leq x$ contains at least three points, and

$(**)$ The space $\overset{\circ}{x}$ of points below x is not discrete.

G is called regular if and only if 1 is a regular flat.

3.2. *If* x *is a regular flat, then*:

a. $\dim x \geq 1$.

b. *If* $\dim x \geq 2$ *then for each line* $l \leq x$, *each point of* l *is a cluster point of* $\overset{\circ}{l}$.

c. *If* $y \leq x$ *and* $\dim y \geq 1$ *then* y *is regular.*

Proof. a: $(**)$. b: [Gr 86a, Theorem 5.4 (2) $\Rightarrow$ (5)]. c. *Case* 1: $\dim x = 1$. Then $y = x$. *Case* 2: $\dim x \geq 2$: $(*)$ clearly carries over to smaller flats. $(**)$: y contains a line l, which by b has a cluster point. □

The following theorem is needed for 3.4. It generalizes [Gr 86 b, 3.2]. It also allows an alternative proof for [Gr 86b, Theorem 3.3] not using [Ka 74, p. 176, Lemma].

3.3. **Theorem.** *Each hyperplane h and each regular j-flat $x \not\leq h$, both passing through a common point, meet in a $(j-1)$-flat.*

Proof. Because of $x \not\leq h$ there exists a flat y satisfying $h \wedge x =: z \lessdot y \leq x$ and hence $h \wedge y = z$. Since $\dim z =: k \geq 0$, by (S) there exists a neighborhood V of y all of whose elements meet h in a k-flat, too. In the topological j-space $\mathbf{G}_x$, the set $W := V \cap \mathbf{G}_x$ is a neighborhood of y all of whose elements pass through z. If our statement were false, i.e. $y < x$, then because of $(*)$, 3.1 would be applicable, implying that $\overset{\circ}{x}$ would be discrete, a contradiction to $(**)$. □

3.4. *If* **G** *contains a regular hyperplane x and at least two points outside then it is regular.*

Proof. We proceed indirectly by induction. By 3.2a, the statement makes sense only for $\dim \mathbf{G} \geq 2$. Theorem 2.5 implies that it is true for $\dim \mathbf{G} = 2$. Assume that it is true for all topological geometries of dimension $n-1$, and that there exists a **G** of dimension n for which it is false. Then, since $(**)$ carries over from x to 1, $(*)$ must be violated. Thus **G** must contain a two-point line l. Since x is regular, $l \not\leq x$.

Case 1: $l \wedge x = 0$. Because $n \geq 3$, there exists a hyperplane $h \geq l$ meeting x.

Case 2: $l \wedge x$ is a *point* p. Then l contains only one further point $p' \not\leq x$. By the assumption of 3.4, there exists a second point $q \not\leq x$. Again, because $n \geq 3$, there exists a hyperplane $h \geq l \vee q$.

In either case, Theorem 3.3 implies that $h \wedge x =: x'$ is a regular (by 3.2c) hyperplane of the topological $(n-1)$-space $\mathbf{G}_h$. Since $\mathbf{G}_h$ contains at least two points outside x', it is regular by our induction hypothesis. In particular, $l \leq h$ contains at least three points, a contradiction. □

3.5. **Corollary.** *If x and y are regular flats with $x \wedge y \neq 0$, then $x \vee y$ is regular.*

Proof. We may assume $y \not\leq x$. Then because of $x \wedge y \neq 0$, there exists a line $l \leq y$ such that $x \wedge l$ is a point. Define $x_1 := x \vee l$. In the topological geometry $\mathbf{G}_{x_1}$, x is a regular hyperplane. Since y is regular, l contains at least two points outside x. Hence by 3.4, x_1 is regular. This proves our statement in case $x_1 = x \vee y$.

