

COMBINATORICS OF TRIANGULATIONS OF 3-MANIFOLDS

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ABSTRACT. In this paper, we study the average edge order of triangulations of closed 3-manifolds and show in particular that the average edge order being less than 4.5 implies that triangulation is on the 3-sphere.

INTRODUCTION

Let K be a triangulation of a closed 3-manifold M with $V_0(K)$, $E_0(K)$, $F_0(K)$, and $T_0(K)$ the numbers of vertices, edges, triangles, and tetrahedra in K , respectively. The order of an edge in K is the number of triangles incident to that edge. The average edge order of K is then $3F_0(K)/E_0(K)$, which we will denote $\mu_0(K)$. The degree of a vertex is the number of edges incident to it in the triangulation, therefore the average vertex degree is $2E_0(K)/V_0(K)$. This quantity is closely related to the average edge order and is occasionally more convenient. The purpose of this note is to show that $\mu_0(K)$ being small implies that the topology of M is fairly simple and restricts the triangulation K . To be more precise, we have

Theorem. *Let K be any triangulation of a closed connected 3-manifold M . Then*

- (a) $3 \leq \mu_0(K) < 6$, equality holds if and only if K is the triangulation of the boundary of a 4-simplex.
- (b) For any $\varepsilon > 0$, there are triangulations K_1 and K_2 of M such that $\mu_0(K_1) < 4.5 + \varepsilon$ and $\mu_0(K_2) > 6 - \varepsilon$.
- (c) If $\mu_0(K) < 4.5$, then K is a triangulation of S^3 .
- (d) If $\mu_0(K) = 4.5$, then K is a triangulation of S^3 , $S^2 \times S^1$, or $S^2 \tilde{\times} S^1$. Furthermore, in the last two cases, the triangulations can be described.

Remark. There are an infinite number of distinct triangulations satisfying (c) in the theorem. However, as we will show later, for any constant $c < 4.5$ there are only finitely many triangulations K with $\mu_0(K) \leq c$.

The motivation for this work comes from the 2-dimensional case. Suppose we have a triangulation of a closed 2-manifold N with v , e , f the numbers of vertices, edges, and triangles respectively. The order of a vertex is the number of triangles incident to it. Thus the average vertex order is $3f/v$ which is the same as $6 - 6\chi(N)/v$ by a Euler characteristic calculation. Therefore, the

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average vertex order being less than 6, equal to 6, or greater than 6 corresponds to N having a spherical, Euclidean, or hyperbolic structure, respectively.

The first two statements of the theorem are very easy. Indeed, since $V_0 - E_0 + F_0 - T_0 = 0$, and $2T_0 = F_0$, we find

$$\mu_0(K) = 3F_0/E_0 = (6E_0 - 6V_0)/E_0 = 6 - 6V_0/E_0.$$

Therefore, $\mu_0(K) < 6$. On the other hand, each edge has at least three triangles incident to it so $\mu_0(K) \geq 3$. The equality case is not hard but we will handle it later as an easy application of the machinery we will develop. As to (b), suppose K has an edge e of order a . Stellar subdividing K by adding vertices in the interior of e n times, we obtain a sequence of triangulations K_n with

$$E_0(K_n) = E_0(K) + (a+1)n \quad \text{and} \quad F_0(K_n) = F_0(K) + 2an.$$

Thus, $\lim_{n \rightarrow \infty} \mu_0(K_n) = 6a/(1+a)$ which is 4.5 if $a = 3$ and approaches 6 as a tends to infinity. Since any 3-manifold M clearly has a triangulation with an edge of order 3 and triangulations with edges of arbitrarily high orders, this establishes (b).

Sketch of the proof of the theorem. Suppose K is a triangulation of M with $\mu_0(K) \leq 4.5$, i.e., $2F_0(K) \leq 3E_0(K)$. Then the average vertex degree satisfies

$$2E_0/V_0 = 2E_0/(E_0 - F_0/2) = 12/(6 - \mu_0) \leq 8.$$

Therefore, by the classification of triangulations of S^2 up to eight vertices, see Oda's book [Od, p. 192], there is a vertex v in K so that $\text{lk}(v)$ is one of the twenty-three triangulations with at most 8 vertices. The goal now is to simplify K to K' so that $\mu_0(K')$ is still less than 4.5. If K is obtained from K' by a stellar subdivision, then $\mu_0(K') \leq 4.5$. To see this simply note that if a stellar subdivision (adding one vertex) introduces E new edges, then it produces at least $3E/2$ new faces (triangles). Therefore, if we could always reduce K at $\text{st}(v)$ by reversing stellar subdivisions, the process would be finished. However, reversing stellar subdivisions alone is not sufficient. New reductions need to be introduced and the structure of the proof becomes more complicated. The first reduction is similar to the sphere decomposition theorem in 3-manifolds (see for example [He]). A subcomplex isomorphic to $\partial\Delta^3$ or the suspension of a triangle $\Sigma\partial\Delta^2$ is the "sphere" in this case. A copy of $\partial\Delta^3$ is inessential in K if it is the boundary of a "ball", a 3-simplex in K . A copy of $\Sigma\partial\Delta^2$ is inessential in K if it is the boundary of a "ball", which is two tetrahedra with a common face in K . Suppose K contains an essential $\partial\Delta^3$ "sphere". We cut K along $\partial\Delta^3$ and add two "balls" to produce a new triangulation (or possibly two new triangulations if the sphere separates M). If there are no more essential $\partial\Delta^3$ type "spheres", we can cut K along an essential $\Sigma\partial\Delta^2$ (if any), and add two "balls" to produce a new triangulation (or two triangulations if the sphere is separate). Note that cutting along an essential $\Sigma\partial\Delta^2$ in general will not result a triangulation if there are essential $\partial\Delta^3$ spheres. It turns out this splitting process will reduce either the Betti number or the number of edges, and the new triangulation(s) still satisfies $\mu_0(\cdot) \leq 4.5$. As a special case, if K has an edge e of order 3, then the boundary of $\text{st}(e)$ is a copy of $\Sigma\partial\Delta^2$, hence K can be reduced. Thus we may assume every edge of K has order at least 4, i.e.,

every vertex of $\text{lk}(v)$ must have degree at least 4 in $\text{lk}(v)$. Further reductions can be introduced:

- (1) contraction of an edge to a vertex and
- (2) "rotation" of an edge of order 4.

One can easily determine when these reductions will produce triangulations. Carefully checking cases (using the classifications of triangulations of S^2 up to eight vertices) gives the following technical result.

