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HIGHER DIMENSIONAL ANALOGUES OF FUCHSIAN SUBGROUPS OF PSL$(2, \mathfrak{o})$

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ABSTRACT. The problem of classification of 2×2 indefinite Hermitian matrices over orders in Clifford algebras is considered. The unit groups of these matrices are analogous to maximal arithmetic Fuchsian subgroups of PSL$(2, \mathfrak{o})$ where $\mathfrak{o}$ is an order in a quadratic number field.

1. INTRODUCTION

In 1902 R. Vahlen [21] described the group of orientation-preserving isometries of the upper half-space model of n-dimensional hyperbolic space H^n in terms of (2×2)-matrices over Clifford numbers. For $n = 2$ and 3, they coincide with PSL$(2, \mathbf{R})$ and PSL$(2, \mathbf{C})$ respectively. Arithmetic subgroups G of Vahlen's groups have been investigated by Maass [12], Elstrodt, Grunewald, and Mennicke [9, 10], and Maclachlan, Waterman, and Wielenberg [17]. The corresponding hyperbolic orbifolds $M_G = G\backslash H^n$ have finite volume [9, 10]. In the present paper we first classify 2×2 indefinite integral Hermitian matrices A over orders in Clifford algebras. The G-unit groups $\mathscr{E}(A)$ of such matrices are analogous to maximal arithmetic Fuchsian subgroups of PSL$(2, \mathfrak{o})$ where $\mathfrak{o}$ is an order in a quadratic number field. In some cases the conjugacy classes of those Fuchsian subgroups in PSL$(2, \mathfrak{o})$ are classified in [24]. A G-unit group $\mathscr{E}(A)$ stabilizes in H^n a hemisphere H_A. The quotient $\mathscr{E}(A)\backslash H_A$ is a geodesic suborbifold of M_G of codimension one and, conversely, any geodesic suborbifold of M_G of codimension one is such a quotient. $\mathscr{E}(A)\backslash H_A$ has finite volume if and only if $A = \lambda B$ where λ is a nonzero real number and B an indefinite integral Hermitian matrix (Corollary 4.3). In §3 the G-orbits of nonzero numerical multiples of indefinite integral Hermitian matrices over $\mathbf{Z}$-orders in Clifford algebras are classified in terms of nonzero minima of associated quadratic forms. For $n > 2$, the spectrum of those minima is discrete with finite multiplicities (Theorem 3.3). It is applied to parametrize, in §4, G-conjugacy classes of G-unit groups of indefinite integral Hermitian matrices mentioned above. In some cases there are obtained explicit results (§3.6). It is

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shown in §5 that $M_G = G\backslash H^4$ contains infinitely many pairwise incommensurable compact hyperbolic 3-suborbifolds.

2. Clifford algebras

In the present section we review some basic facts on Clifford algebras following the lead of Elstrodt, Grunewald, and Mennicke [8, 9, 10]. We refer the reader to [4, 7, and 11] for details.

Let field K be either $\mathbf{R}$ or $\mathbf{Q}$. Let E be an n-dimensional vector space over K. Let $\Phi: E \times E \to K$ be a nondegenerate symmetric bilinear form with associated quadratic form $q(x) = \Phi(x, x)$. Let $\mathfrak{a}_q := x \otimes y + y \otimes x - 2\Phi(x, y)$, $(x, y \in E)$, be the two-sided ideal in the tensor algebra $T(E)$ of E. The Clifford algebra of q is defined as $\mathscr{C}(K, q) := T(E)/\mathfrak{a}_q$; for $n = 0$, $\mathscr{C}(K, q) = K$. We identify K and E with their canonical images in $\mathscr{C}(K, q)$ and define the $(n+1)$-dimensional vector space $V(K, q) := K \cdot 1 + E \subseteq \mathscr{C}(K, q)$.

Let $e_1, \dots, e_n$ be a basis of E orthogonal with respect to q. Then we have

$$e_k^2 = q(e_k), \; e_k e_m = -e_m e_k \qquad (k, m = 1, \dots, n; k \neq m).$$

Let J_n be the set of subsets of $\{1, \dots, n\}$. For $M \in J_n$, $M = \{k_1, \dots, k_r\}$ with $k_1 < \cdots < k_r$ we define $e_M = e_{k_1} \cdot \dots \cdot e_{k_r}$ where $e_0 = 1 \in \mathscr{C}(K, q)$. Then 2^n elements $e_M (M \in J_n)$ constitute a basis of $\mathscr{C}(K, q)$. An element of $\mathscr{C}(K, q)$

$$x = x_0 + x_1 e_1 + \cdots + x_n e_n \in V(K, q) \tag{2.1}$$

is called a *vector* and is identified with $(x_0, \dots, x_n) \in K^{n+1}$. Products of nonzero vectors form a group: the *Clifford group* T_n.

There are three involutions defined on $\mathscr{C}(K, q)$:

(i) the *main involution*, $x \to x'$, obtained by replacing each e_m with $-e_m$, satisfying $(xy)' = x'y'$.

(ii) the *main anti-involution*, $x \to x^*$, obtained by reversing the order of the factors in each term $e_{k_1} \cdot \dots \cdot e_{k_r}$, satisfying $(xy)^* = y^* x^*$.

(iii) the *conjugation*, $x \to \overline{x} = x'^* = x^{*\prime}$, satisfying $(xy) = \overline{y}\overline{x}$. We define the quadratic form $\widehat{Q}: \mathscr{C}(K, q) \to K$ on the vector space $\mathscr{C}(K, q)$ so that

$$x\overline{x} = \widehat{Q}(x) e_0 + \sum_{|M|>0} \alpha_M e_M \tag{2.2}$$

for all $x \in \mathscr{C}(K, q)$ (cf. [8, p. 374]). The restriction of $\widehat{Q}$ to $V(K, q)$ will be denoted by Q. For $x, y \in V(K, q)$, we have

$$x\overline{x} = Q(x), \qquad x\overline{y} + y\overline{x} = 2(x, y) \in \mathbf{R} \tag{2.3}$$

where $2(x, y) = Q(x+y) - Q(x) - Q(y)$. Elements of T_n satisfy

$$x\overline{x} = Q(x_1) \cdots Q(x_r) = \widehat{Q}(x), \tag{2.4}$$

where $x = x_1 \cdots x_r$, $x_i \in V(K, q)$, $(i = 1, \dots, r)$.

Since $e_1, \dots, e_n$ is an orthogonal basis of E with respect to q

$$q(x) = q_d := d_1 x_1^2 + \cdots + d_n x_n^2 \tag{2.5}$$

for $x \in E$, $(d_k = q(e_k), k = 1, \dots, n)$. Let

$$\mathscr{C} := \mathscr{C}(\mathbf{R}, q), \qquad \mathscr{C}_q := \mathscr{C}(\mathbf{Q}, q), \qquad \mathscr{C}_d := \mathscr{C}(\mathbf{Q}, q_d), \tag{2.6}$$

and

$$V := V(\mathbf{R}, q), \qquad V_q := V(\mathbf{Q}, q), \qquad V_d := V(\mathbf{Q}, q_d). \tag{2.7}$$

(Notations $\mathscr{C}_d$ and V_d will be used only in the case when $d_1, \ldots, d_n$ in (2.5) are nonzero integers.)

Lemma 2.1. $\det \widehat{Q} = (\det Q)^{2^{n-1}}$ *for* $n > 0$.

Proof. If $n = 1$, $\widehat{Q} = Q$.

We proceed by induction on n. For a vector x in E, let $q'(x) := q_d(x) - d_n x_n^2$ and $\widehat{Q}'$, Q' be defined according to (2.2). Thus for $x \in V$,

$$Q'(x) := Q(x) + d_n x_n^2.$$

We identify $\mathscr{C}' := \mathscr{C}(R, q')$ with the obvious subalgebra of $\mathscr{C}$. An element z in $\mathscr{C}$ can be uniquely represented as $z = x + ye_n$ with $x, y \in \mathscr{C}'$. Since $e_n\overline{e}_n = -q_d(e_n) = Q(e_n) = d_n$, $z\overline{z} = x\overline{x} + d_n y\overline{y}$ and

$$\widehat{Q}(z) = \widehat{Q}(x + ye_n) = \widehat{Q}'(x) + d_n\widehat{Q}'(y). \tag{2.8}$$

The dimension of the vector space $\mathscr{C}'$ is 2^{n-1} and we have

$$\det \widehat{Q} = (\det \widehat{Q}')^2 d_n^{2^{n-1}}. \tag{2.9}$$

By the induction assumption,

$$\det \widehat{Q}' = (\det Q')^{2^{n-2}} = (d_1 \cdots d_{n-1})^{2^{n-1}}. \tag{2.10}$$

Hence, by (2.9), (2.10),

$$\det \widehat{Q} = (d_1 \cdots d_n)^{2^{n-1}} = (\det Q)^{2^{n-1}}. \qquad \square$$

We say that a quadratic form q is an (n^+, n^-)*-form* if its canonical form contains n^+ pluses and n^- minuses. The number $|n^+ - n^-|$ is called the *signature* of q.