In case $x_1 < x \vee y$, we repeat the above procedure: $x_2 := x_1 \vee l_1, \ldots, x_k := x_{k-1} \vee l_{k-1} = x \vee y$. □

3.6. *Assume* **G** *contains a maximal regular flat $x < 1$. Let h be a hyperplane satisfying $0 < x \wedge h =: y < x$. In case x is a line, assume further:* (c) *y is a cluster point on $\overset{\circ}{x}$. Then* **G** *contains no point outside x and h.*

Proof (indirect; see Figure 1). Assume there exists a point $q \notin \overset{\circ}{x} \cup \overset{\circ}{h}$. By assumption, there exists a point $p \leq y$. Because of $y < x$ there exists a line $l \leq x$ with $l \wedge y = p$. Since x is regular, by 3.2b—in case x is a line by

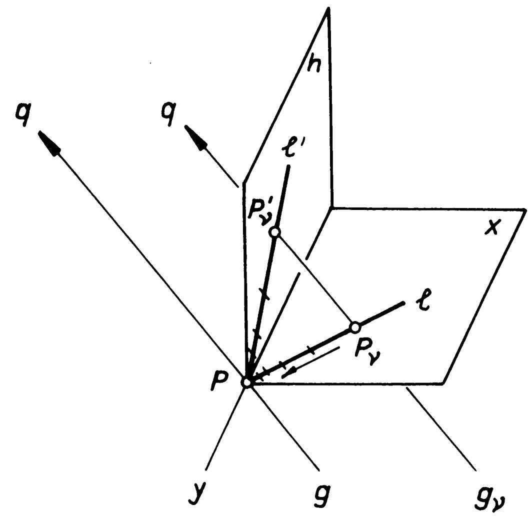

FIGURE 1

(c)—there exists a generalized sequence $p \neq p_\nu \lessdot l$ converging to p. Because of $q \not\leq h$, $q \vee p =: g$ is a line meeting h in a point (namely p). By (C$\vee$) and (S), $q \vee p_\nu =: g_\nu$ is a line finally meeting h in a point p'_ν. Because of $q \not\leq x$ we have $p'_\nu \neq p$. Hence $(q \vee l) \wedge h =: l'$ is a line containing the generalized sequence p'_ν, which by (C$\wedge$) has p as a cluster point. Now $l' \not\leq x$ implies $x' := x \vee l' > x$. By Corollary 3.5, x' is regular, too, a contradiction to the maximality of x. □

3.7. *Assume* **G** *contains a maximal regular flat* $l < 1$ *which is a line. Then there exists a coline* y *satisfying*

(1) $l \wedge y = 0$,
(2) **G** *contains no point outside* l *and* y.

Proof. By (∗∗), $\mathring{l}$ contains a cluster point p. Because of $l < 1$ there exists a hyperplane h with $l \wedge h = p$. Choose $y \lessdot h$ with $p \not\leq y$. Clearly, y satisfies (1). (2) (Indirect; see Figure 2): Assume there exists a point q neither on l nor on y. By 3.6 we have ${}_{\circ}\mathbf{G} = \mathring{l} \cup \mathring{h}$ and hence $q \leq h$.