Claim. Suppose K is a triangulation of a closed 3-manifold and is not the boundary of the 4-simplex. If K cannot be reduced by a sequence of the reductions above, then every neighbor vertex of a vertex of degree at most 7 must have degree at least 9.

From this claim, an easy calculation shows that the average vertex degree must be at least 8, i.e., $\mu_0(K) \geq 4.5$. Handling the case $\mu_0(K) = 4.5$ requires a slight refinement of this claim (which in turn requires the classification of triangulations of S^3 up to nine vertices).

We will use the following terminology. See Spanier's book [Sp] for reference on PL topology and Grunbaum's book [Gr] on triangulations of convex polytopes. A simplex with vertices v_0, \dots, v_n will be denoted by $[v_0, \dots, v_n]$ and $\partial\Delta^3$ denotes the boundary complex of a 3-simplex. For any two triangulations K and L denote by $K\#L$ the connected sum triangulation obtained by removing an open 3-simplex from each of K and L and gluing along the resulting boundaries. Let $\text{lk}(\cdot)$ and $\text{st}(\cdot)$ denote the link and star of a simplex in the ambient complex. Let $\text{lk}(\cdot, A)$ and $\text{st}(\cdot, A)$ denote the link and star of a simplex in the subcomplex A . We will say that a cell decomposition of an n -manifold is a decomposition of the manifold into imbedded simplices such that the intersection of any two simplices is a union of some of their faces. Such a decomposition is a triangulation of the n -manifold if and only if the intersection of any two simplices is actually a face of each of them. We will also find it convenient for technical reasons to define $E_1(K) = E_0(K) - 6$, $F_1(K) = F_0(K) - 4$, and $\mu_1(K) = 3F_1(K)/E_1(K)$. The quantities E_1 and F_1 have the advantage that they are additive under connected sum along a copy of $\partial\Delta^3$. Note that

$$\mu_1(K) = 6 - 6(V_0 - 4)/(E_0 - 6) = \mu_0(K) + 6(\mu_0(K) - 2)/(E_0 - 6)$$

therefore for any triangulation K , $6 > \mu_1(K) > \mu_0(K)$.

THE PROOF OF THE THEOREM

We will first define a number of operations on triangulations. These operations when applied to a triangulation K will always produce a cell decomposition. To understand when they also give triangulations we need the following easy lemma.

Lemma 1. *Let D be a cell decomposition of an n -manifold M , then D fails to be a triangulation if and only if there are two simplices of D with the same boundary.*

Proof. Clearly if two simplices have the same boundary, then D is not a triangulation. Conversely suppose no two simplices of D have the same boundary. Let σ and σ' be simplices of D which intersect in $k+1$ points $\{x_0, x_1, \dots, x_k\}$. We will show that their intersection is actually the k -simplex $[x_0, x_1, \dots, x_k]$.

By definition $\sigma \cap \sigma'$ contains the 0-skeleton of $[x_0, x_1, \dots, x_k]$. Suppose it contains the $(r-1)$ -skeleton. If any $r+1$ points spanned different r -simplices in σ and σ' , then their boundaries would be in the common $(r-1)$ -skeleton and hence would agree. By hypothesis, this cannot happen and hence $\sigma \cap \sigma'$ must contain every r -simplex. By induction it must be the entire k -simplex.

One of our operations will be contraction of an edge of the triangulation. Explicitly, given an edge $[x, y]$ in the triangulation K we let K' be the cell decomposition obtained by removing all simplices containing $[x, y]$ and identifying two r -simplices $[x, v_1, \dots, v_r]$ and $[y, v_1, \dots, v_r]$ if and only if $[x, y, v_1, \dots, v_r]$ is an $(r+1)$ -simplex of K . The lemma above tells us when this procedure produces a triangulation. Call a copy of $\partial\Delta^k$ in a triangulation essential if it does not bound a k -simplex.

Lemma 2. *Contraction of the edge $[x, y]$ in the triangulation K produces a triangulation K' if and only if $[x, y]$ lies in no essential $\partial\Delta^k$.*

Proof. If $[x, y]$ lies in the essential $\partial\Delta^k$ spanned by $\{x, y, v_1, \dots, v_r\}$, then the images of the r -simplices $[x, v_1, \dots, v_r]$ and $[y, v_1, \dots, v_r]$ in K' are two different r -simplices with the same boundary. Therefore K' is not a triangulation.

Conversely if K' is not a triangulation, then there are two simplices of minimal dimension, say r , in K' with the same boundary. These must be the images of two simplices of the form $[x, v_1, \dots, v_r]$ and $[y, v_1, \dots, v_r]$. Since r was minimal, K must have also contained every r -simplex spanned by a subset of $\{x, y, v_1, \dots, v_r\}$. These r -simplices together form an essential $\partial\Delta^{r+1}$ containing $[x, y]$.

Another operation will be “rotation” of an order 4 edge. Explicitly, suppose $[x, y]$ is an order 4 edge, say with link the quadrilateral $ABCD$. We may remove the interior of the octahedron spanned by $\{x, y, A, B, C, D\}$ and replace it with the same figure but rotated so that the edge joins the antipodal vertices A and C . This always produces a cell decomposition as defined above. Further since the only new simplex created whose boundary is in the boundary of the octahedron is $[A, C]$ it gives a triangulation unless $[A, C]$ was already in K .

Our final two operations will be cutting K along a copy of S^2 and gluing in two 3-balls. Specifically, suppose Σ is either a copy of $\partial\Delta^3$ or of the suspension of a triangle $\Sigma\partial\Delta^2$. Then we can cut K along Σ . In the first case, we glue in two copies of the 3-simplex along the resulting boundary components. In the second case, we glue in two copies of a union of two 3-simplices with a common face. Again this always produces a cell decomposition. In the first case, it is a triangulation unless Σ already bounded a 3-simplex in K , i.e. if Σ was inessential. In the second case, it is a triangulation unless the suspended triangle was already inessential in K . Call a copy of $\Sigma\partial\Delta^2$ essential if it does not bound a union of two 3-simplices with a common face.

Theorem 3. *Let K be any triangulation of a 3-manifold M , then (1) $\mu_1(K) \geq 4.5$, with equality if and only if K is a triangulation of S^3 as $\#^k \partial\Delta^4$, for some $k \geq 1$.*

(2) If $M \neq S^3$, then $\mu_0(K) \geq 4.5$, with equality if and only if K is a

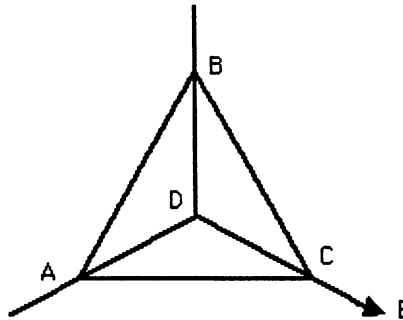


FIGURE 1

triangulation of either $S^2 \times S^1$ or $S^2 \tilde{\times} S^1$ obtained from $\#^k \partial\Delta^4$ by removing two open 3-simplices and gluing along their boundaries.