Lemma 2.2. *If q is a negative definite quadratic form, then $\widehat{Q}$ is positive definite. Otherwise, the signature of $\widehat{Q}$ equals zero.*

Proof. Applying (2.8) one can prove the lemma by induction. $\square$

We now define Vahlen's group of Clifford matrices in the case of q negative definite [1, 2, 8, 9, 12, 21]:

$$\begin{aligned} SV_n := \Bigg\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in M_2(\mathscr{C}),\ \alpha, \beta, \gamma, \delta \in T_n \cup \{0\}, \\ \alpha\overline{\beta}, \overline{\gamma}\delta \in V, \alpha\delta^* - \beta\gamma^* = 1 \Bigg\}. \end{aligned} \tag{2.11}$$

(See also [8, p. 373, or 9, p. 258], for the definition of SV_n in other cases.) Clearly, $SV_0 = SL_2(\mathbf{R})$, and, if $q(x) = -x^2$, $SV_1 = SL_2(\mathbf{C})$.

An arithmetic subgroup of SV_n is defined as follows [10, §2]: A subring $\mathscr{T}$ of a $\mathbf{Q}$-algebra A (with unity element) is called a $\mathbf{Z}$-*order* if $\mathscr{T}$ has the same unity element as A and the additive group of $\mathscr{T}$ is finitely generated and contains a $\mathbf{Q}$-basis of A. A $\mathbf{Z}$-order $\mathscr{T} \in \mathscr{C}_q$, (see (2.5), (2.6)), is called *compatible* if it is stable under the involutions $^-$ and $'$ of $\mathscr{C}_q$. For a compatible $\mathbf{Z}$-order $\mathscr{T}$ let

$$\Lambda := V \cap \mathscr{T}, \tag{2.12}$$

$$SV(\mathscr{C}_q) := SV_n \cap M_2(\mathscr{C}_q), \tag{2.13}$$

$$G = SV(\mathscr{T}) := SV(\mathscr{C}_q) \cap M_2(\mathscr{T}). \tag{2.14}$$

The group G is a discrete subgroup in SV_n. If $n = 0$, then $\mathscr{C}_d = \mathbf{Q}$ and $\mathscr{T} = \mathbf{Z}$, $SV_0(\mathbf{Z}) = SL_2(\mathbf{Z})$. For $n = 1$, $\mathscr{C}_d = \mathbf{Q}(\sqrt{-d})$ is an imaginary quadratic number field, and $\mathscr{T}$ is some order in $\mathbf{Q}(\sqrt{-d})$.

If q is negative definite, SV_n can be identified with the group of orientation-preserving isometries of the $(n+2)$-dimensional hyperbolic space H^{n+2}, G acts discontinuously on H^{n+2}, and the volume of the quotient $G\backslash H^{n+2}$ is finite (see [9, p. 262]).

3. Hermitian matrices

Hermitian matrices over Clifford algebras have been studied in [9]. In this section, generalizing the results obtained in [24] we shall define the spectra of nonzero minima of indefinite Hermitian matrices over $\mathbf{Z}$-orders in Clifford algebras and prove that they are discrete with finite multiplicities. Some explicit results are provided in §3.5.

3.1. **Preliminaries.** For $n > 0$, let

$$\mathscr{H} := \left\{ \begin{pmatrix} a & b \\ \overline{b} & c \end{pmatrix} \in M_2(\mathscr{C}) : a, c \in \mathbf{R}, b \in V \right\}, \tag{3.1}$$

$$\Delta(A) := ac - b\overline{b} = ac - Q(b), \tag{3.2}$$

$$\mathscr{P}(\mathscr{T}) := \left\{ \begin{pmatrix} a & b \\ \overline{b} & c \end{pmatrix} \in \mathscr{H} : a, c \in \mathbf{Z}, b \in \Lambda \right\}. \tag{3.3}$$

The elements of $\mathscr{H}$ will be called *Hermitian matrices* over $\mathscr{C}$ and $\Delta(A)$ the *discriminant* of $A = \left(\begin{smallmatrix} a & b \\ \overline{b} & c \end{smallmatrix}\right) \in \mathscr{H}$. If Q is an (r, s)-form, then the quadratic form Δ is an $(s+1, r+1)$-form defined on an $(n+3)$-dimensional vector space $\mathscr{H}$. We denote

$$\mathscr{H}^+ := \{A \in H : \Delta(A) > 0\}, \quad \mathscr{H}^- := \{A \in \mathscr{H} : \Delta(A) < 0\}. \tag{3.4}$$

The group $\mathbf{R}^*$ of nonzero real numbers acts on $\mathscr{H}$, on $\mathscr{H}^+$, and on $\mathscr{H}^-$ by multiplication. We put

$$\begin{aligned} \overline{\mathscr{H}} &:= \mathscr{H}/\mathbf{R}^*, & \overline{\mathscr{P}}(\mathscr{T}) &:= \mathscr{P}(\mathscr{T})/\mathbf{R}^*, \\ \overline{\mathscr{H}}^+ &:= \mathscr{H}^+/\mathbf{R}^*, & \overline{\mathscr{H}}^- &:= \mathscr{H}^-/\mathbf{R}^*. \end{aligned} \tag{3.5}$$

Let $B = (b_{i,j})$ be an $(2 \times m)$-matrix, $(m = 1$ or $2)$, with $b_{i,j} \in \mathscr{C}$. We put $\overline{B} = (\overline{b}_{i,j})$ and $\overline{B}^t = (\overline{b}_{j,i})$. If $A \in \mathscr{H}$, then

$$A \cdot [B] := \overline{B}^t AB \tag{3.6}$$

is an $(m \times m)$-matrix.

The group SV_n acts on $\mathscr{H}$ as follows. Let $\sigma = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in SV_n$ and $A \in \mathscr{H}$. Then

$$A \cdot [\sigma] := \overline{\sigma}^t A \sigma. \tag{3.7}$$

As in [9, Proposition 3.3], one can show that for $\sigma, \tau \in SV_n$,

$$A \cdot [\sigma] \in \mathscr{H}, \tag{3.8}$$

$$A \cdot [\sigma\tau] = (A \cdot [\sigma]) \cdot [\tau], \tag{3.9}$$

and

$$\Delta(A \cdot [\sigma]) = \Delta(A). \tag{3.10}$$

Hence

$$A \in \mathscr{H}^+ \Rightarrow A \cdot [\sigma] \in \mathscr{H}^+ \quad \text{and} \quad A \in \mathscr{H}^- \Rightarrow A \cdot [\sigma] \in \mathscr{H}^-. \tag{3.11}$$

If q is negative definite, the set $\overline{\mathscr{H}}^+$ may be considered as a model of $(n+2)$-dimensional hyperbolic space [9].

3.2. **Quadratic forms associated to Hermitian matrices.** Suppose that E is an n-dimensional vector space over $\mathbf{Q}$ with nondegenerated quadratic form $q \colon E \times E \to \mathbf{Q}$. The Clifford algebra $\mathscr{C}_q$ in (2.6) is considered as a subalgebra of $\mathscr{C}$. We define the quadratic form $f_A := \mathscr{C} \times \mathscr{C} \to \mathbf{R}$ associated to $A \in \mathscr{H}$ so that

$$A \cdot [(xy)^t] = ax\overline{x} + \overline{y}\overline{b}x + \overline{x}by + cy\overline{y} = f_A(x, y)e_0 + \sum_{|M|>0} \lambda_M e_M \tag{3.12}$$

for all $(x\ y) \in \mathscr{C} \times \mathscr{C}$. Thus, f_A is a quadratic form in 2^{n+1} real variables. Since we are interested in classification of G-orbits of Hermitian matrices in $\mathscr{H}$, we may assume that $a \neq 0$ applying transformation $\left(\begin{smallmatrix} v & 1 \\ -1 & 0 \end{smallmatrix}\right)$, where $v = 0$ or b, if necessary. Then, as one can verify,

$$f_A(x, y) = (\widehat{Q}(ax + by) + \Delta(A)\widehat{Q}(y))/a. \tag{3.13}$$

It follows from Lemma 2.2 that f_A is definite if and only if q is negative definite and $A \in \mathscr{H}^+$. In that case A will be called a *definite* Hermitian matrix. If f_A is indefinite, we say that A is *indefinite*. The subset of indefinite forms in $\mathscr{H}$ and $\mathscr{P}(\mathscr{T})$ will be denoted by $\mathscr{H}'$ and $\mathscr{P}'(\mathscr{T})$ respectively. If q is a negative definite quadratic form, $\mathscr{H}' = \mathscr{H}^-$. Otherwise, $\mathscr{H}' = \mathscr{H}^- \cup \mathscr{H}^+$. By Lemma 2.2 the signature of f_A is zero for any indefinite $A \in \mathscr{H}$.