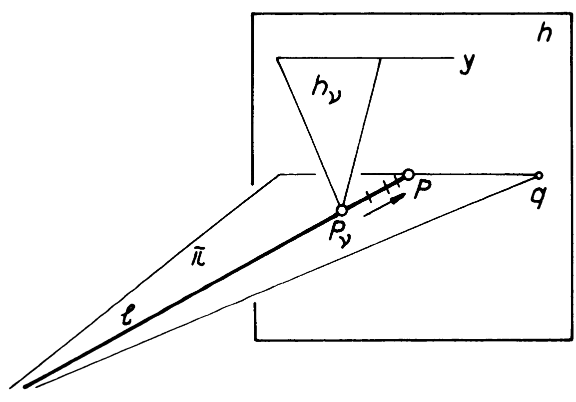

FIGURE 2

Let $p \neq p_\nu \lessdot l$ be a generalized sequence converging to p. Then $h_\nu := y \vee p_\nu$ is a generalized sequence of hyperplanes, which by (C$\vee$) converges to $h = y \vee p$. For the plane $\pi := l \vee q$ we have that $\pi \wedge h = p \vee q$ is a line. Hence by (S), $\pi \wedge h_\nu$ finally is a line, too. On the other hand, since l is maximal regular also in the topological plane $\mathbf{G}_\pi$, 3.4 implies $\mathring{\pi} = \mathring{l} \cup \{q\}$. Since both $l < h_\nu$ and $q < h_\nu$ are impossible, $\pi \wedge h_\nu = p_\nu$ is a point. This is a contradiction. □

Definition (cf. [CR 70, p. 12.1]; [Wh 86, p. 176]). A flat x of a geometric lattice $\mathbf{G}$ is a *separator* if and only if for each "nonincident" pair $p \not\leq h$ of a point p and a hyperplane h, either $p \leq x$ or $x \leq h$ holds.

In each $\mathbf{G}$, 0 is a separator. A separator s is called *minimal* if and only if $s \neq 0$ and there exists no separator s' with $0 < s' < s$.

3.8. **Theorem.** *Maximal regular flats x are minimal separators.*

Proof. Let x be a maximal regular flat.

Claim: x is a *separator*. If $x = 1$, then trivially x is a separator. If $x < 1$, let $0 \lessdot p \not\leq h \lessdot 1$ be given. Assume $x \not\leq h$.

Case 1: $\dim x \geq 2$. Then $x \wedge h \neq 0$. There exists a hyperplane h' satisfying $0 < x \wedge h' < h'$. By 3.6, we have ${}_0\mathbf{G} = \mathring{x} \cup \mathring{h}'$. Hence $x \wedge h = 0$ would imply $\mathring{h} \subseteq \mathring{h}'$, and therefore $h = h'$, a contradiction to $x \wedge h' \neq 0$. Now because of $0 < x \wedge h < x$, 3.6 implies $p \leq x$.

Case 2: $x =: l$ is a line. Then by 3.7 we have ${}_0\mathbf{G} = \mathring{l} \dot\cup \mathring{y}$. From this one can easily determine the hyperplanes of $\mathbf{G}$: $l \not\leq h$ implies $h = q \vee y$ for some $q \lessdot l$. Hence $p \not\leq h$ implies $p \leq l = x$.

Claim: The flat x is *minimal.* For each $0 < s < x$ we construct a point-hyperplane pair $0 \lessdot p \not\leq h \lessdot 1$ such that $p \not\leq s$ and $s \not\leq h$: In the geometry $\mathbf{G}_x$, choose points $p \not\leq s$ and $q \leq s$. By $(*)$ the line $p \vee q$ contains a third point r. Again because of $(*)$, we may apply [Gr 86a, 2.3b] to $x = r$, $y_1 = p$, $y_2 = s$, and obtain a flat $k \lessdot x$ containing neither p nor s. There exists a hyperplane $h \geq k$ of $\mathbf{G}$ not containing x. □

4. Factorization

4.1. *For each separator s, the set $\mathbf{G}_s$ of flats below s is open.*

Proof. By the definition of the topology on $\mathbf{G}$ (see §1.b.2), we must show that for each i the set ${}_i\mathbf{G}_s$ of i-flats below s is open in the set ${}_i\mathbf{G}$ of all i-flats. We use induction on i:

I. $i = 0$: By [CR 70, Proposition 12.4], $\mathbf{G}$ contains a complement t of s such that each point is either on s or on t. Since the set $\mathring{t}$ of all points below any flat t is closed [Gr 86a, 4.1], ${}_0\mathbf{G}_s = \mathring{s} = {}_0\mathbf{G} \setminus \mathring{t}$ is open.