Proof. The proofs of (1) and (2) are parallel. We will present them simultaneously, but the reader should regard (1) as being proved first since the proof of (2) relies on (1). Suppose that there is a triangulation K of M that is a counterexample to either (1) or (2). Assume that K is minimal in that any other counterexample K' either has $b_1(M') > b_1(M)$ or has $b_1(M') = b_1(M)$ and $E_0(K') > E_0(K)$. The move described above, rotation of an order 4 edge, does not change $b_1(M)$ or $E_0(K)$. Therefore we may further insist that among all triangulations obtainable from K by these rotations, K has vertices of minimal degree and the maximum number of vertices of that degree. We will call such a triangulation crowded.

Claim 1. If K is crowded, then it does not contain an essential $\partial\Delta^3$.

Proof. Suppose K contains an essential $\partial\Delta^3$. Then we could cut along this 2-sphere (which may or may not disconnect K) and glue in two copies of the 3-simplex. If this procedure disconnects K , then it divides K into two smaller pieces say K_1 and K_2 .

These pieces have $E_0(K) = E_0(K_1) + E_0(K_2) - 6$ and $F_0(K) = F_0(K_1) + F_0(K_2) - 4$. Therefore

$$\mu_1(K) = \frac{3F_1(K_1) + 3F_1(K_2)}{E_1(K_1) + E_1(K_2)} \geq 4.5.$$

If $M \neq S^3$, then one of the pieces, say K_1 , is not S^3 and therefore

$$\mu_0(K) = \frac{3F_0(K_1) + 3F_1(K_2)}{E_0(K_1) + E_1(K_2)} \geq 4.5.$$

If this cutting does not disconnect, then the cutting produces a triangulation K' with lower b_1 and $E_0(K) = E_0(K') - 6$ and $F_0(K) = F_0(K') - 4$, therefore $\mu_1(K) > \mu_0(K) = \mu_1(K') \geq 4.5$.

Claim 2. If K is crowded, then it does not contain an essential $\Sigma\partial\Delta^2$.

Proof. Label the vertices of any copy of $\Sigma\partial\Delta^2$ in K as in Figure 1. If K contains the 2-simplex $[A, B, C]$, then we are reduced to the case above. If not, then as above we can cut along this 2-sphere (which may or may not disconnect K) and glue in two copies of the union of two 3-simplices. If this procedure

disconnects K , then it divides K into two smaller pieces say K_1 and K_2 . These pieces have $E_0(K) = E_0(K_1) + E_0(K_2) - 9$ and $F_0(K) = F_0(K_1) + F_0(K_2) - 8$. Therefore

$$\mu_1(K) = \frac{3F_1(K_1) + 3F_1(K_2) - 12}{E_1(K_1) + E_1(K_2) - 3} > 4.5.$$

If $M \neq S^3$, then one of the pieces, say K_1 , is not S^3 and therefore

$$\mu_0(K) = \frac{3F_0(K_1) + 3F_1(K_2) - 12}{E_0(K_1) + E_1(K_2) - 3} > 4.5.$$

If this cutting does not disconnect, then it produces a K' with lower first Betti number. Further $E_0(K) = E_0(K') - 9$ and $F_0(K) = F_0(K') - 8$, therefore

$$\mu_1(K) = \frac{3F_1(K') - 24}{E_1(K') - 9} > 4.5 \quad \text{and} \quad \mu_0(K) = \frac{3F_1(K') - 12}{E_1(K') - 3} > 4.5.$$

Note that as a special case of this claim no edge of K can have order three. A rephrasing of this statement is the following claim which will be used extensively below.

Claim 3. Let K be crowded and let v be a vertex of K . If $[A, B, C]$ and $[D, B, C]$ are adjacent faces in $\text{lk}(v)$ in K , then $[D, B, C,]$ is not a face of $\text{lk}(A)$.

Proof. If it were, then $[B, C]$ would have order three being contained in the three 3-simplices $[v, A, B, C]$, $[A, D, B, C]$ and $[D, v, B, C]$.

Claim 4. If K is crowded, then any edge of K must be contained in an essential triangle.