A quadratic form f_A is said to be *rational* (*integral*) over a $\mathbf{Z}$-order $\mathscr{T}$ of Clifford algebra $\mathscr{C}$ if $f_A(x, y) \in \mathbf{Q}$ ($f_A(x, y) \in \mathbf{Z}$) for all $x, y \in \mathscr{T}$. $A \in \mathscr{H}$ will be called *rational* (*integral*) if the associated quadratic form f_A is rational (integral). It can be easily verified that A is rational if and only if

$$a, c \in \mathbf{Q}, \qquad b \in V_q. \tag{3.14}$$

It follows from Lemma 2.1 and (3.13) that the determinant of f_A

$$\Delta(f_A) = (\det \widehat{Q})^2 (\Delta(A))^{2^n} = [(\det Q)\Delta(A)]^{2^n}. \tag{3.15}$$

Hence $\Delta(f_{A\cdot[\sigma]}) = \Delta(f_A)$.

Let $\sigma \in SV_n$. Since $(A \cdot [\sigma]) \cdot [(x\ y)^t] = A \cdot [\sigma(x\ y)^t]$, we have

$$f_{A\cdot[\sigma]}(x\ y) = f_A(\sigma(x\ y)^t). \tag{3.16}$$

Denote

$$\mu(A) = \mu(f_A) := \inf |f_A(x, y)| \tag{3.17}$$

where the infimum is taken over all $x, y \in \mathscr{T}$ such that $f_A(x, y) \neq 0$. By (3.16) and (3.17)

$$\mu(A \cdot [\sigma]) = \mu(A), \quad (\forall \sigma \in G). \tag{3.18}$$

It was shown by G. A. Margulis [18] that for any $\varepsilon > 0$ and any indefinite quadratic form f in $n > 2$ variables there is $x \in \mathbf{Z}^n$ such that $0 < |f(x)| < \varepsilon$ unless f is a multiple of an integral form. This result when applied to the indefinite quadratic form f_A associated to $A \in \mathscr{H}$ leads to the following.

Lemma 3.1. *Let $A \in \mathscr{H}$ be indefinite. Then*

$$\mu(A) > 0 \Leftrightarrow A \in \overline{\mathscr{P}}(\mathscr{T}), \tag{3.19}$$

where $\overline{\mathscr{P}}(\mathscr{T})$ is defined by (3.3) *and* (3.5). □

We shall call

$$\nu(A) := \mu(A)|\Delta(A)|^{-1/2} \tag{3.20}$$

the *nonzero minimum of* A. It follows from (3.15) and (3.18) that

$$\nu(\lambda \cdot A[\sigma]) = \nu(A) \qquad (\forall \sigma \in G, \lambda \in \mathbf{R}^*). \tag{3.21}$$

3.3. **Discreteness.** As in [19], we define

$$\nu(f_A) := \mu(f_A)|\Delta(f_A)|^{-2^{-n-1}}. \tag{3.22}$$

By (3.15), (3.17), (3.20), and (3.22)

$$\nu(A) = |\det Q|^{1/2}\nu(f_A). \tag{3.23}$$

The set

$$\mathscr{S}_n(\mathscr{T}) := \{\nu(A) \colon A \in \mathscr{H}'\} \tag{3.24}$$

will be called the *spectrum of nonzero minima* of indefinite Hermitian matrices over the order $\mathscr{T}$ in the Clifford algebra $\mathscr{C}_q$.

In [23] the spectrum $M(s, n, \mathbf{R})$ of nonzero minima of indefinite quadratic forms with real coefficients in n variables with signature s is defined in a similar way. The spectrum $\mathscr{S}_n(\mathscr{T})$ ($M(s, n, \mathbf{R})$) is said to be *discrete* if for any $\delta > 0$ there is only a finite number of $\nu(A) > \delta$ in $\mathscr{S}_n(\mathscr{T})$ ($\nu(f) > \delta$ in $M(s, n, \mathbf{R})$).

If $n = 0$, $\det Q = 1$, $\mathscr{T} = \mathbf{Z}$, and $\mathscr{S}_0(\mathbf{Z})$ contains the Markov spectrum (see e.g. [6]). Thus $\mathscr{S}_0(\mathbf{Z})$ is not discrete. For any $n > 0$, Margulis' theorem [18] mentioned above and Theorem 1 from [23] show that the spectrum $M(s, n, \mathbf{R})$ is discrete. Since f_A is a quadratic form in 2^{n+1} real variables, by (3.23) and Lemma 3.1,

$$\mathscr{S}_n(\mathscr{T}) \subset |\det Q|^{1/2} M(0, 2^{n+1}, \mathbf{R}) \tag{3.25}$$

which implies the following.

Theorem 3.2. *Let $\mathscr{T}$ be a compatible* **Z**-*order in a Clifford algebra $\mathscr{C}_q$.*

(i) *Let $n > 0$. If $A \in \mathscr{H}'$ and $0 < \nu(A) \in \mathscr{S}_n(\mathscr{T})$, then $A \in \overline{\mathscr{P}}'$. Thus*

$$\mathscr{S}_n(\mathscr{T}) = \{\nu(A) \colon A \in \overline{\mathscr{P}}'(\mathscr{T})\} \cup \{0\}, \tag{3.26}$$

where $\overline{\mathscr{P}}'(\mathscr{T})$ is the subset of nonzero numerical multiples of indefinite Hermitian forms in $\mathscr{P}(\mathscr{T})$. The spectrum $\mathscr{S}_n(\mathscr{T})$ is discrete.

(ii) *The spectrum* $\mathscr{S}_0(\mathscr{T})$ *is not discrete.* □

3.4. **Finiteness.** The spectrum $\mathscr{S}_n(\mathscr{T})$ is said to be *with finite multiplicities* if for any nonzero $\nu \in \mathscr{S}_n(\mathscr{T})$ the number of G-orbits of $A \in \overline{\mathscr{H}}'$ with $\nu(A) = \nu$ is finite.

We shall prove the following refinement of Theorem 3.2 for $n > 0$.

Theorem 3.3. *Let* $n > 0$. *Let* $\mathscr{T}$ *be a* $\mathbf{Z}$*-compatible order in a Clifford algebra* $\mathscr{C}_q$. *Then the spectrum* $\mathscr{S}_n(\mathscr{T})$ *is discrete with finite multiplicities.*

Let $0 < \nu(A) \in \mathscr{S}_n(\mathscr{T})$. By Lemma 3.1 we may assume that $A \in \overline{\mathscr{P}}'(\mathscr{T})$ and that f_A is *primitive* (that is an integral form such that the g.c.d. of its values is one). Consider the subset of $A \in \mathscr{P}'(\mathscr{T})$ with $\nu(A) = \nu = \text{const}$. On using (3.15), (3.17), (3.23), and inequality (25) from [23] applied to f_A we obtain

$$\Delta(A) < c(\nu, q) \tag{3.27}$$

where a constant $c(\nu, q)$ depends on ν and q.

For $D \in \mathbf{Z}$ let

$$\mathscr{P}(\mathscr{T}, D) := (A \in \mathscr{P}(\mathscr{T}) : \Delta(A) = D\}. \tag{3.28}$$

Inequality (3.27) reduces Theorem 3.3 to the following.

Lemma 3.4. *Let* $n > 0$. *Suppose that* $\mathscr{T} \subset \mathscr{C}_q$ *is a compatible* $\mathbf{Z}$*-order, and* $0 \neq D \in \mathbf{Z}$. *Then* $\mathscr{P}(\mathscr{T}, D)$ *splits into finitely many orbits with respect to the action of* G.