II. Assume ${}_i\mathbf{G}_s$ is open. Since ${}_i\overset{\circ}{\bigvee}_{i+1}$ is an open function [Gr 86a, 4.9], ${}_{i+1}\mathbf{G}_s = {}_i\mathbf{G}_s \overset{i+1}{\bigvee} \mathring{s}$ is also open. □

4.2. **Corollary.** *Each separator is isolated.*

Proof. By 4.1, $\{s\} = {}_{\dim s}\mathbf{G}_s$ is open in ${}_{\dim s}\mathbf{G}$. □

Two flats s, t are said to form a *skew pair* if they form a modular pair with $s \wedge t = 0$; see e.g. [CR 70, p. 219].

4.3. *If two flats* s *and* t *form a skew pair, then the restriction of* $\vee$ *to* $\mathbf{G}_s \times \mathbf{G}_t$ *is continuous.*

Proof. Let (x_ν, y_ν), $(x, y) \in \mathbf{G}_s \times \mathbf{G}_t$, where x_ν and y_ν are generalized sequences converging to x resp. y. Then $\dim x_\nu = \dim x =: i$ and $\dim y_\nu = \dim y =: j$ finally. By [MM 70, 1.53], $s \text{ M } t$ implies $x_\nu \text{ M } y_\nu$. Together with $x_\nu \wedge y_\nu = 0$ we obtain

$$\dim x_\nu \vee y_\nu = \dim x_\nu + \dim y_\nu - \dim x_\nu \wedge y_\nu = i + j + 1$$

finally. Hence the continuity of ${}_i\bigvee_k^j$ [Gr 86a, Theorem 4.16] implies $x_\nu \vee y_\nu \to x \vee y$. □

We need the following extension of [CR 70; §12, Theorem 3] to topological geometries:

4.4. *For two elements* s, t *the map*

$$\iota_s \times \iota_t : \mathbf{G}_s \times \mathbf{G}_t \to \mathbf{G}, \quad \textit{defined by } (x, y) \mapsto x \vee y,$$

is an isomorphism of topological geometries if and only if s *and* t *are complementary separators.*

Proof. Because of [CR 70; §12, Theorem 3] it suffices to show: Let s and t be complementary separators. If $\iota_s \times \iota_t$ is an isomorphism of geometric lattices, then it is also a homeomorphism.

Since $\iota_s \times \iota_t$ is bijective, it suffices to show that it is (1) continuous, and (2) open.

(1) Since $\mathbf{G}$ is isomorphic to $\mathbf{G}_s \times \mathbf{G}_t$ as a geometric lattice, we have

$$\dim s \vee t = \dim s + \dim t + 1,$$

and hence (see e.g. [CR 70, Proposition 2.8]) $s \text{ M } t$. We now can apply 4.3.

(2) If A is an open subset of $\mathbf{G}_s$, then by 4.1 it is also open in $\mathbf{G}$. Likewise for $B \subseteq \mathbf{G}_t$. Hence the openness of $\vee$ [Gr 86a, Corollary 4.14] implies that $A \vee B$ is open in $\mathbf{G}$. □

4.5. **Corollary.** *Denote by* $r_1, \ldots, r_k$ *the maximal regular flats of* $\mathbf{G}$, *and let* d *be a complement of* $r_1 \vee \cdots \vee r_k =: s$. *Then* d *is unique, and the map*

$$\iota_1 \times \cdots \times \iota_k \times \iota : \mathbf{G}_{r_1} \times \cdots \times \mathbf{G}_{r_k} \times \mathbf{G}_d \to \mathbf{G}$$

defined by $(x_1, \ldots, x_k, y) \mapsto x_1 \vee \cdots \vee y_k \vee y$ *is an isomorphism of topological geometries.*

Proof. By Theorem 3.8, each r_i is a separator. By [CR 70, §12, Theorem 2] so is each join of these, in particular s. By [CR 70, Proposition 12.4] there is exactly one complement d of s, and d is again a separator. We now use 4.4 for $k = 1, 2, 3, \ldots$ inductively. □