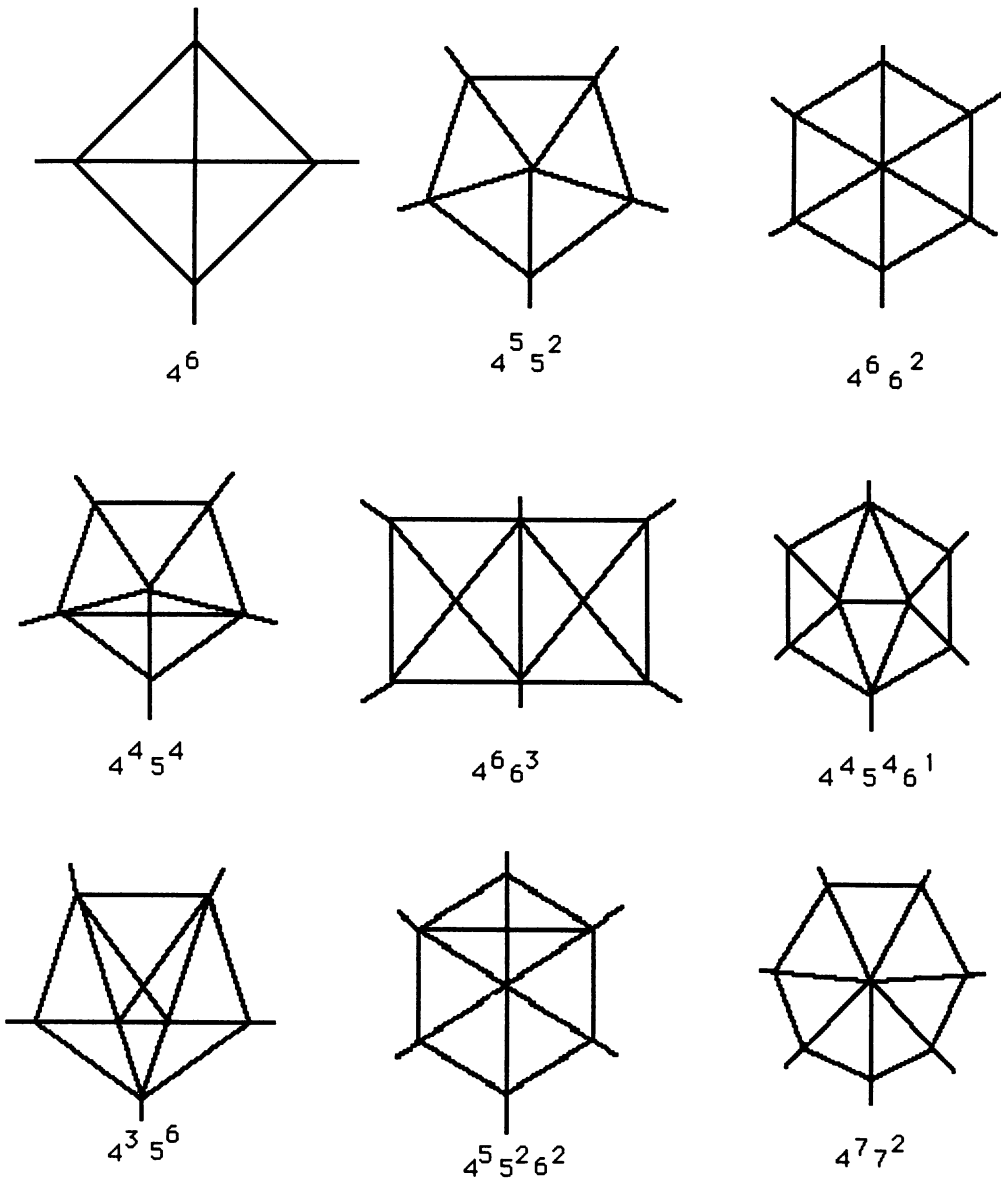
Proof. Suppose the edge $[x, y]$ were not contained in an essential triangle. By Claim 1, it is not contained in an essential $\partial\Delta^3$ nor is it contained in an essential $\partial\Delta^4$ since $K \neq \partial\Delta^4$. Therefore by Lemma 2, we can contract $[x, y]$ producing a smaller triangulation K' . If $[x, y]$ has order p ($p \geq 4$), then this triangulation has $E_i(K) = E_i(K') + p + 1$ and $F_i(K) = F_i(K') + 2p$. Thus for $i = 0$ or 1 ,

$$\mu_i(K) = \frac{3F_i(K') + 6p}{E_i(K') + p + 1} > 4.5$$

Now look at the vertices of K and let $d(v)$ be the degree of the vertex v , that is the number of edges out of v or equivalently the number of vertices in the triangulation of S^2 given by the link $\text{lk}(v)$. If w is a vertex adjacent to v let $d_v(w)$ be the degree of w as a vertex in $\text{lk}(v)$. Note that $d_v(w) = d_w(v)$ is the order of the edge $[v, w]$. Since each edge of K must have order at least four each vertex w in $\text{lk}(v)$ must have $d_v(w)$ at least four. Tabulations of the possible triangulations of S^2 show that v must have degree at least 6 and if $d(v) \leq 9$ then $\text{lk}(v)$ must be one of the triangulations given in Figure 2.

Claim 5. Let v be a vertex of degree at most 7 in K , then every vertex adjacent to v must have degree at least 9.

Proof. Suppose $d(v) = 6$, then from Figure 2, v is the center of an octahedron of vertices which we will label A, B, C, D, E , and F (see Figure 3(a) on



Infinity is a vertex. a^b means b vertices of degree a in the diagram
 List of triangulations of S^2 with no degree three vertex up to nine vertices

FIGURE 2

page 898). By symmetry it is enough to show $d(A) \geq 9$. The edge $[v, A]$ is contained in some essential triangle which clearly must be the triangle $[v, A]$, $[A, C]$, and $[C, v]$. In particular $[A, C]$ is an edge in K . Look at $\text{lk}(A)$; by the above comments $\text{lk}(A)$ must contain the subcomplex shown in Figure 3(b) and the vertex C .