Proof. Let $f(x)$ be a nondegenerated integral quadratic form in k variables. Let

$$O_k^+(\mathbf{Z}, f) := \{\gamma \in SL_k(\mathbf{Z}) : f(\gamma x) = f(x)\}. \tag{3.29}$$

Let $0 \neq m \in \mathbf{Z}$. It is known (see e.g. [5, Chapter 9, Lemma 6.1]) that the set of solutions of equation $f(x) = m$ splits into finitely many orbits with respect to the action of $O_k^+(\mathbf{Z}, f)$. Thus, for $A = (\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}) \in \mathscr{P}(\mathscr{T}, D)$, the solution set of equation $\Delta(A) = D$ consists of a finite number of $O_{n+3}^+(\mathbf{Z}, \Delta)$-orbits. The $\mathbf{Q}$-rational isomorphism between groups $SV_n(\mathbf{Q}, q)$ and $\mathrm{Spin}_{n+3}(\mathbf{Q}, \Delta)$ established in [8] (see also [10]) and commensurability of the groups $\mathrm{Spin}_{n+3}(\mathbf{Z}, \Delta)$ and $O_{n+3}^+(\mathbf{Z}, \Delta)$ (see [11, p. 423]) show that the solution set of the equation $\Delta(A) = D$ also splits into finitely many orbits with respect to the action of the group G. (Groups Γ and Γ' are said to be *commensurable* if $\Gamma \cap \Gamma'$ is of finite index in each of them.)

Remark. In the case of a negative definite q, another proof of this theorem can be found in [9, p. 262].

3.5. **Explicit results.** In this section we shall find a simple set of representatives of $SV_n(\mathscr{T})$-orbits in $\overline{\mathscr{P}}'(\mathscr{T})$ for some quadratic forms q and $\mathbf{Z}$-orders $\mathscr{T}$ in Clifford algebras $\mathscr{C}_q$ (see Theorem 3.6 and Corollary 3.7 below). The results obtained are similar to those from [24].

For the lattice $\Lambda \subset V$ in (2.12) we denote by

$$\Lambda^{\#} := \{x \in V : (x, \Lambda) \in \mathbf{Z}\} \tag{3.30}$$

the dual lattice of Λ.

Let f_A be the quadratic form associated to $A \in \mathscr{H}'$. We define indefinite quadratic form in $n+2$ variables

$$F_A(X) := f_A(x, x_{n+2}), \qquad (x \in \Lambda,\ x_{n+2} \in \mathbf{Z},\ X \in \mathbf{Z}^{n+2}). \tag{3.31}$$

Lemma 3.5. *Let $A \in \mathscr{P}(\mathscr{T})$. The quadratic form $F_A(X)$ is integral if and only if*

$$a, c \in Z, \qquad b \in \tfrac{1}{2}\Lambda^{\#}. \tag{3.32}$$

Proof. By (3.31), (2.3), and (3.12) we have

$$F_A(X) = ax\overline{x} + 2(x, b)x_{n+2} + cx_{n+2}^2. \quad \square$$

Suppose that $F_A(X)$ is primitive. Let p be a fixed prime number. Let $\{u_0, \dots, u_n\}$ be a basis of Λ and $d_i = Q(u_i)$, $(i = 0, \dots, n)$. Then there is $\sigma \in G$ such that $a' = f_{A\cdot[\sigma]}(1, 0)$ is prime to p. Indeed, if $a \equiv c \equiv 0 \pmod p$ for A then at least one of the integers $f_A(u_i, 1) = ad_i + 2(b, u_i) + c$, $(i = 0, \dots, n)$, is not divisible by p. (Otherwise, all the coefficients of $F_A(X)$ are divisible by p and it is not primitive.) If $f_A(u_k, 1)$ is prime to p we choose

$$\sigma := \begin{pmatrix} u_k & d_k - 1 \\ 1 & \overline{u}_k \end{pmatrix}.$$

Theorem 3.6. *Let $n > 1$. Let $\mathscr{T} \subset \mathscr{C}_q$ be a compatible* **Z**-*order and $\Lambda = \mathscr{T} \cap V$. Suppose that*

(i) *the rank of $Q(x)$, $x \in \Lambda$, over the field $\mathbf{F}_p$ is at least 2 for all prime p except possibly one prime p' and*

(ii) *if p' is odd then $p' \equiv 3 \pmod 4$ and the reduction of $Q(x)$ modulo 8 is a binary form of unit discriminant; or $p' = 2$ and the reduction of $Q(x)$ modulo 8 is either a nondegenerated primitive quadratic form in at least three variables or it is a binary form with determinant not equal to 6 or zero.*

Then any indefinite integral Hermitian form A is $SV_n(\mathscr{T})$-equivalent to a numerical multiple of one of the Hermitian matrices

$$\begin{pmatrix} 1 & b \\ \overline{b} & c \end{pmatrix}, \qquad b \in \left(\frac{1}{2}\Lambda^{\#}\right) \cap P,\ c \in \mathbf{Z}, \tag{3.33}$$

where $\Lambda^{\#}$ is the dual lattice of Λ and P a fundamental parallelogram for Λ.

Proof. Let $A = \left(\begin{smallmatrix} a & b \\ \overline{b} & c \end{smallmatrix}\right)$ be an indefinite integral Hermitian form over $\mathscr{T}$. We may suppose that A is primitive.

As mentioned above, for any fixed prime number p, we may suppose that a is prime to p. If p is not the exclusive prime p' from condition (ii) then the conditions (i) and (ii) imply that both congruences $F_A(X) \equiv \pm 1 \pmod{p^\alpha}$, $(\alpha = 2 + (-1)^p)$, are solvable and therefore both equations $F_A(X) = \pm 1$ are solvable in $\mathbf{Z}_p$ (see [19, p. 15]). It follows from condition (ii) that at least one of these congruences is solvable if $p = p'$ (cf. [22, p. 102]) and therefore at least one of equations $F_A(X) = \pm 1$ is solvable in $\mathbf{Z}_{p'}$. Thus at least one of the equations $F_A(X) = \pm 1$ is solvable over $\mathbf{Z}_p$ for all prime p. Since $F_A(X)$ is an indefinite integral quadratic form in $n+2 > 3$ variables, at least one of $F_A(X) = \pm 1$ is solvable over $\mathbf{Z}$ and $\mu(F_A) = 1$ (see [5, p. 131]).

Assume that $F_A(X) = f_A(x, m) = 1$ where $x \in \Lambda$, $m \in \mathbf{Z}$. Then

$$\sigma := \begin{pmatrix} x & -2(x, b) - cm \\ m & \overline{x} \end{pmatrix} \in SV_n(\mathscr{T}) \tag{3.34}$$

since $\det(\sigma) = f_A(x, m) = 1$. Let $A' := A \cdot [\sigma]$. By Lemma 3.5, for some $v \in \Lambda$, $A' \cdot [(\begin{smallmatrix} 1 & v \\ 0 & 1 \end{smallmatrix})]$ satisfies the theorem.

If $F_A(X) = -1$ is solvable, and second column of σ in (3.34) is multiplied by -1. □

Corollary 3.7. *Let $n > 1$. Let $\mathscr{T} \subset \mathscr{C}_q$ be a compatible* **Z***-order. Suppose that $q(y)$ is a primitive quadratic form, $(y \in M := \mathscr{T} \cap E)$, and $\Lambda = \mathbf{Z} \oplus M$. Then $A \in \mathscr{P}'(\mathscr{T})$ is $SV_n(\mathscr{T})$-equivalent to a numerical multiple of a Hermitian matrix in* (3.33).

Proof. In the case under consideration $Q(x) = x_0^2 - q(y)$, $(x = x_0 + y \in \Lambda$, $x_0 \in \mathbf{Z}$, $y \in M)$, and one can easily verify that the hypotheses of Theorem 3.6 are satisfied. □

Example 3.8. Let $n > 1$. Let $d_k = -1$, $(k = 1, \dots, n)$, in (2.5). Let $\mathscr{T} := \bigoplus_{M \in J_n} \mathbf{Z} e_M$ (cf. [17]). Then $Q(x) = x_0^2 + \cdots + x_n^2$ and $\Lambda^{\#} = \Lambda$. By Corollary 3.7 any indefinite primitive Hermitian matrix in $\mathscr{P}(\mathscr{T})$ is $SV_n(\mathscr{T})$-equivalent to $\pm$ matrix in (3.33) where

$$b = 0, \quad \text{or} \quad \tfrac{1}{2}(e_{i_0} + \cdots + e_{i_k}), \qquad 0 \le i_s \le n, \ (s, k = 0, \dots, n), \tag{3.35}$$

$(e_0 := 1)$. The corresponding values of $\Delta(A) = c - Q(b) < 0$ are

$$\Delta(A) = c, \quad \text{or} \quad c - (k+1)/4, \qquad (k = 0, \dots, n), \ c \in \mathbf{Z}. \tag{3.36}$$