4.6. **Main Theorem I.** *Each topological geometry* $\mathbf{G}$ *is isomorphic to the direct product of finitely many*

(1) *open subgeometries of projective spaces over topological fields,*
(2) *topological planes (not necessarily desarguesian or projective),*
(3) *lines over Hausdorff topological spaces, or*
(4) *discrete geometries*

Proof. Because of 4.5 it suffices to know all *regular* topological geometries:

(1) For dimension at least three, these have been classified in [Gr 86b, Corollary 4.5].

(2), (3): Dimensions 2 resp. 1: see [Gr 86a, p. 115, Remarks 2 and 3] (here regularity is not needed). □

5. Varieties

We now take over from Kahn and Kung, just replacing "finite" (combinatorial) by "topological", the following definition motivated in [KK 82, p. 486]:

Definition. A class $\mathscr{T}$ of topological geometries is called a variety if and only if

(H) ("hereditary") $\mathscr{T}$ is closed under the formation of isomorphic copies, (Sub) subgeometries, (C) contractions, and (P) finite direct products; and

(U) $\mathscr{T}$ contains a sequence $\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3, \ldots$ ("universal models") of dimensions $1, 2, 3, \ldots$ such that each $\mathbf{G} \in \mathscr{T}$ of dimension i is isomorphic to a subgeometry of $\mathbf{T}_i$.

Remark. (Sub) means that a subgeometry is also in $\mathscr{T}$ provided it is a *topological* geometry again. Note that this is not always the case: Let $\mathbf{G}$ be the euclidean plane, and M the closed unit disc. Then $\mathbf{G}(M)$ does not satisfy (S).

Notation. Let X, Y be two topological geometries (topological spaces, topological fields). We write

$$X \prec Y$$

if and only if X is isomorphic to a subgeometry (subspace, subfield) of Y.

Remark. $X \prec Y$ and $Y \prec X$ do not imply $X \cong Y$: For the topological spaces $X = \mathbb{R}$ and $Y = [0, 1]$ we have $Y \prec X$ and $X \cong]0, 1[\prec Y$. Nevertheless $X \ncong Y$ (cf. §1.a).

Definition. a. *Projective variety* $\mathscr{L}(F)$: Let F be a topological field. Define $\mathscr{L}(F)$ to be the class of all topological geometries isomorphic to some subgeometry of some topological projective n-space $\mathbf{P}_n(F) =: \mathbf{P}_n$ (see §1).

b. *Matchstick variety* $\mathscr{M}(X)$: Let X be a Hausdorff topological space containing at least two elements. Let $p \in X$. Define $\mathbf{l} :=$ topological line over X (see [Gr 86a, p. 115, Remark 3]), and $\mathbf{p} := \{0, p\}$, $0 \neq p$, ("topological point"). Let $\mathbf{M}_{2k-1} := \mathbf{l}^k$, and $\mathbf{M}_{2k} := \mathbf{l}^k \times \mathbf{p}$. Define $\mathscr{M}(X)$ to be the class of all topological geometries isomorphic to some subgeometry of some $\mathbf{M}_n$.

We hasten to supplement:

5.1. *Each $\mathscr{L}(F)$ and each $\mathscr{M}(X)$ is a variety.*

Proof. Straightforward. □

5.2 **Lemma.** *Let F, F' be topological fields, and let $2 \leq m \leq n$ be integers. Then*