By Claim 3, none of the 2-simplices $[C, B, E]$, $[C, E, D]$, $[C, D, F]$ or

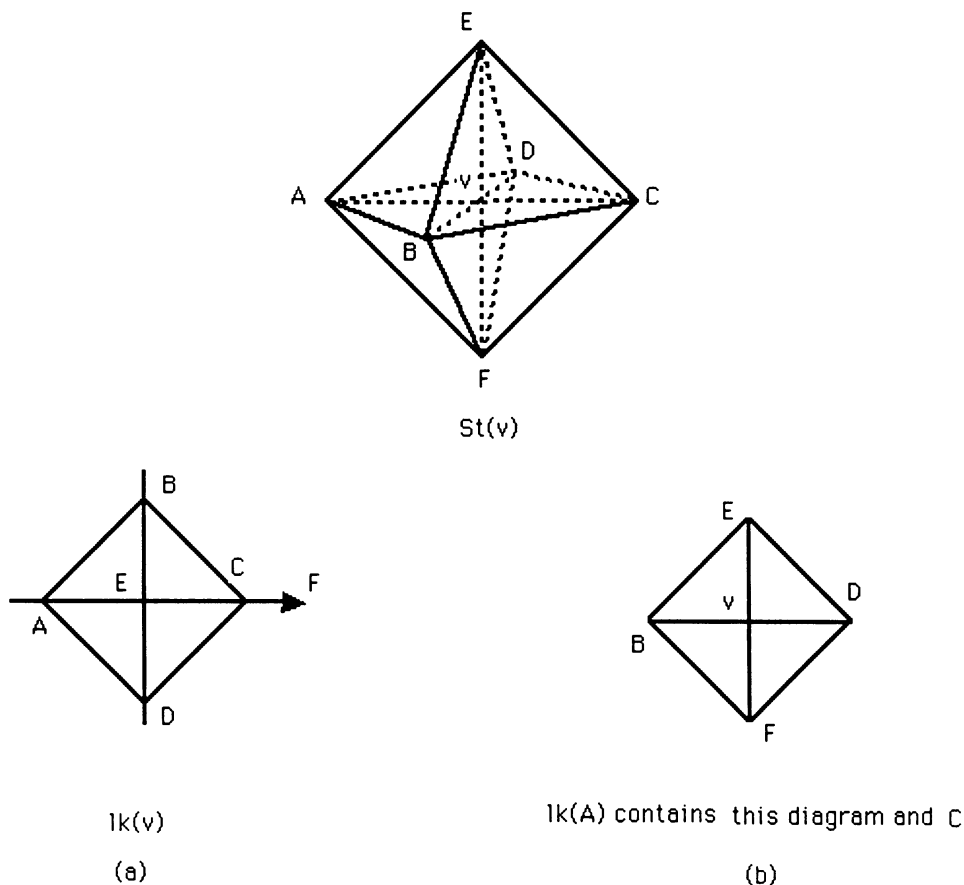


FIGURE 3

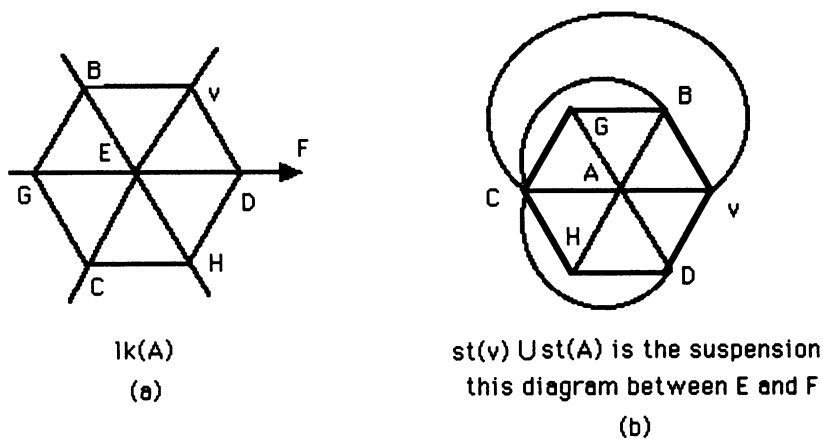


FIGURE 4

$[C, F, B]$ can be in $lk(A)$. Comparing this fact with the diagrams in Figure 2 shows that 4^6 , $4^5 5^2$, and $4^4 5^4$ cannot be $lk(A)$, and for A to have degree below nine $lk(A)$ would have to be, up to symmetry, as shown in Figure

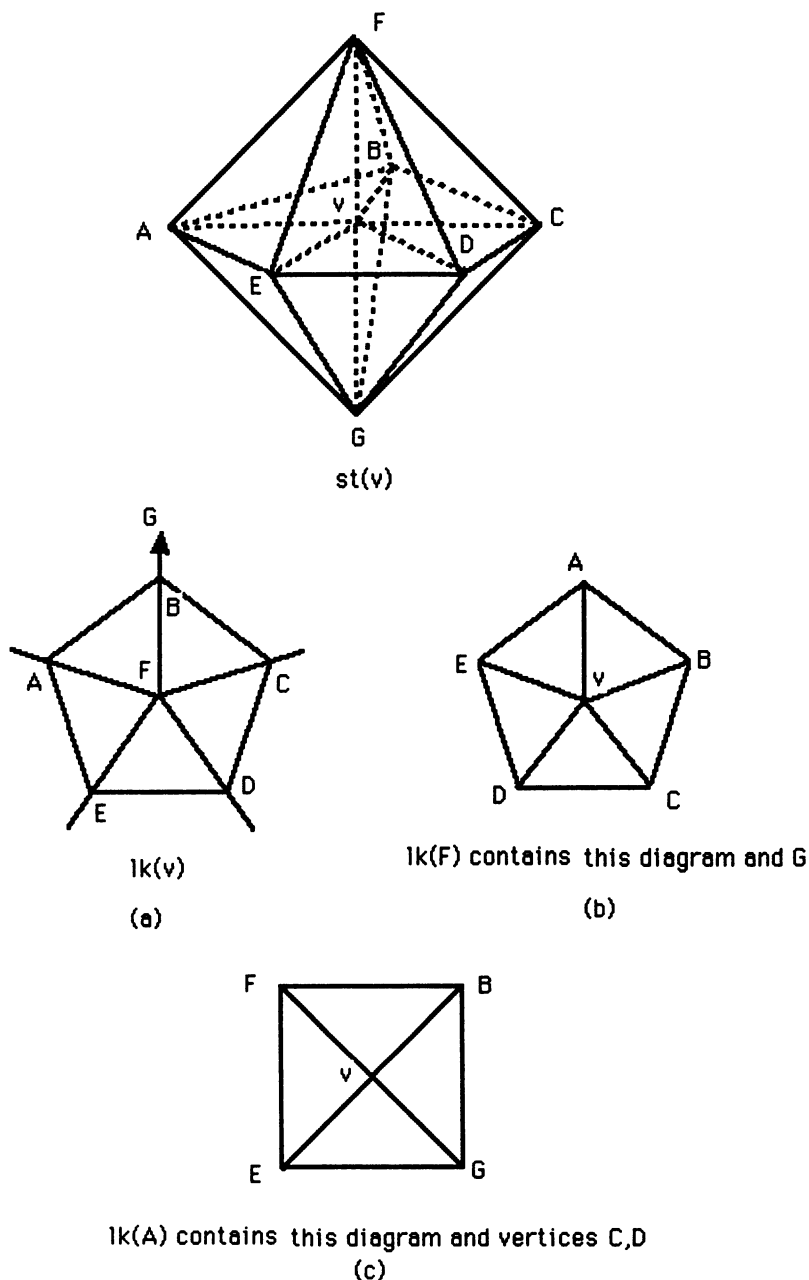


FIGURE 5

4(a). Look at the union $st(v) \cup st(A)$ which is now determined since $st(A)$ and $st(v)$ are as in Figure 4. It is the suspension between E and F of the complex shown in Figure 4(b) since both $st(v)$ and $st(A)$ are suspensions between E and F . But in this case the vertices $\{E, F, B, C, G\}$ would give a copy of $\Sigma\partial\Delta^2$ as discussed in Claim 2. Hence K would have to contain the simplex $[F, B, C, G]$ and G would be a vertex of degree three in $lk(F)$, a contradiction. Therefore $d(A) \geq 9$.