It has been shown in the proof of Theorem 3.6 that $\mu(A) = 1$. Hence by (3.20) the spectrum $\mathscr{S}_n(\mathscr{T})$ coincides with the set $\{(-\Delta(A))^{-1/2}\}$ where $\Delta(A)$ runs through all the negative values in (3.36). Thus for any $n > 1$ we have

$$\mathscr{S}_n(\mathscr{T}) = \{2m^{-1/2}, m \in \mathbf{N}\} \cup \{0\}. \tag{3.37}$$

The fundamental parallelogram P of $\Lambda = \Lambda^{\#}$ is the $(n+1)$-cube $0 \le x_i < 1$, $(i = 0, \dots, n)$. b's in (3.35) are the vertices of $\frac{1}{2}P$. The $(n+1)$-cube possesses the group of fixing the origin symmetries W generated by reflections. Modulo the action of this group there is only one b for each value of k in (3.35) and we can choose the following representatives

$$b = 0, \tfrac{1}{2}, \quad \text{or} \quad \tfrac{1}{2}(1 + e_1 + \cdots + e_k), \qquad (k = 1, \dots, n). \tag{3.38}$$

In the case of $n = 2$, this leads to a 1-1 correspondence between the nonzero points of the spectrum $\mathscr{S}_2(\mathscr{T})$ in (3.37) and the G'-orbits of the elements of $\overline{\mathscr{P}}'(\mathscr{T})$ where G' is the extension of $SV_2(\mathscr{T})$ by W.

Remarks. 1. In the case when Theorem 3.6 is applicable Theorem 3.3 follows from it and Lemma 3.1.

2. Notice that if $Q(x) = x_0^2 + p(x_1^2 + \cdots + x_n^2)$ where $p \equiv 1 \pmod 4$ is a prime number, then there is A in $\mathscr{P}'(\mathscr{T})$ such that $F_A(X) \equiv a x_0^2 \pmod p$ and

$\mu(F_A) > 1$ (cf. [22, p. 105]). Thus the conditions (i) and (ii) in the statement of Theorem 3.6 are essential.

4. G-UNIT GROUPS OF HERMITIAN MATRICES

In [24] the set of maximal nonelementary Fuchsian subgroups of $\mathrm{PSL}_2(\mathfrak{o})$, where $\mathfrak{o}$ is an order in a certain imaginary quadratic field, is identified with the unit groups of binary indefinite Hermitian forms which leads to the classification of the former. Here, using the classification of the binary indefinite integral Hermitian matrices obtained in §3 we generalize this result to the cofinite stabilizers in G of n-spheres in V_{n+1}. The notation established in §§2 and 3 is maintained in this section.

Let

$$q_0(x) := -x_1^2 - \cdots - x_n^2. \tag{4.1}$$

Let $\mathscr{C}_n = \mathscr{C}(\mathbf{R}, q_0)$ be the Clifford algebra over $\mathbf{R}$ associated with q_0. $\mathscr{C}_n$ is generated by $i_1, \dots, i_n$ which satisfy the following relations in $\mathscr{C}_n$:

$$i_k^2 = -1, \qquad i_k i_m = -i_m i_k \qquad (k, m = 1, \dots, n, k \neq m). \tag{4.2}$$

The involutions $'$, *, and $^-$ are defined as in §2. For $x = \sum_M \lambda_M i_M \in \mathscr{C}_n$,

$$|x|^2 := \widehat{Q}(x) = \sum_M \lambda_M^2 \tag{4.3}$$

where the quadratic form $\widehat{Q}$ is defined by (2.2) and $|x|$ denotes the Euclidean norm of x. Let V_{n+1} stand for the set of vectors in $\mathscr{C}_n$. Formula (2.4) yields

$$|x|^2 = x\overline{x} = \overline{x}x \qquad (\forall x \in V_{n+1}). \tag{4.4}$$

The upper half-space

$$H^{n+2} := \{z + t i_{n+1} \colon z \in V_{n+1}, t > 0\} \tag{4.5}$$

is a model of $(n+2)$-dimensional hyperbolic space. Vahlen's group of projective Clifford matrices $PSV_n(\mathscr{C}_n) := SV_n/\{\pm 1\}$ acts on H^{n+1} by

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} x := (\alpha x + \beta)(\gamma x + \delta)^{-1} \tag{4.6}$$

preserving the Poincaré metric. It is isomorphic to the group of orientation-preserving hyperbolic isometries [1, 2, 8, 9].

Let E be an n-dimensional vector space. Let $q \colon E \to \mathbf{Q}$ be negative definite. Let $\mathscr{T}$ be a compatible $\mathbf{Z}$-order in $\mathscr{C}_q$ (see (2.6)). The group $G = SV_n(\mathscr{T})$ acts discontinuously on H^{n+2} and the quotient $G\backslash H^{n+2}$ has finite volume (see [10, §2]).

Let $A \in \mathscr{H}'$ and let f_A be the associated quadratic form. Then

$$f_A(z, 1) = 0, \qquad (z \in V_{n+1}), \tag{4.7}$$

is an equation of a sphere S_A with center $-b/a$ and radius $|\Delta(A)/a^2|^{1/2}$. (A hyperplane in V_{n+1} will be regarded as a sphere with infinite radius.)

For $A \in \mathscr{H}$, we define

$$\mathscr{E}(A) := \{\sigma \in G \colon A \cdot [\sigma] = \pm A\}, \tag{4.8}$$

to be *the group of G-units* of A. Thus

$$\mathrm{Stab}(S_A, G) = \mathscr{E}(A). \tag{4.9}$$

The group $\mathscr{E}(A)$ stabilizes also the hemisphere H_A on S_A in H^{n+2} with equation

$$f_A(z, 1) + at^2 = 0. \tag{4.10}$$

Conversely, an equation of any sphere (or hyperplane) in V_{n+1} can be written in the form (4.7) with some $A \in \mathscr{H}'$. A hemisphere H_A in H^{n+2} is a hyperbolic $(n+1)$-space under the restriction of the hyperbolic metric in H^{n+2}.

Let $s < n+2$. Let Γ be a subgroup of G. We denote by

$$\mathscr{H}_\Gamma := \{A \in \mathscr{H} : A \cdot [\sigma] = \pm A, \ \forall \sigma \in \Gamma\}. \tag{4.11}$$

Thus $\mathscr{H}_\Gamma$ is the solution set of the system of linear homogeneous equations in $n+3$ real variables $a, c \in \mathbf{R}$, $b \in V_{n+1}$. If the dimension of $\mathscr{H}_\Gamma$ equals s, we shall call Γ an $(n-s+1)$-*subgroup* of G Thus G is an $(n+1)$-subgroup of itself. Let $n = 1$. Then $G = SL_2(\mathfrak{o})$ where $\mathfrak{o}$ is an order in an imaginary quadratic number field $\mathbf{Q}(\sqrt{-d})$. A 0-subgroup Γ of G is an elementary Fuchsian subgroup of G. It is either an infinite cyclic group generated by a hyperbolic element or such a group is a subgroup of index two in Γ. A 1-subgroup of G is a nonelementary Fuchsian subgroup in G.

A k-subgroup of G is said to be *maximal* if it is not a subgroup of any other k-subgroup in G.

Lemma 4.1. *Let Γ be an $(n-s+1)$-subgroup of G. Then there are $A_1, \dots, A_s \in \mathscr{P}(\mathscr{T})$ which form a basis of $\mathscr{H}_\Gamma$ in* (4.11). *If Γ is a maximal $(n-s+1)$-subgroup of G, then $\Gamma = \mathscr{E}(A_1) \cap \dots \cap \mathscr{E}(A_s)$.*

Proof. Since the coefficients of an equation $A \cdot [\sigma] = \pm A$ are rational numbers, a basis of the subspace $\mathscr{H}_\Gamma$ in $\mathscr{H}$ in (4.11) can be chosen to belong to $\mathscr{P}(\mathscr{T})$. □

Theorem 4.2. *Let Γ be a maximal k-subgroup of G. Let S be the k-sphere in V_{n+1} fixed by Γ and H_S the $(k+1)$-hemisphere on S in H^{n+2}. Then Γ is finitely generated and the quotient $\Gamma \backslash H_S$ has finite volume.*

Proof. We may assume that q is a diagonal form and a $\mathbf{Z}$-order $\mathscr{T} \subset \mathscr{C}_d$ is the module of integer combinations of $1, e_1, \dots, e_1 \cdot \dots \cdot e_n$ (see [10, §§1–3]).