$$\mathbf{P}_m(F) \prec \mathbf{P}_n(F') \Leftrightarrow F \prec F'.$$

Proof. $(\Rightarrow)$ Because of $m \geq 2$ we can conclude $\mathbf{P}_2(F) \prec \mathbf{P}_2(F') =: \mathbf{P}$. Thus $\mathbf{P}_2(F) \cong \mathbf{P}(M)$ for some point set $M \subseteq {}_\circ\mathbf{P}$. Since $\mathbf{P}$ induces on M a (topological) projective plane, M contains a quadrangle. Any corresponding ordered quadrangle (o, u, v, i) induces operations $+$ and $\cdot$ on the points $\neq v$ of the line $o \vee v =: l$, making it a (ternary) field K' (see e.g. [HP 73, V.2]; [Ha 59, 20.3]; [Pi 75, 1.5]). Since $+$, $\cdot$, and $^{-1}$ are multiple compositions of the functions ${}_\circ\overset{1}{\underset{\circ}{\bigvee}}{}_\circ$ and ${}^1\bigwedge^1$, by (C$\vee$) and (C$\wedge$), K' is a *topological* field. One verifies that each $\mathbf{P}$-automorphism in $PGL_2(F')$ is continuous. Hence the transitivity of the group $PGL_2(F')$ on the set of ordered quadrangles (see e.g. [HP 73, Theorem 2.12]) implies $K' \cong F'$ (as topological fields). Furthermore, $\overset{\circ}{l} \cap M \setminus \{v\}$ becomes a subfield $K \leq K'$. In $\mathbf{P}_2(F)$, the ordered quadrangle corresponding to (o, u, v, i) under the isomorphism $\mathbf{P}(M) \cong \mathbf{P}_2(F)$ yields a topological field $K^* \cong K$. Analogously to the above, the transitivity of $PGL_2(F)$ implies $K^* \cong K$. To sum up, $F \cong K^* \cong K \leq K' \cong F'$ implies $F \prec F'$.

$(\Leftarrow)$ Verification, using [Gr 86a, 4.22]. □

5.3. *Let $\mathscr{S}$, $\mathscr{T}$ be varieties containing sequences $\mathbf{S}_n$ resp. $\mathbf{T}_n$ of universal models such that there exists an integer k with $\mathbf{S}_n \prec \mathbf{T}_n$ for all $n \geq k$. Then $\mathscr{S} \subseteq \mathscr{T}$.*

Proof. (U), transitivity of $\prec$, (Sub). For $\dim \mathbf{G} < k$: (P). □

5.4. **Main Theorem II.** *Let $\mathscr{T}$ be a variety of topological geometries containing a nondiscrete one. Then $\mathscr{T}$ is either*

A. *a projective variety $\mathscr{L}(F)$ over some topological field F, or*

B. *a matchstick variety $\mathscr{M}(X)$ over some Hausdorff topological space X.*

Proof. Since $\mathscr{T}$ contains a nondiscrete geometry, by the Main Theorem I, [Gr 86a, 5.4(2) $\Rightarrow$ (6)], (Sub), and (U), $\mathbf{T}_1$ must be a regular line.

Case A: $\mathbf{T}_2$ *regular.* Then all $\mathbf{T}_n$ are regular (cf. [KK 82, Lemma 5]): From (Sub), (P) and (U) we deduce $\mathbf{T}_1, \mathbf{T}_{n-2}, \mathbf{T}_{n-1} \prec \mathbf{T}_n$. Assume $\mathbf{T}_n$ is not regular, but all preceding $\mathbf{T}_i$ are. Then the Main Theorem I and $\mathbf{T}_{n-1} \prec \mathbf{T}_n$, as well as the regularity of $\mathbf{T}_{n-1}$, imply $\mathbf{T}_n \cong \mathbf{H} \times \mathbf{p}$. Here $\mathbf{H}$ is a regular topological $(n-1)$-space, and $\mathbf{p}$ is a point. Now from (P) and (U) we get $\mathbf{T}_{n-2} \times \mathbf{T}_1 \prec \mathbf{H} \times \mathbf{p}$. Because of the regularity of $\mathbf{T}_{n-2}$ and $\mathbf{T}_1$, this implies $\mathbf{T}_{n-2} \times \mathbf{T}_1 \prec \mathbf{H}$, a contradiction to $\dim \mathbf{H} = n - 1$.