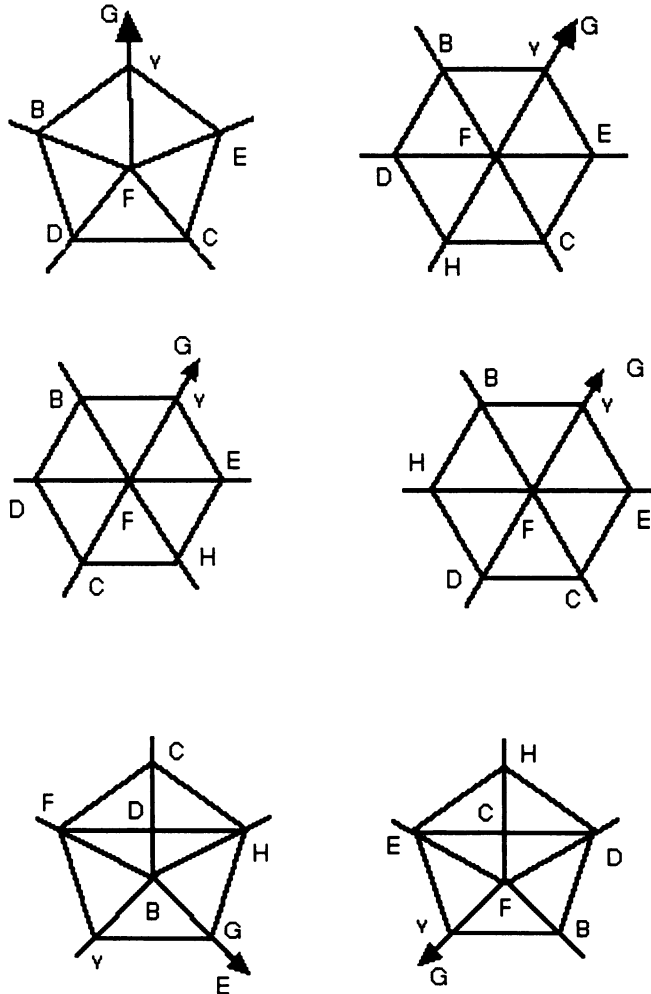
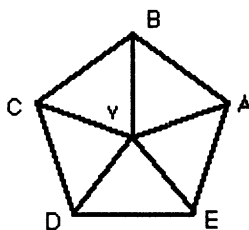


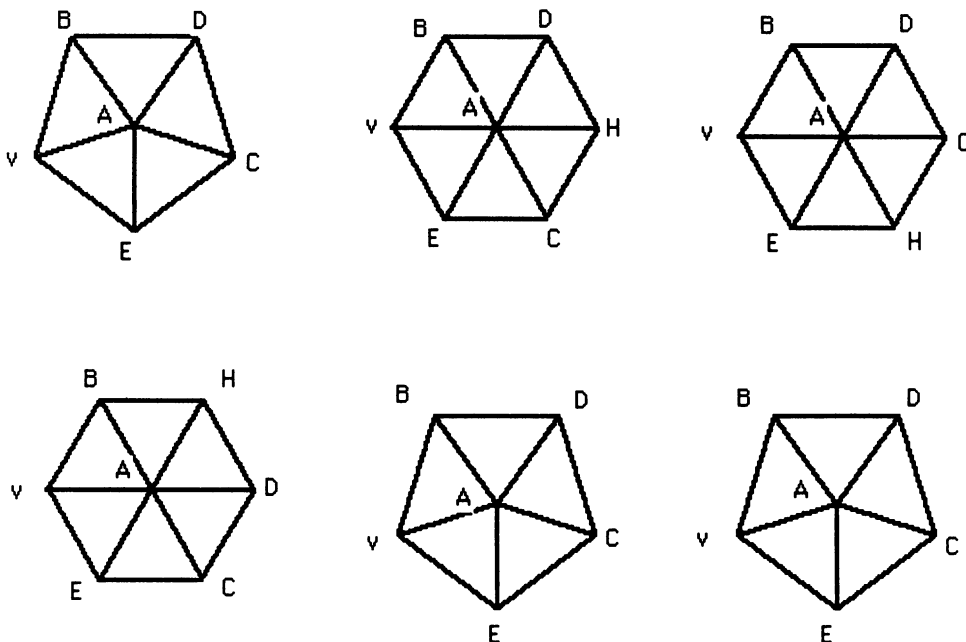
FIGURE 6

Suppose $d(v) = 7$, then from Figure 2, v is at the center of a suspension of a pentagon as shown in Figure 5(a). As above $[v, F]$ must lie in an essential triangle and hence $[F, G]$ must be an edge of K . Therefore $\text{lk}(F)$ must contain the subcomplex shown in Figure 5(b) and the vertex G . Further by Claim 3, none of the simplices $[G, A, B]$, $[G, B, C]$, $[G, C, D]$, $[G, D, E]$, $[G, E, A]$ can be in $\text{lk}(F)$. Comparing this fact with the diagrams in Figure 2 shows that $d(F) \geq 9$ and symmetrically $d(G) \geq 9$.

By symmetry we need only show that $d(A) \geq 9$. Suppose the edge $[A, C]$ (or symmetrically $[A, D]$) were not in K . Then contrary to assumption, we could rotate the edge $[v, B]$ to $[A, C]$ in $\text{st}([B, v])$ obtaining a new triangulation with more degree six vertices. Thus $\text{lk}(A)$ must contain the subcomplex shown in Figure 5(c) and the vertices C and D . By Claim 3, none of the 2-simplices $[C, B, F]$, $[C, B, G]$, $[D, E, F]$, $[D, E, G]$ can lie in $\text{lk}(A)$. Comparing these facts with the diagrams in Figure 2 shows that if $d(A) < 9$ then $\text{lk}(A)$ must be up to symmetry one of the diagrams shown in Figure 6.



$\text{lk}(F)$ contains this diagram



star of A in $\text{lk}(F)$

FIGURE 7

To exclude all these possibilities look in $\text{lk}(F)$. The link of v inside $\text{lk}(F)$ is the same as the link of F inside $\text{lk}(v)$ since both are $\text{lk}([F, v])$. Therefore the star of v in $\text{lk}(F)$ is given by Figure 7(a). Similarly the six possibilities for the star of A in $\text{lk}(F)$ corresponding to the six diagrams in Figure 6 are given in Figure 7(b). These subsets of $\text{lk}(F)$ must be identified along the common 2-simplices $[B, A, v]$ and $[A, v, E]$. In any such case $\text{lk}(F)$ is forced to contain two closed loops, namely $CvAC$ and $DEAD$, that intersect in a single point A (see Figure 7). This is a contradiction since $\text{lk}(F)$ must be a 2-sphere.