Let $k = n-s+1$. Let $A_i \in \mathscr{P}(\mathscr{T})$, $(i = 1, \dots, s)$, be the set of Hermitian matrices from Lemma 4.1. Applying the theorem of Borel and Harish-Chandra [3] one can prove Theorem 4.2 following the approach of Maclachlan, Waterman, and Wielenberg [17, p. 744]. For it is sufficient to add to the set of equations considered there the finite number of polynomial equations with integer coefficients $A_i \cdot [\sigma] = \pm A_i$, $(i = 1, \dots, s)$, for the entries of σ in Γ. □

Corollary 4.3. *Let $n > 0$. Let S_A be an n-sphere in V_{n+1} with equation $f_A(z, 1) = 0$ where $A \in \mathscr{H}'$. Then $\mathscr{E}(A) = \mathrm{Stab}(S_A, G)$ is cofinite if and only if $A \in \overline{\mathscr{P}}'(\mathscr{T})$.*

Proof. If $A \in \overline{\mathscr{P}}'(\mathscr{T})$, then by Theorem 4.2 the group $\mathrm{Stab}(S_A, G)$ is cofinite.

Conversely, assume that $\Gamma = \mathrm{Stab}(S_A, G)$ is cofinite. Then the closure of the limit set of Γ on S_A is S_A. Hence, $\Gamma = \mathscr{E}(A)$ is an n-subgroup in G and $A \in \overline{\mathscr{P}}'(\mathscr{T})$ by (4.9) and Lemma 4.1. □

Corollary 4.3 shows that the cofinite stabilizers of n-spheres in V_{n+1} can be identified with the G-unit groups of binary indefinite integral Hermitian matrices. Therefore the results of §3 can be applied to classify the conjugacy classes of these subgroups in G. If $k < n$, then there are k-spheres in V_{n+1} whose stabilizers being cofinite do not contain k-subgroups of G. Indeed, let $n = 1$. Then a 0-sphere S is a pair of points in the complex plane $\mathbf{C}$. Assume that S is fixed by a loxodromic element γ in $G = SL_2(\mathfrak{o})$ where $\mathfrak{o}$ is an order in an imaginary quadratic field $\mathbf{Q}(\sqrt{-d})$. Then $\Gamma = \operatorname{Stab}(S, G)$ is a cofinite subgroup of $\operatorname{Stab}(S, SL_2(\mathbf{C}))$. But no circle through S is fixed by γ. Hence Γ does not contain a 0-subgroup of G.

Let $A \in \mathscr{H}'$ and $\Gamma = \mathscr{E}(A)$. We denote

$$\rho(\Gamma) := |\Delta(A)|^{1/2}/\mu(A) \tag{4.12}$$

where we put $\rho(\Gamma) = \infty$ if $\nu(A) = 0$. By (3.20) and (3.21), $\rho(\Gamma) = 1/\nu(A)$ and $\rho(\sigma\Gamma\sigma^{-1}) = \rho(\Gamma)$ for every σ in G. We shall call

$$R_n(\mathscr{T}) := \{\rho(\Gamma) \colon \Gamma = \mathscr{E}(A),\ A \in \mathscr{P}'(\mathscr{T})\} \tag{4.13}$$

the radius n-spectrum of G. Now the results of §3 can be represented as follows.

Theorem 4.4. *Let $n > 0$. Let $\mathscr{T}$ be a compatible $\mathbf{Z}$-order in Clifford algebra $\mathscr{C}_q$ and G defined by* (2.14).

(i) *If two maximal n-subgroups Γ and Γ' are conjugate in G, then $\rho(\Gamma) = \rho(\Gamma')$.*

(ii) *Suppose that $\rho(\Gamma) < \infty$. Then $\Gamma = \mathscr{E}(A)$ for some $A \in \mathscr{P}'(\mathscr{T})$. Conversely, if $\Gamma = \mathscr{E}(A)$, $A \in \mathscr{P}'(\mathscr{T})$, then $\rho(\Gamma) < \infty$.*

(iii) *The radius n-spectrum of G, $R_n(\mathscr{T})$, is discrete with finite multiplicities.*

(iv) *Let S be an n-sphere in V_{n+1}. Under the hypotheses of Theorem* 3.6, *the group* $\operatorname{Stab}(S, G)$ *is cofinite if and only if it is conjugate in G to $\mathscr{E}(A)$ with A being one of the Hermitian matrices in* (3.33). □

Example 4.5. Let q and $\mathscr{T}$ be as in Example 3.8. Continuing this example we obtain the explicit description of the radius n-spectrum of G $R_n(\mathscr{T})$. By (4.12) and (4.13), $R_n(\mathscr{T}) = \{\sqrt{-\Delta(A)}\}$ where $\Delta(A)$ runs through all the negative values in (3.36). Hence for any $n > 1$

$$R_n(\mathscr{T}) = \{\tfrac{1}{2}\sqrt{m} \colon m \in \mathbf{N}\}.$$

Let $n = 2$. Let the stabilizers Γ and Γ' in $G = SV_2(\mathscr{T})$ of two 2-spheres in V_3 be cofinite and let G' be the extension of G by the group W defined in Example 3.8. Then

$$\Gamma \textit{ and } \Gamma' \textit{ are conjugate in } G' \Leftrightarrow \rho(\Gamma) = \rho(\Gamma').$$

An element $\sigma = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in G$ is said to be *parabolic* if it has one and only one fixed point in V_{n+1} and σ is *strictly parabolic* if it is conjugate to $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. We define the *trace* of σ by $\tau := \alpha + \delta^*$. Then we have the following.

Lemma 4.6 [2, 17]. *If $\sigma = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in G$ is strictly parabolic, then σ fixes $(\alpha - \delta^*)/2\gamma$.*

σ is strictly parabolic if and only if $\tau = \pm 2$ and $\gamma = \gamma^*$.

$\gamma \in V_{n+1}$, and such τ are conjugacy invariant. □

Theorem 4.7. *Let $n > 0$. Suppose that S is an n-sphere in V_{n+1} with equation $f_A(z, 1) = 0$ for some $A \in \mathscr{H}'$.*

Group $\Gamma = \operatorname{Stab}(S, G)$ is cofinite if and only if $\rho(\Gamma) < \infty$.

Let $A \in \mathscr{P}'(\mathscr{T})$. Then $\operatorname{Stab}(S, G)$ is noncocompact if and only if the form $F_A(X)$ defined by (3.31) is isotropic.

Proof. We have to prove only the last assertion. For $A \in \mathscr{P}'(\mathscr{T})$, the quotient $\mathscr{E}(A)\backslash H_A$ has finite volume and, by Proposition 6.3 in [9] (see also [10, §2]), it is compact if and only if $\mathscr{E}(A)$ does not contain strictly parabolic elements. Assume that $\sigma \in \mathscr{E}(A)$ is strictly parabolic. By Lemma 4.6 z, the fixed point of σ, belongs to $S \cap V_q$ (see (2.7)), i.e. $f_A(z, 1) = 0$ where $z = x/x_{n+2}$ with some $x \in \Lambda$ and $x_{n+2} \in \mathbf{Z}$. Then $f_A(x, x_{n+2}) = F_A(X) = 0$, $X = (x, x_{n+2})$, and F_A is isotropic. Conversely, let $f_A(x, m) = 0$, $(x \in \Lambda,\ m \in \mathbf{Z})$. By (3.12) and (2.3)

$$f_A(x, m) = a\overline{x}x + 2m(b, x) + cm^2 = 0 \tag{4.14}$$

which can be written as

$$ux + vm = 0 \tag{4.15}$$

where

$$u := a\overline{x} + m\overline{b}, \qquad v := \overline{x}b + cm. \tag{4.16}$$

For any $s \in \Lambda$ satisfying the condition

$$(s, u) = 0 \tag{4.17}$$

or the equivalent condition

$$\overline{xs}v + \overline{v}sx = \overline{x}(s, u)x = 0, \tag{4.18}$$

$$\sigma_s := \begin{pmatrix} 1 - mxs & xsx \\ -m^2 s & 1 + msx \end{pmatrix} \in \mathscr{E}(A), \tag{4.19}$$

and $\sigma_s(x\ m)^t = (x\ m)^t$. To verify (4.19) one applies relations (4.14)–(4.18). For example, by (4.14) and (4.17), $f_A(1 - mxs, -m^2 s) = a + m^2\overline{s}f(x, m)s - m(s, u) = a$. Since x and s are vectors, $x^* = x$ and $s^* = s$ and the trace of σ_s $\tau = \alpha + \delta^* = 2 - m(xs - x^*s^*) = 2$. By Lemma 4.6 σ_s is strictly parabolic. Thus, for some $\sigma \in \mathscr{E}(A)$, $\mathscr{E}(A)\backslash H_A$ has a cusp $\sigma(x/m)$ and therefore it is not compact. □

Corollary 4.8. *Let $A \in \mathscr{P}'(\mathscr{T})$. If $n > 2$, then $\mathscr{E}(A)$ is noncocompact.*

Proof. The form $F_A(X)$ in Theorem 4.7 is an indefinite integral form in at least five variables. By Meyer's theorem (see e.g. [5]), $F_A(X)$ represents zero rationally. □

Example 4.9. Let $n = 2$. Let q and $\mathscr{T}$ be as in Example 3.8. For A as in (3.33) with b in (3.38), the form $F_A(X)$ is isotropic if and only if $\rho^2(\Gamma) = |\Delta(A)|$ is represented in $\mathbf{Q}$ by the sum of three squares (see (3.13), (3.31)). Hence $F_A(X)$ is isotropic if and only if $\rho^2(\Gamma)$ is of the form $4^t(8s - 1)$ with $s, t \in \mathbf{Z}$, $t \geq -1$ (see e.g. [19, p. 45]). It follows that the G-unit group $\mathscr{E}(A)$ is cocompact if and only if either $b = 0$, $c = 4^t(8s - 1)$ or $b = (1 + e_1 + e_2)/2$, $c = 2r + 1$, where $r, s, t, c \in \mathbf{Z}$, $c \leq 0$.