The regularity of the $\mathbf{T}_n$ implies for $n \geq 3$, by [Gr 86b, Corollary 4.5], the existence of a topological field F_n with $\mathbf{T}_n \prec \mathbf{P}_n(F_n)$. Let $F =: F_3$. We will show $\mathscr{T} = \mathscr{L}(F)$ by a sequence of steps:

(1) $\mathbf{T}_n^p \cong \mathbf{P}_{n-1}(F_n)$ for each point p of $\mathbf{T}_n$: Since $\mathbf{T}_n$ may be embedded as an *open* subgeometry of $\mathbf{P}_n(F_n) =: \mathbf{P}$ [Gr 86b, Corollary 4.5], we have by [Gr 86a, Corollary 5.6c] that $\mathbf{T}_n^p = \mathbf{P}^p \cong \mathbf{P}_{n-1}(F_n)$.

(2) $F \prec F_n$: Since $\prec$ is transitive, it suffices to show $F_n \prec F_{n+1}$: Because of (C), (P), and (U) we have $\mathbf{T}_n^p \times \mathbf{T}_1 \prec \mathbf{T}_{n+1} \prec \mathbf{P}_{n+1}(F_{n+1})$. Together with (1) we deduce $\mathbf{P}_{n-1}(F_n) \prec \mathbf{P}_{n+1}(F_{n+1})$. By Lemma 5.2 $(\Rightarrow)$ this implies $F_n \prec F_{n+1}$.

(3) $F_n \prec F$: Because of (1), (C), and (U) we have $\mathbf{P}_{n-1}(F_n) \cong \mathbf{T}_n^p \prec \mathbf{T}_{n-1} \prec \mathbf{P}_{n-1}(F_{n-1})$. By Lemma 5.2 $(\Rightarrow)$ this implies $F_n \prec F_{n-1}$.

(4) $\mathbf{P}_n(F) \prec \mathbf{P}_n(F_{n+1}) \cong \mathbf{T}_{n+1}^p \prec \mathbf{T}_n$ by (2), Lemma 5.2 $(\Leftarrow)$, (1), (C), and (U).

(5) $\mathbf{T}_n \prec \mathbf{P}_n(F_n) \prec \mathbf{P}_n(F)$ by (3) and Lemma 5.2 $(\Leftarrow)$.

(6) $\mathscr{T} = \mathscr{L}(F)$ because of (4), (5), and 5.3.

Case B: $\mathbf{T}_2$ *not regular.* Then by (Sub) and (U), no $\mathbf{T}_n$ contains a regular plane. For an arbitrary variety, (P) and (U) imply:

$$\mathbf{M}_{2k-1} := \mathbf{T}_1^k \prec \mathbf{T}_{2k-1} \quad \text{and} \quad \mathbf{M}_{2k} := \mathbf{T}_1^k \times \mathbf{p} \prec \mathbf{T}_{2k},$$

where $\mathbf{p} \in \mathscr{T}$ is a point. Combining all this with the regularity of $\mathbf{T}_1$ and the Main Theorem I, we obtain

$$\mathbf{T}_{2k-1} = \mathbf{l}_{1,2k-1} \times \cdots \times \mathbf{l}_{k,2k-1} \quad \text{and} \quad \mathbf{T}_{2k} = \mathbf{l}_{1,2k} \times \cdots \times \mathbf{l}_{k,2k} \times \mathbf{p},$$

where $\mathbf{l}_{ij}$ are regular lines. (Sub) and (U) imply $\mathbf{l}_{ij} \prec \mathbf{T}_1$. Hence $\mathbf{T}_n \prec \mathbf{S}_n$. In summary, 5.3 implies $\mathscr{T} = \mathscr{M}(X)$, where X is the point space of $\mathbf{T}_1$. □

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