Now we will show that any crowded K has $\mu_1(K) > \mu_0(K) \geq 4.5$. One easily calculates that $4F_0(K) - 6E_0(K) = 2E_0(K) - 8V_0(K) = \sum_v (d(v) - 8)$. Let $n(v)$ be the number of neighbors of v with degree at most seven, which

we will call deficient vertices. Then

$$\begin{aligned} 4F_0(K) - 6E_0(K) &= \sum_v (d(v) - 8) = \sum_{d(v) \geq 8} \left(d(v) - 8 - \frac{1}{3}n(v) \right) \\ &\quad + \sum_{d(v) \leq 7} \left(\frac{4}{3}d(v) - 8 \right) \geq \sum_{d(v) \geq 8} \left(d(v) - 8 - \frac{1}{3}n(v) \right). \end{aligned}$$

If $d(v) = 8$, then by claim 5, $n(v) = 0$. If $d(v) = 9$, then the $n(v)$ deficient vertices are pairwise nonadjacent in $\text{lk}(v)$. In particular, Figure 2 shows that $n(v) \leq 3$. If $d(v) \geq 10$, then since each deficient vertex is adjacent to at least four nondeficient vertices in $\text{lk}(v)$ we have $d(v) \geq n(v) + 4$, furthermore equality cannot be attained for a planar graph. These comments show that in any case $d(v) - 8 - n(v)/3 \geq 0$. Hence $4F_0(K) - 6E_0(K) \geq 0$ or $\mu_1(K) > \mu_0(K) \geq 4.5$.

For the equality cases, note that the only reductions that could give equality were the separations along an essential $\partial\Delta^3$. Therefore we obtain $\mu_1(K) = 4.5$ if and only if $k = \#^k \partial\Delta^4$ for some $k \geq 1$. Similarly $\mu_0(K) = 4.5$ only occurs either for a triangulation of $S^2 \times S^1$ or $S^2 \tilde{\times} S^1$ obtained by gluing along two 3-simplices of $\#^k \partial\Delta^4$ or if K is crowded and equality occurs in the calculation above. The latter case can happen in only two ways, either

(1) Every vertex has degree 6 or 9 and each degree 9 vertex is adjacent to three vertices of degree 6, or

(2) Every vertex has degree 8.

We will argue that neither of these cases can actually occur.

Suppose we are in Case (1). Let v be a vertex of degree 9, then $\text{lk}(v)$ must be one of the five 9 vertex triangulations of S^2 shown in Figure 2. The three degree 6 vertices adjacent to v must have degree 4 in $\text{lk}(v)$ and must be pairwise nonadjacent. This excludes the diagrams $4^6 6^3$ and $4^4 5^4 6^1$ for which at most two such vertices can be chosen. Suppose $\text{lk}(v)$ is $4^7 7^2$. Then v is adjacent to two vertices u and w with $d_v(u) = d_v(w) = 7$ and since $[v, w]$ must be in an essential triangle $[u, w]$ must be an edge of K . Since $d_w(v) = 7$, $\text{lk}(w)$ must also be $4^7 7^2$ and u must be the other degree 7 vertex in $\text{lk}(w)$. This however contradicts Claim 3 so $4^7 7^2$ does not occur.

Suppose $\text{lk}(v)$ is $4^3 5^6$. The three degree 4 vertices in $\text{lk}(v)$ (A, B , and C in Figure 8(a)) must all have degree 6 in K . Therefore $\text{lk}(B)$ must be as shown in Figure 8(b). Further since K is crowded $[x, v]$ must be in K , hence x is either H or I . Either case contradicts Claim 3. This leaves only the case $\text{lk}(v)$ is $4^5 5^2 6^2$. Up to symmetry the three degree 6 vertices adjacent to v are A, B , and C in Figure 9. The edges $[G, E]$ and $[G, I]$ must be in K . Otherwise we could rotate one of the order 4 edges $[v, B]$ or $[v, A]$ to produce a vertex of degree 5, a contradiction. Therefore $\text{lk}(G)$ (which is also $4^5 5^2 6^2$) must contain E, I and the subcomplex shown in Figure 10(a). Claim 3 says that $\text{lk}(G)$ cannot contain any of the faces $[E, D, B]$, $[E, B, F]$, $[I, F, C]$, $[I, C, H]$, $[I, H, A]$, $[I, A, D]$, therefore $\text{lk}(G)$ must be as shown in Figure 10(b). Hence $\text{lk}(D)$ must contain the Möbius band shown in Figure 11 on page 904, a contradiction. Therefore Case (1) cannot occur.

Suppose we are in Case (2) every vertex has degree 8. If $\text{lk}(v)$ is the triangulation $4^6 6^2$, then as for the link $4^7 7^2$ above the two degree 6 vertices in $\text{lk}(v)$

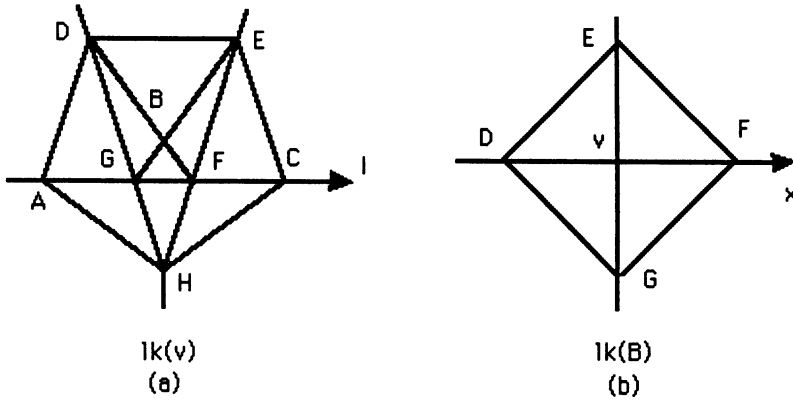


FIGURE 8

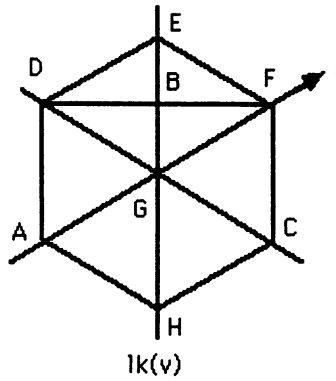


FIGURE 9

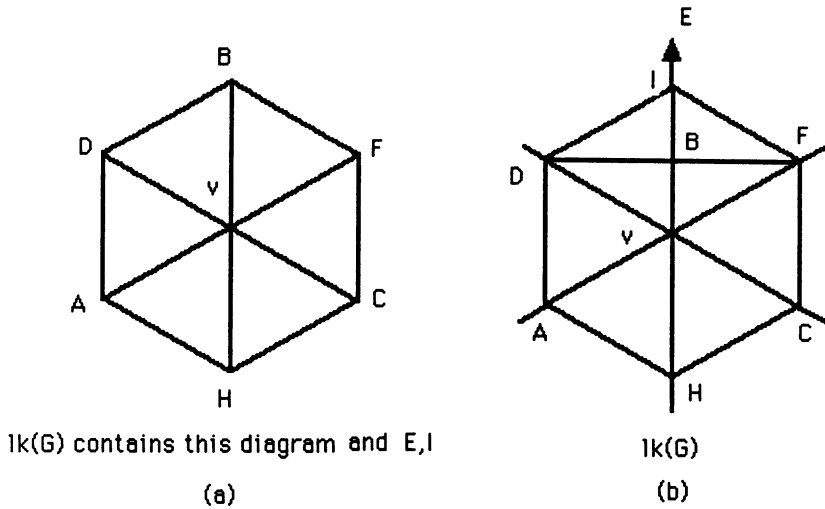


FIGURE 10

are adjacent in K , contradicting Claim 3. Thus $lk(v)$ must be 4^45^4 . Label the vertices of $lk(v)$ as in Figure 12. The edges $[C, G]$ and $[C, H]$ must be in K otherwise we could rotate one of $[v, E]$ or $[v, A]$ producing vertices of