Remark. Let $S \in V_{n+1}$ be an n-sphere with center c and radius r. Let H_S be a hemisphere in H^{n+2} on S. As mentioned above, H_S is a hyperbolic $(n+1)$-space under the restriction of the hyperbolic metric in H^{n+2}. Group $\Gamma = \mathrm{Stab}(S, G)$ acts discontinuously on H_S. As in [24], one can show that the region in H_S satisfying the inequalities

$$|\mu^* z - \lambda^*|^2 + |\mu|^2 t^2 > 1 \tag{4.20}$$

for all $(\lambda\ \mu)^t = \sigma(1\ 0)^t$, $\sigma \in \Gamma$, is the Dirichlet region $D(e)$ for Γ with center $e = (c, r)$. Notice that $|\mu^* z - \lambda^*| = 1$ is an isometric sphere of σ^{-1} in V_{n+1} [2], hence the $(n+1)$-hemisphere $S(\zeta)$, $(\zeta := \lambda\mu^{-1} \in V_{n+1})$, in H^{n+2}, the boundary in (4.20), with center ζ and radius $|\mu|^{-1}$ is orthogonal to H_S. Similarly a fundamental domain for $\Gamma = \mathrm{Stab}(S, G)$ where S is any k-sphere in V_{n+1}, $0 < k < n$, can be described. Furthermore the region in H^{n+2} satisfying (4.20) for all $(\lambda\ \mu)^t = \sigma(1\ 0)^t$, $\sigma \in G = SV_n(\mathscr{T})$, is the fundamental domain for G (cf. [20]). Since by (4.20) the region $t > 1$ in S belongs to $D(e)$, one can derive the following inequality for the hyperbolic volume of $D(e)$: $\mathrm{vol}(\Gamma\backslash H_S) > v(S^{k-1}) I_k(r)$ where $k = \dim H_S$, $v(S^{k-1})$ is the Euclidean volume of the unit $(k-1)$-sphere, and $I_k(r) = \int_1^r (t^2-1)^{k/2-1}\,dt \geq (r-1)^{k-1}/(k-1)$.

Let now S_A be an n-sphere in V_{n+1} with equation (4.7) where $A \in \mathscr{P}'(\mathscr{T})$. The Dirichlet region $D(e)$ mentioned above has a finite volume and the $(n+1)$-balls $|\mu^* z - \lambda^*| \leq 1$ in V_{n+1} cover S_A. Thus $N_n(D)$, the number of faces of $D(e)$, is larger than the Euclidean volume of S_A divided by the Euclidean volume of the unit n-ball. (The latter is larger than the Euclidean volume of the intersection of S_A with any of the $(n+1)$-balls mentioned above.) Hence

$$N(D) > K\frac{n!!}{(n-1)!!} r^n \tag{4.21}$$

where $K = 2$ or π for n even or odd respectively. Thus, $N_1(D) > \pi r$ and $N_2(D) > 4r^2$. As $n \to \infty$, $K\frac{n!!}{(n-1)!!} \sim \sqrt{2\pi n}$.

Under the hypotheses of Theorem 3.6, r can be replaced by $\rho(\Gamma)$ in (4.21) and $\mathrm{vol}(\Gamma\backslash H_A) > v(S^n)(\rho(\Gamma)-1)^n/n$. In particular, when $n = 1$ or 2, $\mathrm{vol}(\Gamma\backslash H_A) > 2\pi(\rho(\Gamma)-1)^n$.

5. Commensurability classes of hyperbolic 3-orbifolds

In this section we maintain the notation of §4. Two subgroups Γ and Γ' of a group G are *commensurable* if their intersection is of finite index in both Γ and Γ'. They are said to be commensurable *in wide sense* if Γ and $\sigma\Gamma'\sigma^{-1}$ for some $\sigma \in SV_n$ are commensurable. A. M. MacBeath [13] (see also [15]) showed that there are infinitely many wide commensurability classes of cocompact Kleinian groups, and hence of compact hyperbolic 3-manifolds. Let $n = 2$. It follows from Theorem 4.7 that the group G contains infinitely many distinct G-conjugacy classes of cocompact three-dimensional hyperbolic subgroups. Here, applying the results of C. Maclachlan and A. W. Reid [14–17], we shall show that these subgroups are distributed in infinitely many commensurability classes.

Let S_A be the n-sphere in V_{n+1} with finite radius, (i.e. $a \neq 0$), defined by equation (4.7). The interior of S_A,

$$D_A := \{z = x_0 + x_1 i_1 + \cdots + x_n i_n \in V_{n+1} : a f_A(z, 1) < 0\}$$

with the metric $ds = 4|\Delta(A)|(f_A(z, 1))^{-2}|dz|^2$, ($|dz|^2$ is the Euclidean metric on V_{n+1}), represents the disc model of $(n+1)$-dimensional hyperbolic space H^{n+1} (cf. [8, p. 381]). The group $\mathscr{E}(A)\otimes_{\mathbf{Q}}\mathbf{R}$, $A\in\mathscr{P}'(\mathscr{T})$, acts as on the hemisphere model H_A as on the disc model D_A of H^{n+1} as the group of isometries.

Assume that $A\in\mathscr{P}'(\mathscr{T})$. If $t := \left(\begin{smallmatrix}\sqrt{a} & b/\sqrt{a}\\ 0 & 1/\sqrt{a}\end{smallmatrix}\right)$ then $t(S_A)$ is the sphere in V^{n+1} with equation $|z|^2 = |\Delta(A)|$. One can show that $\mathscr{E}(A)$ and $\mathscr{E}(A\cdot[t^{-1}])$ are commensurable subgroups of G (cf. [14, p. 307]). Thus we may confine ourselves to the integral diagonal Hermitian matrices with $a = 1$ and in the sequel we suppose that $A = \left(\begin{smallmatrix}1 & 0\\ 0 & -c\end{smallmatrix}\right)$, $c\in\mathbf{N}$. Then (see [8, 21 or 12]),

$$\mathscr{E}(A) = \left\{\begin{pmatrix}\alpha & c\gamma'\\ \gamma & \alpha'\end{pmatrix}\in G\right\}. \tag{5.1}$$

The relation between the upper half-space model and the disc model is described as follows [8, p. 383].

Lemma 5.1. *For $z\in D_A$, $\pi_0(z) := (z+\sqrt{c}i_n)(\sqrt{c}i_n z + c)^{-1}\in H^{n+1}$. The map π_0 is an isometry between D_A and H^{n+1}. Let $\pi := \frac{1}{\sqrt{2c}}\left(\begin{smallmatrix}1 & \sqrt{c}i_n\\ \sqrt{c}i_n & c\end{smallmatrix}\right)$. If $\sigma\in\mathscr{E}(A)\otimes_{\mathbf{Q}}\mathbf{R}$ then $\pi(\sigma) := \pi\sigma\pi^{-1}\in SV_{n-1}$. The map π is a polynomial isomorphism between $\mathscr{E}(A)\otimes_{\mathbf{Q}}\mathbf{R}$ and SV_{n-1}. π_0 is π-equivariant.*

Now let $n = 2$. Then $SV_{n-1} = SL_2(\mathbf{C})$. We shall show that $\pi(\mathscr{E}(A))$ is an arithmetic Kleinian group which contains a nonelementary Fuchsian subgroup.

Let E be a two-dimensional vector space over $\mathbf{Q}$ with negative definite quadratic form $q\colon E\to\mathbf{Q}$. Let $\mathscr{T}$ be a compatible $\mathbf{Z}$-order in $\mathscr{C}_q$. Since $\mathscr{T}$ contains a suborder of finite index with orthogonal basis, we may assume that the $\mathscr{T}$ itself has an orthogonal basis $\{1, e_1, e_2, e_1e_2\}$ where $e_k^2 = q(e_k) = -d_k$, $d_k\in\mathbf{N}$, $(k = 1, 2)$, $e_1e_2 = -e_2e_1$. We embed the Clifford algebra $\mathscr{C}_q = \mathscr{C}_d$ in (2.6) as subalgebra of $\mathscr{C}$ via the map $e_k\to\sqrt{d_k}i_k$, $(k = 1, 2)$. $\mathscr{C}_d$ can be regarded as the division algebra $K + Ke_2$ where $K := \mathbf{Q}(\sqrt{-d_1})$. The order $\mathscr{T}_0 := \mathfrak{o} + \mathfrak{o}e_2$, where $\mathfrak{o} = \mathbf{Z} + \mathbf{Z}\sqrt{-d_1}$, is a compatible $\mathbf{Z}$-order in $\mathscr{C}_d$. Since $G = SV(\mathscr{T})$ and $G' := SV(\mathscr{T}_0)$ are cofinite subgroups of $SV(\mathscr{C}_d)$, G and G' are commensurable. Thus we may suppose that $\mathscr{T} = \mathfrak{o} + \mathfrak{o}e_2$. Let $\sigma = \left(\begin{smallmatrix}\alpha & c\gamma'\\ \gamma & \alpha'\end{smallmatrix}\right)\in\mathscr{E}(A)$ where $\alpha = (a_1+\omega a_2) + (a_3+\omega a_4)e_2$, $\gamma = (b_1+\omega b_2) + (b_3+\omega b_4)e_2$, $(a_j, b_j\in Z, k = 1, 2, 3, 4; \omega = \sqrt{-d_1})$. Let $\delta := \sqrt{-cd_1d_2}$. Then

$$\pi(\sigma) = \begin{pmatrix} x_1+\omega x_2 & x_3-\omega x_4\\ c(x_3+\omega x_4) & x_1-\omega x_2\end{pmatrix}, \tag{5.2}$$

where

$$x_1 = a_1 + \delta b_4, \qquad x_2 = a_2 + \delta b_3/d_1,$$
$$x_3 = b_1 + \delta a_4/c, \qquad x_4 = b_2 + \delta a_3/(cd_1).$$

Let $\mathscr{O}$ be an order of the quaternion algebra $B := \left(\frac{-d_1, c}{\mathbf{Q}}\right)$ with $\mathscr{O}_1$ denoting the group of elements of $\mathscr{O}$ of reduced norm 1. Then by (5.2) the subgroup of $\pi(\mathscr{E}(A))$ with $x_k\in\mathbf{Z}$, $(k = 1, 2, 3, 4)$, is a nonelementary Fuchsian group which is commensurable with $P\eta(\mathscr{O}_1)$ where η is a representation of B into $M_2(\mathbf{C})$ [14]. Let $\mathscr{O}'$ be an order of the quaternion algebra $\mathscr{A} := B\otimes_{\mathbf{Q}}\mathbf{Q}(\delta)$.

Then by (5.2) the group $\pi(\mathscr{E}(A))$ is commensurable with $P\eta'(\mathscr{O}'_1)$ where $\mathscr{O}'_1$ is the group of elements in $\mathscr{O}'$ of norm 1 and η' a representation of $\mathscr{A}$ into $M_2(\mathbf{C})$. Hence $\pi(\mathscr{E}(A))$ is an arithmetic Kleinian group [15].

Theorem 5.2. *Let E be a 2-dimensional vector space over $\mathbf{Q}$. Let $q\colon E \to \mathbf{Q}$ be negative definite. Let $\mathscr{T}$ be a compatible $\mathbf{Z}$-order in a Clifford algebra $\mathscr{C}_q$. Suppose that A and A' are indefinite integral Hermitian matrices over $\mathscr{T}$. Then the $SV(\mathscr{T})$-unit groups $\mathscr{E}(A)$ and $\mathscr{E}(A')$ are commensurable in the wide sense if and only if $\Delta(A)\Delta(A')$ is a square or, equivalently, $\rho(\mathscr{E}(A)) = \lambda\rho(\mathscr{E}(A'))$ for some $\lambda \in \mathbf{Q}$.*

Proof. As above, let $\mathscr{A} = (\frac{-d_1, c}{\mathbf{Q}(\delta)})$ with basis $\{1, i, j, ij\}$ where

$$i^2 = -d_1, \qquad j^2 = c, \qquad ij = -ji.$$

We define the conjugate-linear involution τ on $\mathscr{A}$ by [16]

$$\tau(a_0 + a_1 + a_2 j + a_3 ij) := \overline{a}_0 - \overline{a}_1 i - \overline{a}_2 j - \overline{a}_3 ij.$$

Let $V_\tau := \{x \in \mathscr{A} : \tau(x) = x\}$. Then V_τ is a four-dimensional $\mathbf{Q}$-vector space with basis $\{1, \delta i, \delta j, \delta ij\}$. The reduced norm of $\mathscr{A}$ restricted to V_τ defines a $(3, 1)$-quadratic form n which, with respect to the above basis, is given by the matrix $\mathrm{Diag}\{1, \delta^2 d_1, -\delta^2 c, -\delta^2 c d_1\}$. Let $\mathscr{A}^1$ denote the elements of $\mathscr{A}$ of norm 1. Then for each $y \in \mathscr{A}^1$ define ϕ_y on V_τ by $\phi_y(x) = yx\tau(y)$. Then $\phi_y(x) \in V_\tau$ and $n(\phi_y(x)) = n(x)$ and we obtain an isomorphism $\Phi\colon \mathscr{A}^1 \to O(V_\tau, n)$ defined by $\Phi(y) = \phi_y$ [16]. The norm n of $\mathscr{A}$ restricted to V_τ is equivalent over $\mathbf{Q}$ to the quadratic form F_A given by $\mathrm{Diag}\{1, d_1, d_2, -c\}$ (see (3.31)).

Let $A_i = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -c_i \end{smallmatrix}\right) \in P'(T)$, $i = 1, 2$. Let $\Gamma_i = \mathscr{E}(A_i)$ and let $\pi_i(\Gamma_i)$ be the corresponding arithmetic Kleinian group. Denote $\delta_i = \sqrt{-c_i d_1 d_2}$. $\mathscr{A}_i = (\frac{-d_1, c_i}{Q(\delta_i)})$ is the quaternion algebra related to $\pi_i(\Gamma_i)$ with the associated involution τ_i, $(i = 1, 2)$. Let V_i be the four-dimensional subspace of $\mathscr{A}_i$ invariant with respect to τ_i. Let $G_i := O(V_i, F_{A_i})$, $(i = 1, 2)$. The quaternion algebras $\mathscr{A}_1$ and $\mathscr{A}_2$ are said to be *isomorphic* if there exists a ring isomorphism $\phi\colon \mathscr{A}_1 \to \mathscr{A}_2$ such that $\phi|_{Z(\mathscr{A}_1)}$ is the identity or the complex conjugate embedding. $\pi_1(\Gamma_1)$ and $\pi_2(\Gamma_2)$ are commensurable if and only if $\mathscr{A}_1$ and $\mathscr{A}_2$ are isomorphic [15] or, equivalently, if and only if G_1 and G_2 are $\mathbf{Q}$-isomorphic [16], hence if and only if $\Delta(A_1)\Delta(A_2)$ is a square. □

Let $A_1, A_2 \in \mathscr{P}'(\mathscr{T})$. Suppose that H_1 and H_2 are the three-hemispheres in H^4 stabilized by $\mathscr{E}(A_1)$ and $\mathscr{E}(A_2)$ respectively. The suborbifolds $\mathscr{E}(A_1)\backslash H_1$ and $\mathscr{E}(A_2)\backslash H_2$ of $SV(\mathscr{T})\backslash H^4$ are said to be *commensurable* if $\mathscr{E}(A_1)$ and $\mathscr{E}(A_2)$ are commensurable. Theorem 4.7 implies the following (cf. [13, 15]).

Corollary 5.3. *Let $n = 2$. Let q be negative definite. Let $\mathscr{T}$ be a compatible $\mathbf{Z}$-order in the Clifford algebra $\mathscr{C}_q$. The quotient $SV(\mathscr{T})\backslash H^4$ contains infinitely many pairwise incommensurable compact hyperbolic three-suborbifolds.* □

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