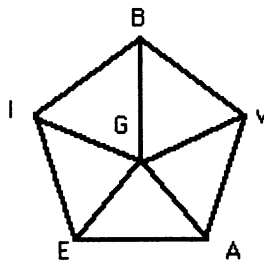
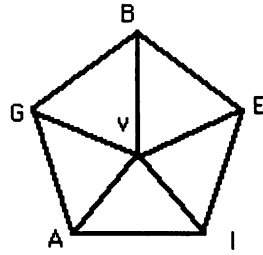
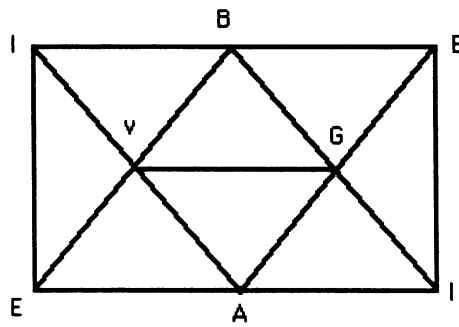
link of G in $lk(D)$  $lk(D)$ contains this diagram $lk(D)$

FIGURE 11

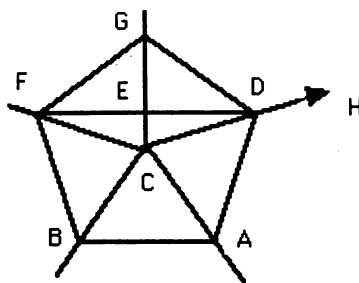
 $lk(v)$

FIGURE 12

degree 7. Now we proceed exactly as in the case $4^5 5^2 6^2$ above. By Claim 3, $lk(C)$ must be as in Figure 13(a), therefore $lk(D)$ contains the Möbius band in Figure 13(b). Thus this case cannot occur either and no crowded triangulation attains equality.

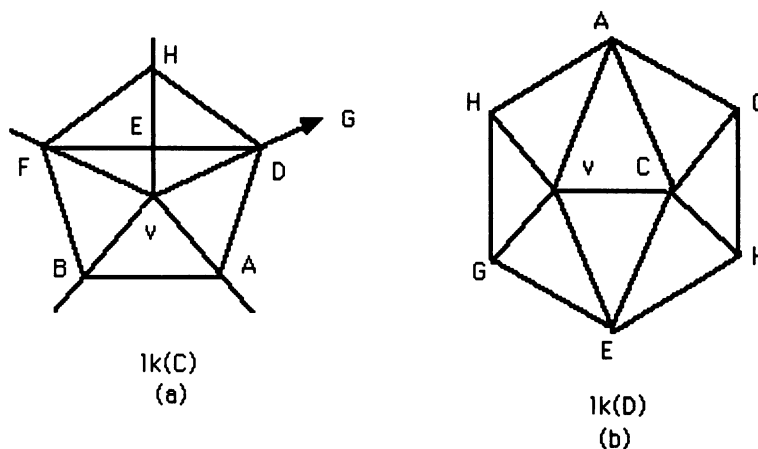


FIGURE 13

TRIANGULATIONS OF THE 3-SPHERE

The results above simplify considerably if we restrict to triangulations of S^3 . First suppose K is a triangulation, necessarily of S^3 , with $\mu_0(K) < 4.5$. Since $\mu_1(K) \geq 4.5$ we have the following lower bound on μ_0

$$\mu_0(K) = \mu_1(K)(1 - 6/E_0) + 12/E_0 \geq 4.5 - 15/E_0.$$

In particular, for any constant $c < 4.5$ this inequality gives a bound on the number of edges in any triangulation of S^3 with $\mu_0 \leq c$. Therefore there are only finitely many such triangulations proving our earlier remark. Furthermore, suppose K is a triangulation with $\mu_0(K) = 3$, then the inequality above says that $E_0 \leq 10$. Since every vertex has degree at least four we always have $2E_0/V_0 \geq 4$, therefore for such a K we have $V_0 \leq 5$. The only triangulation of a 3-manifold with only five vertices is clearly $\partial\Delta^4$. This gives the equality case for the lower bound on μ_0 .

Suppose next that we start with a triangulation K of S^3 with $\mu_0(K) \leq 4.5$. By the results above this triangulation cannot be crowded, therefore we can reduce the number of edges by a sequence of the moves above. Suppose we reduce the triangulation by splitting it along a copy of $\partial\Delta^3$ or $\Sigma\partial\Delta^2$, say into pieces K_1 and K_2 , then the argument above shows that

$$\mu_0(K) = \frac{3F_0(K_1) + 3F_1(K_2)}{E_0(K_1) + E_1(K_2)}, \quad \text{or} \quad \mu_0(K) = \frac{3F_0(K_1) + 3F_1(K_2) - 12}{E_0(K_1) + E_1(K_2) - 3}.$$

In either case, since $\mu_0(K) \leq 4.5$ and $\mu_1(K_2) \geq 4.5$ we have $\mu_0(K_1) \leq 4.5$. Symmetrically we also have $\mu_0(K_2) \leq 4.5$. Therefore successively applying the moves above to K will eventually produce some number of copies of $\partial\Delta^4$.

A similar remark applies to any triangulation of S^3 . Applying the moves above as long as possible will always produce some number of copies of $\partial\Delta^4$ and possibly some crowded triangulations of S^3 . It seems that crowded triangulations of S^3 must exist but the authors are unaware of any.

One possible application of these results would be to classification of triangulations of S^3 with few vertices [Ba]. A crowded triangulation of S^3 would

have to have average vertex degree larger than 8. Thus it must have at least 10 vertices (further checking of cases shows that 10 cannot be attained). Therefore any triangulation of S^3 with at most 10 vertices can be built by the reverses of our moves. In principle a search could be carried out by computer.

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